

Chapter 2

Stochastic Stability of Semi-Markovian Jump Systems

Abstract This chapter is concerned with the stochastic stability for a class of S-MJS with time-variant delays. By the Lyapunov function approach, together with a piecewise analysis method, sufficient conditions are proposed to guarantee the stochastic stability of the S-MJS. As more time-delay information is used, our results are much less conservative than some existing ones in literature. Finally, two examples are given to show the effectiveness and advantages of the proposed techniques.

2.1 Introduction

The presence of time delay is quite common in practical dynamical systems, which is frequently the main cause of instability and poor performance of the systems. As an extension of the delay-partitioning method, a piecewise analysis method (PAM) was considered in the stability analysis of the time-delay systems. The idea of PAM has initially appeared in [1], and then by this approach, some stability criteria for linear time-varying delay systems with less conservativeness were presented.

In this chapter, the stochastic stability problem has investigated for S-MJS. Specifically, the concepts of S-MJS, and mode-dependent time delays are introduced together for the stochastically stable problem in order to reflect a more realistic environment. By Lyapunov function approach, together with PAM, conditions are proposed to ensure the stochastic stability of the underlying S-MJS with time-delays. Finally, numerical examples are given to illustrate the effectiveness of the proposed control scheme.

2.2 System Description and Preliminaries

Consider a semi-Markovian jump time-delay system described by:

$$\begin{cases} \dot{x}(t) = A(\eta_t)x(t) + A_\tau(\eta_t)x(t - \tau_{\eta_t}(t)), \\ x(t) = \phi(t), \quad r(0) = r_0, \quad t \in [-\tau_M, 0], \end{cases} \quad (2.1)$$

where $\{\eta_t, t \geq 0\}$ is a continuous-time Markov process on the probability space which has been defined in Definition 1.18 of Chap. 1, and $x(t) \in \mathbb{R}^n$ is the system state, $A(\eta_t)$ and $A_\tau(\eta_t)$ are matrix functions of the random jumping process $\{\eta_t, t \geq 0\}$. For notation simplicity, when the system operates in the i th mode, $A(\eta_t)$ and $A_\tau(\eta_t)$ are denoted by $A(i)$ and $A_\tau(i)$, respectively. $\phi(t)$ is a vector-valued initial continuous function defined on the interval $[-\tau_M, 0]$, and r_0 is the initial conditions of the continuous state and the mode. $\tau_{\eta_i}(t)$ is the mode-dependent time-varying delay (denoted by $\tau_i(t)$ for simplicity), and it satisfies

$$0 \leq \underline{\tau}_i \leq \tau_i(t) \leq \bar{\tau}_i, \quad \dot{\tau}_i(t) \leq \mu < 1, \quad (2.2)$$

where $\underline{\tau}_i$, $\bar{\tau}_i$ and μ are known scalars.

Remark 2.1 As mentioned in [2], we relax the probability distribution of sojourn-time from exponential distribution to Weibull distribution, so the transition rate in S-MJS will be time-varying instead of constant in MJS. In practice, the transition rate $\pi_{ij}(h)$ is general bounded by $\underline{\pi}_{ij} \leq \pi_{ij}(h) \leq \bar{\pi}_{ij}$, with $\underline{\pi}_{ij}$ and $\bar{\pi}_{ij}$ are real constant scalars. In this case, $\pi_{ij}(h)$ can always be described by $\pi_{ij}(h) = \pi_{ij} + \Delta\pi_{ij}$, where $\pi_{ij} = \frac{1}{2}(\bar{\pi}_{ij} + \underline{\pi}_{ij})$ and $|\Delta\pi_{ij}| \leq \kappa_{ij}$ with $\kappa_{ij} = \frac{1}{2}(\bar{\pi}_{ij} - \underline{\pi}_{ij})$. ♦

Parallel to full MJS, We give the following stochastic stability definition for system (2.1).

Definition 2.2 System (2.1) is said to be stochastically stable if, for any finite $\phi(t)$ defined on $[-\tau_M, 0]$, and $r_0 \in \mathcal{M}$, the following condition is satisfied,

$$\lim_{t \rightarrow \infty} \mathbf{E} \left\{ \int_0^t \|x(s)\|^2 ds | (\phi, r_0) \right\} \leq \infty. \quad (2.3)$$

Remark 2.3 The above definition is consistent with that of stochastic stability of delayed S-MJS. It should be pointed out that Definition 2.2 has similar form with the stochastic stability definition in [3, 4], however, the MJS and the S-MJS are governed by different stochastic processes. The stochastic stability of S-MJS in Definition 2.2 mean that the trajectory of the state $x(t)$ along the semi-Markovian process from the initial state (ϕ, r_0) , in the mean-square sense, asymptotically to the origin. ♦

2.3 Main Results

In this section, we will use the PAM combined with the Lyapunov–Krasovskii function technique to analyze the stochastic stability for system (2.1). To this end, firstly, let $\tau_m = \min\{\underline{\tau}_i, i \in \mathcal{M}\}$ and $\tau_M = \max\{\bar{\tau}_i, i \in \mathcal{M}\}$, then, the variation interval $[\tau_m, \tau_M]$ of time-delay can be divided into l parts with equal length. So $[\tau_m, \tau_M] = \bigcup_{q=0}^{l-1} [\tau_q, \tau_{q+1}]$, where $\tau_q = \tau_m + q(\tau_M - \tau_m)/l$, $q = 0, 1, 2, \dots, l$, and $\delta = \tau_q - \tau_{q-1}$. For a given l , a Lyapunov functional $V(\cdot)$ is constructed, and

then by checking the variation of derivative of $V(\cdot)$ for the case $\tau_i(t) \in [\tau_q, \tau_{q+1}]$, $q = 0, 1, 2, \dots, l-1$, thus some delay-dependent conditions with less conservativeness will be obtained. Our first result in this chapter is as follows.

Theorem 2.4 *For given scalars $\bar{\tau}_i$, $\underline{\tau}_i$ and μ , system (2.1) is stochastically stable if there exist a positive integer l , and matrices $P(i) > 0$, $Q(i) > 0$, $W > 0$, $S_q > 0$ and $R_q > 0$ ($i \in \mathcal{M}$; $q = 1, 2, \dots, l+1$), such that the following inequalities hold:*

$$\begin{bmatrix} \Gamma(i) + \Phi_k & \mathcal{A}(i)M \\ * & -M \end{bmatrix} < 0, \quad k = 0, 1, \dots, l-1, \quad (2.4)$$

$$\sum_{j=1}^N \bar{\pi}_{ij} Q_j - W \leq 0, \quad (2.5)$$

where

$$\begin{aligned} \Gamma(i) &\triangleq \mathcal{A}^T(i)P(i)\mathbb{I}_1 + \mathbb{I}_1^T P(i)\mathcal{A}(i) - \sum_{q=0}^l \mathbb{I}_{3i}^T S_{q+1} \mathbb{I}_{3i} - (1-\mu)\mathbb{I}_2^T Q(i)\mathbb{I}_2 \\ &\quad + \mathbb{I}_1^T \left(Q(i) + \tau_M W + \sum_{q=0}^l S_{q+1} + \sum_{j=1}^N \pi_{ij}(h)P(j) \right) \mathbb{I}_1 - \mathbb{I}_4^T R_1 \mathbb{I}_4, \end{aligned}$$

$$\mathcal{A}(i) \triangleq \begin{bmatrix} A(i) & A_\tau(i) & 0_{m \times (l+1)m} \end{bmatrix}, \quad M \triangleq \tau_m^2 R_1 + \sum_{q=1}^l \delta R_{q+1},$$

$$\Phi_k \triangleq -\frac{1}{\delta} \sum_{\substack{q=1 \\ q \neq k+1}}^l \mathbb{I}_{5i}^T R_{q+1} \mathbb{I}_{5i} - \frac{1}{\delta} \mathbb{I}_{6k}^T R_{k+2} \mathbb{I}_{6k} - \frac{1}{\delta} \mathbb{I}_{7k}^T R_{k+2} \mathbb{I}_{7k}, \quad k = 0, \dots, l-1,$$

$$\mathbb{I}_1 \triangleq \begin{bmatrix} I_m & 0_{m \times (l+2)m} \end{bmatrix}, \quad \mathbb{I}_2 = \begin{bmatrix} 0_m & I_m & 0_{m \times (l+1)m} \end{bmatrix},$$

$$\mathbb{I}_{3q} \triangleq \begin{bmatrix} 0_{m \times (q+2)m} & I_m & 0_{m \times (l-q)m} \end{bmatrix}, \quad q = 0, 1, 2, \dots, l,$$

$$\mathbb{I}_4 \triangleq \begin{bmatrix} I_m & 0_m & -I_m & 0_{m \times lm} \end{bmatrix},$$

$$\mathbb{I}_{5q} \triangleq \begin{bmatrix} 0_{m \times (q+1)m} & I_m & -I_m & 0_{m \times (l-q)m} \end{bmatrix}, \quad q = 1, 2, \dots, l,$$

$$\mathbb{I}_{6k} \triangleq \begin{bmatrix} 0_m & I_m & 0_{m \times km} & -I_m & 0_{m \times (l-k)m} \end{bmatrix}, \quad k = 0, 1, \dots, l-1,$$

$$\mathbb{I}_{7k} \triangleq \begin{bmatrix} 0_m & I_m & 0_{m \times (k+1)m} & -I_m & 0_{m \times (l-k-1)m} \end{bmatrix}, \quad k = 0, 1, \dots, l-1.$$

Proof Let $\mathbb{C}[-\tau_M, 0]$ be the space of continuous functions. Define a process $x(t) \in \mathbb{C}[-\tau_M, 0]$, $t \geq 0$ by $x(t) = x(t + \theta)$, $\theta \in [-\tau_M, 0]$. Then $\{(x(t), r_t), t \geq 0\}$ is a semi-Markovian process with initial state $(\phi(t), r_0)$.

Consider the following Lyapunov–Krasovskii function:

$$V(x(t), \eta_t) \triangleq \sum_{i=1}^6 V_i(x(t), \eta_t), \quad (2.6)$$

where

$$\begin{aligned}
V_1(x(t), \eta_t) &\triangleq x^T(t)P(\eta_t)x(t), \\
V_2(x(t), \eta_t) &\triangleq \int_{-\tau_i(t)}^0 x^T(t+\theta)Q(\eta_t)x(t+\theta)d\theta, \\
V_3(x(t), t) &\triangleq \int_{-\tau_i(t)}^0 \int_{t+\theta}^t x^T(s)Wx(s)dsd\theta, \\
V_4(x(t), t) &\triangleq \sum_{q=0}^l \int_{t-\tau_q}^t x^T(s)S_{q+1}x(s)ds, \\
V_5(x(t), t) &\triangleq \tau_m \int_{t-\tau_m}^t \int_s^t \dot{x}^T(\nu)R_1\dot{x}(\nu)d\nu ds, \\
V_6(x(t), t) &\triangleq \sum_{q=1}^l \int_{t-\tau_q}^{t-\tau_{q-1}} \int_s^t \dot{x}^T(\nu)R_{q+1}\dot{x}(\nu)d\nu ds,
\end{aligned}$$

with $P(i) > 0$, $Q(i) > 0$, $W > 0$, $S_q > 0$ and $R_q > 0$, $i \in \mathcal{M}$; $q = 0, 1, \dots, l$, are real matrices to be determined.

The infinitesimal generator $\mathcal{L}$ can be considered as a derivative of the Lyapunov function $V(x(t), \eta_t)$ along the trajectory of the semi-Markovian process $\{\eta_t, t > 0\}$ at the point $\{x(t), \eta_t\}$ at time t . The MJS and the S-MJS are governed by different stochastic processes, so the infinitesimal generator of the Lyapunov function for the S-MJS is essentially different from the one for general MJS. Firstly, we need to derive the infinitesimal generator $\mathcal{L}$. According to the definition of $\mathcal{L}$, we have

$$\mathcal{L}V(x(t), \eta_t) \triangleq \lim_{\Delta \rightarrow 0} \frac{\mathbf{E}\left\{V(x(t+\Delta), \eta_{t+\Delta})|x(t), \eta_t\right\} - V(x(t), \eta_t)}{\Delta}, \quad (2.7)$$

where Δ is a small positive number. Applying the law of total probability and conditional expectation with $\eta_t = i$, we have

$$\begin{aligned}
\mathcal{L}V_1(x(t), \eta_t) &= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left[\sum_{j=1, j \neq i}^N \Pr(\eta_{t+\Delta} = j | \eta_t = i) x^T(t+\Delta)P(j)x(t+\Delta) \right. \\
&\quad \left. + \Pr(\eta_{t+\Delta} = i | \eta_t = i) x^T(t+\Delta)P(i)x(t+\Delta) - x^T(t)P(i)x(t) \right] \\
&= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left[\sum_{j=1, j \neq i}^N \frac{q_{ij}(G_i(h+\Delta) - G_i(h))}{1 - G_i(h)} x^T(t+\Delta)P(j)x(t+\Delta) \right. \\
&\quad \left. + \frac{1 - G_i(h+\Delta)}{1 - G_i(h)} x^T(t+\Delta)P(i)x(t+\Delta) - x^T(t)P(i)x(t) \right], \quad (2.8)
\end{aligned}$$

where h is the time elapsed when the system stays at mode i from the last jump; $G_i(t)$ is the cumulative distribution function of the sojourn-time when the system remains in mode i , and q_{ij} is the probability intensity of the system jump from mode i to mode j . Given that Δ is small, the first order approximation of $x(t + \Delta)$ is

$$\begin{aligned} x(t + \Delta) &= x(t) + \dot{x}(t)\Delta + o(\Delta) \\ &= [A(i)\Delta + I]x(t) + A_\tau(i)\Delta x(t - \tau_i(t)) + o(\Delta). \end{aligned}$$

Then, (2.8) becomes

$$\begin{aligned} \mathcal{L}V_1(x(t), \eta_t) &= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left\{ \sum_{j=1, j \neq i}^N \frac{q_{ij}(G_i(h + \Delta) - G_i(h))}{1 - G_i(h)} \xi_1^T(t) \begin{bmatrix} A^T(i)\Delta + I \\ A_\tau^T(i)\Delta \end{bmatrix} \right. \\ &\quad \times P(j) \begin{bmatrix} A^T(i)\Delta + I \\ A_\tau^T(i)\Delta \end{bmatrix}^T \xi_1(t) + \frac{1 - G_i(h + \Delta)}{1 - G_i(h)} \xi_1^T(t) \begin{bmatrix} A^T(i)\Delta + I \\ A_\tau^T(i)\Delta \end{bmatrix} \\ &\quad \left. \times P(i) \begin{bmatrix} A^T(i)\Delta + I \\ A_\tau^T(i)\Delta \end{bmatrix}^T \xi_1(t) - x^T(t)P(i)x(t) \right\}. \end{aligned}$$

Using the condition of $\lim_{\Delta \rightarrow 0} \frac{G_i(h + \Delta) - G_i(h)}{1 - G_i(h)} = 0$, we have

$$\mathcal{L}V_1(x(t), \eta_t) = \xi_1^T(t) \begin{bmatrix} A(i) \lim_{\Delta \rightarrow 0} \frac{1 - G_i(h + \Delta)}{1 - G_i(h)} P(i) A(i) \\ * \quad 0 \end{bmatrix} \xi_1(t)$$

where $\xi_1(t) \triangleq [x^T(t) \ x^T(t - \tau_i(t))]^T$, and

$$\begin{aligned} A(i) &\triangleq A^T(i)P(i) + P(i)A(i) + \lim_{\Delta \rightarrow 0} \sum_{j=1, j \neq i}^N \frac{q_{ij}(G_i(h + \Delta) - G_i(h))}{\Delta(1 - G_i(h))} P(j) \\ &\quad + \lim_{\Delta \rightarrow 0} \frac{G_i(h) - G_i(h + \Delta)}{\Delta(1 - G_i(h))} P(i). \end{aligned}$$

By the same techniques used in [5], we have

$$\lim_{\Delta \rightarrow 0} \frac{1 - G_i(h + \Delta)}{1 - G_i(h)} = 1 \quad \text{and} \quad \lim_{\Delta \rightarrow 0} \frac{G_i(h) - G_i(h + \Delta)}{\Delta(1 - G_i(h))} = \pi_i(h),$$

where $\pi_i(h)$ is the transition rate of the system jumping from mode i .

Define

$$\pi_{ij}(h) \triangleq q_{ij}\pi_i(h), i \neq j \quad \text{and} \quad \pi_{ii}(h) \triangleq - \sum_{j=1, j \neq i}^N \pi_{ij}(h),$$

then we obtain

$$\begin{aligned}
\mathcal{L}V_1(x(t), \eta_t) &= x^T(t) \left(\sum_{j=1}^N P(j) \pi_{ij}(h) \right) x(t) + 2x^T(t) P(i) \\
&\quad \times \left[A(i)x(t) + A_\tau(i)x(t - \tau_i(t)) \right] \\
&= 2\xi^T(t) \mathcal{A}^T(i) P(i) \mathbb{I}_1 \xi(t) \\
&\quad + \xi^T(t) \mathbb{I}_1^T \left(\sum_{j=1}^N \pi_{ij}(h) P(j) \right) \mathbb{I}_1 \xi(t), \tag{2.9}
\end{aligned}$$

where $\mathcal{A}(i)$ and $\mathbb{I}_1$ are defined as in Theorem 2.4, and

$$\xi(t) \triangleq \left[x^T(t) \ x^T(t - \tau_i(t)) \ x^T(t - \tau_m) \ \dots \ x^T(t - \tau_{l-1}) \ x^T(t - \tau_M) \right]^T.$$

On the other hand,

$$\begin{aligned}
&\mathbf{E} \{ V_2(x(t + \Delta), \eta_{t+\Delta}) | x(t), \eta_t \} \\
&= \mathbf{E} \left\{ \int_{-\tau_i(t+\Delta)}^0 x^T(t + \Delta + \theta) Q(\eta_{t+\Delta}) x(t + \Delta + \theta) d\theta | x(t), \eta_t \right\} \\
&= \mathbf{E} \left\{ \int_{-\tau_i(t)}^0 x^T(t + \theta) Q(\eta_{t+\Delta}) x(t + \theta) d\theta | x(t), \eta_t \right\} \\
&\quad + \mathbf{E} \left\{ \int_{-\tau_i(t+\Delta)}^0 x^T(t + \Delta + \theta) Q(\eta_{t+\Delta}) x(t + \Delta + \theta) d\theta \right. \\
&\quad \left. - \int_{-\tau_i(t)}^0 x^T(t + \theta) Q(\eta_{t+\Delta}) x(t + \theta) d\theta | x(t), \eta_t \right\} \\
&= \sum_{j=1, j \neq i}^N \Pr(\eta_{t+\Delta} = j | \eta_t = i) \int_{-\tau_i(t)}^0 x^T(t + \theta) Q(j) x(t + \theta) d\theta \\
&\quad + \Pr(\eta_{t+\Delta} = i | \eta_t = i) \int_{-\tau_i(t)}^0 x^T(t + \theta) Q(i) x(t + \theta) d\theta \\
&\quad + \mathbf{E} \left\{ \int_{\Delta - \tau_i(t+\Delta)}^{\Delta} x^T(t + \theta) Q(\eta_{t+\Delta}) x(t + \theta) d\theta \right. \\
&\quad \left. - \int_{-\tau_i(t)}^0 x^T(t + \theta) Q(\eta_{t+\Delta}) x(t + \theta) d\theta | x(t), \eta_t \right\} \\
&= \sum_{j=1, j \neq i}^N \frac{q_{ij}(G_i(h + \Delta) - G_i(h))}{1 - G_i(h)} \int_{-\tau_i(t)}^0 x^T(t + \theta) Q(j) x(t + \theta) d\theta \\
&\quad + \frac{1 - G_i(h + \Delta)}{1 - G_i(h)} \int_{-\tau_i(t)}^0 x^T(t + \theta) Q(i) x(t + \theta) d\theta
\end{aligned}$$

$$\begin{aligned}
& + \mathbf{E} \left\{ \int_0^\Delta x^T(t+\theta) Q(\eta_{t+\Delta}) x(t+\theta) d\theta \right. \\
& \quad \left. - \int_{-\tau_i(t)}^{\Delta - (\tau_i(t) + \dot{\tau}_i \Delta + o(\Delta))} x^T(t+\theta) Q(\eta_{t+\Delta}) x(t+\theta) d\theta | x(t), \eta_t \right\} \\
& = \sum_{j=1, j \neq i}^N \frac{q_{ij}(G_i(h+\Delta) - G_i(h))}{1 - G_i(h)} \int_{-\tau_i(t)}^0 x^T(t+\theta) Q(j) x(t+\theta) d\theta \\
& \quad + \frac{1 - G_i(h+\Delta)}{1 - G_i(h)} \int_{-\tau_i(t)}^0 x^T(t+\theta) Q(i) x(t+\theta) d\theta + \Delta x^T(t) Q(i) x(t) \\
& \quad - \Delta(1 - \dot{\tau}_i(t)) x^T(t - \tau_i(t)) Q(i) x(t - \tau_i(t)) + o(\Delta).
\end{aligned}$$

The above, together with (2.7), yields

$$\begin{aligned}
\mathcal{L}V_2(x(t), \eta_t) & = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left\{ \mathbf{E} \{ V_2(x(t+\Delta), \eta_{t+\Delta}) | x(t), \eta_t \} - V_2(x(t), \eta_t) \right\} \\
& = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left\{ \sum_{j=1, j \neq i}^N \frac{q_{ij}(G_i(h+\Delta) - G_i(h))}{1 - G_i(h)} \int_{-\tau_i(t)}^0 x^T(t+\theta) Q(j) x(t+\theta) d\theta \right. \\
& \quad + \frac{1 - G_i(h+\Delta)}{1 - G_i(h)} \int_{-\tau_i(t)}^0 x^T(t+\theta) Q(i) x(t+\theta) d\theta + \Delta x^T(t) Q(i) x(t) \\
& \quad \left. + o(\Delta) - V_2(x(t), \eta_t) - \Delta(1 - \dot{\tau}_i(t)) x^T(t - \tau_i(t)) Q(i) x(t - \tau_i(t)) \right\}.
\end{aligned}$$

Then, similar to the development of (2.9) it follows

$$\begin{aligned}
\mathcal{L}V_2(x(t), \eta_t) & = \sum_{j=1}^N \pi_{ij}(h) \int_{-\tau_i(t)}^0 x^T(t+\theta) Q(j) x(t+\theta) d\theta \\
& \quad - (1 - \dot{\tau}_i(t)) x^T(t - \tau_i(t)) Q(i) x(t - \tau_i(t)) + x^T(t) Q(i) x(t) \\
& \leq \xi^T(t) \mathbb{I}_1^T Q(i) \mathbb{I}_1 \xi(t) - (1 - \mu) \xi^T(t) \mathbb{I}_2^T Q(i) \mathbb{I}_2 \xi(t) \\
& \quad + \sum_{j=1}^N \pi_{ij}(h) \int_{-\tau_i(t)}^0 x^T(t+\theta) Q(j) x(t+\theta) d\theta. \tag{2.10}
\end{aligned}$$

Moreover, from (2.7), it can be shown that

$$\begin{aligned}
\mathcal{L} \left(\sum_{i=3}^6 V_i(x(t), \eta_t) \right) & = \xi^T(t) \mathcal{A}^T(i) M \mathcal{A}(i) \xi(t) - \int_{t-\tau_i(t)}^t x^T(\theta) W x(\theta) d\theta \\
& \quad + \xi^T(t) \mathbb{I}_1^T \left(\tau_M W + \sum_{q=0}^l S_{i+1} \right) \mathbb{I}_1 \xi(t)
\end{aligned}$$

$$\begin{aligned}
& - \sum_{q=0}^l \xi^T(t) \mathbb{I}_{3i}^T S_{q+1} \mathbb{I}_{3i} \xi(t) - \tau_m \int_{t-\tau_m}^t \dot{x}^T(s) R_1 \dot{x}(s) ds \\
& - \sum_{q=1}^l \int_{t-\tau_q}^{t-\tau_{q-1}} \dot{x}^T(s) R_{q+1} \dot{x}(s) ds. \tag{2.11}
\end{aligned}$$

It follows by Jensen's inequality that

$$- \tau_m \int_{t-\tau_m}^t \dot{x}^T(s) R_1 \dot{x}(s) ds \leq -\xi^T(t) \mathbb{I}_4^T R_1 \mathbb{I}_4 \xi(t), \tag{2.12}$$

$$- \sum_{\substack{q=1 \\ q \neq k+1}}^l \int_{t-\tau_q}^{t-\tau_{q-1}} \dot{x}^T(s) R_{q+1} \dot{x}(s) ds \leq -\frac{1}{\delta} \sum_{\substack{q=1 \\ q \neq k+1}}^l \xi^T(t) \mathbb{I}_{5i}^T R_{q+1} \mathbb{I}_{5i} \xi(t). \tag{2.13}$$

Next, by checking $\tau_i(t) \in [\tau_k, \tau_{k+1}]$ for some positive integer k with $0 \leq k \leq l-1$, one has

$$\begin{aligned}
& - \int_{t-\tau_{k+1}}^{t-\tau_k} \dot{x}^T(s) R_{k+2} \dot{x}(s) ds \\
& = - \int_{t-\tau_i(t)}^{t-\tau_k} \dot{x}^T(s) R_{k+2} \dot{x}(s) ds - \int_{t-\tau_{k+1}}^{t-\tau_i(t)} \dot{x}^T(s) R_{k+2} \dot{x}(s) ds \\
& \leq -\frac{1}{\delta} \xi^T(t) \mathbb{I}_{6k}^T R_{k+2} \mathbb{I}_{6k} \xi(t) - \frac{1}{\delta} \xi^T(t) \mathbb{I}_{7k}^T R_{k+2} \mathbb{I}_{7k} \xi(t). \tag{2.14}
\end{aligned}$$

By combining (2.9)–(2.14), we have

$$\begin{aligned}
\mathcal{L}V(x(t), \eta_t) & = \xi^T(t) \left(\Gamma(i) + \Phi_k + \mathcal{A}^T(i) M \mathcal{A}(i) \right) \xi(t) \\
& \quad - \int_{t-\tau_i(t)}^t x^T(\theta) W x(\theta) d\theta \\
& \quad + \sum_{j=1}^N \pi_{ij}(h) \int_{t-\tau_i(t)}^t x^T(\theta) Q(j) x(\theta) d\theta,
\end{aligned}$$

where $\xi(t)$, $\Gamma(i)$, Φ_k , and $\mathcal{A}(i)$ are defined as before. From (2.4)–(2.5) and the Schur complement, it can be seen that $\mathcal{L}V(x(t), \eta_t) < 0$ for any $i \in \mathcal{M}$.

Next, we set

$$\Theta_i \triangleq \Gamma(i) + \Phi_k + \mathcal{A}^T(i) M \mathcal{A}(i),$$

and

$$\lambda_1 \triangleq \min_{i \in \mathcal{M}} \{\lambda_{\min}(-\Theta_i)\},$$

then for any $t \geq \tau$ we have

$$\mathcal{L}V(x(t), \eta_t) \leq -\lambda_1 \|\xi(t)\|^2 \leq -\lambda_1 \|x(t)\|^2.$$

By Dynkin's formula, we have

$$\mathbf{E}\{V(x(t), \eta_t) | \phi, r_0\} \leq \mathbf{E}\{V(\phi, r_0)\} - \lambda_1 \mathbf{E}\left\{\int_0^t \|x(s)\|^2 ds\right\}. \quad (2.15)$$

So, from (2.6), it follows that for any $t \geq 0$

$$\mathbf{E}\{V(x(t), \eta_t)\} \geq \lambda_2 \mathbf{E}\{\|x(t)\|^2\}, \quad (2.16)$$

where $\lambda_2 = \min_{i \in \mathcal{M}} \{\lambda_{\min}(-P(i))\} > 0$. From (2.15) and (2.16), we have

$$\mathbf{E}\{\|x(t)\|^2\} \leq \varepsilon_2 V(x(t), \eta_t) - \varepsilon_1 \mathbf{E}\left\{\int_0^t \|x(s)\|^2 ds\right\},$$

with $\varepsilon_1 = \lambda_1 \lambda_2^{-1}$ and $\varepsilon_2 = \lambda_2^{-1}$. By Gronwell–Bellman lemma, one has

$$\mathbf{E}\{\|x(t)\|^2\} \leq \varepsilon_2 V(x(t), \eta_t) e^{-\varepsilon_1 t}, \quad (2.17)$$

then

$$\mathbf{E}\left\{\int_0^t \|x(s)\|^2 ds | \phi, r_0\right\} \leq -\varepsilon_1^{-1} \varepsilon_2 V(x(t), \eta_t) (e^{-\varepsilon_1 t} - 1).$$

Let $t \rightarrow \infty$, we have

$$\lim_{t \rightarrow \infty} \mathbf{E}\left\{\int_0^t \|x(s)\|^2 ds | \phi, r_0\right\} \leq \varepsilon_1^{-1} \varepsilon_2 V(x(t), \eta_t).$$

Then there always exists a scalar $c > 0$, such that

$$\lim_{t \rightarrow \infty} \mathbf{E}\left\{\int_0^t \|x(s)\|^2 ds | \phi, r_0\right\} \leq c \sup_{-\tau \leq s \leq 0} \|\phi(s)\|^2,$$

which is the desired result, by Definition 2.2, and the proof is completed. ■

Remark 2.5 Based on the idea of PAM and the piecewise Lyapunov function technique, a new delay-dependent sufficient condition is proposed in Theorem 2.4 for the stochastic stability of S-MJS (2.1). Due to the instrumental idea of PAM, the proposed stability condition is much less conservative than the existing results. Moreover, the conservatism reduction becomes more obvious with the partitioning getting thinner (i.e., l becoming bigger). In addition, from the proof of Theorem 2.4, we can see that the conditions to guarantee $\mathcal{L}V(x(t), \eta_t) < 0$ are derived by checking the delay variation in the sub-intervals. So, the PAM for stability analysis in this chapter, actually, is quite different from the delay partitioning method used in [6, 7]. Compared with the technique of dividing the lower bounded of the time-delay into m parts with equal length in [6, 7], in our work, the variation interval of time-delay has divided into l parts with equal length. So the PAM can be seen as an extension of the delay-partitioning method, which may further reduce the conservativeness (see Example 2.7). ♦

Due to the time-varying term $\pi_{ij}(h)$, it is difficult to solve inequalities (2.4), since they contain infinite number of inequalities, which limits the use of Theorem 2.4 in practice. To overcome this shortcoming, we present our next result which reduces infinite number of inequalities in Theorem 2.4 to finitely many ones, which will be simple to solve.

Theorem 2.6 For given scalars $\bar{\tau}_i$, $\underline{\tau}_i$ and μ , system (2.1) is stochastically stable if there exists a positive integer l , and matrices $P(i) > 0$, $Q(i) > 0$, $Q > 0$, $S_q > 0$, $R_q > 0$, and T_{ij} , ($i, j \in \mathcal{M}$; $q = 1, 2, \dots, l+1$), such that (2.5) and the following inequalities hold:

$$\begin{bmatrix} \tilde{F}(i) + \Phi_k & \mathcal{A}(i)M & \mathcal{M}_i \\ * & -M & 0 \\ * & * & -\Lambda_i \end{bmatrix} < 0, \quad k = 0, 1, \dots, l-1, \quad (2.18)$$

where

$$\begin{aligned} \tilde{F}(i) &\triangleq \mathbb{I}_1^T \left(Q(i) + \tau_M W + \sum_{i=0}^l S_{i+1} + \sum_{j=1}^N \pi_{ij} P(j) + \sum_{j=1, j \neq i}^N \frac{\kappa_{ij}^2}{4} T_{ij} \right) \mathbb{I}_1 \\ &\quad + \mathcal{A}^T(i) P(i) \mathbb{I}_1 + \mathbb{I}_1^T P(i) \mathcal{A}(i) - \sum_{i=0}^l \mathbb{I}_{3i}^T S_{i+1} \mathbb{I}_{3i} \\ &\quad - \mathbb{I}_4^T R_1 \mathbb{I}_4 - (1 - \mu) \mathbb{I}_2^T Q(i) \mathbb{I}_2, \\ \mathcal{M}_i &\triangleq [P_i - P_1 \dots P_i - P_{i-1} \quad P_i - P_{i+1} \dots P_i - P_s], \\ \Lambda_i &\triangleq \text{diag}\{T_{i1}, \dots, T_{i(i-1)}, T_{i(i+1)}, \dots, T_{is}\}, \end{aligned}$$

and Φ_k , $\mathcal{A}(i)$, $\mathbb{I}_1$, $\mathbb{I}_2$, $\mathbb{I}_{3i}$, and $\mathbb{I}_4$ are defined as in Theorem 2.4.

Proof According to Theorem 2.4, system (2.1) is stochastically stable if the following inequalities hold for all $i \in \mathcal{M}$,

$$\begin{bmatrix} \Omega_i + \mathbb{I}_1^T \left(\sum_{j=1}^N (\pi_{ij} + \Delta\pi_{ij}) P_j \right) \mathbb{I}_1 & \mathcal{A}(i)M \\ * & -M \end{bmatrix} < 0, \quad (2.19)$$

where Ω_i is defined as below,

$$\begin{aligned} \Omega_i &\triangleq \mathcal{A}^T(i)P(i)\mathbb{I}_1 + \mathbb{I}_1^T P(i)\mathcal{A}(i) + \mathbb{I}_1^T \left(Q(i) + \tau_M W + \sum_{i=0}^l S_{i+1} \right) \mathbb{I}_1 \\ &\quad - \sum_{i=0}^l \mathbb{I}_{3i}^T S_{i+1} \mathbb{I}_{3i} - \mathbb{I}_4^T R_1 \mathbb{I}_4 - (1 - \mu) \mathbb{I}_2^T Q(i) \mathbb{I}_2 + \Phi_k, \end{aligned}$$

while (2.19) can be further rewritten as

$$\begin{bmatrix} \Psi(i) & \mathcal{A}(i)M \\ * & -M \end{bmatrix} < 0.$$

where

$$\begin{aligned} \Psi(i) &\triangleq \Omega_i + \mathbb{I}_1^T \left\{ \sum_{j=1}^N \pi_{ij} P_j + \sum_{j=1, j \neq i}^N \left[\frac{1}{2} \Delta\pi_{ij} (P(j) - P(i)) \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \Delta\pi_{ij} (P(j) - P(i)) \right] \right\} \mathbb{I}_1. \end{aligned}$$

From Lemma 1.27, the above inequality holds for all $|\Delta\pi_{ij}| \leq \kappa_{ij}$, if there exist matrices T_{ij} , $j \in \mathcal{M}$, such that

$$\begin{bmatrix} \hat{\Psi}(i) & \mathcal{A}(i)M \\ * & -M \end{bmatrix} < 0, \quad (2.20)$$

with

$$\begin{aligned} \hat{\Psi}(i) &\triangleq \Omega_i + \mathbb{I}_1^T \left\{ \sum_{j=1}^N \pi_{ij} P_j + \sum_{j=1, j \neq i}^N \left[\frac{\kappa_{ij}^2}{4} T_{ij} \right. \right. \\ &\quad \left. \left. + (P(j) - P(i)) T_{ij}^{-1} (P(j) - P(i)) \right] \right\} \mathbb{I}_1. \end{aligned}$$

In view of Schur complement lemma, it can be seen that (2.20) is equivalent to inequality (2.18). Then, by this fact together with Theorem 2.4, we can conclude that system (2.1) is stochastically stable. The proof is completed. $\blacksquare$

2.4 Illustrative Example

In this section, two examples will be presented to demonstrate the effectiveness and superiority of the methods developed previously.

Example 2.7 Consider S-MJS in (2.1) with two modes and the following parameters:

$$A(1) = \begin{bmatrix} -2.0 & 0.0 \\ 0.0 & -0.9 \end{bmatrix}, \quad A_\tau(1) = \begin{bmatrix} -1.0 & 0.0 \\ -1.0 & -1.0 \end{bmatrix},$$

$$A(2) = \begin{bmatrix} -1.0 & 0.5 \\ 0.0 & -1.0 \end{bmatrix}, \quad A_\tau(2) = \begin{bmatrix} -1.0 & 0.0 \\ -0.1 & -1.0 \end{bmatrix}.$$

In this example, we choose $\tau_m = 1$, $\mu = 0.5$, $\pi_{11} = -0.1$ and $\pi_{22} = -0.8$. The maximum allowed τ_M , obtained by the criteria in [6, 8–10] and Theorem 2.4 in this chapter, is shown in Table 2.1.

From Table I, it is obvious that, when the delay partitioning number l becomes larger, the conservatism of the result is further reduced, while the computational cost increases. This is reasonable since l is related to the decision variables. Thus, the larger l indicates that the solution can be searched in a wider space and a larger maximum allowable delay τ_M can be obtained. Table I shows that for no partitioning ($l = 1$), the new criterion given in Theorem 2.4 has the same conservatism with the previous result in [10] and [6] ($m = 1$). Moreover, for $l > 1$, the conservatism reduction by the PAM proved to be quite effective. Moreover, we can see that even with the case $l = 3$ our results still outperform those in [6] with $m = 5$.

Table 2.1 Maximum upper delay bound τ_M with $\tau_m = 1$ and $\mu = 0.5$

Different results	Maximum allowed τ_M
Theorem 3.1 of [8]	0.2237
Theorem 1 of [9]	1.4701
Theorem 1 of [10]	1.6602
Theorem 1 of [6] ($m = 1$)	1.6600
Theorem 1 of [6] ($m = 2$)	1.6902
Theorem 1 of [6] ($m = 3$)	1.7318
Theorem 1 of [6] ($m = 4$)	1.7456
Theorem 1 of [6] ($m = 5$)	1.7527
Theorem 1 ($l = 1$)	1.6602
Theorem 1 ($l = 2$)	1.7344
Theorem 1 ($l = 3$)	1.8023
Theorem 1 ($l = 4$)	1.8072

Table 2.2 Maximum upper delay bound τ_M with different μ

Theorem 2.6	$\mu = 0.1$	$\mu = 0.2$	$\mu = 0.5$	$\mu = 0.9$
$l = 3$	$\tau_M = 5.405$	$\tau_M = 4.001$	$\tau_M = 2.068$	$\tau_M = 1.437$
$l = 4$	$\tau_M = 5.476$	$\tau_M = 4.072$	$\tau_M = 2.102$	$\tau_M = 1.500$
$l = 5$	$\tau_M = 5.512$	$\tau_M = 4.101$	$\tau_M = 2.125$	$\tau_M = 1.542$
$l = 6$	$\tau_M = 5.541$	$\tau_M = 4.112$	$\tau_M = 2.161$	$\tau_M = 1.568$

Example 2.8 Consider S-MJS in (2.1) with two modes and the following parameters:

$$A(1) = \begin{bmatrix} -1.0 & 0.4 \\ 0.0 & -0.8 \end{bmatrix}, \quad A_\tau(1) = \begin{bmatrix} -0.6 & 0.0 \\ -1.0 & -1.0 \end{bmatrix},$$

$$A(2) = \begin{bmatrix} -1.1 & 1.5 \\ 0.0 & -1.2 \end{bmatrix}, \quad A_\tau(2) = \begin{bmatrix} -1.1 & 0.0 \\ -0.1 & -0.9 \end{bmatrix}.$$

In this example, the transition rates in the model are $\pi_{11}(h) \in (-2.2, -1.8)$, $\pi_{12}(h) \in (1.8, 2.2)$, $\pi_{21}(h) \in (2.6, 3.4)$ and $\pi_{22}(h) \in (-3.4, -2.6)$, so we can always set $\pi_{11} = -2$, $\pi_{12} = 2$, $\pi_{21} = -3$, $\pi_{22} = 3$ and $\kappa_{1j} = 0.2$, $\kappa_{2j} = 0.4$ for the stochastic stability analysis. Our purpose is to find the allowable maximum time delay τ_M such that the delayed S-MJS is stochastic stable. The obtained allowable maximum delays by testing the conditions in Theorem 2.6 for various partitioning number l are shown in Table 2.2. It can be seen from Theorem 2.6 that the maximum allowable delay is becoming larger as the partitioning becomes thinner. Moreover, it should be pointed out that although conservatism is reduced as the partitioning becomes thinner, and there is no significant improvement after $l = 5$.

When $\tau_m = 1$, $\mu = 0.5$ and $l = 3$, we can obtain feasible solutions to the inequalities in Theorem 2.6 as follows:

$$P(1) = \begin{bmatrix} 0.65 & -1.8 \\ -0.28 & 1.37 \end{bmatrix}, \quad P(2) = \begin{bmatrix} 0.49 & -1.23 \\ -1.13 & 1.29 \end{bmatrix}, \quad Q(1) = \begin{bmatrix} 1.52 & -0.25 \\ 1.15 & 0.07 \end{bmatrix},$$

$$Q(2) = \begin{bmatrix} 0.42 & -0.25 \\ -0.15 & 1.06 \end{bmatrix}, \quad W = \begin{bmatrix} 0.02 & -0.02 \\ -0.02 & 0.02 \end{bmatrix}, \quad T_{12} = \begin{bmatrix} 0.27 & -1.34 \\ -0.74 & 1.35 \end{bmatrix},$$

$$T_{21} = \begin{bmatrix} 1.88 & -0.42 \\ 1.32 & 0.23 \end{bmatrix}, \quad S_1 = \begin{bmatrix} -1.48 & 0.16 \\ 0.36 & -0.46 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 1.24 & 0 \\ 0 & 1.24 \end{bmatrix},$$

$$S_3 = \begin{bmatrix} 0.20 & -0.02 \\ -0.02 & 0.60 \end{bmatrix}, \quad S_4 = \begin{bmatrix} 0.50 & 0 \\ 0 & 0.05 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 0.35 & 0.04 \\ 0.04 & 0.18 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} 0.29 & 0.05 \\ 0.05 & 0.47 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 0.46 & 0.05 \\ 0.05 & 0.17 \end{bmatrix}, \quad R_4 = \begin{bmatrix} 0.53 & 0.02 \\ 0.02 & 0.33 \end{bmatrix},$$

and we can obtain the maximum allowed time-delay $\tau_M = 2.18$. When setting the initial condition as $\phi(0) = 0$, $r_0 = 1$, the simulation result is shown in Figs. 2.1 and 2.2, which illustrates that the S-MJS with time-varying transition rates is stable.

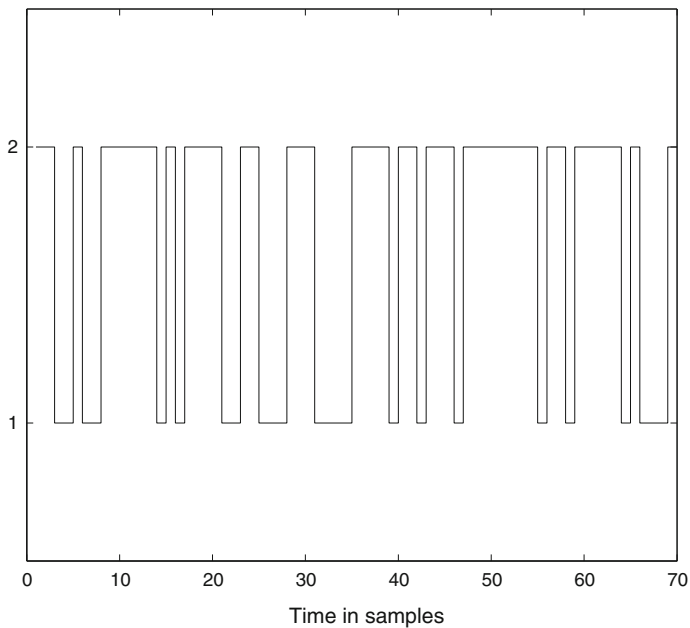


Fig. 2.1 Switched signal of the system in Example 2.8

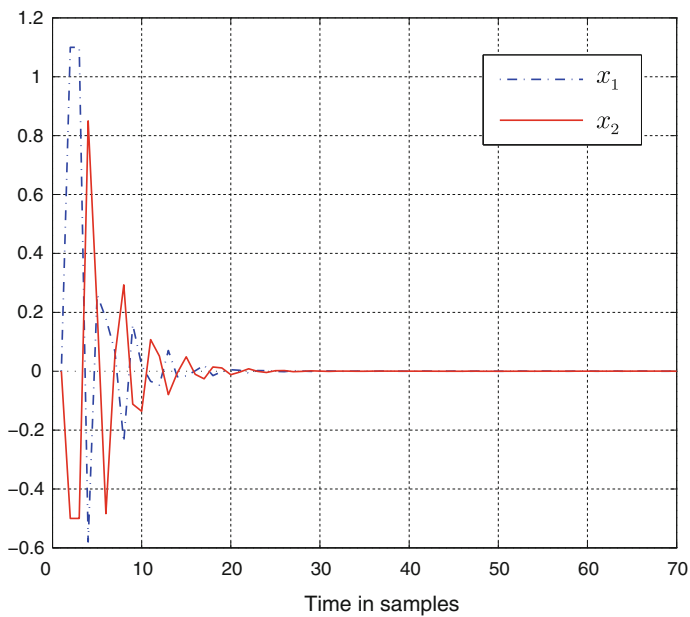


Fig. 2.2 States of the system in Example 2.8

2.5 Conclusion

In this chapter, the problem of stochastic stability for a class of linear S-MJS with time-delays has been addressed. Inspired by piecewise analysis method, a new stochastic stability condition has been established for the S-MJS, which is less conservative than some existing results. Moreover, by linear matrix inequality approach, a criterion has been presented to reduced to infinite number of inequalities to finitely many ones which are practically solvable. Two numerical examples have been given to demonstrate the potential of the new design method.

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