Chapter 2
Equations of Motion of an Aircraft

2.1 Nomenclature

In this chapter, we derive and analyze the equations of motion of an aircraft and, to meet this goal, hereafter we establish the necessary notation. Specifically, in this section we define inertia and body reference frames, we describe the forces and the moment of the forces acting on an airplane, and finally we present the aircraft state and control vectors. Figure 2.1 provides a pictorial representation of the variables characterizing the dynamics of an aircraft.

2.1.1 Body and Inertial Reference Frames

As discussed in Chap. 1, the notions of displacement, velocity, and acceleration concern the relative motion of two rigid bodies or, equivalently, two reference frames. The concept of inertial reference frame, illustrated in Sect. 1.7, responds to the need of identifying an *ideal* coordinate system, which is not accelerated with respect to any other reference frame.

In the following, we consider any reference frame $\mathbb{I} = \{O; X, Y, Z\}$, where O, X, Y, and Z are fixed with the Earth, as an inertial reference frame. Since our planet spins about its rotation axis and orbits around the sun, it follows from Remark 1.9 that a reference frame fixed with the Earth is *not* inertial. However, this assumption is sufficiently realistic for a wide range of aeronautical applications, such as aircraft flying at Mach number 3 or less and at an altitude lower than 10 km; the case whereby the Earth is modeled as a spinning sphere is discussed in [12, Chap. 5].

The direction of the gravitational acceleration provides a privileged direction to orientate the axes of the inertial reference frame $\mathbb{I}$. In this brief, we assume that the axis Z is aligned with the gravitational force, so that the gravitational force acting on the center of mass of an aircraft is expressed as

A. L'Afflitto, *A Mathematical Perspective on Flight Dynamics and Control*,
SpringerBriefs in Applied Sciences and Technology,
DOI 10.1007/978-3-319-47467-0_2

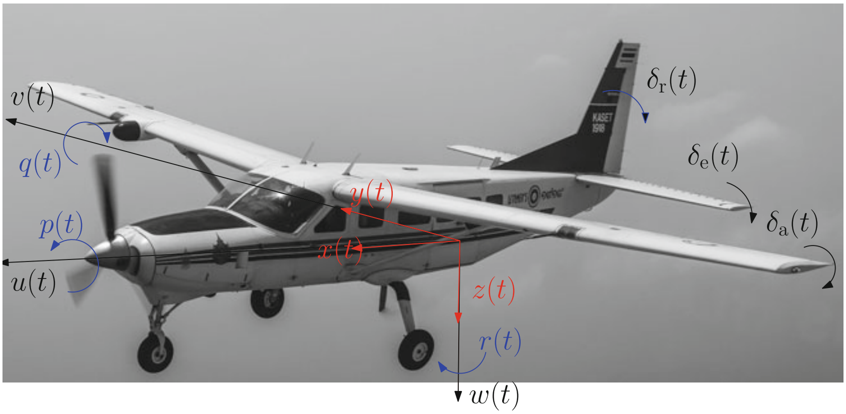

Fig. 2.1 Aircraft state and control variables

$$F_g \triangleq mgZ, \tag{2.1}$$

where g denotes the gravitational acceleration and m denotes the mass of the aircraft, which we assume constant.

The position of the aircraft center of mass in the inertial reference frame $\mathbb{I}$ is denoted by

$$r_c(t) = [x_c(t), y_c(t), z_c(t)]^T, \qquad t \geq 0, \tag{2.2}$$

and we denote an orthonormal reference frame fixed with the aircraft by $\mathbb{J} = \{r_c(\cdot); x(\cdot), y(\cdot), z(\cdot)\}$, where $x, y, z : [0, \infty) \to \mathbb{R}^3$. In this brief, we model aircraft as symmetric rigid bodies and hence $r_c(t)$ is constant if the aircraft is fixed in $\mathbb{I}$. Since aircraft have several moving parts, such as the propellers, the ailerons, the elevators, and the rudder, modeling an aircraft as a rigid body implies assuming that the movement of the propellers and the deflection of the aircraft control surfaces do not substantially alter the inertia properties of the vehicle. The case of time-varying inertia properties is discussed in Chap. 5 of [12].

The choice of the axes $\{x(\cdot), y(\cdot), z(\cdot)\}$ strongly depends on the problem considered. For example, it may be convenient to set $x(t)$, $t \geq 0$, parallel to the axis of the aircraft fuselage. In this brief, we set $\{x(\cdot), y(\cdot), z(\cdot)\}$ as the *stability axes of the aircraft*, that is, the aircraft flies at equilibrium in symmetric flight conditions at $t = 0$, $x(0)$ is contained in the aircraft plane of symmetry, $x(0)$ is in the same direction as the velocity of the aircraft center of mass with respect to the wind, $z(0)$ is contained in the aircraft plane of symmetry, $z(0)$ points "down", that is, toward the floor of the cabin in regular flight conditions, and $y(0) = z^{\times}(0)x(0)$.

As thoroughly discussed in Sect. 2.4, if an aircraft is not disturbed by exogenous forces, such as a wind gust, the aircraft controls are fixed, and the velocity of the center of mass with respect to the wind, the angular position, and the angular velocity of the aircraft are *constant* in time, then the aircraft is flying at *equilibrium*. If an aircraft flies at equilibrium, the angular velocity of $\mathbb{J}$ with respect to $\mathbb{I}$ is zero, and the velocity of the aircraft center of mass with respect to the wind in the direction orthogonal to the plane of symmetry is zero, then the aircraft flies at *equilibrium in symmetric flight conditions*.

2.1.2 The Aircraft State Vector

Given the inertial reference frames $\mathbb{I}$ and the body reference frame $\mathbb{J}$, the attitude of an aircraft can be described by the Tait–Bryan angles $\phi, \theta, \psi : [0, \infty) \to \mathbb{R}$, named *roll*, *pitch*, and *yaw angles*, respectively. The *velocity of the aircraft center of mass* with respect to the wind is denoted by $[u, v, w]^{\mathrm{T}} : [0, \infty) \to \mathbb{R}^3$ and the *aircraft angular velocity* with respect to the inertial reference frame $\mathbb{I}$ is denoted by $[p, q, r]^{\mathrm{T}} : [0, \infty) \to \mathbb{R}^3$; both $[u(\cdot), v(\cdot), w(\cdot)]^{\mathrm{T}}$ and $[p(\cdot), q(\cdot), r(\cdot)]^{\mathrm{T}}$ are expressed in the body reference frame $\mathbb{J}$.

In this brief, we define

$$\chi \triangleq [x_{\mathrm{c}}, y_{\mathrm{c}}, z_{\mathrm{c}}, \phi, \theta, \psi, u, v, w, p, q, r]^{\mathrm{T}} \tag{2.3}$$

as the *aircraft state vector*. As discussed in Sect. 1.5, the Euler parameters q_1, $q_\imath$, $q_\jmath$, and q_κ provide a useful alternative to Tait–Bryan angles. In this case, the aircraft state vector can be defined as

$$\chi = [x_{\mathrm{c}}, y_{\mathrm{c}}, z_{\mathrm{c}}, q_1, q_\imath, q_\jmath, q_\kappa, u, v, w, p, q, r]^{\mathrm{T}}. \tag{2.4}$$

The components u and w of the state vector are known as *symmetric variables*, v is called *asymmetric variable*, q is known as *longitudinal variable*, and p and r are called *lateral-directional variables*.

2.1.3 The Aircraft Control Vector

Aircraft are maneuvered by controlling the *engines*, the *ailerons*, the *elevator*, and the *rudder*, which are designed to steer the propulsive and aerodynamic forces and moments acting on a vehicle. Usually, the aerodynamic force is expressed by mean of three components along mutually orthogonal axes, namely, the *drag*, which is parallel and opposed to the the velocity of the aircraft center of mass with respect to the wind, the *lift*, which is orthogonal to both the drag and the $y(\cdot)$ axis, and the *side force*.

The main goal of aircraft engines is to produce *thrust* $T(\cdot) \in \mathbb{R}^3$ and move the aircraft forward. The motion of the aircraft induces an airflow around the aircraft, which generates lift. This airflow also induces drag and the side force, which are compensated by the thrust. If the thrust increases, then the velocity of the aircraft center of mass with respect to the wind increases and hence also the lift increases and the aircraft altitude increases. However, if the thrust increases, then also the drag increases. In this brief, we assume that $T(\cdot)$ is functions of a parameter $\delta_{\mathrm{T}} : [0, \infty) \to [0, 1]$, which captures the *throttle setting*. If $\delta_{\mathrm{T}}(t) = 0$, $t \geq 0$, then the thrust is zero. Furthermore, we assume that $T(\delta_{\mathrm{T}}(t))$ is aligned to $x(t)$ for all $t \geq 0$.

Ailerons are mobile surfaces hinged at the trailing edge of the aircraft wings and are symmetric with respect to the aircraft plane of symmetry. The main scope of the ailerons is to induce a *roll moment* $L(\cdot)$ on the aircraft by increasing or decreasing the lift generated by a wing. For example, if the left aileron is deflected downward by an angle δ_{A}, then the left wing produces more lift than the right wing and the aircraft starts rolling. Ailerons deflect independently or, alternatively, symmetrically, that is, if the left aileron is deflected by an angle δ_{A}, then the right aileron is deflected by an angle $-\delta_{\mathrm{A}}$. In this brief, we assume that ailerons deflect symmetrically and we consider the deflection angle δ_{A} positive if it induces a positive roll moment. In general, the deflection of the ailerons induces also a yaw moment. Specifically, if the left aileron is deflected by an angle δ_{A} and the right aileron is deflected by an angle $-\delta_{\mathrm{A}}$, then the left wing generates more induced drag than the right wing. This difference in induced drag between the left and the right wings generates an undesired yaw moment that is usually compensated activating the rudder.

The rudder is a mobile surface hinged at the trailing edge of the aircraft vertical stabilizer. The main scope of the rudder is to induce a *yaw moment* $N(\cdot)$ on the aircraft by varying the aerodynamic force generated by the vertical stabilizer. For example, if the rudder is deflected toward left by an angle δ_{R}, then the aircraft tail produces a side force. Since the center of pressure of the vertical tail is not on the $z(\cdot)$ axis, this side force generates a yaw moment. However, since the center of pressure of the vertical tail is not on the $x(\cdot)$ axis, this side force induces also an undesired roll moment, which is usually compensated by the use of the ailerons. In this brief, we consider the deflection angle δ_{R} positive if it induces a *negative* yaw moment.

Elevators are mobile surfaces hinged at the trailing edge of the aircraft horizontal stabilizer and are symmetric with respect to the aircraft plane of symmetry. The main scope of the elevators is to induce a *pitch moment* $M(\cdot)$ on the aircraft by varying the aerodynamic force generated by the horizontal stabilizer. For example, if the elevators are deflected downward by an angle δ_{E}, then the horizontal stabilizer produces more lift and the aircraft pitches down. Elevators deflect symmetrically, that is, both deflect simultaneously of an angle δ_{E}. In this brief, we consider the deflection angle δ_{E} positive if it induces a positive pitch moment. We define

$$\eta \triangleq [\delta_{\mathrm{E}}, \delta_{\mathrm{T}}, \delta_{\mathrm{A}}, \delta_{\mathrm{R}}]^{\mathrm{T}} \tag{2.5}$$

as the *aircraft control vector*.

2.1.4 Aerodynamic Angles

The angle of attack and the sideslip angle play key roles in the study of aircraft dynamics.

Definition 2.1 (*Angle of attack*) Consider a symmetric aircraft and let $\mathbb{J} = \{r_c(\cdot); x(\cdot), y(\cdot), z(\cdot)\}$ be the body reference frame. The *angle of attack at the aircraft center of mass* is defined as

$$\alpha(u, w) \triangleq \begin{cases} \tan^{-1} \frac{w}{u}, & u \geq 0, \\ \pi + \tan^{-1} \frac{w}{u}, & w \geq 0, u < 0, \\ -\pi + \tan^{-1} \frac{w}{u}, & w < 0, u < 0, \end{cases} \tag{2.6}$$

where u and w denote respectively the first and third component of the velocity of the aircraft center of mass with respect to the wind in the reference frame $\mathbb{J}$.

Per definition, the angle of attack at the aircraft center of mass is such that $\alpha(u, w) \in (-\pi, \pi]$ for all $u, w \in \mathbb{R}$. As shown in Fig. 2.2, the angle of attack is the angle between the axis $x(\cdot)$ and the projection of the velocity of the aircraft center of mass with respect to the wind on the aircraft plane of symmetry.

Exercise 2.1 Consider Definition 2.1 and discuss why the angle of attack at the aircraft center of mass (2.6) should not be defined as

$$\alpha(u, w) = \tan^{-1} \frac{w}{u}, \tag{2.7}$$

for all $u, w \in \mathbb{R}$. △

Fig. 2.2 Aircraft angle of attack

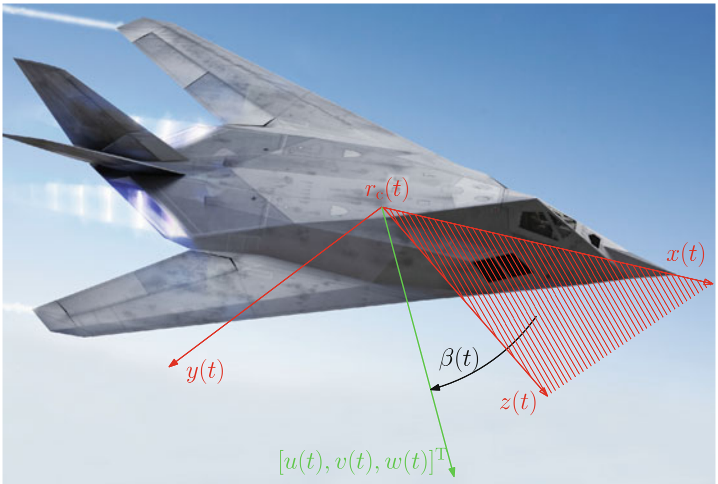

Fig. 2.3 Aircraft sideslip angle

Definition 2.2 (*Sideslip angle*) Consider a symmetric aircraft and let $\mathbb{J} = \{r_c(\cdot); x(\cdot), y(\cdot), z(\cdot)\}$ be the body reference frame. The *sideslip angle at the aircraft center of mass* is defined as

$$\beta(u, v, w) \triangleq \sin^{-1} \frac{v}{\sqrt{u^2 + v^2 + w^2}} \tag{2.8}$$

where $[u, v, w]^T$ denotes the velocity of the aircraft center of mass with respect to the wind in the reference frame $\mathbb{J}$.

The sideslip angle at the aircraft center of mass is such that $\beta(u, v, w) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for all $u, v, w \in \mathbb{R}$. As shown in Fig. 2.3, the sideslip angle at the aircraft center of mass is the angle between the vector $[u(\cdot), v(\cdot), w(\cdot)]^T$ and the aircraft plane of symmetry measured in the plane containing both $[u(\cdot), v(\cdot), w(\cdot)]^T$ and $y(\cdot)$.

Exercise 2.2 Consider a symmetric aircraft and let $\mathbb{J} = \{r_c(\cdot); x(\cdot), y(\cdot), z(\cdot)\}$ be the body reference frame. Let $[u, v, w]^T : [0, \infty) \to \mathbb{R}^3$ denote the velocity of the aircraft center of mass with respect to the wind in the reference frame $\mathbb{J}$ and $V(t) \triangleq \sqrt{u^2(t) + v^2(t) + w^2(t)}$, $t \geq 0$. Prove that

$$u(t) = V(t) \cos \alpha(t) \cos \beta(t), \qquad t \geq 0, \tag{2.9}$$

$$v(t) = V(t) \sin \beta(t), \tag{2.10}$$

$$w(t) = V(t) \sin \alpha(t) \cos \beta(t). \tag{2.11}$$

$\triangle$

Exercise 2.2 allows introducing an alternative body reference frame, which axes are known as *wind axes*; for details, see [46, Chap. 2, 3] and [12, Chap. 4, 5]. Given the components u and w of the velocity of the aircraft center of mass with respect to the wind, (2.8) provides a bijection between the sideslip angle $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and the component $v \in (-\infty, \infty)$ of the velocity of the aircraft center of mass. Similarly, given u, (2.6) provides a bijective correspondence between the angle of attack $\alpha \in (-\pi, \pi)$ and the component $w \in (-\infty, \infty)$ of the velocity of the aircraft center of mass. Therefore, in some application it is convenient to express the aircraft state vector as

$$\chi = [x_{\mathrm{c}}, y_{\mathrm{c}}, z_{\mathrm{c}}, \phi, \theta, \psi, u, \alpha, \beta, p, q, r]^{\mathrm{T}},$$

where w and v have been replaced by α and β, respectively.

2.2 Forces and Moments Acting on an Aircraft

The study of the aerodynamic, gravitational, and propulsive forces and moment of the forces plays a fundamental role both in the analysis of the aircraft dynamics and the synthesis of effective control actions. The next exercise allows expressing the gravitational force (2.1) in the reference frame $\mathbb{J}$.

Exercise 2.3 Apply Theorem 1.6 to show that the gravitational force $F_{\mathrm{g}} = mgZ$ can be expressed as

$$F_{\mathrm{g}} = mg \begin{bmatrix} -\sin\theta(t) \\ \cos\theta(t)\sin\phi(t) \\ \cos\theta(t)\cos\phi(t) \end{bmatrix}, \qquad t \geq 0, \tag{2.12}$$

in the reference frame $\mathbb{J}$. Then proceed as in Example 1.2 and Exercises 1.10 and 1.11 to achieve the same results. △

In general, the aerodynamic and propulsive forces acting on an aircraft and their moments explicitly depend on χ, $\dot{\chi}$, and η, but rarely depend on higher derivatives of the state vector; for details, see [12, Chap. 6, 8]. Therefore, in this brief we express the aerodynamic and propulsive forces in the reference frame $\mathbb{J}$ by

$$[F_x^{\mathrm{T}}(\chi, \dot{\chi}, \eta), F_y^{\mathrm{T}}(\chi, \dot{\chi}, \eta), F_z^{\mathrm{T}}(\chi, \dot{\chi}, \eta)]^{\mathrm{T}}, \qquad (\chi, \dot{\chi}, \eta) \in \mathbb{R}^{12} \times \mathbb{R}^{12} \times \mathbb{R}^4, \tag{2.13}$$

and we express the moment of the forces in the reference frame $\mathbb{J}$ by

$$[L^{\mathrm{T}}(\chi, \dot{\chi}, \eta), M^{\mathrm{T}}(\chi, \dot{\chi}, \eta), N^{\mathrm{T}}(\chi, \dot{\chi}, \eta)]^{\mathrm{T}}. \tag{2.14}$$

The components $F_x(\cdot)$ and $F_z(\cdot)$ are called *symmetric forces*, $M(\cdot)$ is known as *longitudinal moment of the forces*, $F_y(\cdot)$ is the *asymmetric force*, and $L(\cdot)$ and $N(\cdot)$ are called *lateral-directional moments of the forces*.

It follows from (2.13) and (2.14) that aerodynamic and propulsive forces as well as the moment of the forces depend on *twenty-eight parameters*. However, several observations allow simplifying these functional dependencies. For instance, $F_x(\cdot)$, $F_z(\cdot)$, and $M(\cdot)$ are even functions of v, p, r, $\dot{v}$, $\dot{p}$, and $\dot{r}$. Therefore, in the neighborhood of the equilibrium condition, the symmetric forces and moment of the forces are invariant with respect to changes in asymmetric and lateral-directional variables. Moreover, in this brief we consider symmetric vehicles only and, in this case, it is realistic to assume that the asymmetric force and the lateral-directional moments of the forces are invariant with respect to changes in symmetric or longitudinal variables [12, pp. 159–160]. In addition, we assume that aerodynamic and propulsive forces and moment of the forces do not vary with the aircraft position and attitude. Lastly, we make the following assumption, which holds in most cases of practical interest.

Assumption 2.1 It holds that

(i) $F_x(\cdot)$ and $F_z(\cdot)$ do not explicitly depend on $\dot{q}$, $\dot{u}$, $\dot{w}$, δ_{A}, and δ_{R},
(ii) $F_x(\cdot)$ does not explicitly depend on q,
(iii) $M(\cdot)$ does not explicitly depend on $\dot{q}$, $\dot{u}$, δ_{A}, and δ_{R},
(iv) $F_y(\cdot)$, $L(\cdot)$, and $N(\cdot)$ do not explicitly depend on $\dot{v}$, $\dot{p}$, $\dot{r}$, δ_{E}, and δ_{T},
(v) $F_y(\cdot)$ does not explicitly depend on δ_{A}.

In light of these considerations and noting that $[\dot{x}(t), \dot{y}(t), \dot{z}(t)]^{\mathrm{T}}$, $t \geq 0$, is a function of $[u(t), v(t), w(t)]^{\mathrm{T}}$ and $[\dot{\phi}(t), \dot{\theta}(t), \dot{\psi}(t)]^{\mathrm{T}}$ is a function of $[p(t), q(t), r(t)]^{\mathrm{T}}$, we assume that the following result always holds true.

Proposition 2.1 *Consider a symmetric aircraft. Then,*

(i) $F_x(\cdot)$ *explicitly depends on* u, w, δ_{E}, *and* δ_{T},
(ii) $F_y(\cdot)$, $L(\cdot)$, *and* $N(\cdot)$ *explicitly depend on* v, p, r, *and* δ_{R},
(iii) $L(\cdot)$ *and* $N(\cdot)$ *explicitly depend on* δ_{A},
(iv) $F_z(\cdot)$ *and* $M(\cdot)$ *explicitly depend on* u, w, q, δ_{E}, *and* δ_{T},
(v) $M(\cdot)$ *explicitly depends on* $\dot{w}$.

Detailed analyses of the forces and moment of the forces acting on an aircraft is provided by [12, Chap. 5], [46, Chap. 2] and [47, Chap. 2].

2.3 Equations of Motion of an Aircraft

Modeling an aircraft as a rigid body, the equations of motion follow from the results proven in Chap. 1. Specifically, it directly follows from Exercise 1.12 that the position of the center of mass of an aircraft can be computed integrating

$$\frac{d}{dt}\begin{bmatrix} x_{\mathrm{c}}(t) \\ y_{\mathrm{c}}(t) \\ z_{\mathrm{c}}(t) \end{bmatrix} = \begin{bmatrix} \cos\psi(t) & -\sin\psi(t) & 0 \\ \sin\psi(t) & \cos\psi(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta(t) & 0 & \sin\theta(t) \\ 0 & 1 & 0 \\ -\sin\theta(t) & 0 & \cos\theta(t) \end{bmatrix}$$
$$\cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi(t) & -\sin\phi(t) \\ 0 & \sin\phi(t) & \cos\phi(t) \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \\ w(t) \end{bmatrix}, \quad \begin{bmatrix} x_{\mathrm{c}}(0) \\ y_{\mathrm{c}}(0) \\ z_{\mathrm{c}}(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}, \quad t \geq 0, \tag{2.15}$$

in the reference frame $\mathbb{I}$. Moreover, it follows from (1.77) that the angular position of an aircraft is captured by

$$\frac{d}{dt}\begin{bmatrix} \phi(t) \\ \theta(t) \\ \psi(t) \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi(t)\tan\theta(t) & \cos\phi(t)\tan\theta(t) \\ 0 & \cos\phi(t) & -\sin\phi(t) \\ 0 & \sin\phi(t)\sec\theta(t) & \cos\phi(t)\sec\theta(t) \end{bmatrix} \begin{bmatrix} p(t) \\ q(t) \\ r(t) \end{bmatrix}, \quad \begin{bmatrix} \phi(0) \\ \theta(0) \\ \psi(0) \end{bmatrix} = \begin{bmatrix} \phi_0 \\ \theta_0 \\ \psi_0 \end{bmatrix}, \quad t \geq 0. \tag{2.16}$$

Lastly, it follows from Theorems 1.3 and 1.11 and Exercise 2.3 that

$$\frac{d}{dt}\begin{bmatrix} u(t) \\ v(t) \\ w(t) \end{bmatrix} = \frac{1}{m}\begin{bmatrix} F_x(u(t), w(t), \delta_{\mathrm{E}}(t), \delta_{\mathrm{T}}(t)) \\ F_y(v(t), p(t), r(t), \delta_{\mathrm{R}}(t)) \\ F_z(u(t), w(t), q(t), \delta_{\mathrm{E}}(t), \delta_{\mathrm{T}}(t)) \end{bmatrix} + g\begin{bmatrix} -\sin\theta(t) \\ \cos\theta(t)\sin\phi(t) \\ \cos\theta(t)\cos\phi(t) \end{bmatrix}$$
$$-\begin{bmatrix} p(t) \\ q(t) \\ r(t) \end{bmatrix}^{\times} \begin{bmatrix} u(t) \\ v(t) \\ w(t) \end{bmatrix}, \quad \begin{bmatrix} u(0) \\ v(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix}, \quad t \geq 0, \tag{2.17}$$

and it follows from Theorems 1.3 and 1.12 that

$$\frac{d}{dt}\begin{bmatrix} p(t) \\ q(t) \\ r(t) \end{bmatrix} = \begin{bmatrix} I_x & 0 & -I_{xz} \\ 0 & I_y & 0 \\ -I_{xz} & 0 & I_z \end{bmatrix}^{-1} \left(\begin{bmatrix} L(v(t), p(t), r(t), \delta_{\mathrm{A}}(t), \delta_{\mathrm{R}}(t)) \\ M(u(t), w(t), \dot{w}(t), q(t), \delta_{\mathrm{E}}(t), \delta_{\mathrm{T}}(t)) \\ N(v(t), p(t), r(t), \delta_{\mathrm{A}}(t), \delta_{\mathrm{R}}(t)) \end{bmatrix} \right.$$
$$\left. -\begin{bmatrix} p(t) \\ q(t) \\ r(t) \end{bmatrix}^{\times} \begin{bmatrix} I_x & 0 & -I_{xz} \\ 0 & I_y & 0 \\ -I_{xz} & 0 & I_z \end{bmatrix} \begin{bmatrix} p(t) \\ q(t) \\ r(t) \end{bmatrix} \right), \quad \begin{bmatrix} p(0) \\ q(0) \\ r(0) \end{bmatrix} = \begin{bmatrix} p_0 \\ q_0 \\ r_0 \end{bmatrix}. \tag{2.18}$$

Equations (2.15)–(2.18) are the *equations of motion of a symmetric aircraft*, expressed as functions of the state vector (2.3). The equations of motion of an aircraft can be also expressed as functions of the state vector (2.4), which is comprised of the position of the aircraft center of mass, the Euler parameters characterizing the rotational dynamics, the velocity of the aircraft center of mass with respect to the wind, and the aircraft angular velocity. In this case, it follows from (1.121) that

$$\frac{d}{dt}\begin{bmatrix} x_{\mathrm{c}}(t) \\ y_{\mathrm{c}}(t) \\ z_{\mathrm{c}}(t) \end{bmatrix} = \rho_{q_\gamma}(t)\begin{bmatrix} u(t) \\ v(t) \\ w(t) \end{bmatrix}, \qquad \begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}, \qquad t \geq 0, \tag{2.19}$$

where $\rho_{q_\gamma}(\cdot)$ is given by (1.119), it follows from (1.139) that

$$\begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_\iota(t) \\ \dot{q}_J(t) \\ \dot{q}_\kappa(t) \end{bmatrix} = \begin{bmatrix} -q_\iota(t) & -q_J(t) & -q_\kappa(t) \\ q_1(t) & q_\kappa(t) & -q_J(t) \\ -q_\kappa(t) & q_1(t) & q_\iota(t) \\ q_J(t) & -q_\iota(t) & -q_1(t) \end{bmatrix}\begin{bmatrix} p(t) \\ q(t) \\ r(t) \end{bmatrix}, \qquad \begin{bmatrix} q_1(0) \\ q_\iota(0) \\ q_J(0) \\ q_\kappa(0) \end{bmatrix} = \begin{bmatrix} q_{10} \\ q_{\iota 0} \\ q_{J0} \\ q_{\kappa 0} \end{bmatrix}, \tag{2.20}$$

and it follows from (1.125) that

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} u(t) \\ v(t) \\ w(t) \end{bmatrix} &= \frac{1}{m}\begin{bmatrix} F_x(u(t), w(t), \delta_{\mathrm{E}}(t), \delta_{\mathrm{T}}(t)) \\ F_y(v(t), p(t), r(t), \delta_{\mathrm{R}}(t)) \\ F_z(u(t), w(t), q(t), \delta_{\mathrm{E}}(t), \delta_{\mathrm{T}}(t)) \end{bmatrix} + g\begin{bmatrix} 2\left[q_\kappa(t)q_\iota(t) - q_J(t)q_1(t)\right] \\ 2\left[q_J(t)q_\kappa(t) + q_\iota(t)q_1(t)\right] \\ 1 - 2\left[q_\iota^2(t) + q_J^2(t)\right] \end{bmatrix} \\ &\quad - \begin{bmatrix} p(t) \\ q(t) \\ r(t) \end{bmatrix}^\times \begin{bmatrix} u(t) \\ v(t) \\ w(t) \end{bmatrix}, \qquad \begin{bmatrix} u(0) \\ v(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix}, \end{aligned} \tag{2.21}$$

It follows from (2.15)–(2.18) that the aircraft equations of motion do not depend on the position of the aircraft center of mass. Furthermore, note that (2.16)–(2.18) do not explicitly depend on the yaw angle $\psi(\cdot)$. Therefore, in order to fully describe the motion of an aircraft, firstly we need to integrate

$$\frac{d}{dt}\begin{bmatrix} \phi(t) \\ \theta(t) \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi(t)\tan\theta(t) & \cos\phi(t)\tan\theta(t) \\ 0 & \cos\phi(t) & -\sin\phi(t) \end{bmatrix}\begin{bmatrix} p(t) \\ q(t) \\ r(t) \end{bmatrix}, \quad \begin{bmatrix} \phi(0) \\ \theta(0) \end{bmatrix} = \begin{bmatrix} \phi_0 \\ \theta_0 \end{bmatrix}, \quad t \geq 0, \tag{2.22}$$

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} u(t) \\ v(t) \\ w(t) \end{bmatrix} &= \frac{1}{m}\begin{bmatrix} F_x(u(t), w(t), \delta_{\mathrm{E}}(t), \delta_{\mathrm{T}}(t)) \\ F_y(v(t), p(t), r(t), \delta_{\mathrm{R}}(t)) \\ F_z(u(t), w(t), q(t), \delta_{\mathrm{E}}(t), \delta_{\mathrm{T}}(t)) \end{bmatrix} + g\begin{bmatrix} -\sin\theta(t) \\ \cos\theta(t)\sin\phi(t) \\ \cos\theta(t)\cos\phi(t) \end{bmatrix} \\ &\quad - \begin{bmatrix} p(t) \\ q(t) \\ r(t) \end{bmatrix}^\times \begin{bmatrix} u(t) \\ v(t) \\ w(t) \end{bmatrix}, \qquad \begin{bmatrix} u(0) \\ v(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix}, \end{aligned} \tag{2.23}$$

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} p(t) \\ q(t) \\ r(t) \end{bmatrix} &= \begin{bmatrix} I_x & 0 & -I_{xz} \\ 0 & I_y & 0 \\ -I_{xz} & 0 & I_z \end{bmatrix}^{-1} \left(\begin{bmatrix} L(v(t), p(t), r(t), \delta_{\mathrm{A}}(t), \delta_{\mathrm{R}}(t)) \\ M(u(t), w(t), \dot{w}(t), q(t), \delta_{\mathrm{E}}(t), \delta_{\mathrm{T}}(t)) \\ N(v(t), p(t), r(t), \delta_{\mathrm{A}}(t), \delta_{\mathrm{R}}(t)) \end{bmatrix} \right. \\ &\quad \left. - \begin{bmatrix} p(t) \\ q(t) \\ r(t) \end{bmatrix}^\times \begin{bmatrix} I_x & 0 & -I_{xz} \\ 0 & I_y & 0 \\ -I_{xz} & 0 & I_z \end{bmatrix}\begin{bmatrix} p(t) \\ q(t) \\ r(t) \end{bmatrix} \right), \qquad \begin{bmatrix} p(0) \\ q(0) \\ r(0) \end{bmatrix} = \begin{bmatrix} p_0 \\ q_0 \\ r_0 \end{bmatrix}. \end{aligned} \tag{2.24}$$

The aircraft position $[x(t), y(t), z(t)]^{\mathrm{T}}$, $t \geq 0$, and the yaw angle $\psi(t)$ can be found integrating (2.15) and (2.16), respectively. In light of this consideration, in the remainder of this brief we will refer to (2.22)–(2.24) as the *aircraft equations of motion* and

$$\chi = [\phi, \theta, u, v, w, p, q, r]^{\mathrm{T}} \tag{2.25}$$

as the *aircraft state vector*.

2.4 Flight at Equilibrium

As discussed in Sect. 2.1.1, if an aircraft is not disturbed by exogenous forces, such as a wind gust, the aircraft controls are fixed, and the velocity of the center of mass with respect to the wind, the angular position, and the angular velocity of the aircraft are constant in time, then the aircraft is flying at equilibrium. This notion is formalized by the next definition. For the statement of the next result, let $F(\chi, \eta)$ denote the right-hand side of (2.22)–(2.24).

Definition 2.3 (*Trim (equilibrium) condition*) Consider the aircraft equations of motion (2.22)–(2.24). If $\chi(t) = \chi_{\mathrm{e}}$, $t \geq 0$, $\eta(t) = \eta_{\mathrm{e}}$, and

$$0 = F(\chi_{\mathrm{e}}, \eta_{\mathrm{e}}), \tag{2.26}$$

where $\chi_{\mathrm{e}} \triangleq [\phi_{\mathrm{e}}, \theta_{\mathrm{e}}, u_{\mathrm{e}}, v_{\mathrm{e}}, w_{\mathrm{e}}, p_{\mathrm{e}}, q_{\mathrm{e}}, r_{\mathrm{e}}]^{\mathrm{T}}$, and $\eta_{\mathrm{e}} \triangleq [\delta_{\mathrm{Ee}}, \delta_{\mathrm{Te}}, \delta_{\mathrm{Ae}}, \delta_{\mathrm{Re}}]^{\mathrm{T}}$, then the aircraft is flying in *trim condition* or, equivalently, the *aircraft is trimmed* or the *aircraft is at equilibrium*. An aircraft flies *at equilibrium in symmetric flight conditions* if $\phi_{\mathrm{e}} = 0$, $v_{\mathrm{e}} = 0$, $p_{\mathrm{e}} = q_{\mathrm{e}} = r_{\mathrm{e}} = 0$.

As discussed in Sect. 2.1, we can set the body axis $x(t)$, $t \geq 0$, at equilibrium in the same direction as the velocity of the aircraft center of mass with respect to the wind, so that $w_{\mathrm{e}} = 0$. It follows from (2.22)–(2.24) and (2.26) that if an aircraft flies in a condition of symmetric equilibrium, then

$$0 = \begin{bmatrix} F_x(u_{\mathrm{e}}, 0, \delta_{\mathrm{Ee}}, \delta_{\mathrm{Te}}) \\ F_y(0, 0, 0, \delta_{\mathrm{Re}}) \\ F_z(u_{\mathrm{e}}, 0, 0, \delta_{\mathrm{Ee}}, \delta_{\mathrm{Te}}) \end{bmatrix} + mg \begin{bmatrix} -\sin\theta_{\mathrm{e}} \\ 0 \\ \cos\theta_{\mathrm{e}} \end{bmatrix}, \tag{2.27}$$

$$0 = \begin{bmatrix} L(0, 0, 0, \delta_{\mathrm{Ae}}, \delta_{\mathrm{Re}}) \\ M(u_{\mathrm{e}}, 0, 0, 0, \delta_{\mathrm{Ee}}, \delta_{\mathrm{Te}}) \\ N(0, 0, 0, \delta_{\mathrm{Ae}}, \delta_{\mathrm{Re}}) \end{bmatrix}. \tag{2.28}$$

Hence, in case of symmetric equilibrium, *the forces acting on the aircraft center of mass are equal to zero and the moment of the forces about the aircraft center of mass is zero*.

As will become clear in the next section, the knowledge of the trim condition is fundamental to characterize the stability of aircraft and design their controls. This task is particularly demanding and is usually accomplished resorting to numerical tools for the simulation of the aircraft dynamics; for details, see [9, 46].

2.5 Linearization of the Aircraft Equations of Motion

It is common practice in nonlinear systems theory to analyze a nonlinear system by observing the behavior of the linearized system in a neighborhood of a conveniently chosen point, such as an equilibrium point $[\chi^{\mathrm{T}}, \eta^{\mathrm{T}}]^{\mathrm{T}} = [\chi_{\mathrm{e}}^{\mathrm{T}}, \eta_{\mathrm{e}}^{\mathrm{T}}]^{\mathrm{T}}$. Linearizing the equations of motion of a symmetric aircraft is advantageous because linear dynamical systems are undoubtedly easier to analyze. Moreover, it follows from Taylor's theorem that the behavior of a nonlinear dynamical system in a neighborhood of the equilibrium points could be extrapolated from the dynamics of the linearized system; for details, see Theorem A.1. Lastly, *Lyapunov indirect method* proves that an equilibrium point of a nonlinear dynamical system is asymptotically stable (respectively, unstable) if and only if the linearized system is asymptotically stable (unstable) [27, Th. 4.7].

Let $F(\chi, \eta)$ denote the right-hand side of (2.22)–(2.24). It follows from Theorem A.1 and Remark A.1 that

$$\dot{\xi}_{\chi}(t) \approx A\xi_{\chi}(t) + B\xi_{\eta}(t), \qquad \xi(0) = \chi_0 - \chi_{\mathrm{e}}, \qquad t \geq 0, \tag{2.29}$$

where

$$A = \left.\frac{\partial F(\chi, \eta)}{\partial \chi}\right|_{[\chi^{\mathrm{T}}, \eta^{\mathrm{T}}]^{\mathrm{T}} = [\chi_{\mathrm{e}}^{\mathrm{T}}, \eta_{\mathrm{e}}^{\mathrm{T}}]^{\mathrm{T}}}, \qquad B = \left.\frac{\partial F(\chi, \eta)}{\partial \eta}\right|_{[\chi^{\mathrm{T}}, \eta^{\mathrm{T}}]^{\mathrm{T}} = [\chi_{\mathrm{e}}^{\mathrm{T}}, \eta_{\mathrm{e}}^{\mathrm{T}}]^{\mathrm{T}}}, \tag{2.30}$$

$\xi_{\chi}(t) = \chi(t) - \chi_{\mathrm{e}}$, and $\xi_{\eta} = \eta(t) - \eta_{\mathrm{e}}$. It is customary in flight dynamics to ***rewrite*** (2.29) as

$$\dot{\chi}(t) = A\chi(t) + B\eta(t), \qquad \chi(0) = \chi_0 - \chi_{\mathrm{e}}, \qquad t \geq 0, \tag{2.31}$$

where A and B are given by (2.39) and (2.40), respectively. In the following, we refer to (2.31) as the *aircraft linearized equations of motion*. It is important to remember that the partial derivatives in (2.39) and (2.40) are evaluated at equilibrium.

Since $M(\cdot)$ explicitly depends on $\dot{w}$, (2.22)–(2.24) is a set of ***implicit*** nonlinear differential equations. The next example illustrates how the linearized equations of motion have been transformed in ***explicit*** form.

Example 2.1 It follows from (2.24) that

$$\begin{aligned}
I_y \frac{dq(t)}{dt} &= M(u(t), w(t), \dot{w}(t), q(t), \delta_{\mathrm{E}}(t), \delta_{\mathrm{T}}(t)) + I_{xz}\left[r^2(t) - p^2(t)\right] \\
&\quad + (I_x - I_z)\, p(t)r(t) \\
&\approx \left.\frac{\partial M(u, w, \dot{w}, q, \delta_{\mathrm{E}}, \delta_{\mathrm{T}})}{\partial u}\right|_{[\chi^{\mathrm{T}},\eta^{\mathrm{T}}]^{\mathrm{T}}=[\chi_{\mathrm{e}}^{\mathrm{T}},\eta_{\mathrm{e}}^{\mathrm{T}}]^{\mathrm{T}}} u(t) \\
&\quad + \left.\frac{\partial M(u, w, \dot{w}, q, \delta_{\mathrm{E}}, \delta_{\mathrm{T}})}{\partial w}\right|_{[\chi^{\mathrm{T}},\eta^{\mathrm{T}}]^{\mathrm{T}}=[\chi_{\mathrm{e}}^{\mathrm{T}},\eta_{\mathrm{e}}^{\mathrm{T}}]^{\mathrm{T}}} w(t) \\
&\quad + \left.\frac{\partial M(u, w, \dot{w}, q, \delta_{\mathrm{E}}, \delta_{\mathrm{T}})}{\partial \dot{w}}\right|_{[\chi^{\mathrm{T}},\eta^{\mathrm{T}}]^{\mathrm{T}}=[\chi_{\mathrm{e}}^{\mathrm{T}},\eta_{\mathrm{e}}^{\mathrm{T}}]^{\mathrm{T}}} \dot{w}(t) \\
&\quad + \left.\frac{\partial M(u, w, \dot{w}, q, \delta_{\mathrm{E}}, \delta_{\mathrm{T}})}{\partial q}\right|_{[\chi^{\mathrm{T}},\eta^{\mathrm{T}}]^{\mathrm{T}}=[\chi_{\mathrm{e}}^{\mathrm{T}},\eta_{\mathrm{e}}^{\mathrm{T}}]^{\mathrm{T}}} q(t) \\
&\quad + \left.\frac{\partial M(u, w, \dot{w}, q, \delta_{\mathrm{E}}, \delta_{\mathrm{T}})}{\partial \delta_{\mathrm{E}}}\right|_{[\chi^{\mathrm{T}},\eta^{\mathrm{T}}]^{\mathrm{T}}=[\chi_{\mathrm{e}}^{\mathrm{T}},\eta_{\mathrm{e}}^{\mathrm{T}}]^{\mathrm{T}}} \delta_{\mathrm{E}}(t) \\
&\quad + \left.\frac{\partial M(u, w, \dot{w}, q, \delta_{\mathrm{E}}, \delta_{\mathrm{T}})}{\partial \delta_{\mathrm{T}}}\right|_{[\chi^{\mathrm{T}},\eta^{\mathrm{T}}]^{\mathrm{T}}=[\chi_{\mathrm{e}}^{\mathrm{T}},\eta_{\mathrm{e}}^{\mathrm{T}}]^{\mathrm{T}}} \delta_{\mathrm{T}}(t), \quad q(0) = q_0, \quad t \geq 0,
\end{aligned} \tag{2.32}$$

and, similarly, it follows from (2.23) that

$$\begin{aligned}
\frac{dw(t)}{dt} &\approx \frac{1}{m} \left.\frac{\partial F_z(u, w, q, \delta_{\mathrm{E}}, \delta_{\mathrm{T}})}{\partial u}\right|_{[\chi^{\mathrm{T}},\eta^{\mathrm{T}}]^{\mathrm{T}}=[\chi_{\mathrm{e}}^{\mathrm{T}},\eta_{\mathrm{e}}^{\mathrm{T}}]^{\mathrm{T}}} u(t) \\
&\quad + \frac{1}{m} \left.\frac{\partial F_z(u, w, q, \delta_{\mathrm{E}}, \delta_{\mathrm{T}})}{\partial w}\right|_{[\chi^{\mathrm{T}},\eta^{\mathrm{T}}]^{\mathrm{T}}=[\chi_{\mathrm{e}}^{\mathrm{T}},\eta_{\mathrm{e}}^{\mathrm{T}}]^{\mathrm{T}}} w(t) \\
&\quad + \left[\frac{1}{m} \left.\frac{\partial F_z(u, w, q, \delta_{\mathrm{E}}, \delta_{\mathrm{T}})}{\partial q}\right|_{[\chi^{\mathrm{T}},\eta^{\mathrm{T}}]^{\mathrm{T}}=[\chi_{\mathrm{e}}^{\mathrm{T}},\eta_{\mathrm{e}}^{\mathrm{T}}]^{\mathrm{T}}} + u_{\mathrm{e}}\right] q(t) \\
&\quad + \frac{1}{m} \left.\frac{\partial F_z(u, w, q, \delta_{\mathrm{E}}, \delta_{\mathrm{T}})}{\partial \delta_{\mathrm{E}}}\right|_{[\chi^{\mathrm{T}},\eta^{\mathrm{T}}]^{\mathrm{T}}=[\chi_{\mathrm{e}}^{\mathrm{T}},\eta_{\mathrm{e}}^{\mathrm{T}}]^{\mathrm{T}}} \delta_{\mathrm{E}}(t) \\
&\quad + \frac{1}{m} \left.\frac{\partial F_z(u, w, q, \delta_{\mathrm{E}}, \delta_{\mathrm{T}})}{\partial \delta_{\mathrm{T}}}\right|_{[\chi^{\mathrm{T}},\eta^{\mathrm{T}}]^{\mathrm{T}}=[\chi_{\mathrm{e}}^{\mathrm{T}},\eta_{\mathrm{e}}^{\mathrm{T}}]^{\mathrm{T}}} \delta_{\mathrm{T}}(t) - g \sin\theta_{\mathrm{e}}\theta(t), \\
&\qquad w(0) = w_0, \quad t \geq 0.
\end{aligned} \tag{2.33}$$

Hence, the linearized equation of motion regulating the pitch angular velocity can be expressed as an explicit differential equation by substituting (2.33) in (2.32). △

It follows from (2.39) and (2.40) that

$$\frac{d\theta(t)}{dt} = q(t), \qquad \theta(0) = \theta_0 - \theta_e, \qquad t \geq 0, \tag{2.34}$$

in a neighborhood of the equilibrium point (χ_e, η_e), where $\chi_e = [0, \theta_e, u_e, 0, 0, 0, 0, 0]^T$. Furthermore,

$$\frac{d\phi(t)}{dt} = p(t) + \tan\theta_e r(t), \qquad \phi(0) = \phi_0, \qquad t \geq 0, \tag{2.35}$$

and if $\theta_e = 0$, then

$$\frac{d\phi(t)}{dt} = p(t), \qquad \phi(0) = \phi_0, \qquad t \geq 0. \tag{2.36}$$

Hence, if $\chi_e = [0, 0, u_e, 0, 0, 0, 0, 0]^T$, then in a small neighborhood of the equilibrium point (χ_e, η_e) the roll rate and the pitch rate are equal to the first and second components of the angular velocity vector, respectively; in general, however, this relation is not satisfied.

2.6 Decoupling of the Linearized Equations of Motion

It follows from (2.39) and (2.40) that (2.31) can be equivalently written as two *decoupled* sets of linear differential equations, namely

$$\frac{d}{dt}\begin{bmatrix}\theta(t)\\u(t)\\w(t)\\q(t)\end{bmatrix} = A_{\rm long}\begin{bmatrix}\theta(t)\\u(t)\\w(t)\\q(t)\end{bmatrix} + B_{\rm long}\begin{bmatrix}\delta_{\rm E}(t)\\\delta_{\rm T}(t)\end{bmatrix}, \qquad \begin{bmatrix}\theta(0)\\u(0)\\w(0)\\q(0)\end{bmatrix} = \begin{bmatrix}\theta_0-\theta_e\\u_0-u_e\\w_0\\q_0\end{bmatrix}, \qquad t \geq 0, \tag{2.37}$$

$$\frac{d}{dt}\begin{bmatrix}\phi(t)\\v(t)\\p(t)\\r(t)\end{bmatrix} = A_{\rm lat}\begin{bmatrix}\phi(t)\\v(t)\\p(t)\\r(t)\end{bmatrix} + B_{\rm lat}\begin{bmatrix}\delta_{\rm A}(t)\\\delta_{\rm R}(t)\end{bmatrix}, \qquad \begin{bmatrix}\phi(0)\\v(0)\\p(0)\\r(0)\end{bmatrix} = \begin{bmatrix}\phi_0\\v_0\\p_0\\r_0\end{bmatrix}, \tag{2.38}$$

where $A_{\rm long}$, $B_{\rm long}$, $A_{\rm lat}$, and $B_{\rm lat}$ are given by (2.45)–(2.48) respectively. In this brief, we refer to (2.37) as the *longitudinal equations of motion* and (2.38) as the *lateral-directional equations of motion*.

$$
A = \left[\begin{array}{c||cc||cc}
 & \phi & \theta & u & v \\
\hline
\dot\phi & 0 & 0 & 0 & 0 \\
\dot\theta & 0 & 0 & 0 & 0 \\
\hline
\dot u & 0 & -g\cos\theta_{\mathrm{e}} & \frac{\partial F_x(u,w,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{m\partial u} & 0 \\
\dot v & g\cos\theta_{\mathrm{e}} & 0 & 0 & \frac{\partial F_y(v,p,r,\delta_{\mathrm{R}})}{m\partial v} \\
\dot w & 0 & -g\sin\theta_{\mathrm{e}} & \frac{\partial F_z(u,w,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{m\partial u} & 0 \\
\hline
\dot p & 0 & 0 & 0 & \frac{I_{xz}}{I_{xz}^2-I_xI_z}\frac{\partial N(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial v} - \frac{I_z}{I_{xz}^2-I_xI_z}\frac{\partial L(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial v} \\
\dot q & 0 & -g\sin\theta_{\mathrm{e}}\frac{\partial M(u,w,\dot w,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{I_y\partial\dot w} & \frac{\partial M(u,w,\dot w,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{I_y\partial u} + \frac{\partial M(u,w,\dot w,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{I_y\partial\dot w}\frac{\partial F_z(u,w,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{m\partial u} & 0 \\
\dot r & 0 & 0 & 0 & \frac{I_x}{I_{xz}^2-I_xI_z}\frac{\partial N(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial v} - \frac{I_{xz}}{I_{xz}^2-I_xI_z}\frac{\partial L(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial v}
\end{array}\right.\cdots
$$

$$
\begin{array}{c||c||c}
 & \ldots & w \\
\hline
\dot\phi & \ldots & 0 \\
\dot\theta & \ldots & 0 \\
\hline
\dot u & \ldots & \frac{\partial F_x(u,w,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{m\partial w} \\
\dot v & \ldots & 0 \\
\dot w & \ldots & \frac{\partial F_z(u,w,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{m\partial w} \\
\hline
\dot p & \ldots & 0 \\
\dot q & \ldots & \frac{\partial M(u,w,\dot w,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{I_y\partial w} + \frac{\partial M(u,w,\dot w,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{I_y\partial\dot w}\frac{\partial F_z(u,w,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{m\partial w} \\
r & \ldots & 0
\end{array}\;\cdots
$$

$$
\begin{array}{c||c||cc}
 & \ldots & p & q \\
\hline
\dot\phi & \ldots & 1 & 0 \\
\dot\theta & \ldots & 0 & 1 \\
\hline
\dot u & \ldots & 0 & 0 \\
\dot v & \ldots & \frac{\partial F_y(v,p,r,\delta_{\mathrm{R}})}{m\partial p} & 0 \\
\dot w & \ldots & 0 & \frac{\partial F_z(u,w,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{m\partial q} + u_{\mathrm{e}} \\
\hline
\dot p & \ldots & \frac{I_{xz}}{I_{xz}^2-I_xI_z}\frac{\partial N(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial p} - \frac{I_z}{I_{xz}^2-I_xI_z}\frac{\partial L(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial p} & 0 \\
\dot q & \ldots & 0 & \frac{\partial M(u,w,\dot w,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{I_y\partial q} + \frac{\partial M(u,w,\dot w,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{I_y\partial\dot w}\frac{\partial F_z(u,w,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{m\partial q} \\
\dot r & \ldots & \frac{I_x}{I_{xz}^2-I_xI_z}\frac{\partial N(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial p} - \frac{I_{xz}}{I_{xz}^2-I_xI_z}\frac{\partial L(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial p} & 0
\end{array}\;\cdots
$$

$$
\left.\begin{array}{c||c|c}
 & \ldots & r \\
\hline
\dot\phi & \ldots & \tan\theta_{\mathrm{e}} \\
\dot\theta & \ldots & 0 \\
\hline
\dot u & \ldots & 0 \\
\dot v & \ldots & \frac{\partial F_y(v,p,r,\delta_{\mathrm{R}})}{m\partial r} - u_{\mathrm{e}} \\
\dot w & \ldots & 0 \\
\hline
\dot p & \ldots & \frac{I_{xz}}{I_{xz}^2-I_xI_z}\frac{\partial N(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial r} - \frac{I_z}{I_{xz}^2-I_xI_z}\frac{\partial L(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial r} \\
\dot q & \ldots & 0 \\
\dot r & \ldots & \frac{I_x}{I_{xz}^2-I_xI_z}\frac{\partial N(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial r} - \frac{I_{xz}}{I_{xz}^2-I_xI_z}\frac{\partial L(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial r}
\end{array}\right], \tag{2.39}
$$

$$
B = \left[\begin{array}{c||ccc}
 & \delta_{\mathrm{E}} & \delta_{\mathrm{T}} & \cdots \\
\hline\hline
\dot{\phi} & 0 & 0 & \cdots \\
\dot{\theta} & 0 & 0 & \cdots \\
\hline
\dot{u} & \frac{\partial F_x(u,w,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{m\partial\delta_{\mathrm{E}}} & \frac{\partial F_x(u,w,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{m\partial\delta_{\mathrm{T}}} & \cdots \\
\dot{v} & 0 & 0 & \cdots \\
\dot{w} & \frac{\partial F_z(u,w,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{m\partial\delta_{\mathrm{E}}} & \frac{\partial F_z(u,w,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{m\partial\delta_{\mathrm{R}}} & \cdots \\
\hline
\dot{p} & 0 & 0 & \cdots \\
\dot{q} & \frac{\partial M(u,w,\dot{w},q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{I_y\partial\delta_{\mathrm{E}}} + \frac{\partial M(u,w,\dot{w},q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{I_y\partial\dot{w}} \frac{\partial F_z(u,w,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{m\partial\delta_{\mathrm{E}}} & \frac{\partial M(u,w,\dot{w},q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{I_y\partial\delta_{\mathrm{T}}} + \frac{\partial M(u,w,\dot{w},q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{I_y\partial\dot{w}} \frac{\partial F_z(u,w,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{m\partial\delta_{\mathrm{T}}} & \cdots \\
\dot{r} & 0 & 0 & \cdots
\end{array}\right.
$$

$$
\left.\begin{array}{c||ccc}
 & \cdots & \delta_{\mathrm{A}} & \delta_{\mathrm{R}} \\
\hline\hline
\dot{\phi} & \cdots & 0 & 0 \\
\dot{\theta} & \cdots & 0 & 0 \\
\hline
\dot{u} & \cdots & 0 & 0 \\
\dot{v} & \cdots & 0 & \frac{\partial F_y(v,p,r,\delta_{\mathrm{R}})}{m\partial\delta_{\mathrm{R}}} \\
\dot{w} & \cdots & 0 & 0 \\
\hline
\dot{p} & \cdots & \frac{I_{xz}}{I_{xz}^2 - I_x I_z} \frac{\partial N(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\delta_{\mathrm{A}}} - \frac{I_z}{I_{xz}^2 - I_x I_z} \frac{\partial L(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\delta_{\mathrm{A}}} & \frac{I_{xz}}{I_{xz}^2 - I_x I_z} \frac{\partial N(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\delta_{\mathrm{R}}} - \frac{I_z}{I_{xz}^2 - I_x I_z} \frac{\partial L(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\delta_{\mathrm{R}}} \\
\dot{q} & \cdots & 0 & 0 \\
\dot{r} & \cdots & \frac{I_x}{I_{xz}^2 - I_x I_z} \frac{\partial N(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\delta_{\mathrm{A}}} - \frac{I_{xz}}{I_{xz}^2 - I_x I_z} \frac{\partial L(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\delta_{\mathrm{A}}} & \frac{I_x}{I_{xz}^2 - I_x I_z} \frac{\partial N(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\delta_{\mathrm{R}}} - \frac{I_{xz}}{I_{xz}^2 - I_x I_z} \frac{\partial L(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\delta_{\mathrm{R}}}
\end{array}\right], \tag{2.40}
$$

Exercise 2.4 Equations (2.37) and (2.38) have been derived assuming that the component of the aerodynamic and propulsive forces along the z body axis is a function of u, w, q, δ_{E}, and δ_{T}. Rewrite the linearized equations of motion (2.37) and (2.38) assuming that $F_z(\cdot)$ depends on u, w, q, δ_{E}, δ_{T}, *and* $\dot{w}$. △

The matrices A_{long}, B_{long}, A_{lat}, and B_{lat} in (2.45)–(2.48) clearly depend on the derivatives of the aerodynamic and propulsive forces and moments with respect to the control and state vectors, evaluated at equilibrium, that is for $[\chi^{\mathrm{T}}, \eta^{\mathrm{T}}]^{\mathrm{T}} = [\chi_{\mathrm{e}}^{\mathrm{T}}, \eta_{\mathrm{e}}^{\mathrm{T}}]^{\mathrm{T}}$. These partial derivatives are know as *stability derivatives* and, in most cases of practical interest, it is not possible to measure the stability derivatives of an aircraft. However, it is common practice to *compute* the stability derivatives as functions of the *aerodynamic coefficients*, which are measured in wind tunnels testing reduced-scale aircraft models or aircraft components, such as the wings. Remarkably, aerodynamic coefficients are *dimensionless quantities*, since these are computed dividing forces and moments of the forces by some reference quantity, and, in general, do not vary by scaling the dimensions of an aircraft. A study of the stability derivatives and the aerodynamic coefficients is beyond the scopes of this brief; for details, see [35, Chap. 3] and [12, Chap. 7, 8].

It follows from Theorem A.1, (2.6), and (2.8) that

$$\alpha(u, w) = \frac{w}{u_{\mathrm{e}}} + r_{1,\alpha}(u, w), \tag{2.41}$$

$$\beta(u, v, w) = \frac{v}{|u_{\mathrm{e}}|} + r_{1,\beta}(u, v, w) = \frac{v}{u_{\mathrm{e}}} + r_{1,\beta}(u, v, w) \tag{2.42}$$

where $\alpha(\cdot)$ denotes the angle of attack at the aircraft center of mass, $\beta(\cdot)$ denotes the sideslip angle at the aircraft center of mass, $[u_{\mathrm{e}}, 0, 0]^{\mathrm{T}}$ denotes the velocity of the aircraft center of mass with respect to the wind at equilibrium, and $r_{1,\alpha}(\cdot)$ and $r_{1,\beta}(\cdot)$ denote the remainders. Therefore, in a neighborhood of the equilibrium point $[\chi_{\mathrm{e}}^{\mathrm{T}}, \eta_{\mathrm{e}}^{\mathrm{T}}]^{\mathrm{T}}$ it holds that

$$w(\alpha) \approx u_{\mathrm{e}}\alpha, \qquad v(\beta) \approx u_{\mathrm{e}}\beta,$$

and (2.37) and (2.38) are equivalent to

$$\frac{d}{dt}\begin{bmatrix}\theta(t)\\ u(t)\\ \alpha(t)\\ q(t)\end{bmatrix} = \hat{A}_{\mathrm{long}}\begin{bmatrix}\theta(t)\\ u(t)\\ \alpha(t)\\ q(t)\end{bmatrix} + \hat{B}_{\mathrm{long}}\begin{bmatrix}\delta_{\mathrm{E}}(t)\\ \delta_{\mathrm{T}}(t)\end{bmatrix}, \qquad \begin{bmatrix}\theta(0)\\ u(0)\\ \alpha(0)\\ q(0)\end{bmatrix} = \begin{bmatrix}\theta_0 - \theta_{\mathrm{e}}\\ u_0 - u_{\mathrm{e}}\\ \alpha_0\\ q_0\end{bmatrix}, \qquad t \geq 0, \tag{2.43}$$

$$\frac{d}{dt}\begin{bmatrix}\phi(t)\\ \beta(t)\\ p(t)\\ r(t)\end{bmatrix} = \hat{A}_{\mathrm{lat}}\begin{bmatrix}\phi(t)\\ \beta(t)\\ p(t)\\ r(t)\end{bmatrix} + \hat{B}_{\mathrm{lat}}\begin{bmatrix}\delta_{\mathrm{A}}(t)\\ \delta_{\mathrm{R}}(t)\end{bmatrix}, \qquad \begin{bmatrix}\phi(0)\\ \beta(0)\\ p(0)\\ r(0)\end{bmatrix} = \begin{bmatrix}\phi_0\\ \beta_0\\ p_0\\ r_0\end{bmatrix}, \tag{2.44}$$

where $\hat{A}_{\mathrm{long}}$, $\hat{B}_{\mathrm{long}}$, $\hat{A}_{\mathrm{lat}}$, and $\hat{B}_{\mathrm{lat}}$ are given by (2.53)–(2.56) respectively.

$$
A_{\text{long}} \triangleq \left[\begin{array}{c||ccc}
 & \theta & u & \dots \\
\hline\hline
\dot\theta & 0 & 0 & \dots \\
\dot u & -g\cos\theta_{\text{e}} & \frac{\partial F_x(u,w,\delta_{\text{E}},\delta_{\text{T}})}{m\partial u} & \dots \\
\dot w & -g\sin\theta_{\text{e}} & \frac{\partial F_z(u,w,q,\delta_{\text{E}},\delta_{\text{T}})}{m\partial u} & \dots \\
\dot q & -g\sin\theta_{\text{e}}\frac{\partial M(u,w,\dot w,q,\delta_{\text{E}},\delta_{\text{T}})}{I_y\partial\dot w} & \frac{\partial M(u,w,\dot w,q,\delta_{\text{E}},\delta_{\text{T}})}{I_y\partial u}+\frac{\partial M(u,w,\dot w,q,\delta_{\text{E}},\delta_{\text{T}})}{I_y\partial\dot w}\frac{\partial F_z(u,w,q,\delta_{\text{E}},\delta_{\text{T}})}{m\partial u} & \dots
\end{array}\right.
$$

$$
\left.\begin{array}{c||ccc}
 & \dots & w & q \\
\hline\hline
\dot\theta & \dots & 0 & 1 \\
\dot u & \dots & \frac{\partial F_x(u,w,\delta_{\text{E}},\delta_{\text{T}})}{m\partial w} & 0 \\
\dot w & \dots & \frac{\partial F_z(u,w,q,\delta_{\text{E}},\delta_{\text{T}})}{m\partial w} & \frac{\partial F_z(u,w,q,\delta_{\text{E}},\delta_{\text{T}})}{m\partial q}+u_{\text{e}} \\
\dot q & \dots & \frac{\partial M(u,w,\dot w,q,\delta_{\text{E}},\delta_{\text{T}})}{I_y\partial w}+\frac{\partial M(u,w,\dot w,q,\delta_{\text{E}},\delta_{\text{T}})}{I_y\partial\dot w}\frac{\partial F_z(u,w,q,\delta_{\text{E}},\delta_{\text{T}})}{m\partial w} & \frac{\partial M(u,w,q,\delta_{\text{E}},\delta_{\text{T}})}{I_y\partial q}+\frac{\partial M(u,w,\dot w,q,\delta_{\text{E}},\delta_{\text{T}})}{I_y\partial\dot w}\left(\frac{\partial F_z(u,w,q,\delta_{\text{E}},\delta_{\text{T}})}{m\partial q}+u_{\text{e}}\right)
\end{array}\right], \tag{2.45}
$$

$$
B_{\text{long}} \triangleq \left[\begin{array}{c||cc}
 & \delta_{\text{E}} & \delta_{\text{T}} \\
\hline\hline
\dot\theta & 0 & 0 \\
\dot u & \frac{\partial F_x(u,w,\delta_{\text{E}},\delta_{\text{T}})}{m\partial\delta_{\text{E}}} & \frac{\partial F_x(u,w,\delta_{\text{E}},\delta_{\text{T}})}{m\partial\delta_{\text{T}}} \\
\dot w & \frac{\partial F_z(u,w,q,\delta_{\text{E}},\delta_{\text{T}})}{m\partial\delta_{\text{E}}} & \frac{\partial F_z(u,w,q,\delta_{\text{E}},\delta_{\text{T}})}{m\partial\delta_{\text{T}}} \\
\dot q & \frac{\partial M(u,w,\dot w,q,\delta_{\text{E}},\delta_{\text{T}})}{I_y\partial\delta_{\text{E}}}+\frac{\partial M(u,w,\dot w,q,\delta_{\text{E}},\delta_{\text{T}})}{I_y\partial\dot w}\frac{\partial F_z(u,w,q,\delta_{\text{E}},\delta_{\text{T}})}{m\partial\delta_{\text{E}}} & \frac{\partial M(u,w,\dot w,q,\delta_{\text{E}},\delta_{\text{T}})}{I_y\partial\delta_{\text{T}}}+\frac{\partial M(u,w,\dot w,q,\delta_{\text{E}},\delta_{\text{T}})}{I_y\partial\dot w}\frac{\partial F_z(u,w,q,\delta_{\text{E}},\delta_{\text{T}})}{m\partial\delta_{\text{T}}}
\end{array}\right], \tag{2.46}
$$

$$
A_{\mathrm{lat}} \triangleq \left[\begin{array}{c||ccc}
 & \phi & v & \cdots \\
\hline\hline
\dot{\phi} & 0 & 0 & \cdots \\
\dot{v} & g\cos\theta_{\mathrm{e}} & \frac{\partial F_y(v,p,r,\delta_{\mathrm{R}})}{m\partial v} & \cdots \\
\dot{p} & 0 & -\frac{I_{xz}}{I_{xz}^2-I_xI_z}\frac{\partial N(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial v}-\frac{I_z}{I_{xz}^2-I_xI_z}\frac{\partial L(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial v} & \cdots \\
\dot{r} & 0 & -\frac{I_x}{I_{xz}^2-I_xI_z}\frac{\partial N(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial v}-\frac{I_{xz}}{I_{xz}^2-I_xI_z}\frac{\partial L(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial v} & \cdots
\end{array}\right.
$$

$$
\left.\begin{array}{c||ccc}
 & \cdots & p & r \\
\hline\hline
\dot{\phi} & \cdots & 1 & \tan\theta_{\mathrm{e}} \\
\dot{v} & \cdots & \frac{\partial F_y(v,p,r,\delta_{\mathrm{R}})}{m\partial p} & \frac{\partial F_y(v,p,r,\delta_{\mathrm{R}})}{m\partial r}-u_{\mathrm{e}} \\
\dot{p} & \cdots & -\frac{I_{xz}}{I_{xz}^2-I_xI_z}\frac{\partial N(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial p}-\frac{I_z}{I_{xz}^2-I_xI_z}\frac{\partial L(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial p} & -\frac{I_{xz}}{I_{xz}^2-I_xI_z}\frac{\partial N(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial r}-\frac{I_z}{I_{xz}^2-I_xI_z}\frac{\partial L(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial r} \\
\dot{r} & \cdots & -\frac{I_{xz}}{I_{xz}^2-I_xI_z}\frac{\partial L(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial p}-\frac{I_x}{I_{xz}^2-I_xI_z}\frac{\partial N(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial p} & -\frac{I_x}{I_{xz}^2-I_xI_z}\frac{\partial N(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial r}-\frac{I_{xz}}{I_{xz}^2-I_xI_z}\frac{\partial L(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial r}
\end{array}\right], \tag{2.47}
$$

$$
B_{\mathrm{lat}} \triangleq \left[\begin{array}{c||cc}
 & \delta_{\mathrm{A}} & \delta_{\mathrm{R}} \\
\hline\hline
\dot{\phi} & 0 & 0 \\
\dot{v} & 0 & \frac{\partial F_y(v,p,r,\delta_{\mathrm{R}})}{m\partial\delta_{\mathrm{R}}} \\
\dot{p} & -\frac{I_{xz}}{I_{xz}^2-I_xI_z}\frac{\partial N(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\delta_{\mathrm{A}}}-\frac{I_z}{I_{xz}^2-I_xI_z}\frac{\partial L(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\delta_{\mathrm{A}}} & \frac{I_{xz}}{I_{xz}^2-I_xI_z}\frac{\partial N(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\delta_{\mathrm{R}}}-\frac{I_z}{I_{xz}^2-I_xI_z}\frac{\partial L(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\delta_{\mathrm{R}}} \\
\dot{r} & -\frac{I_x}{I_{xz}^2-I_xI_z}\frac{\partial N(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\delta_{\mathrm{A}}}-\frac{I_{xz}}{I_{xz}^2-I_xI_z}\frac{\partial L(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\delta_{\mathrm{A}}} & -\frac{I_x}{I_{xz}^2-I_xI_z}\frac{\partial N(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\delta_{\mathrm{R}}}-\frac{I_{xz}}{I_{xz}^2-I_xI_z}\frac{\partial L(v,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\delta_{\mathrm{R}}}
\end{array}\right]. \tag{2.48}
$$

2.7 Analysis of the Longitudinal Dynamics of an Aircraft

Consider the *uncontrolled* linear dynamical system

$$\frac{d}{dt}\begin{bmatrix}\theta(t)\\ u(t)\\ \alpha(t)\\ q(t)\end{bmatrix} = \hat{A}_{\mathrm{long}}\begin{bmatrix}\theta(t)\\ u(t)\\ \alpha(t)\\ q(t)\end{bmatrix}, \qquad \begin{bmatrix}\theta(0)\\ u(0)\\ \alpha(0)\\ q(0)\end{bmatrix} = \begin{bmatrix}\theta_0-\theta_{\mathrm{e}}\\ u_0-u_{\mathrm{e}}\\ \alpha_0\\ q_0\end{bmatrix}, \qquad t\geq 0, \tag{2.49}$$

where $\hat{A}_{\mathrm{long}}$ is given by (2.53). It follows from Theorem A.3 that

$$\begin{bmatrix}\theta(t)\\ u(t)\\ \alpha(t)\\ q(t)\end{bmatrix} = e^{\hat{A}_{\mathrm{long}}t}\begin{bmatrix}\theta_0-\theta_{\mathrm{e}}\\ u_0-u_{\mathrm{e}}\\ \alpha_0\\ q_0\end{bmatrix}, \qquad t\geq 0, \tag{2.50}$$

and it follows from Theorems A.7 and A.32 and Definitions A.15 and A.34 that

$$\mathcal{L}[e^{\hat{A}_{\mathrm{long}}t}] = \frac{1}{\chi_{A_{\mathrm{long}}(s)}}C^{\mathrm{T}}_{(sI-A_{\mathrm{long}})}, \qquad s\in\mathbb{C}, \tag{2.51}$$

where $\mathcal{L}[\cdot]$ denotes the Laplace transform operator, $\chi_{\hat{A}_{\mathrm{long}}}(s)$, denotes the characteristic polynomial of $\hat{A}_{\mathrm{long}}$, and $C_{(sI-\hat{A}_{\mathrm{long}})}$ denotes the cofactor matrix of $(sI-\hat{A}_{\mathrm{long}})$.

Since $\hat{A}_{\mathrm{long}}\in\mathbb{R}^{4\times 4}$, $\chi_{\hat{A}_{\mathrm{long}}}(s)$, $s\in\mathbb{C}$, is a fourth order polynomial and it follows from Remark A.3 that $\chi_{\hat{A}_{\mathrm{long}}}(s)$ can be expressed as

(i) the product of two second-order polynomials, whose roots are complex, or
(ii) the product of a second-order polynomial, whose roots are complex, and two monomials, which roots are real, or
(iii) the product of four monomials, whose roots are real.

In most cases of practical interest, $\hat{A}_{\mathrm{long}}$ has two pairs of complex conjugate eigenvalues with negative real part. As discussed in Sect. A.8.2, the effect of those eigenvalues of $\hat{A}_{\mathrm{long}}$ with smaller real part fades before the effect of eigenvalues with larger negative real part. Indeed, Frederick Lanchester distinguished between the "short-period" and the "long period" longitudinal dynamics of an aircraft.

The short-period dynamics is characterized by the pair of complex conjugate eigenvalues, whose real part is negative and larger in absolute value. The long-period dynamics is characterized by the pair of complex conjugate eigenvalues, whose real part is negative and smaller in absolute value. The pair of eigenvalues with smaller real part is generally associated to $[\alpha(\cdot), q(\cdot)]^{\mathrm{T}}$ and the corresponding eigenvectors have imaginary part *almost* equal to zero. Thus, the short-period dynamics is

$$\frac{d}{dt}\begin{bmatrix}\alpha(t)\\ q(t)\end{bmatrix} = A_{\mathrm{long,sp}}\begin{bmatrix}\alpha(t)\\ q(t)\end{bmatrix} + B_{\mathrm{long,sp}}\begin{bmatrix}\delta_{\mathrm{E}}(t)\\ \delta_{\mathrm{T}}(t)\end{bmatrix}, \qquad \begin{bmatrix}\alpha(0)\\ q(0)\end{bmatrix} = \begin{bmatrix}\alpha_0\\ q_0\end{bmatrix}, \qquad t\geq 0, \tag{2.52}$$

where $A_{\mathrm{long,sp}}$ is given by (2.63) and $B_{\mathrm{long,sp}}$ is given by (2.64). Specifically, (2.52)

$$\hat{A}_{\text{long}} = \left[\begin{array}{c||ccc}
 & \theta & u & \dots \\
\hline\hline
\dot{\theta} & 0 & 0 & \dots \\
\dot{u} & -g\cos\theta_{\text{e}} & \frac{\partial F_x(u,\alpha,\delta_{\text{E}},\delta_{\text{T}})}{m\partial u} & \dots \\
\dot{\alpha} & -\frac{g\sin\theta_{\text{e}}}{u_{\text{e}}} & \frac{\partial F_z(u,\alpha,q,\delta_{\text{E}},\delta_{\text{T}})}{mu_{\text{e}}\partial u} & \dots \\
\dot{q} & -\frac{g\sin\theta_{\text{e}}}{u_{\text{e}}}\frac{\partial M(u,\alpha,\dot{\alpha},q,\delta_{\text{E}},\delta_{\text{T}})}{I_y\partial\dot{\alpha}} & \frac{\partial M(u,\alpha,\dot{\alpha},q,\delta_{\text{E}},\delta_{\text{T}})}{I_y\partial u} + \frac{\partial M(u,\alpha,\dot{\alpha},q,\delta_{\text{E}},\delta_{\text{T}})}{I_y u_{\text{e}}\partial\dot{\alpha}}\frac{\partial F_z(u,\alpha,q,\delta_{\text{E}},\delta_{\text{T}})}{m\partial u} & \dots
\end{array}\right.$$

$$\left.\begin{array}{c||ccc}
 & \dots & \alpha & q \\
\hline\hline
\dot{\theta} & \dots & 0 & 1 \\
\dot{u} & \dots & \frac{\partial F_x(u,\alpha,\delta_{\text{E}},\delta_{\text{T}})}{m\partial\alpha} & 0 \\
\dot{\alpha} & \dots & \frac{\partial F_z(u,\alpha,q,\delta_{\text{E}},\delta_{\text{T}})}{mu_{\text{e}}\partial\alpha} & \frac{\partial F_z(u,\alpha,q,\delta_{\text{E}},\delta_{\text{T}})}{mu_{\text{e}}\partial q}+1 \\
\dot{q} & \dots & \frac{\partial M(u,\alpha,\dot{\alpha},q,\delta_{\text{E}},\delta_{\text{T}})}{I_y\partial\alpha} + \frac{\partial M(u,\alpha,\dot{\alpha},q,\delta_{\text{E}},\delta_{\text{T}})}{I_y\partial\dot{\alpha}}\frac{\partial F_z(u,\alpha,q,\delta_{\text{E}},\delta_{\text{T}})}{mu_{\text{e}}\partial\alpha} & \frac{\partial M(u,\alpha,q,\delta_{\text{E}},\delta_{\text{T}})}{I_y\partial q} + \frac{\partial M(u,\alpha,\dot{\alpha},q,\delta_{\text{E}},\delta_{\text{T}})}{I_y u_{\text{e}}\partial\dot{\alpha}}\left(\frac{\partial F_z(u,\alpha,q,\delta_{\text{E}},\delta_{\text{T}})}{m\partial q}+u_{\text{e}}\right)
\end{array}\right], \tag{2.53}$$

$$\hat{B}_{\text{long}} = \left[\begin{array}{c||cc}
 & \delta_{\text{E}} & \delta_{\text{T}} \\
\hline\hline
\dot{\theta} & 0 & 0 \\
\dot{u} & \frac{\partial F_x(u,\alpha,\delta_{\text{E}},\delta_{\text{T}})}{m\partial\delta_{\text{E}}} & \frac{\partial F_x(u,\alpha,\delta_{\text{E}},\delta_{\text{T}})}{m\partial\delta_{\text{T}}} \\
\dot{\alpha} & \frac{\partial F_z(u,\alpha,q,\delta_{\text{E}},\delta_{\text{T}})}{mu_{\text{e}}\partial\delta_{\text{E}}} & \frac{\partial F_z(u,\alpha,q,\delta_{\text{E}},\delta_{\text{T}})}{mu_{\text{e}}\partial\delta_{\text{T}}} \\
\dot{q} & \frac{\partial M(u,\alpha,\dot{\alpha},q,\delta_{\text{E}},\delta_{\text{T}})}{I_y\partial\delta_{\text{E}}} + \frac{\partial M(u,\alpha,\dot{\alpha},q,\delta_{\text{E}},\delta_{\text{T}})}{I_y u_{\text{e}}\partial\dot{\alpha}}\frac{\partial F_z(u,\alpha,q,\delta_{\text{E}},\delta_{\text{T}})}{m\partial\delta_{\text{E}}} & \frac{\partial M(u,\alpha,\dot{\alpha},q,\delta_{\text{E}},\delta_{\text{T}})}{I_y\partial\delta_{\text{T}}} + \frac{\partial M(u,\alpha,\dot{\alpha},q,\delta_{\text{E}},\delta_{\text{T}})}{I_y u_{\text{e}}\partial\dot{\alpha}}\frac{\partial F_z(u,\alpha,q,\delta_{\text{E}},\delta_{\text{T}})}{m\partial\delta_{\text{T}}}
\end{array}\right], \tag{2.54}$$

$$
\hat{A}_{\mathrm{lat}} = \left[\begin{array}{c||ccc}
 & \phi & \beta & \dots \\
\hline\hline
\dot\phi & 0 & 0 & \dots \\
\dot\beta & \frac{g\cos\theta_{\mathrm{e}}}{u_{\mathrm{e}}} & \frac{\partial F_y(\beta,p,r,\delta_{\mathrm{R}})}{mu_{\mathrm{e}}\partial\beta} & \dots \\
\dot p & 0 & -\frac{I_{xz}}{I_{xz}^2-I_xI_z}\frac{\partial N(\beta,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\beta}-\frac{I_z}{I_{xz}^2-I_xI_z}\frac{\partial L(\beta,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\beta} & \dots \\
\dot r & 0 & -\frac{I_x}{I_{xz}^2-I_xI_z}\frac{\partial N(\beta,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\beta}-\frac{I_{xz}}{I_{xz}^2-I_xI_z}\frac{\partial L(\beta,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\beta} & \dots
\end{array}\right.
$$

$$
\left.\begin{array}{c||ccc}
 & \dots & p & r \\
\hline\hline
\dot\phi & \dots & 1 & \tan\theta_{\mathrm{e}} \\
\dot\beta & \dots & \frac{\partial F_y(\beta,p,r,\delta_{\mathrm{R}})}{mu_{\mathrm{e}}\partial p} & \frac{\partial F_y(\beta,p,r,\delta_{\mathrm{R}})}{mu_{\mathrm{e}}\partial r}-1 \\
\dot p & \dots & -\frac{I_{xz}}{I_{xz}^2-I_xI_z}\frac{\partial N(\beta,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial p}-\frac{I_z}{I_{xz}^2-I_xI_z}\frac{\partial L(\beta,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial p} & -\frac{I_{xz}}{I_{xz}^2-I_xI_z}\frac{\partial N(\beta,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial r}-\frac{I_z}{I_{xz}^2-I_xI_z}\frac{\partial L(\beta,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial r} \\
\dot r & \dots & -\frac{I_{xz}}{I_{xz}^2-I_xI_z}\frac{\partial L(\beta,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial p}-\frac{I_x}{I_{xz}^2-I_xI_z}\frac{\partial N(\beta,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial p} & -\frac{I_x}{I_{xz}^2-I_xI_z}\frac{\partial N(\beta,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial r}-\frac{I_{xz}}{I_{xz}^2-I_xI_z}\frac{\partial L(\beta,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial r}
\end{array}\right], \tag{2.55}
$$

$$
\hat{B}_{\mathrm{lat}} = \left[\begin{array}{c||cc}
 & \delta_{\mathrm{A}} & \delta_{\mathrm{R}} \\
\hline\hline
\dot\phi & 0 & 0 \\
\dot\beta & 0 & \frac{\partial F_y(\beta,p,r,\delta_{\mathrm{R}})}{mu_{\mathrm{e}}\partial\delta_{\mathrm{R}}} \\
\dot p & -\frac{I_{xz}}{I_{xz}^2-I_xI_z}\frac{\partial N(\beta,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\delta_{\mathrm{A}}}-\frac{I_z}{I_{xz}^2-I_xI_z}\frac{\partial L(\beta,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\delta_{\mathrm{A}}} & \frac{I_{xz}}{I_{xz}^2-I_xI_z}\frac{\partial N(\beta,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\delta_{\mathrm{R}}}-\frac{I_z}{I_{xz}^2-I_xI_z}\frac{\partial L(\beta,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\delta_{\mathrm{R}}} \\
\dot r & -\frac{I_x}{I_{xz}^2-I_xI_z}\frac{\partial N(\beta,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\delta_{\mathrm{A}}}-\frac{I_{xz}}{I_{xz}^2-I_xI_z}\frac{\partial L(\beta,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\delta_{\mathrm{A}}} & -\frac{I_x}{I_{xz}^2-I_xI_z}\frac{\partial N(\beta,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\delta_{\mathrm{R}}}-\frac{I_{xz}}{I_{xz}^2-I_xI_z}\frac{\partial L(\beta,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\delta_{\mathrm{R}}}
\end{array}\right]. \tag{2.56}
$$

follows from (2.43) assuming that the variations with respect to u are negligible and $\theta_{\mathrm{e}} = 0$. Furthermore, in (2.63) we assumed that $\frac{\partial F_z(u,\alpha,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{\partial q} = 0$ for simplicity of exposition. If $\frac{\partial F_z(u,\alpha,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{m\partial \alpha} \neq 0$, then the uncontrolled short-period dynamics of an aicraft is *not* reducible to a double integrator, that is, a system in the form

$$\frac{d}{dt}\begin{bmatrix}\alpha(t)\\ q(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ k_1 & k_2\end{bmatrix}\begin{bmatrix}\alpha(t)\\ q(t)\end{bmatrix}, \qquad \begin{bmatrix}\alpha(0)\\ q(0)\end{bmatrix} = \begin{bmatrix}\alpha_0\\ q_0\end{bmatrix}, \qquad t \geq 0, \tag{2.57}$$

where $k_1, k_2 \in \mathbb{R}$. Although the natural frequency is defined for dynamical systems in the form (2.57), it is customary to define the natural frequency for the aircraft longitudinal dynamics as

$$\omega_{\mathrm{n,sp}} = \left[\frac{\partial F_z(u,\alpha,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{m u_{\mathrm{e}}\partial \alpha}\frac{\partial M(u,\alpha,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{I_y \partial q} - \frac{\partial M(u,\alpha,\dot{\alpha},q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{I_y \partial \alpha}\right]^{\frac{1}{2}} \tag{2.58}$$

and the damping ratio as

$$\zeta_{\mathrm{sp}} = -\frac{1}{2\omega_{\mathrm{n,sp}}}\left[\frac{\partial F_z(u,\alpha,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{m u_{\mathrm{e}}\partial \alpha} + \frac{\partial M(u,\alpha,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{I_y \partial q} + \frac{\partial M(u,\alpha,\dot{\alpha},q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{I_y \partial \dot{\alpha}}\right]. \tag{2.59}$$

If the short-period longitudinal dynamics of an aircraft is characterized by high natural frequency and high damping ratio, then this aircraft responses faster to elevator deflections without excessive overshoot. Conversely, if the short-period natural frequency and damping ratio are small, then an aircraft is difficult to control.

The pair of eigenvalues with larger real part is generally associated to $[u(\cdot), \theta(\cdot)]^{\mathrm{T}}$ and the corresponding eigenvectors are *almost* orthogonal. Long period dynamics is also referred to *phugoid*.

Exercise 2.5 Prove that the long-period dynamics is captured by

$$\frac{d}{dt}\begin{bmatrix}u(t)\\ \theta(t)\end{bmatrix} = A_{\mathrm{long,phugoid}}\begin{bmatrix}u(t)\\ \theta(t)\end{bmatrix} + B_{\mathrm{long,phugoid}}\begin{bmatrix}\delta_{\mathrm{E}}(t)\\ \delta_{\mathrm{T}}(t)\end{bmatrix}, \quad \begin{bmatrix}u(0)\\ \theta(0)\end{bmatrix} = \begin{bmatrix}u_0 - u_{\mathrm{e}}\\ \theta_0\end{bmatrix}, \quad t \geq 0, \tag{2.60}$$

where $A_{\mathrm{long,phugoid}}$ is given by (2.65) and $B_{\mathrm{long,phugoid}}$ is given by (2.66) assuming that variations with respect to α, $\dot{\alpha}$, and q are negligible, and $\theta_{\mathrm{e}} = 0$. *Hint:* Note that $q(t) = \dot{\theta}(t)$, $t \geq 0$. △

It follows from Exercise A.16 that (2.60) is equivalent to a second-order linear dynamical system, whose natural frequency is given by

$$\omega_{\mathrm{n,phugoid}} = \left[-g\frac{\partial F_z(u,\alpha,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{m u_{\mathrm{e}}\partial u}\right]^{\frac{1}{2}} \tag{2.61}$$

and whose damping ratio is given by

$$\zeta_{\mathrm{phugoid}} = -\frac{1}{2\omega_{\mathrm{n,phugoid}}}\frac{\partial F_x(u,\alpha,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{m\partial u}. \tag{2.62}$$

$$A_{\mathrm{long,sp}} = \begin{bmatrix} \frac{\partial F_z(u,\alpha,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{mu_{\mathrm{e}}\partial\alpha} & 1 \\ \frac{\partial M(u,\alpha,\dot{\alpha},q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{I_y\partial\alpha} + \frac{\partial M(u,\alpha,\dot{\alpha},q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{I_y\partial\dot{\alpha}} \frac{\partial F_z(u,\alpha,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{mu_{\mathrm{e}}\partial\alpha} & \frac{\partial M(u,\alpha,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{I_y\partial q} + \frac{\partial M(u,\alpha,\dot{\alpha},q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{I_y\partial\dot{\alpha}} \end{bmatrix}, \tag{2.63}$$

$$B_{\mathrm{long,sp}} = \begin{bmatrix} \frac{\partial F_z(u,\alpha,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{mu_{\mathrm{e}}\partial\delta_{\mathrm{E}}} & \frac{\partial F_z(u,\alpha,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{mu_{\mathrm{e}}\partial\delta_{\mathrm{T}}} \\ \frac{\partial M(u,\alpha,\dot{\alpha},q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{I_y\partial\delta_{\mathrm{E}}} + \frac{\partial M(u,\alpha,\dot{\alpha},q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{I_y u_{\mathrm{e}}\partial\dot{\alpha}} \frac{\partial F_z(u,\alpha,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{m\partial\delta_{\mathrm{E}}} & \frac{\partial M(u,\alpha,\dot{\alpha},q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{I_y\partial\delta_{\mathrm{T}}} + \frac{\partial M(u,\alpha,\dot{\alpha},q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{I_y u_{\mathrm{e}}\partial\dot{\alpha}} \frac{\partial F_z(u,\alpha,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{m\partial\delta_{\mathrm{T}}} \end{bmatrix}, \tag{2.64}$$

$$A_{\mathrm{long,phugoid}} = \begin{bmatrix} \frac{\partial F_x(u,\alpha,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{m\partial u} & -g \\ -\frac{\partial F_z(u,\alpha,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{mu_{\mathrm{e}}\partial u} & 0 \end{bmatrix}, \tag{2.65}$$

$$B_{\mathrm{long,phugoid}} = \begin{bmatrix} \frac{\partial F_x(u,\alpha,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{m\partial\delta_{\mathrm{E}}} & \frac{\partial F_x(u,\alpha,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{m\partial\delta_{\mathrm{T}}} \\ -\frac{\partial F_z(u,\alpha,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{mu_{\mathrm{e}}\partial\delta_{\mathrm{E}}} & -\frac{\partial F_z(u,\alpha,q,\delta_{\mathrm{E}},\delta_{\mathrm{T}})}{mu_{\mathrm{e}}\partial\delta_{\mathrm{T}}} \end{bmatrix}, \tag{2.66}$$

$$A_{\mathrm{lat,dr}} = \begin{bmatrix} \frac{\partial F_y(\beta,p,r,\delta_{\mathrm{R}})}{mu_{\mathrm{e}}\partial\beta} & \frac{\partial F_y(\beta,p,r,\delta_{\mathrm{R}})}{mu_{\mathrm{e}}\partial r} - 1 \\ -\frac{I_x}{I_{xz}^2 - I_x I_z} \frac{\partial N(\beta,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\beta} - \frac{I_{xz}}{I_{xz}^2 - I_x I_z} \frac{\partial L(\beta,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\beta} & -\frac{I_x}{I_{xz}^2 - I_x I_z} \frac{\partial N(\beta,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial r} - \frac{I_{xz}}{I_{xz}^2 - I_x I_z} \frac{\partial L(\beta,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial r} \end{bmatrix}, \tag{2.67}$$

$$B_{\mathrm{lat,dr}} = \begin{bmatrix} -\frac{I_x}{I_{xz}^2 - I_x I_z} \frac{\partial N(\beta,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\delta_{\mathrm{A}}} - \frac{I_{xz}}{I_{xz}^2 - I_x I_z} \frac{\partial L(\beta,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\delta_{\mathrm{A}}} & -\frac{I_x}{I_{xz}^2 - I_x I_z} \frac{\partial N(\beta,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\delta_{\mathrm{R}}} - \frac{I_{xz}}{I_{xz}^2 - I_x I_z} \frac{\partial L(\beta,p,r,\delta_{\mathrm{A}},\delta_{\mathrm{R}})}{\partial\delta_{\mathrm{R}}} \\ 0 & \frac{\partial F_y(\beta,p,r,\delta_{\mathrm{R}})}{mu_{\mathrm{e}}\partial\delta_{\mathrm{R}}} \end{bmatrix}. \tag{2.68}$$

Exercise 2.6 Find the linear differential equation that captures the long period longitudinal dynamics of an aircraft for the case $\theta_e \neq 0$. Compute the natural frequency and the damping ratio for this second-order linear dynamical system. △

Remark 2.1 Let λ_i, $i = 1, 2, 3, 4$, be the eigenvalues of the matrix $\hat{A}_{\text{long}} \in \mathbb{R}^{4\times4}$, which captures the linearized longitudinal dynamics of an aircraft in a neighborhood of the trim condition. The eigenvalues of $\hat{A}_{\text{long}}$ come in complex conjugate pairs, that is, $\lambda_1 = \lambda_2^*$ and $\lambda_3 = \lambda_4^*$, which implies that $\sigma_1 = \sigma_2$, $\sigma_3 = \sigma_4$, $\omega_1 = -\omega_2$, and $\omega_3 = -\omega_4$, where σ_i denotes the real part of λ_i, that is, $\sigma_i = \Re(\lambda_i)$, $i = 1, 2, 3, 4$, and ω_i denotes the imaginary part of λ_i, that is, $\omega_i = \Im(\lambda_i)$. Since $\chi_{A_{\text{long}}}(\cdot)$ is a fourth order polynomial, finding λ_i, $i = 1, 2, 3, 4$, is sometimes a demanding task. Now, let $\sigma_1 < \sigma_3 < 0$. In most cases of practical interest, it holds that $\omega_1 \approx \omega_{\text{n,sp}}$ and $\omega_3 \approx \omega_{\text{n,phugoid}}$. Hence, we can estimate ω_1 and ω_3 by computing the natural frequencies of two second-order linear dynamical systems, that is, (2.52) and (2.60). This task is definitely simpler as it is reduces to finding the roots the two second-order characteristic polynomials $\chi_{A_{\text{long,sp}}}(\cdot)$ and $\chi_{A_{\text{long,phugoid}}}(\cdot)$.

The analysis of the longitudinal dynamics of an aircraft is one of the central topics in flight dynamics, which has been addressed both from a system-theoretic perspective and a physical point of view. For details, see [8, 12, 13, 35, 37, 39, 42, 43, 46–48], to mention but a few of the most authoritative sources on this topic.

2.8 Analysis of the Lateral-Directional Dynamics of an Aircraft

Consider the linear dynamical system

$$\frac{d}{dt}\begin{bmatrix}\phi(t)\\ \beta(t)\\ p(t)\\ r(t)\end{bmatrix} = \hat{A}_{\text{lat}}\begin{bmatrix}\phi(t)\\ \beta(t)\\ p(t)\\ r(t)\end{bmatrix}, \quad \begin{bmatrix}\phi(0)\\ \beta(0)\\ p(0)\\ r(0)\end{bmatrix} = \begin{bmatrix}\phi_0\\ \beta_0\\ p_0\\ r_0\end{bmatrix}, \quad t \geq 0, \tag{2.69}$$

where $\hat{A}_{\text{lat}}$ is given by (2.55). It follows from Theorem A.3 that

$$\begin{bmatrix}\phi(t)\\ \beta(t)\\ p(t)\\ r(t)\end{bmatrix} = e^{\hat{A}_{\text{lat}}t}\begin{bmatrix}\phi_0\\ \beta_0\\ p_0\\ r_0\end{bmatrix}, \quad t \geq 0, \tag{2.70}$$

and it follows from Theorems A.7 and A.32 and Definitions A.15 and A.34 that

$$\mathcal{L}[e^{\hat{A}_{\text{lat}}t}] = \frac{1}{\chi_{\hat{A}_{\text{lat}}(s)}} C^{\mathrm{T}}_{(sI-\hat{A}_{\text{lat}})}, \quad s \in \mathbb{C}, \tag{2.71}$$

where $\chi_{\hat{A}_{\rm lat}}(s)$, denotes the characteristic polynomial of $\hat{A}_{\rm lat}$ and $C_{(sI-\hat{A}_{\rm lat})}$ denotes the cofactor matrix of $(sI - \hat{A}_{\rm lat})$.

Since $\hat{A}_{\rm lat} \in \mathbb{R}^{4\times 4}$, $\chi_{\hat{A}_{\rm lat}}(s)$, $s \in \mathbb{C}$, is a fourth order polynomial and it follows from Remark A.3 that $\chi_{\hat{A}_{\rm lat}}(s)$ can be expressed as

(i) the product of two second-order polynomials, whose roots are complex, or
(ii) the product of a second-order polynomial, whose roots are complex, and two monomials, which roots are real, or
(iii) the product of four monomials, whose roots are real.

In most cases of practical interest, $\hat{A}_{\rm lat}$ has one pair of complex conjugate eigenvalues with negative real part, one negative eigenvalue, which is large in absolute value, and one real eigenvalue, which is either positive or negative and small in absolute value. The complex conjugate pair characterizes the aircraft "Dutch roll" dynamics, the real eigenvalue with small negative real part characterizes the "roll" dynamics, and the remaining eigenvalue characterizes the "spiral" dynamics.

The pair of complex conjugate eigenvalues of $\hat{A}_{\rm lat}$ is generally associated to $[\beta(\cdot), r(\cdot)]^{\rm T}$ and the Dutch dynamics roll is captured by

$$\frac{d}{dt}\begin{bmatrix}\beta(t)\\ r(t)\end{bmatrix} = A_{\rm lat,dr}\begin{bmatrix}\beta(t)\\ r(t)\end{bmatrix} + B_{\rm lat,dr}\begin{bmatrix}\delta_{\rm A}(t)\\ \delta_{\rm R}(t)\end{bmatrix}, \qquad \begin{bmatrix}\beta(0)\\ r(0)\end{bmatrix} = \begin{bmatrix}\beta_0\\ r_0\end{bmatrix}, \qquad t \geq 0, \tag{2.72}$$

where $A_{\rm lat,dr}$ is given by (2.67) and $B_{\rm lat,dr}$ is given by (2.68). Equation (2.72) follows from (2.44) assuming that the variations with respect to p are negligible, $\theta_{\rm e} = 0$, and $u_{\rm e}$ is such that $\frac{g}{u_{\rm e}} \approx 0$.

If $\frac{\partial F_y(\beta, p, r, \delta_{\rm R})}{m u_{\rm e} \partial\beta} \neq 0$, then the Dutch roll uncontrolled dynamics is *not* reducible to a double integrator. However, it is customary to define the natural frequency for the aircraft Dutch roll as

$$\begin{aligned}\omega_{\rm n,dr} = \Bigg[&-\frac{\partial F_y(\beta, p, r, \delta_{\rm R})}{m u_{\rm e}\partial\beta}\left(\frac{I_x}{I_{xz}^2 - I_x I_z}\frac{\partial N(\beta, p, r, \delta_{\rm A}, \delta_{\rm R})}{\partial r} + \frac{I_{xz}}{I_{xz}^2 - I_x I_z}\frac{\partial L(\beta, p, r, \delta_{\rm A}, \delta_{\rm R})}{\partial r}\right)\\ &+\left(\frac{\partial F_y(\beta, p, r, \delta_{\rm R})}{m u_{\rm e}\partial r} - 1\right)\left(\frac{I_x}{I_{xz}^2 - I_x I_z}\frac{\partial N(\beta, p, r, \delta_{\rm A}, \delta_{\rm R})}{\partial\beta}\right.\\ &\left.+\frac{I_{xz}}{I_{xz}^2 - I_x I_z}\frac{\partial L(\beta, p, r, \delta_{\rm A}, \delta_{\rm R})}{\partial\beta}\right)\Bigg]^{\frac{1}{2}}\end{aligned} \tag{2.73}$$

and the damping ratio as

$$\begin{aligned}\zeta_{\rm dr} = \frac{1}{2\omega_{\rm n,dr}}\Bigg[&\frac{I_x}{I_{xz}^2 - I_x I_z}\frac{\partial N(\beta, p, r, \delta_{\rm A}, \delta_{\rm R})}{\partial r} + \frac{I_{xz}}{I_{xz}^2 - I_x I_z}\frac{\partial L(\beta, p, r, \delta_{\rm A}, \delta_{\rm R})}{\partial r}\\ &-\frac{\partial F_y(\beta, p, r, \delta_{\rm R})}{m u_{\rm e}\partial\beta}\Bigg].\end{aligned} \tag{2.74}$$

Fig. 2.4 Dihedral angle Γ of an aircraft. This angle is measured positive *upward*

If $\dfrac{\partial F_y(\beta, p, r, \delta_R)}{m u_e \partial\beta} \approx 0$, as in most cases of practical interest, then

$$\omega_{n,dr} = \left[-\left(\frac{I_x}{I_{xz}^2 - I_x I_z} \frac{\partial N(\beta, p, r, \delta_A, \delta_R)}{\partial\beta} + \frac{I_{xz}}{I_{xz}^2 - I_x I_z} \frac{\partial L(\beta, p, r, \delta_A, \delta_R)}{\partial\beta} \right) \right]^{\frac{1}{2}}. \tag{2.75}$$

The term $\frac{\partial N(\beta,p,r,\delta_A,\delta_R)}{\partial\beta}$, which is known as *yaw stiffness*, strongly depends on the vertical stabilizer and is always positive. The term $\frac{\partial L(\beta,p,r,\delta_A,\delta_R)}{\partial\beta}$, which is known as *dihedral effect*, strongly depends on the location of the wing with respect to the fuselage, the dihedral angle, and the sweep angle. The *dihedral angle*, which is shown in Fig. 2.4, is the angle between the plane containing the left wing and the plane containing the $x(\cdot)$ and $y(\cdot)$ axes. The *sweep angle*, which is shown in Fig. 2.5, is the angle between the the left wing leading edge and the $y(\cdot)$ axis. Since I_{xz} is generally positive and small compared to I_x and I_y, it holds that $\frac{I_x}{I_{xz}^2 - I_x I_z} \frac{\partial N(\beta,p,r,\delta_A,\delta_R)}{\partial\beta}$ is large and negative, whereas $\frac{I_{xz}}{I_{xz}^2 - I_x I_z} \frac{\partial L(\beta,p,r,\delta_A,\delta_R)}{\partial\beta}$ is negligible. Hence, (2.75) is well-defined.

Similarly, if $\dfrac{\partial F_y(\beta, p, r, \delta_R)}{m u_e \partial\beta} \approx 0$, then

$$\zeta_{dr} = \frac{1}{2\omega_{n,dr}} \left[\frac{I_x}{I_{xz}^2 - I_x I_z} \frac{\partial N(\beta, p, r, \delta_A, \delta_R)}{\partial r} + \frac{I_{xz}}{I_{xz}^2 - I_x I_z} \frac{\partial L(\beta, p, r, \delta_A, \delta_R)}{\partial r} \right]. \tag{2.76}$$

The term $\frac{\partial N(\beta,p,r,\delta_A,\delta_R)}{\partial r}$, which is known as *damping-in-yaw*, is always negative and large in absolute value. Thus, the Dutch roll damping ratio is well-defined.

Fig. 2.5 Sweep angle Λ of an aircraft. This angle is measured positive *backward*

The small negative eigenvalue of $\hat{A}_{\text{lat}}$ is generally associated to $p(\cdot)$ and the uncontrolled roll dynamics is captured by

$$\frac{dp(t)}{dt} = -\left(\frac{I_{xz}}{I_{xz}^2 - I_x I_z} \frac{\partial N(\beta, p, r, \delta_{\text{A}}, \delta_{\text{R}})}{\partial p} + \frac{I_z}{I_{xz}^2 - I_x I_z} \frac{\partial L(\beta, p, r, \delta_{\text{A}}, \delta_{\text{R}})}{\partial p} \right) p(t),$$
$$p(0) = p_0, \qquad t \geq 0, \tag{2.77}$$

which follows from (2.44) assuming that the variations with respect to β and r are negligible and $\theta_{\text{e}} = 0$. Note that in most cases of practical interest, the *damping-in-roll* derivative $\frac{\partial L(\beta,p,r,\delta_{\text{A}},\delta_{\text{R}})}{\partial p}$ is *constant and negative*. Now, since I_{xz} is generally positive and small compared to I_x and I_y, it holds that $\frac{I_z}{I_{xz}^2 - I_x I_z} \frac{\partial L(\beta,p,r,\delta_{\text{A}},\delta_{\text{R}})}{\partial p}$ is positive, which implies that the roll dynamics is usually stable.

The eigenvalue of A_{lat} that is closest to the imaginary axis is generally associated to $\phi(\cdot)$. In this case, assuming that variations with respect to p are negligible and $\theta_{\text{e}} = 0$, it follows from (2.44) that

$$\frac{d\phi(t)}{dt} = p_0, \qquad \phi(0) = \phi_0, \qquad t \geq 0. \tag{2.78}$$

Since (2.78) is a *kinematic* relation, (2.78) is *not* sufficient to describe the dynamics of an aircraft. To resolve this impasse, we make the following consideration. It follows from (2.16) that

$$\frac{d\psi(t)}{dt} = r(t), \qquad \psi(0) = \psi_0, \qquad t \geq 0, \tag{2.79}$$

in any arbitrarily small neighborhood equilibrium point and it follows from (2.69) that

$$\begin{aligned}\frac{dr(t)}{dt} &= -\left[\frac{I_x}{I_{xz}^2 - I_x I_z}\frac{\partial N(\beta, p, r, \delta_\mathrm{A}, \delta_\mathrm{R})}{\partial \beta} + \frac{I_{xz}}{I_{xz}^2 - I_x I_z}\frac{\partial L(\beta, p, r, \delta_\mathrm{A}, \delta_\mathrm{R})}{\partial \beta}\right]\beta(t) \\ &\quad - \left[\frac{I_x}{I_{xz}^2 - I_x I_z}\frac{\partial N(\beta, p, r, \delta_\mathrm{A}, \delta_\mathrm{R})}{\partial r} + \frac{I_{xz}}{I_{xz}^2 - I_x I_z}\frac{\partial L(\beta, p, r, \delta_\mathrm{A}, \delta_\mathrm{R})}{\partial r}\right] r(t), \\ &\qquad r(0) = r_0, \quad t \geq 0. \end{aligned} \tag{2.80}$$

Now, if $\frac{dr(t)}{dt} = 0$, $t \geq 0$, then

$$\begin{aligned}0 &= -\left[\frac{I_{xz}}{I_{xz}^2 - I_x I_z}\frac{\partial N(\beta, p, r, \delta_\mathrm{A}, \delta_\mathrm{R})}{\partial \beta} + \frac{I_z}{I_{xz}^2 - I_x I_z}\frac{\partial L(\beta, p, r, \delta_\mathrm{A}, \delta_\mathrm{R})}{\partial \beta}\right]\beta(t) \\ &\quad - \left[\frac{I_{xz}}{I_{xz}^2 - I_x I_z}\frac{\partial N(\beta, p, r, \delta_\mathrm{A}, \delta_\mathrm{R})}{\partial r} + \frac{I_x}{I_{xz}^2 - I_x I_z}\frac{\partial L(\beta, p, r, \delta_\mathrm{A}, \delta_\mathrm{R})}{\partial r}\right] r(t). \end{aligned} \tag{2.81}$$

Hence, solving (2.81) for $\beta(\cdot)$, we obtain

$$\begin{aligned}\frac{dr(t)}{dt} &= \frac{I_{xz}\frac{\partial N(\beta,p,r,\delta_\mathrm{A},\delta_\mathrm{R})}{\partial r} + I_x\frac{\partial L(\beta,p,r,\delta_\mathrm{A},\delta_\mathrm{R})}{\partial r}}{I_{xz}\frac{\partial N(\beta,p,r,\delta_\mathrm{A},\delta_\mathrm{R})}{\partial \beta} + I_z\frac{\partial L(\beta,p,r,\delta_\mathrm{A},\delta_\mathrm{R})}{\partial \beta}} \\ &\quad \cdot \left[\frac{I_x}{I_{xz}^2 - I_x I_z}\frac{\partial N(\beta, p, r, \delta_\mathrm{A}, \delta_\mathrm{R})}{\partial \beta} + \frac{I_{xz}}{I_{xz}^2 - I_x I_z}\frac{\partial L(\beta, p, r, \delta_\mathrm{A}, \delta_\mathrm{R})}{\partial \beta}\right] r(t) \\ &\quad - \left[\frac{I_x}{I_{xz}^2 - I_x I_z}\frac{\partial N(\beta, p, r, \delta_\mathrm{A}, \delta_\mathrm{R})}{\partial r} + \frac{I_{xz}}{I_{xz}^2 - I_x I_z}\frac{\partial L(\beta, p, r, \delta_\mathrm{A}, \delta_\mathrm{R})}{\partial r}\right] r(t), \\ &\qquad r(0) = r_0, \quad t \geq 0. \end{aligned} \tag{2.82}$$

Remark 2.2 Let λ_i, $i = 1, 2, 3, 4$, be the eigenvalues of the matrix $\hat{A}_\mathrm{lat} \in \mathbb{R}^{4\times 4}$, which captures the linearized lateral dynamics of an aircraft in a neighborhood of the trim condition. Two eigenvalues of $\hat{A}_\mathrm{lat}$ are real and two eigenvalues are complex conjugate, that is, $\lambda_1 \in \mathbb{R}$, $\lambda_2 \in \mathbb{R}$ and $\lambda_3 = \lambda_4^* \in \mathbb{C}$, which implies that $\sigma_3 = \sigma_4$ and $\omega_3 = -\omega_4$ where σ_i denotes the real part of λ_i, that is, $\sigma_i = \Re(\lambda_i)$, $i = 3, 4$, and ω_i denotes the imaginary part of λ_i, that is, $\omega_i = \Im(\lambda_i)$. Since $\chi_{\hat{A}_\mathrm{lat}}(\cdot)$ is a fourth order polynomial, finding λ_i, $i = 1, 2, 3, 4$, is sometimes a demanding task. However, in most cases of practical interest it holds that

$$\lambda_1 \approx -\left(\frac{I_{xz}}{I_{xz}^2 - I_x I_z}\frac{\partial N(\beta, p, r, \delta_\mathrm{A}, \delta_\mathrm{R})}{\partial p} + \frac{I_z}{I_{xz}^2 - I_x I_z}\frac{\partial L(\beta, p, r, \delta_\mathrm{A}, \delta_\mathrm{R})}{\partial p}\right), \tag{2.83}$$

which is the eigenvalue associated with the roll dynamics,

$$\lambda_2 \approx \frac{I_{xz}\frac{\partial N(\beta,p,r,\delta_A,\delta_R)}{\partial r} + I_x\frac{\partial L(\beta,p,r,\delta_A,\delta_R)}{\partial r}}{I_{xz}\frac{\partial N(\beta,p,r,\delta_A,\delta_R)}{\partial \beta} + I_z\frac{\partial L(\beta,p,r,\delta_A,\delta_R)}{\partial \beta}} \cdot \left[\frac{I_x}{I_{xz}^2 - I_x I_z}\frac{\partial N(\beta, p, r, \delta_A, \delta_R)}{\partial \beta} + \frac{I_{xz}}{I_{xz}^2 - I_x I_z}\frac{\partial L(\beta, p, r, \delta_A, \delta_R)}{\partial \beta}\right] - \left[\frac{I_x}{I_{xz}^2 - I_x I_z}\frac{\partial N(\beta, p, r, \delta_A, \delta_R)}{\partial r} + \frac{I_{xz}}{I_{xz}^2 - I_x I_z}\frac{\partial L(\beta, p, r, \delta_A, \delta_R)}{\partial r}\right], \tag{2.84}$$

which is the eigenvalue associated with the spiral dynamics, and $\omega_3 \approx \omega_{n,dr}$.

The analysis of the lateral-directional dynamics of an aircraft is one of the central topics in flight dynamics, which has been addressed both from a system-theoretic perspective and a more physical point of view. For details, see [8, 12, 13, 35, 37, 39, 42, 43, 46–48], to mention but a few of the most authoritative sources on this topic.

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http://www.springer.com/978-3-319-47466-3

A Mathematical Perspective on Flight Dynamics and Control
L'Afflitto, A.
2017, XII, 122 p. 19 illus., 14 illus. in color., Softcover
ISBN: 978-3-319-47466-3