

Chapter 2

Option Pricing and Hedging with Liquidity Costs and Market Impact

F. Abergel and G. Loeper

Abstract We study the influence of taking liquidity costs and market impact into account when hedging a contingent claim. In the continuous time setting and under the assumption of perfect replication, we derive a fully non-linear pricing partial differential equation, and characterize its parabolic nature according to the value of a numerical parameter interpreted as a *relaxation coefficient* for market impact. We also investigate the case of stochastic volatility models with pseudo-optimal strategies.

2.1 Introduction

2.1.1 Position of the Problem

There is a long history of studying the effect of transaction costs and liquidity costs in the context of derivative pricing and hedging. Transaction costs due to the presence of a Bid-Ask spread are well understood in discrete time, see (Lamberton et al. 1997). In continuous time, they lead to quasi-variational inequalities, see e.g. (Zakamouline 2006), and to imperfect claim replication due to the infinite cost of hedging continuously over time. In this work, the emphasis is put rather on **liquidity costs**, that is, the extra price one has to pay over the theoretical price of a tradable asset, due to the finiteness of available liquidity at the best possible price. A reference work for the modelling and mathematical study of liquidity in the context of a dynamic hedging strategy is (Cetin et al. 2004), see also (Roch 2009), and our results can be seen as partially building on the same approach.

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It is however unfortunate that a major drawback occurs when adding liquidity costs: as can easily be seen in (Cetin et al. 2004; Millot and Abergel 2011; Roch 2009), the pricing and hedging equation are not unconditionally parabolic anymore. Note that this sometimes dramatic situation can already be inferred from the early heuristics in Leland (1985): the formula suggested by Leland makes perfectly good sense for small perturbation of the initial volatility, but is meaningless when the modified volatility becomes negative. An answer to this problem is proposed in Cetin et al. (2010), where the authors introduce *super-replicating* strategies and show that the minimal cost of a super-replicating strategy solves a well-posed parabolic equation. In such a case, a perfectly replicating strategy, provided that it exists, may not be the *optimal* strategy, as there may exist a strategy with cheaper initial wealth that super-replicates the payoff at maturity. It appears however that such a situation, where liquidity costs lead to an imperfect replication, is dependent on the assumption one is making regarding the **market impact** of the delta-hedger, as some recent work of one of the author (Loeper 2013) already shows. In this work, we provide necessary and sufficient conditions that ensure the parabolicity of the pricing equation and hence, the existence and uniqueness of a self-financing, perfectly replicating strategy—at least in the complete market case.

Motivated by the need for quantitative approaches to algorithmic trading, the study of market impact in order-driven markets has become a very active research subject in the past decade. In a very elementary way, there always is an instantaneous market impact—termed *virtual impact* in Weber and Rosenow (2005)—whenever a transaction takes place, in the sense that the best available price immediately following a transaction may be modified if the size of the transaction is larger than the quantity available at the best limit in the order book. As many empirical works show, see e.g. (Almgren et al. 2005; Weber and Rosenow 2005), a relaxation phenomenon then takes place: after a trade, the instantaneous impact decreases to a smaller value, the permanent impact. This phenomenon is named **resilience** in Weber and Rosenow (2005), it can be interpreted as a rapid, negatively correlated response of the market to large price changes due to liquidity effects. In the context of derivative hedging, it is clear that there are realistic situations—e.g., a large option on an illiquid stock—where the market impact of an option hedging strategy is significant. This situation has already been addressed by several authors, see in particular (Schönbucher and Wilmott 2000; Frey and Stremme 1997; Frey 1998; Platen and Schweizer 1998), where various hypothesis on the dynamics, the market impact and the hedging strategy are proposed and studied. One may also refer to (Liu and Yong 2005; Roch 2009) for more recent related works. It is however noteworthy that in these references, liquidity costs and market impact are not considered jointly, whereas in fact, the latter is a rather direct consequence of the former. As we shall demonstrate, the level of permanent impact plays a fundamental role in the well-posedness of the pricing and hedging equation, a fact that was overlooked in previous works on liquidity costs and impact. Also, from a practical point of view, it seems relevant to us to relate the well-posedness of the modified Black-Scholes equation to a parameter that can be measured empirically using high frequency data.

2.1.2 Main Results

This paper aims at contributing to the field by laying the grounds for a reasonable yet complete model of liquidity costs and market impact for derivative hedging. Liquidity costs are modelled by a simple, stationary order book, characterized by its shape around the best price, and the permanent market impact is measured by a numerical parameter γ , $0 \leq \gamma \leq 1$: $\gamma = 0$ means no permanent impact, so the order book goes back to its previous state after the transaction is performed, whereas $\gamma = 1$ means no relaxation, the liquidity consumed by the transaction is shifted around the final transaction price. This simplified representation of market impact rests on the realistic hypothesis that the characteristic time of the derivative hedger, although comparable to, may be different from the relaxation time of the order book.

What we consider as our main result is Theorem 2.1, which states that, in the complete market case, the range of parameter for which the pricing equation is unconditionally parabolic is $\frac{2}{3} \leq \gamma \leq 1$. This result, which we find quite nice in that it is explicit in terms of the parameter γ , gives necessary and sufficient conditions for the perfectly replicating strategy to be optimal. It also sheds some interesting light on the ill-posedness of the pricing equations in the references (Cetin et al. 2004; Millot and Abergel 2011) corresponding to the case $\gamma = 0$, or (Liu and Yong 2005) corresponding to the case $\gamma = \frac{1}{2}$ within our formulation. In particular, Theorem 2.1 implies that when re-hedging occurs at the same frequency as that at which liquidity is provided to the order book—that is, when $\gamma = 1$ —the pricing equation is well-posed. Note that there are some recent empirical evidence (Bershova and Rakhlin 2013) as well as a theoretical justification (Farmer et al. 2013) of the fact that the level of permanent impact should actually be equal to $\frac{2}{3}$, in striking compliance with the constraints Theorem 2.1 imposes!

It is of course interesting and important to thoroughly address the case where this condition is violated. If this is the case, see Sect. 2.8.1, one can build an option portfolio having the following property: there exist two european-style claims with terminal payoffs ϕ_1, ϕ_2 such that $\phi_1 \leq \phi_2$ but the perfect replication price of ϕ_1 is strictly greater than that of ϕ_2 . The way out of this paradox should be via an approach similar to that developed in (Çetin et al. 2010), based on super-replication, but the situation is made much more complicated by the fact that, in our model, the dynamics is modified by the strategy, a feature not present in Çetin et al. (2010). We do find it interesting however that the perfect replication is not optimal, and are intrigued by a market where the value of γ would lead to imperfect replication.

Another interesting question is the comparison between our approach and that of (Almgren 2012), where the delta-hedging strategy of a large option trader is addressed. We want to point out that the two problems are tackled under very different sets of hypotheses: essentially, we consider strategies with infinite variation, whereas (Almgren 2012) refers on the contrary, to strategies with bounded variation. From a physical point of view, we deal with re-hedging that occurs at roughly the same frequency as that of the arrival of liquidity in the book, whereas (Almgren 2012)

considers two different time scales, a slow one for the change in the optimal delta, and a fast one for the execution strategy. Hence, our results and models are significantly different.

The paper is organized as follows: after recalling some classical notations and concepts, Sect. 2.4 presents the continuous time model under scrutiny. The pricing and hedging equations are then worked out and characterized in the case of a complete market, in the single asset case in Sect. 2.5, and in the multi-asset case in Sect. 2.6. Section 2.7 touches upon the case of stochastic volatility models, for which partial results are presented. Finally, a short discussion of the two main conditions for Theorem 2.1, viz market impact level and Gamma-constraint, is presented in the concluding Sect. 2.8.

2.2 Basic Notations and Definitions

To ease notations, we will assume throughout the paper that the risk-free interest rate is always 0, and that the assets pay no dividend.

2.2.1 Discrete Time Setting

The tradable asset price is modelled by a positive stochastic process $S = (S_k)_{k=0,\dots,T}$ on a probability space (Ω, \mathcal{F}, P) . The process S is adapted to the filtration $(\mathcal{F}_k)_{k=0,\dots,T}$, where \mathcal{F}_k denotes the σ -field of events observable up to and including time k . Moreover, \mathcal{F}_0 is trivial and $\mathcal{F}_T = \mathcal{F}$.

A contingent claim is a random variable H of the following form $H = \delta^H S_T + \beta^H$ with δ^H and β^H , \mathcal{F}_T -measurable random variables.

A trading strategy Φ is given by two stochastic processes δ and β . δ_k (resp. β_k) is the amount of stock (resp. cash) held during period k , ($= [t_k, t_{k+1})$) and is fixed at the beginning of that period, i.e. we assume that δ_k (resp. β_k) is \mathcal{F}_k -measurable ($k = 0, \dots, T$).

The theoretical value of the portfolio at time k is given by

$$V_k = \delta_k S_k + \beta_k, (k = 1, \dots, T).$$

In order to avoid dealing with several rather involved cases, we assume that no transaction on the stock takes place at maturity: the claim will be settled with whatever position there is in stock, plus a cash adjustment to match its theoretical value (see the discussion in Lamberton et al. 1997, Sect. 4).

For the model to be specified, one must specify some integrability conditions on the various random variables just introduced, see e.g. (Millot 2012; Abergel and Millot 2011). However, since market impact is considered, the dynamics of S is not independent from that of the strategy (δ, β) , so that this set of assumptions can only

be verified a posteriori, once a strategy is chosen. Since our purpose is to use the discrete case as an illustrative example laying the ground for the continuous-time setting, we will not make such conditions more explicit.

2.2.2 Continuous Time Setting

In the continuous case, (Ω, \mathcal{F}, P) is a probability space with a filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying the usual conditions of right-continuity and completeness. $T \in \mathbb{R}^{**+}$ denotes a fixed and finite time horizon. As before, \mathcal{F}_0 is trivial and $\mathcal{F}_T = \mathcal{F}$.

The risky asset $S = (S_t)_{0 \leq t \leq T}$ is a strictly positive, continuous \mathcal{F}_t -semimartingale, and a trading strategy Φ is a pair of càdlàg and adapted processes $\delta = (\delta_t)_{0 \leq t \leq T}$, $\beta = (\beta_t)_{0 \leq t \leq T}$, while a contingent claim is described by a random variable H of the form $H = \delta^H S_T + \beta^H$, δ^H and β^H being \mathcal{F}_T -measurable random variables.

As in the discrete case, some further admissibility conditions must be imposed. One of the important consequences of our main result, Theorem 2.1, will be precisely to give sufficient conditions ensuring that *perfectly replicating* trading strategies are admissible.

2.2.3 Order Book, Liquidity Cost and Impact

Let us first emphasize that we are not pretending to use a realistic order book model here, but rather, a stylized version which can be considered a much simplified yet useful approximation of the way liquidity is provided to the market.

A stationary, symmetric order-book profile is considered around the **logarithm** of the price \hat{S}_t of the asset S at a given time t **before** the option position is delta-hedged—think of \hat{S}_t as a theoretical price in the absence of the option hedger. The relative density $\mu(x) \geq 0$ of the order book is the derivative of the function $M(x) \equiv \int_0^x \mu(t) dt \equiv$ number of shares one can buy (resp. sell) between the prices \hat{S}_t and $\hat{S}_t e^x$ for positive (resp. negative) x .

This choice of representation in logarithmic scale is intended to avoid inconsistencies for large sell transactions.

The instantaneous—*virtual* in the terminology of (Weber and Rosenow 2005)—market impact of a transaction of size ε is then

$$I_{\text{virtual}}(\varepsilon) = \hat{S}_t (e^{M^{-1}(\varepsilon)} - 1), \quad (2.1)$$

it is precisely the difference between the price before and immediately after the transaction is completed.

The level of permanent impact is then measured via a parameter γ :

$$I_{\text{permanent}}(\varepsilon) = \hat{S}_t(e^{\gamma M^{-1}(\varepsilon)} - 1). \quad (2.2)$$

The actual cost of the same transaction is

$$C(\varepsilon) = \hat{S}_t \int_0^\varepsilon e^{M^{-1}(y)} dy. \quad (2.3)$$

Denote by κ the function M^{-1} . Since some of our results in **discrete time** depend on the simplifying assumption that κ is a linear function:

$$\kappa(\varepsilon) \equiv \lambda \varepsilon \quad (2.4)$$

for some $\lambda \in \mathbf{R}$, the computations are worked out explicitly in this setting.

$$I_{\text{virtual}}(\varepsilon) = \hat{S}_t(e^{\lambda \varepsilon} - 1), \quad (2.5)$$

$$I_{\text{permanent}}(\varepsilon) = \hat{S}_t(e^{\gamma \lambda \varepsilon} - 1), \quad (2.6)$$

and

$$C(\varepsilon) = \hat{S}_t \int_0^\varepsilon e^{M^{-1}(y)} dy \equiv \hat{S}_t \frac{(e^{\lambda \varepsilon} - 1)}{\lambda}. \quad (2.7)$$

This simplifying assumption is necessary for the derivation of the dynamic programming principle satisfied by local-risk minimizing strategies, see Sect. 2.3. Note however that this assumption plays no role in the continuous-time case, where the infinitesimal market impact becomes linear, see Eq. (2.24), and only the shape of the order book around 0 is relevant.

2.3 Cost Process with Market Impact in Discrete Time

In this section, we focus on the discrete time case. As said above, the order book is now assumed to be *flat*, so that κ is a linear function as in (2.4).

2.3.1 The Observed Price Dynamics

The model for the dynamics of the observed price—that is, the price S_k that the market can see at every time t_k after the re-hedging is complete—is now presented.

A natural modelling assumption is that the price moves according to the following sequence of events:

- First, it changes under the action of the “market” according to some (positive) stochastic dynamics for the theoretical price increment $\Delta \hat{S}_k$

$$\hat{S}_k \equiv S_{k-1} + \Delta \hat{S}_k \equiv S_{k-1} e^{\Delta M_k + \Delta A_k}, \quad (2.8)$$

where ΔM_k (resp. ΔA_k) is the increment of an \mathcal{F} -martingale (resp. an \mathcal{F} -predictable process).

- Then, the hedger applies some extra pressure by re-hedging her position, being thereby subject to liquidity costs and market impact as introduced in Sect. 2.2. As a consequence, the dynamics of the observed price is

$$S_k = S_{k-1} e^{\Delta M_k + \Delta A_k} e^{\gamma \lambda (\delta_k - \delta_{k-1})}. \quad (2.9)$$

Since this model is “exponential-linear”—a consequence of the assumption that κ is linear—this expression can be simplified to give

$$S_k = S_0 e^{M_k + A_k} e^{\gamma \lambda \delta_k}. \quad (2.10)$$

with the convention that M, A, δ are equal to 0 for $k = 0$.

2.3.2 Incremental Cost and Optimal Hedging Strategy

Following the approach developed in Millot and Abergel (2011), the incremental cost ΔC_k of re-hedging at time t_k is now studied. The strategy associated to the pair of processes β, δ consists in buying $\delta_k - \delta_{k-1}$ shares of the asset and rebalancing the cash account from β_{k-1} to β_k at the beginning of each hedging period $[t_k, t_{k+1})$. With the notations just introduced in Sect. 2.3.1, there holds

$$\Delta C_k = \hat{S}_k \frac{(e^{\lambda(\delta_k - \delta_{k-1})} - 1)}{\lambda} + (\beta_k - \beta_{k-1}). \quad (2.11)$$

Upon using a quadratic criterion, and under some assumptions ensuring the convexity of the quadratic risk, see e.g. (Millot and Abergel 2011), one easily derives the two (pseudo-)optimality conditions for local risk minimization

$$E(\Delta C_k | \mathcal{F}_{k-1}) = 0 \quad (2.12)$$

and

$$E((\Delta C_k)(\hat{S}_k(\gamma + (1 - \gamma)e^{\lambda(\delta_k - \delta_{k-1})}))) | \mathcal{F}_{k-1}) = 0,$$

where one must be careful to differentiate \hat{S}^k with respect to δ_{k-1} , see (2.10).

This expression is now transformed—using the martingale condition (2.12) and the observed price (2.10)—into

$$E((\Delta C_k)(S_k e^{-\lambda \gamma (\delta_k - \delta_{k-1})} (\gamma + (1 - \gamma) e^{\lambda (\delta_k - \delta_{k-1})})) | \mathcal{F}_{k-1}) = 0 \quad (2.13)$$

Equation (2.13) can be better understood—especially when passing to the continuous time limit—by introducing a modified price process accounting for the cumulated effect of liquidity costs and market impact, as in (Milot and Abergel 2011; Cetin et al. 2004). To this end, we introduce the

Definition 2.1 The supply price \bar{S} is the process defined by

$$\bar{S}_0 = S_0 \quad (2.14)$$

and, for $k \geq 1$,

$$\bar{S}_k - \bar{S}_{k-1} = S_k e^{-\lambda \gamma (\delta_k - \delta_{k-1})} (\gamma + (1 - \gamma) e^{\lambda (\delta_k - \delta_{k-1})}) - S_{k-1}. \quad (2.15)$$

Then, the orthogonality condition (2.13) is equivalent to

$$E((\Delta C_k)(\bar{S}_k - \bar{S}_{k-1}) | \mathcal{F}_{k-1}) = 0. \quad (2.16)$$

It is classical—and somewhat more natural—to use the portfolio value process

$$V_k = \beta_k + \delta_k S_k, \quad (2.17)$$

so that one can then rewrite the incremental cost in (2.11) as

$$\Delta C_k = (V_k - V_{k-1}) - (\delta_k S_k - \delta_{k-1} S_{k-1}) + \hat{S}_k \frac{(e^{\lambda (\delta_k - \delta_{k-1})} - 1)}{\lambda}, \quad (2.18)$$

or equivalently

$$\Delta C_k = (V_k - V_{k-1}) - \delta_{k-1} (S_k - S_{k-1}) + S_k \left(\frac{e^{\lambda (\delta_k - \delta_{k-1})} - 1}{\lambda e^{\gamma \lambda (\delta_k - \delta_{k-1})}} - (\delta_k - \delta_{k-1}) \right). \quad (2.19)$$

To ease the notations, let us define, for $x \in \mathbf{R}$,

$$\mathbf{g}(x) \equiv \frac{e^{\lambda x} - 1}{\lambda e^{\gamma \lambda x}} - x. \quad (2.20)$$

The function \mathbf{g} is smooth and satisfies

$$\mathbf{g}(0) = \mathbf{g}'(0) = 0, \quad \mathbf{g}''(0) = (1 - 2\gamma)\lambda. \quad (2.21)$$

As a consequence, the incremental cost of implementing a hedging strategy at time t_k has the following expression

$$\Delta C_k = (V_k - V_{k-1}) - \delta_{k-1} (S_k - S_{k-1}) + S_k \mathbf{g}(\delta_k - \delta_{k-1}), \quad (2.22)$$

and Eq. (2.13) can be rewritten using the value process V and the supply price process \bar{S} as

$$E((V_k - V_{k-1} - \delta_{k-1}(S_k - S_{k-1}) + S_k \mathbf{g}(\delta_k - \delta_{k-1}))(\bar{S}_k - \bar{S}_{k-1}) | \mathcal{F}_{k-1}) = 0. \quad (2.23)$$

One can easily notice that Eqs. (2.12) and (2.13) reduce exactly to Eq. (2.1) in (Milot and Abergel 2011) when market impact is neglected ($\gamma = 0$) and the risk function is quadratic.

2.4 The Continuous-Time Setting

This section is devoted to the characterization of the limiting equation for the value and the hedge parameter when the time step goes to zero. Since the proofs are identical to those given in (Abergel and Milot 2011; Milot and Abergel 2011), we shall only provide formal derivations, limiting ourselves to the case of (continuous) Itô semi-martingales for the driving stochastic equations. However, in the practical situations considered in this paper, in particular those covered in Theorem 2.1, necessary and sufficient conditions are given that ensure the well-posedness in the classical sense of the strategy-dependent stochastic differential equations determining the price, value and cost processes, so that the limiting arguments can be made perfectly rigorous.

2.4.1 The Observed Price Dynamics

A first result concerns the dynamics of the observed price. Assuming that the underlying processes are continuous and taking limits in *ucp* topology, one shows that the continuous-time equivalent of (2.10) is

$$dS_t = S_t(dX_t + dA_t + \gamma \lambda d\delta_t) \quad (2.24)$$

where X is a continuous martingale and A is a continuous, predictable process of bounded variation.

Equation (2.24) is fundamental in that it contains the information on the strategy-dependent volatility of the observed price that will lead to fully non-linear parabolic pricing equation. In fact, the following result holds true:

Lemma 2.1 *Consider a hedging strategy δ which is a function of time and the observed price S at time t : $\delta_t \equiv \delta(S_t, t)$. Then, the observed price dynamics (2.24) can be rewritten as*

$$(1 - \gamma \lambda S_t \frac{\partial \delta}{\partial S}) \frac{dS_t}{S_t} = dX_t + dA'_t, \quad (2.25)$$

where A' is another predictable, continuous process of bounded variation.

Proof Use Itô's lemma in Eq. (2.24).

2.4.2 Cost of a Strategy and Optimality Conditions

At this stage, we are not concerned with the actual optimality—with respect to local-risk minimization—of pseudo-optimal solutions, but rather, with pseudo-optimality in continuous time. Hence, we shall use Eqs. (2.12) and (2.23) as a starting point when passing to the continuous time limit.

Thanks to $\mathbf{g}'(0) = 0$, there holds the

Proposition 2.1 *The cost process of an admissible hedging strategy (δ, V) is given by*

$$C_t \equiv \int_0^t (dV_u - \delta dS_u + \frac{1}{2} S_u \mathbf{g}''(0) d) < \delta, \delta >_u). \quad (2.26)$$

Moreover, an admissible strategy is (pseudo-)optimal iff it satisfies the two conditions

- C is a martingale
- C is orthogonal to the supply price process \bar{S} , with

$$d\bar{S}_t = dS_t + S_t((1 - 2\gamma)\lambda d\delta_t + \mu d) < \delta, \delta >_t) \quad (2.27)$$

$$\text{and } \mu = \frac{1}{2}(\lambda^2(\gamma^3 + (1 - \gamma)^3)).$$

In particular, if C is pseudo-optimal, there holds that

$$d < C, \bar{S} >_t \equiv d < V, S >_t - \delta d < S, S >_t + (1 - 2\gamma)\lambda S_t d < V, \delta >_t - \delta S_t(1 - 2\gamma)\lambda d < \delta, S >_t = 0. \quad (2.28)$$

2.5 Complete Market: The Single Asset Case

It is of course interesting and useful to fully characterize the hedging and pricing strategy in the case of a complete market. Hence, we assume in this section that the driving factor X is a one-dimensional Wiener process W and that \mathcal{F} is its natural filtration, so that the increment of the observed price is simply

$$dS_t = S_t(\sigma dW_t + \gamma\lambda d\delta_t + dA_t) \quad (2.29)$$

where the “unperturbed” volatility σ is supposed to be constant. We also make the Markovian assumption that the strategy is a function of the state variable S and of time.

Under this set of assumptions, perfect replication is considered: the cost process C has to be identically 0, and Eq. (2.26) yields the two conditions

$$\frac{\partial V}{\partial S} = \delta, \quad (2.30)$$

and

$$\frac{\partial V}{\partial t} + \frac{1}{2} \left(\frac{\partial^2 V}{\partial S^2} + S_t \mathbf{g}''(0) \left(\frac{\partial^2 V}{\partial S^2} \right)^2 \right) \frac{d < S, S >_t}{dt} = 0. \quad (2.31)$$

Applying Lemma 2.1 yields

$$(1 - \gamma \lambda S_t \frac{\partial \delta}{\partial S}) \frac{d S_t}{S_t} = \sigma d W_t + d A'_t \quad (2.32)$$

leading to

$$\frac{d < S, S >_t}{dt} = \frac{\sigma^2 S_t^2}{(1 - \gamma \lambda S_t \frac{\partial \delta}{\partial S})^2}. \quad (2.33)$$

Hence, taking (2.30) into account, there holds

$$\frac{\partial V}{\partial t} + \frac{1}{2} \left(\frac{\partial^2 V}{\partial S^2} + \mathbf{g}''(0) S_t \left(\frac{\partial^2 V}{\partial S^2} \right)^2 \right) \frac{\sigma^2 S_t^2}{(1 - \gamma \lambda S_t \frac{\partial \delta}{\partial S})^2} = 0 \quad (2.34)$$

or, using (2.30) and the identity $\mathbf{g}''(0) = (1 - 2\gamma)\lambda$:

$$\frac{\partial V}{\partial t} + \frac{1}{2} \left(\frac{\partial^2 V}{\partial S^2} \left(1 + (1 - 2\gamma)\lambda S_t \frac{\partial^2 V}{\partial S^2} \right) \right) \frac{\sigma^2 S_t^2}{(1 - \gamma \lambda S_t \frac{\partial^2 V}{\partial S^2})^2} = 0. \quad (2.35)$$

Equation (2.35) can be seen as the pricing equation in our model: any contingent claim can be perfectly replicated at zero cost, as long as one can exhibit a solution to (2.35). Consequently, of the utmost importance is the parabolicity of the pricing equation (2.35).

For instance, the case $\gamma = 1$ corresponding to a full market impact (no relaxation) yields the following equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \frac{\sigma^2 S^2}{(1 - \gamma \lambda S \frac{\partial^2 V}{\partial S^2})} = 0, \quad (2.36)$$

which can be shown to be parabolic, see (Loeper 2013). In fact, there holds the sharp result

Theorem 2.1 *Let us assume that $\frac{2}{3} \leq \gamma \leq 1$. Then, there holds:*

- *The non-linear backward partial differential operator*

$$V \rightarrow \frac{\partial V}{\partial t} + \frac{1}{2} \left(1 + (1 - 2\gamma)\lambda S \frac{\partial^2 V}{\partial S^2} \right) \frac{\sigma^2 S^2}{(1 - \gamma \lambda S \frac{\partial^2 V}{\partial S^2})^2} \frac{\partial^2 V}{\partial S^2} \quad (2.37)$$

is parabolic.

- Every European-style contingent claim with payoff Φ satisfying the terminal constraint

$$\sup_{S \in \mathbf{R}^+} \left(S \frac{\partial^2 \Phi}{\partial S^2} \right) < \frac{1}{\gamma \lambda} \quad (2.38)$$

can be perfectly replicated via a δ -hedging strategy given by the unique, smooth away from T , solution to Eq. (2.35).

Proof The parabolic nature of the operator is determined by the monotonicity with respect to p of the function

$$p \rightarrow F(p) = \frac{p(1 + (1 - 2\gamma)p)}{(1 - \gamma p)^2}. \quad (2.39)$$

A direct computation shows that $F'(p)$ has the sign of $1 + (2 - 3\gamma)p$, so that F is globally monotonic increasing on its domain of definition whenever $\frac{2}{3} \leq \gamma \leq 1$. Now, given that the payoff satisfies the terminal constraint, some deep results on the maximum principle for the **second derivative** of the solution of nonlinear parabolic equations, see e.g. (Wang 1992a, b), ensure that the same constraint is satisfied globally for $t \leq T$, and therefore, (2.36) is globally well-posed. As a consequence, the stochastic differential equation determining the price of the asset has a classical, strong solution up to time T .

In order to keep this paper self-contained, we provide a straightforward proof of the maximum principle for the second derivative of V in the more general case where the volatility can be state- and time-dependent, as follows: differentiating twice (2.36) with respect to S yields the following equation

$$\frac{\partial U}{\partial t} + \frac{\partial^2}{\partial S^2} \left(\frac{\sigma^2 S}{2\lambda} F(U) \right) = 0, \quad (2.40)$$

where $U \equiv \lambda S \frac{\partial^2 V}{\partial S^2}$. Assuming for the moment that this is legitimate, we introduce a new unknown function $Z = \frac{\sigma^2 S}{2\lambda} F(U)$, so that Z is formally the solution to

$$\frac{\partial}{\partial t} \left(F^{-1} \left(\frac{2\lambda Z}{\sigma^2 S} \right) \right) + \frac{\partial^2 Z}{\partial S^2} = 0, \quad (2.41)$$

rewritten under the form

$$\frac{\partial Z}{\partial t} + \frac{\sigma^2 S}{2\lambda} F'(F^{-1}(Z)) \frac{\partial^2 Z}{\partial S^2} - \frac{\partial \sigma^2}{\partial t} Z = 0. \quad (2.42)$$

As a final change of unknown function, let us introduce $Y \equiv \frac{Z}{S}$, a solution to

$$\frac{\partial Y}{\partial t} + \frac{\sigma^2 S}{2\lambda} F'(F^{-1}(SY)) \frac{\partial^2 Y}{\partial S^2} + \frac{\sigma^2 S}{\lambda} F'(F^{-1}(SY)) \frac{\partial Y}{\partial S} - \frac{\partial \sigma^2}{\partial t} Y = 0. \quad (2.43)$$

At this stage, and under the only natural and trivial assumption that the coefficient $\frac{\partial \sigma^2}{\partial r}$ of the 0th term is bounded, one can apply the classical maximum principle for a smooth solution of (2.43): upon multiplying the unknown function Y by some exponential time-dependent function $e^{\alpha(T-t)}$, α large enough, one easily shows that a solution of (2.43) cannot have a local positive maximum or negative minimum; hence, it is uniformly bounded over any time interval $[0, T]$ if its terminal condition is. Once this a priori estimate is proven, the method of continuity allows one to obtain a unique, smooth classical solution (Ladyzhenskaya et al. 1968; Gilbarg and Trudinger 1998). Then, applying in reverse order the various changes of unknown function, one constructs the unique smooth, classical solution to the original equation (2.35), satisfying by construction the constraint (2.38) everywhere.

As a consequence, there exists a classical, strong solution to the SDE (2.38)—since the denominator is bounded away from 0—and the cost process introduced in Proposition 2.1 is well-defined, and identically 0. Hence, the perfect replication is possible.

Clearly, the constraint on the second derivative is binding, in that it is necessary to ensure the existence of the asset price itself. See however Sect. 2.8 for a discussion of other situations.

2.6 Complete Market: The Multi-asset Case

Consider a complete market described by d state variables $X = X_1, \dots, X_d$: one can think for instance of a stochastic volatility model with $X_1 = S$ and $X_2 = \sigma$ when option-based hedging is available. Using tradable market instruments, one is able to generate d hedge ratio $\delta = \delta_1, \dots, \delta_d$ with respect to the independent variables X_1, \dots, X_d , that is, one can buy a combination of instruments whose price $P(t, X)$ satisfies

$$\partial_{X_i} P = \delta_i. \quad (2.44)$$

We now introduce two matrices, Λ_1 and Λ_2 . Λ_1 accounts for the liquidity costs, so that its entry Λ_{ij}^1 measures the virtual impact on Asset i of a transaction on Asset j : according to the simplified view of the order book model presented in Sect. 2.2.3, it would be natural to assume that Λ_1 is diagonal, but it is not necessary, and we will not make this assumption in the derivations that follow.

As for Λ_2 , it measures the permanent impact, and need not be diagonal.

When $d = 1$, Λ_1 and Λ_2 are linked to the notations in Sect. 2.4 by

$$\Lambda_1 = \lambda S, \Lambda_2 = \gamma \lambda S.$$

Note that here, we proceed directly in the continuous time case, so that the actual shape of the order book plays a role only through its Taylor expansion around 0; hence, the use of the “linearized” impact via the matrices Λ_i .

The pricing equation is derived along the same lines as in Sect. 2.4: the dynamics of the observed price change can be written as

$$dX_t = d\hat{X}_t + dA_t + \Lambda_2 d\delta_t, \quad (2.45)$$

the d -dimensional version of (2.24).

Again, a straightforward application of Itô's formula in a Markovian setting yields the dynamics of the observed price

$$dX_t = (I - \Lambda_2 D\delta)^{-1} d\hat{X}_t + dA'_t. \quad (2.46)$$

where $D\delta$ contains the first-order terms in the differential of δ , in matrix form $(D\delta)_{ij} = \frac{\partial \delta_i}{\partial \delta_j}$.

Denote by V the value of the hedging portfolio. The d -dimensional version of Proposition 2.1 for the incremental cost of hedging is

$$dC_t = dV_t - \sum_{i=1}^d \delta_i dX_t^i + \frac{1}{2} \text{Trace}((\Lambda_1 - 2\Lambda_2) d < \delta, \delta >_t). \quad (2.47)$$

The market being complete, the perfect hedge condition $dC_t = 0$ yields the usual delta-hedging strategy

$$\frac{\partial V}{\partial X_i} = \delta_i, \quad (2.48)$$

so that one can now write $D\delta = \Gamma$, where Γ is the Hessian of V , and therefore, the pricing equation is

$$\partial_t V + \frac{1}{2} \text{Trace} \left(\Gamma \frac{d < X, X >_t}{dt} \right) = \text{Trace} \left(\Gamma \left(\Lambda_2 - \frac{1}{2} \Lambda_1 \right) \Gamma \frac{d < X, X >_t}{dt} \right). \quad (2.49)$$

Using (2.46), one obtains

$$\partial_t V + \frac{1}{2} \text{Trace} \left[(\Gamma (I - (2\Lambda_2 - \Lambda_1) \Gamma)) (M \Sigma M^T) \right] = 0. \quad (2.50)$$

where we have set $\Sigma = \frac{d < \hat{X}, \hat{X} >_t}{dt}$, $M = (I - \Lambda_2 \Gamma)^{-1}$ and M^T is the transpose of the matrix M .

In the particular case where $\Lambda_1 = \Lambda_2$ (i.e. no relaxation), the pricing equation becomes

$$\partial_t V + \frac{1}{2} \text{Trace}(\Gamma \Sigma ((I - \Lambda \Gamma)^{-1})^T) = 0 \quad (2.51)$$

or, after a few trivial manipulations using the symmetry of the matrices M and Γ ,

$$\partial_t V + \frac{1}{2} \text{Trace}(\Gamma(I - \Lambda\Gamma)^{-1} \Sigma) = 0. \quad (2.52)$$

In particular, the 1-dimensional case yields the equation already derived in Loeper (2013)

$$\partial_t V + \frac{1}{2} \frac{\Gamma}{1 - \lambda S \Gamma} S^2 \sigma^2 = 0, \quad (2.53)$$

a particular case of Eq. (2.35) with $\gamma = 1$. The assessment of well-posedness in a general setting is related to the monotonicity of the linearized operator, and it may be cumbersome—if not theoretically challenging—to seek explicit conditions. In the case of full market impact $\Lambda_1 = \Lambda_2 \equiv \Lambda$, there holds the

Proposition 2.2 *Assume that the matrix Λ is symmetric. Then, Eq. (2.51) is parabolic on the connected component of $\{\det(I - \Lambda\Gamma) > 0\}$ that contains $\{\Gamma = 0\}$.*

Proof Let

$$F(\Gamma) = \text{Trace}(\Gamma(I - \Lambda\Gamma)^{-1} \Sigma_t),$$

and

$$H(\Gamma) = \Gamma(I - \Lambda\Gamma)^{-1}.$$

Denoting by \mathbb{S}_d^+ the set of d -dimensional symmetric positive matrices, we need to show that for all $d\Gamma \in \mathbb{S}_d^+$, for all covariance matrix $\Sigma \in \mathbb{S}_d^+$, there holds

$$F(\Gamma + d\Gamma) \geq F(\Gamma).$$

Performing a first order expansion yields

$$H(\Gamma + d\Gamma) - H(\Gamma) = \Gamma(I - \Lambda\Gamma)^{-1} \Lambda d\Gamma(I - \Lambda\Gamma)^{-1} + d\Gamma(I - \Lambda\Gamma)^{-1} \quad (2.54)$$

$$= (\Gamma(I - \Lambda\Gamma)^{-1} \Lambda + I) d\Gamma(I - \Lambda\Gamma)^{-1}. \quad (2.55)$$

Using the elementary Lemma 2.2—stated below without proof—there immediately follows that

$$F(\Gamma + d\Gamma) - F(\Gamma) = \text{Trace}((I - \Gamma\Lambda)^{-1} d\Gamma(I - \Lambda\Gamma)^{-1} \Sigma) \quad (2.56)$$

$$= \text{Trace}(d\Gamma(I - \Lambda\Gamma)^{-1} \Sigma(I - \Gamma\Lambda)^{-1}). \quad (2.57)$$

Then, the symmetry condition on Λ allows to conclude the proof of Proposition 2.2.

Lemma 2.2 *The following identity holds true for all matrices Γ, Λ :*

$$\Gamma(I - \Lambda\Gamma)^{-1} \Lambda + I = (I - \Gamma\Lambda)^{-1}.$$

2.7 The Case of an Incomplete Market

In this section, stochastic volatility is now considered. Clearly, the results obtained in Sect. 2.6 could apply in this context whenever the market were assumed to be **completed** via an option-based hedging strategy. However, it is well-known that such an assumption is equivalent to a very demanding hypothesis on the realization of the options dynamics and their associated risk premia, and it may be more realistic to assume that the market remains incomplete, and then, study a hedging strategy based on the underlying asset only. As we shall see below, such a strategy leads to more involved pricing and hedging equations.

Let then the observed price process be a solution to the following set of SDE's

$$dS_t = S_t(\sigma_t dW_t^1 + \gamma \lambda d\delta_t + \mu_t dt) \quad (2.58)$$

$$d\sigma_t = v_t dt + \Sigma_t dW_t^2 \quad (2.59)$$

where (W^1, W^2) is a two-dimensional Wiener process under \mathcal{P} with correlation ρ :

$$d \langle W^1, W^2 \rangle_t = \rho dt,$$

and the processes μ_t , v_t and Σ_t are actually functions of the state variables S, σ .

Consider again a Markovian framework, thereby looking for the value process V and the optimal strategy δ as smooth functions of the state variables

$$\begin{aligned} \delta_t &= \delta(S_t, \sigma_t, t) \\ V_t &= V(S_t, \sigma_t, t). \end{aligned}$$

Then, the dynamics of the observed price becomes

$$dS_t = \frac{S_t}{1 - \gamma \lambda S_t \frac{\partial \delta}{\partial S}} \left(\sigma_t dW_t^1 + \gamma \lambda \frac{\partial \delta}{\partial \sigma} d\sigma_t + dQ_t \right), \quad (2.60)$$

the orthogonality condition reads

$$\left(\frac{\partial V}{\partial S} - \delta \right) d \langle S, \bar{S} \rangle_t + \frac{\partial V}{\partial \sigma} d \langle \sigma, \bar{\sigma} \rangle_t = 0 \quad (2.61)$$

and the pricing equation for the value function V is

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2} \left(\frac{\partial^2 V}{\partial S^2} - \gamma \lambda S_t \left(\frac{\partial \delta}{\partial S} \right)^2 \right) \frac{d \langle S, S \rangle_t}{dt} + \frac{1}{2} \left(\frac{\partial^2 V}{\partial \sigma^2} - \gamma \lambda S_t \left(\frac{\partial \delta}{\partial \sigma} \right)^2 \right) \frac{d \langle \sigma, \sigma \rangle_t}{dt} + \\ + \left(\frac{\partial^2 V}{\partial \sigma \partial S} - \gamma \lambda S_t \frac{\partial \delta}{\partial \sigma} \frac{\partial \delta}{\partial S} \right) \frac{d \langle S, \sigma \rangle_t}{dt} + \mathcal{L}_1 V = 0, \end{aligned} \quad (2.62)$$

where \mathcal{L}_1 is a first-order partial differential operator.

Equations (2.61) and (2.62) are quite complicated. In the next paragraph, we focus on a particular case that allows one to fully assess their well-posedness.

2.7.1 The Case $\gamma = 1$, $\rho = 0$

When $\gamma = 1$, the martingale component of the supply price does not depend on the strategy anymore. As a matter of fact, the supply price dynamics is given by

$$d\bar{S}_t = dS_t + S_t \left((1 - 2\gamma)\lambda d\delta_t + \frac{1}{2}\mu d < \delta, \delta >_t \right),$$

see (2.27), and therefore, using (2.58), there holds that

$$d\bar{S}_t = S_t(\sigma_t dW_t^1 + \lambda(1 - \gamma)d\delta_t + dR_t) \equiv S_t(\sigma_t dW_t^1 + dR_t), \quad (2.63)$$

where R is a process of bounded variation. If, in addition, the Wiener processes for the asset and the volatility are supposed to be uncorrelated: $\rho = 0$, the tedious computations leading to the optimal hedge and value function simplify, and one can study in full generality the well-posedness of the pricing and hedging equations (2.61) and (2.62).

First and foremost, the orthogonality condition (2.61) simply reads in this case

$$\delta = \frac{\partial V}{\partial S}, \quad (2.64)$$

exactly as in the complete market case. This is a standard result in local-risk minimization with stochastic volatility when there is no correlation.

As for the pricing equation (2.62), one first works out using (2.64) the various brackets in (2.62) and finds that

$$\frac{d < S, S >_t}{dt} = \left(1 - \lambda S_t \frac{\partial^2 V}{\partial S^2} \right)^{-2} (\sigma_t^2 S_t^2 + \lambda^2 S_t^2 \left(\frac{\partial^2 V}{\partial S \partial \sigma} \right)^2 \Sigma_t^2), \quad (2.65)$$

$$\frac{d < \sigma, \sigma >_t}{dt} = \Sigma_t^2 \quad (2.66)$$

and

$$\frac{d < S, \sigma >_t}{dt} = \left(1 - \lambda S_t \frac{\partial^2 V}{\partial S^2} \right)^{-1} \lambda S_t \Sigma_t^2 \frac{\partial^2 V}{\partial S \partial \sigma}. \quad (2.67)$$

Plugging these expressions in (2.62) yields the pricing equation for V

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \left(1 - \lambda S \left(\frac{\partial^2 V}{\partial S^2} \right) \right)^{-1} \left(\sigma_t^2 S^2 + \lambda^2 S^2 \left(\frac{\partial^2 V}{\partial S \partial \sigma} \right)^2 \Sigma_t^2 \right) + \frac{1}{2} \left(\frac{\partial^2 V}{\partial \sigma^2} - \lambda S \left(\frac{\partial^2 V}{\partial S \partial \sigma} \right)^2 \right) \Sigma^2 + \\ \lambda S \Sigma_t^2 \left(\frac{\partial^2 V}{\partial S \partial \sigma} \right)^2 + \mathcal{L}_1 V = 0, \end{aligned} \quad (2.68)$$

or, after a few final rearrangements,

$$\frac{\partial V}{\partial t} + \frac{\sigma_t^2 S^2}{2(1 - \lambda S(\frac{\partial^2 V}{\partial S^2}))} \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} \frac{\partial^2 V}{\partial \sigma^2} \Sigma^2 + \frac{1}{2} \frac{\lambda S \Sigma^2}{(1 - \lambda S(\frac{\partial^2 V}{\partial S^2}))} \left(\frac{\partial^2 V}{\partial \sigma \partial S} \right)^2 + \mathcal{L}_1 V = 0. \quad (2.69)$$

The main result of this section is the

Proposition 2.3 *Equation (2.69) is of parabolic type.*

Proof One has to study the monotocity of the operator

$$\mathcal{L} : V \rightarrow \mathcal{L}(V) \equiv \frac{\sigma_t^2 S^2}{2(1 - \lambda S(\frac{\partial^2 V}{\partial S^2}))} \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} \frac{\partial^2 V}{\partial \sigma^2} \Sigma^2 + \frac{1}{2} \frac{\lambda S \Sigma^2}{(1 - \lambda S(\frac{\partial^2 V}{\partial S^2}))} \left(\frac{\partial^2 V}{\partial \sigma \partial S} \right)^2. \quad (2.70)$$

Introducing the classical notations

$$p \equiv \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \quad (2.71)$$

with $p_{11} = \frac{\partial^2 V}{\partial S^2}$, $p_{12} = p_{21} = \frac{\partial^2 V}{\partial S \partial \sigma}$ and $p_{22} = \frac{\partial^2 V}{\partial \sigma^2}$ and defining

$$\mathbf{L}(S, \mathbf{p}) \equiv \frac{\sigma_t^2 S^2 p_{11}}{(1 - \lambda S p_{11})} + \Sigma^2 p_{22} + \frac{\lambda S \Sigma^2}{(1 - \lambda S p_{11})} p_{12}^2, \quad (2.72)$$

one is led to study the positivity of the 2×2 matrix

$$\begin{pmatrix} \frac{\partial \mathbf{L}}{\partial p_{11}} & \frac{1}{2} \frac{\partial \mathbf{L}}{\partial p_{12}} \\ \frac{1}{2} \frac{\partial \mathbf{L}}{\partial p_{12}} & \frac{\partial \mathbf{L}}{\partial p_{22}} \end{pmatrix}. \quad (2.73)$$

Setting $F(p_{11}) = \frac{\sigma_t^2 S^2 p_{11}}{1 - \lambda S p_{11}}$ and $D(p_{11}) = 1 - \lambda S p_{11}$, one needs to show that the matrix $\mathbf{H}(\mathbf{p})$

$$\begin{pmatrix} F'(p_{11}) + (\lambda S \Sigma)^2 \frac{p_{12}^2}{D^2} \lambda S \Sigma^2 \frac{p_{12}}{D} \\ \lambda S \Sigma^2 \frac{p_{12}}{D} & \Sigma^2 \end{pmatrix} \quad (2.74)$$

is positive. This result is trivially shown to be true by computing the trace and determinant of $\mathbf{H}(\mathbf{p})$:

$$Tr(\mathbf{H}(\mathbf{p})) = F'(p_{11}) + \Sigma^2 + (\lambda S \Sigma)^2 \frac{p_{12}^2}{D^2} \quad (2.75)$$

and

$$Det(\mathbf{H}(\mathbf{p})) = \Sigma^2 F'(p_{11}) \quad (2.76)$$

and using the fact that F is a monotonically increasing function.

This ends the proof of Proposition 2.3.

As a final remark, we point out that the condition on the payoff for (2.69) to have a global, smooth solution, is exactly the same as in the one-dimensional case: stochastic volatility does not impose further constraints, except the now imperfect replication strategy.

2.8 Concluding Remarks

In this work, we model the effect of liquidity costs and market impact on the pricing and hedging of derivatives, using a static order book description and introducing a numerical parameter measuring the level of asymptotic market impact. In the complete market case, a structural result characterizing the well-posedness of the strategy-dependent diffusion is proven. Extensions to incomplete markets and non-linear hedging strategies are also considered.

We conclude with a discussion of the two conditions that play a fundamental role in our results.

2.8.1 The Condition $\gamma \in [\frac{2}{3}, 1]$

Of interest is the interpretation of the condition on the resilience parameter: $\frac{2}{3} \leq \gamma \leq 1$.

The case $\gamma > 1$ is rather trivial to understand, as one can easily see that it leads to arbitrage by a simple round-trip trade. The case $\gamma < \frac{2}{3}$ is not so simple. The loss of monotonicity of the function $F(p) = \frac{p(1+(1-2\gamma)p)}{(1-\gamma p)^2}$ for $\gamma < \frac{2}{3}$ yields the existence of p_1, p_2 such that $p_1 < p_2$ but $F(p_1) > F(p_2)$, which will lead to an inconsistency in the perfectly replicating strategies, as we now show.

Recall that the price of the replicating strategy solves the equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S F \left(S \frac{\partial^2 V}{\partial S^2} \right) = 0, \quad (2.77)$$

and assume that there exists $p \in \mathbb{R}$ with $F'(p) < 0$. One can then find two values $p_1 < p_2$ such that $F(p_1) > F(p_2)$. Consider now two contingent claims Φ_1, Φ_2 satisfying $S \frac{\partial^2 \Phi_i}{\partial S^2} \equiv p_i, i = 1, 2$, together with $\frac{\partial \Phi}{\partial S}(S_0) = 0, \Phi_i(S_0) = 0$ for some given $S_0 > 0$. Under these assumptions, $\Phi_2(S) \geq \Phi_1(S)$ for all S . Then, there exist explicit solutions $V_i(t, S)$ to (2.77) with terminal conditions $\Phi_i, i = 1, 2$, given simply by translations in time of the terminal payoff:

$$V_i(t, S) = \Phi_i(S) + (T - t) \frac{\sigma^2}{2} S F(p_i). \quad (2.78)$$

Consider the following strategy: sell the terminal payoff Φ_1 at price $V_1(0, S_0)$, without hedging, and hedge Φ_2 following the replicating strategy given by (2.77).

The final wealth of such a strategy is given by

$$\text{Wealth}(T) = \underbrace{(\Phi_2(S_T) - V_2(0, S_0))}_{\text{hedge strategy}} + \underbrace{(V_1(0, S_0) - \Phi_1(S_T))}_{\text{option sold}}. \quad (2.79)$$

Using (2.78), one obtains

$$\text{Wealth}(T) = T \frac{\sigma^2}{2} S_0 (F(p_1) - F(p_2)) + (\Phi_2(S_T) - \Phi_2(S_0)) - (\Phi_1(S_T) - \Phi_1(S_0)), \quad (2.80)$$

which is always positive, given the conditions on Φ_1, Φ_2 , and thereby generates what may be interpreted as an arbitrage opportunity.

Note that this arbitrage exists both for $\gamma > 1$ and $\gamma < 2/3$, since it just requires that F be locally decreasing. However, in the case $\gamma > 1$, round-trip trades generate money and the price dynamics create actual arbitrage opportunities, whereas in the case $\gamma < 2/3$, it is the option prices generated by exact replication strategies that lead to a potential arbitrage: in order to make a profit, one should find a counterparty willing to buy an option at its exact replication price.

It is clear that such a “counterexample” is not an arbitrage opportunity per se, as one has to find a counterparty to this contract—what this means is simply that the price of the perfect hedge is not the right price for the option.

2.8.2 The Condition $S \frac{\partial^2 V}{\partial S^2} < \frac{1}{\gamma \lambda}$

Another important question has been left aside so far: the behaviour of the solution to the pricing equation when the constraint is violated at maturity—after all, this is bound to be the case for a real-life contingent claim such as a call option! From a mathematical point of view, see the discussion in Loeper (2013), there is a solution which amounts to replace the pricing equation $\mathcal{P}(D)(V) = 0$ by

$\text{Max}(\mathcal{P}(D)(V), S \frac{\partial^2 V}{\partial S^2} - \frac{1}{\gamma \lambda}) = 0$, but of course, in this case, the perfect replication does not exist any longer—one should use a super-replicating strategy as introduced originally in Soner and Touzi (2000) exactly for this purpose.

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