Chapter 2
Quantum Field Theory Set Up

Some standard introductions to quantum field theory are [1–3], for a particularly diagrammatic approach see [4]. For the reader who is not familiar with these ideas we will briefly go over the intuition of what quantum field theory is along with some of the key vocabulary. Many readers would be safe skipping this chapter either because they are familiar with this material or because they are more interested in the problems which appear later than in their motivation.

Quantum field theory is a framework in which we can understand arbitrary numbers of interacting particles quantum mechanically. It is the standard way to unify quantum mechanics and special relativity. The particles in question can be subatomic particles in high energy physics in which case quantum field theory, through the standard model, describes all known particles extremely well. The particles can also be quasiparticles in condensed matter physics and so quantum field theory is a useful tool for understanding condensed matter systems and the mathematician or mathematical physicist gets new theories to play with.

In either case, the fundamental thing a quantum field theory describes is how particles interact and scatter, so one imagines an idealized experiment where some known particles are sent in, collide and interact in some way, and then what comes out is detected. Since we don't know what happened in the collision we, in the spirit of quantum mechanics, take a weighted sum over all possibilities. Any particular story of what the particles did traces out a graph in spacetime with the interactions as vertices and the edges as particles propagating. Combining together those possibilities which after forgetting the spacetime embedding give the same graph, we obtain Feynman graphs,[1] see Fig. 2.1. See Chap. 5 for precise definitions.

The weight of the graph in the sum is its Feynman integral. The weighted sum itself is a ***perturbative expansion*** for the ***scattering amplitude*** in question. We'll also see this kind of sum, over appropriate graphs, as ***Green functions*** when we come to Dyson-Schwinger equations.

Feynman integrals are, in general, very difficult to compute and there is a whole part of high energy physics devoted to the technique and practice of computing

[1]Feynman graphs drawn with tikz-feynman [5].

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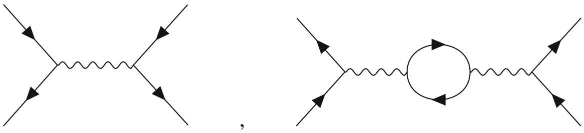

Fig. 2.1 Example Feynman graphs

them, with the practical aim of computing backgrounds for accelerator experiments and making predictions, see for example the proceedings [6]. For the purposes of this brief there are four things which will be important about Feynman integrals. First, one contribution to the Feynman integral is the strength of each interaction which is captured in one or more ***coupling constants***. The coupling constants can be reinterpreted as counting variables. Second, the Feynman integrand expression can be read off the graph with each edge and vertex contributing a factor. The rules to do this are called ***Feynman rules***. Third, in interesting cases these integrals are divergent and so to extract physically meaningful quantities from them they must be ***renormalized***, see Sect. 4.3 for more on renormalization. Finally, the sums of Feynman integrals contributing to a given process are expected to be divergent for all interesting cases.

From a discrete math perspective, taking a Feynman-graphs-first approach to quantum field theory is quite appealing, as we have graphs playing a central role. Furthermore we have series indexed by graphs which are divergent and hence as a first step are reasonably thought of as formal. There are other less apparent reasons why this is a nice perspective for those with discrete tastes: the structure of the renormalization process is captured with a combinatorial Hopf algebra and important integral and differential equations come from decompositions of combinatorial objects, all of which we will investigate over the course of this brief.

There is a downside to a Feynman-graphs-first approach. The series in question are expected to be divergent in the cases that matter and so they can only be asymptotic series for the presumed functions which describe the physical processes in question. That is, a Feynman-graph-first approach is a perturbative approach. By itself a perturbative approach does not have access to any phenomenon which is asymptotically flat at the point around which we are expanding, that is it cannot see the ***instantons*** in the theory or any other nonperturbative phenomenon. Fortunately, we can access these things by the back door: a Feynman-graphs-first approach doesn't mean a Feynman-graphs-only approach. The way to do this is as follows. The recursive structure of the Feynman graphs and the perturbative expansion give us functional equations for the perturbative expansions. Since these underlying structures are not mere combinatorial happenstance but reflect the physics, they also hold non-perturbatively and so the functional equations can be upgraded to non-perturbative equations where they, potentially at least, can see nonperturbative effects. The functional equations of this type we understand best are Dyson-Schwinger equations. That is why Dyson-Schwinger equations are very important in this approach. To date this is a mere sketch and a lot of work remains before these ideas could be used foundationally for quantum field theory.

More traditionally, quantum field theorists escape the limitations of perturbation theory by beginning with non-perturbative definitions and from there deriving Feynman graphs and the perturbative expansion. One popular and important way to do this is via the path integral, see [3] for an introduction. The initial intuition is very much the same—sum over all possibilities—but here we think of the possibilities as arbitrary paths and so the space of possibilities is continuous and infinite dimensional making the "sum" an integral and, because of the infinite dimensionality, not one which is well defined in general. None-the-less it is an approach which captures the physical intuition well and works in practice, so it's important and interesting even without a complete mathematical foundation.

If spacetime is zero dimensional then the path integral is well defined and we get the zero dimensional field theory approach to counting graphs which is used both by physicists and mathematicians, see for example [7, 8].

In higher dimensions the path integral is still a good candidate for viewing combinatorially simply by temporarily forgetting the analytic difficulties and treating it formally. Jackson, Morales and Kempf have been looking at the enumerative combinatorics of quantum field theory from this perspective. So far this collaboration has resulted in [9, 10] with a comprehensive treatment in the works.

In any case, even purely perturbative quantum field theory is extremely useful and full of interesting mathematics, a small part of which we will investigate in what follows.

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