

Chapter 2

A Dialogue Concerning Contradiction and Reasoning

Enric Trillas

To Claudio Moraga, who, joyously for me, is more than a colleague, is a loved friend.

Carla. The understanding of an idea, once it is linguistically expressed, requires moving back up to reduce it to something that is currently considered clear enough. For instance, and concerning precise predicates, the Aristotle's principle of contradiction expressed by 'nothing can and cannot be simultaneously', is often understood by mathematicians [1] passing through the following steps:

1. Question: Can it be that x is P and is not P , for some elements x into consideration?
2. Suppose that X is the set containing all those elements x .
3. Let it be $\mathbf{P}$ the subset of X specified by the use of P .
4. The subset $\mathbf{P}^c$ of X , the complement of $\mathbf{P}$, just contains those elements in X to which P is not applicable.
5. The principle of contradiction is just represented by the set's law $\mathbf{P} \cap \mathbf{P}^c = \emptyset$;
6. This law is 'identified' with the contradiction principle.

But this is only acceptable for predicates specifying a set, and fuzzy logic deals with those that can't specify a set where the principle, if identified in an analogous form, fails.

Karl. Well, it depends on the forms of expressing both the 'and', and the 'not'. For instance, if the 'and' is represented by a continuous t-norm T , and the 'not' by a strong negation function N , the corresponding functional equation $T(a, N(a)) = 0$, with a in $[0, 1]$, has the solutions $T = W_f$, $N \leq N_f$, with f any order-automorphism of the unit interval. Hence, it is not true that the principle always fails, but it is true

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Fig. 2.1 Claudio Moraga, Enric Trillas, and Fatima Zohra Hadjam at the “1. International Symposium Fuzziness, Philosophy and Medicine” at the *European Centre for Soft Computing*, Mieres (Asturias), Spain, March 23–24, 2011; in the background, Mila Kwiatkowska; photograph by Rudolf Seising

that it fails in many cases; it holds with $f = id$, $T = W$ and $N = 1 - id$, and fails with $f = id$, $T = prod$, $T = min$, and the same negation $1 - id$ [2].

Carla. Yes, and without considering strong negations there are also positive and negative cases. For instance, with the order-automorphism $f(x) = x^2$, and the non-strong but continuous negation $N(x) = 1 - x^2$, it is $W_f(a, 1 - a^2) = [\max(0, a^4 - a^2)]^{1/2} = 0$, but with $f = id$, it is $W(a, 1 - a^2) = \max(0, a - a^2) = 0 \iff a \leq a^2 \iff a = 0$, or $a = 1$. In the first case the principle holds, but it does not hold in the second.

Karl. Yes, by keeping a continuous t-norm for representing the ‘and’, and independently of the strong character of the negation, the solutions of the functional equation $T(a, N(a)) = 0$, are $T = W_f$ and $N \leq N_f$, with independence on the properties of N . Obviously, with $f(x) = x^2$, it is $N(x) = 1 - x^2 \leq N_f(x) = [1 - x^2]^{1/2}$. Actually, even without strong negation, there are cases in which the principle fails, and cases in which it holds.

Carla. Even if it is true that the preservation of the Aristotle’s principle is bounded, in the standard algebras of fuzzy sets, to expressing the conjunction by a ‘ f -modification’ of the Łukasiewicz t-norm $W(a, b) = \max(0, a + b - 1)$, that is, to continuous t-norms in the ‘family’ of Łukasiewicz, it is also true that the principle not always fails. Hence, it cannot be properly said that fuzzy sets violate the principle [3].

Karl. Let me add that all that comes from understanding the principle *more classico*, that is, in the form $\mu \cdot \mu' = \mu_0$ that, when $\mu \in \{0, 1\}^X$, reduces to the classical, or crisp, ‘empty intersection’. It is a view directly coming from the theory of crisp sets. Are there other views?

Carla. Yes, of course. For instance, by understanding “it is $\mu \cdot \mu'$ impossible” as “ $\mu \cdot \mu'$ is not deducible from μ ” and the relation of deduction as the pointwise ordering between fuzzy sets, if it were $\mu \cdot \mu'$ deducible from μ , or $\mu \leq \mu \cdot \mu'$, since it is clear that the principle holds. But notice that this validity depends on a particular form of interpreting deduction, and only for those μ such that its intersection with μ' is not coincidental with μ ; certainly, for a lot of fuzzy sets [4].

Karl. Ok, but I doubt that Aristotle could think on ‘impossible’ as a synonym of ‘deducible’. Even if, perhaps, he could thought on the absurdity of a syllogism with premises p , and $p \& p'$; that any conclusion of such syllogism seems to be contradictory.

Carla. Yes, this is the right concept, self-contradictory! The greatest ‘sin’ that can be considered. Something that nobody will accept.

Karl. Although Aristotle did state that the principle cannot be submitted to proof, you should have just proven in which cases it holds in the standard algebras of fuzzy sets, and also its validity under the equivalence of impossible and non-deducible. Is it possible to prove it under the interpretation of impossible as self-contradictory?

Carla. Well, it is indeed possible by taking into account some definitions and by supposing some properties of what is defined. What is interesting to note is that the proof can be achieved under a simple symbolism verifying very few and soft suppositions.

Karl. It seems to mean that the principle holds in very general settings. Go ahead, please!

Carla. Suppose it exists a ‘natural’ relationship of inference that, even produced in the brain as a part of the natural phenomenon called *thinking*, can be represented by the symbol $\leq$, with which $p \leq q$ represents that the statements p and q are linked in the form ‘ q is inferred from p ’. Two statements p and q are called inferentially equivalent whenever $p \leq q$ and $q \leq p$, and can be written $p = q$. Of this symbol we need to just supposing it is, in itself, transitive, that is, verifies: $p \leq q, q \leq r \implies p \leq r$. In addition, it is required to count with the symbols ($\&$) for ‘and’, and ($'$) for ‘not’, of which it will be supposed they verify the simple laws $p \& q \leq p$, $p \& q \leq q$, and $p \leq (p')$. Finally, a definition: p is contradictory with q if and only if $p \leq q'$; hence, p is self-contradictory if and only if $p \leq p'$.

Karl. Wow, I see the proof! It is very simple [5]:

$$\begin{aligned} p \& p' &\leq p, \\ p \& p' &\leq p' \implies (p')' \leq (p \& p')', \\ p &\leq (p')' \implies p \leq (p \& p')', \\ p &\leq p' \leq (p \& p')'. \end{aligned}$$

That is, $p \& p'$ is self-contradictory, or impossible!

Carla. Yes, that's all. Under very few hypotheses on the behavior of the conjunction, and the negation, a symbolic proof of the principle follows. And such proof holds when the symbols $\&$ and $'$ are interpreted in different forms; for instance, when taking the relation $\leq$ as the natural ordering of the lattice, as the connectives in an Ortholattice, or a De Morgan algebra, and of course, in a Orthomodular lattice, a Boolean algebra, and, of course, as those in a Basic algebra of fuzzy sets and, in particular, in a standard algebra of them with, always, the inference relation identified with the pointwise ordering of fuzzy sets. Actually, the proof covers a big number of frameworks where statements are typically represented, and, in particular, all cases where the negation is strong, that is, verifies $p \leq (p')'$ and $(p')' \leq p$, or p and $(p')'$ are inferentially equivalent.

Karl. Really nice! All that means that in classical, quantum, intuitionistic, and fuzzy logics, the contradiction principle actually holds, and thinkers can be relaxed for what concerns the solid character of their grounds. The fact that the principle can be proven is, in my view, something new.

Carla. Notice that neither commutative nor associative laws for $\&$, no duality law for both $\&$ and $'$, no functional expressibility of the connectives in the fuzzy case, etc., are supposed. The theorem is free from most of the constraints that are typically considered. Anyway, the theorem can fail in those cases in which it is not $p \leq (p')'$, that is, the double negation of a single statement p cannot be inferentially obtained from it; nevertheless, in any case, the principle will fail for this statement p , or for all those statements whose double negation will be not reachable from them.

Karl. Turning back to the concept of contradiction, it should be noticed that in the setting of Ortholattices it is $p \leq q' \implies p \cdot q = 0$, since the inequality implies $p \cdot q \leq q' \cdot q = 0$. In the case of Orthomodular lattices, and provided p and q commute, that is, $p = p \cdot q + p \cdot q'$, the reciprocal holds since $p \cdot q = 0$ implies $p = p \cdot q'$, or $p \leq q'$. In particular, since in Boolean algebras all pair of elements commute, it is $p \leq q' \iff p \cdot q = 0$. Thus, in Boolean algebras, it is $p \leq p' \iff p = 0$: the only self-contradictory element is 0; this is perhaps a reason for which it is difficult, at least for children, to accept the empty set, $\emptyset$, as a set.

Carla. But this is not the case in De Morgan algebras, since, for instance, in the case of the De Morgan algebra in the unit interval $[0, 1]$ with $\cdot = \min$, $+$ = $\max$, and $' = 1 - id$, it is $0.3' = 1 - 0.3 = 0.7$, but $0.3 \cdot 0.7 = 0.3 \neq 0$, even if $0.3 < 0.7 = 0.3'$. In this example, there are a lot of self-contradictory elements since $p \leq p' = 1 - p \iff p < 0.5$, that is, all numbers in the sub-interval $[0, 0.5]$, and only they, are self-contradictory. Anyway, if $p \cdot q = \min(p, q) = 0$, since it implies $p = 0$, or $q = 0$, it follows $0 = p \leq q'$, or $0 = q \leq p'$.

Karl. Contradiction is a symmetrical property, $p \leq q' \iff q \leq p'$, provided the negation verifies $q \leq (q')'$, since $p \leq q' \implies (q')' \leq p' \implies q \leq p'$.

Carla. Boolean algebras are endowed with so a lot of laws that many theorems can be proven in them. Boolean algebras cannot allow, for instance, to distinguish



Fig. 2.2 Claudio Moraga and Ramon López de Mántaras during the First “Alfredo Dean” Seminar on Ordinary Reasoning, to be held at the Cultural Center “Muralla Romana” of Gijón (Asturias), Spain, April 29, 2011.; photograph by Rudolf Seising

between incompatibility and contradiction. In the classical setting contradiction and incompatibility are indistinguishable concepts, but not in the quantum, nor in the fuzzy case, and less again in language. That is, Boolean algebras are not well suited for representing commonsense reasoning; they have too much constrictions, and it analogously happens with Ortholattices even if the particular case of Orthomodular lattices are sometimes taken to represent and to study some aspects of quantum reasoning. But in commonsense reasoning there are laws common to both Ortholattices and De Morgan algebras, that are not always valid like it appears, for instance, when studying the validity of the laws $(\mu \cdot \sigma)' = \sigma + (\mu' \cdot \sigma')$, and $\mu = \mu \cdot \sigma + \mu \cdot \sigma'$, with fuzzy sets [6]. Let me say that what cannot be supposed for the second law when dealing with imprecise predicates is duality!

Karl. And what on the principle of excluded middle? Since it does not hold in intuitionistic logic, I suppose no straightforward proof is available.

Carla. Well, excluded middle, or *tertium non datur*, was not so clearly explicitly posed by Aristotle as it was contradiction, but, and from very early in the Middle

Age, it is understood by ‘either the affirmative, or the negative, always hold’, that in set theory is represented by $\mathbf{P} \cup \mathbf{P}^c = X$, if the statements are on a universe of discourse X , and are specified by its subsets. In this case, it straightforwardly follows from contradiction because of duality: $\mathbf{P} \cup \mathbf{P}^c = (\mathbf{P} \cap \mathbf{P}^c)^c = \emptyset^c = X$, and reciprocally. In classical logic both principles are equivalent. They correspond to stating “‘ x is P ’ or ‘ x is not P ’” for all x in X .

Karl. In the case of lattices with a negation it corresponds to $p + p' = 1$, with the disjunction $+$ and the maximum, 1, of the natural order, and that, in Ortholattices is a structural law, but not in De Morgan algebras. Of course, since in these lattices the negation is strong, the disjunction is commutative, and the duality law $p' + q' = (p \cdot q)'$ holds, it is clear that, like with sets, it is $p + p' = p'' + p' = (p' \cdot p)' = (p \cdot p')' = 0' = 1$. The two laws of contradiction and excluded middle are equivalent, and both do hold.

Carla. With fuzzy sets in a Basic algebra, the principle can be posed in the same form $\mu + \mu' = \mu_1$, and in the very particular case of a standard algebra its validity comes from solving the functional equation $S(a, N(a)) = 1$, for all a in $[0, 1]$, with S a continuous t-conorm, N a negation either strict, or strong, and whose solution is $S = W *_{f,}$ and $N_f \leq N$, for all order-automorphisms f of the unit interval. Hence, there are cases in which the principle holds and others that not as, for instance and respectively, with $S = W*$, or with $S = \max$, and in both cases with $N = 1 - id^{[2]}$.

Karl. Thus, the two principles jointly hold in the standard algebras of fuzzy sets with $T = W_f$, $S = W *_{g,}$ and $N_g \leq N \leq N_f$, where f and g are order-automorphisms of the unit interval ^[2]. Of course, if S is the N -dual of T , $S = N \circ T \circ (N \times N)$, the two principles are equivalent like it happens in lattices.

Carla. Analogously to the first principle, the second can be analogously posed by asking if “ p or p' ” is deductible from the single premise p , and each time the disjunction $+$ verifies $p \leq p + q$, it is $p \leq p + p'$. That is, $p + q$ is actually deductible from p .

Karl. By interpreting $\leq$ as the ‘natural’ relation of inference, and accepting it verifies $p \leq p + q$, what can be simply said is that $p + q$ is inferable from p . Is not it?

Carla. Yes, it is, and ‘inferable from p' ’ either can mean that $p + q$ can be conjectured from p , or that it can be deduced from p depending from which properties can have the relation $\leq$.

Karl. And, concerning the beforehand interpretation of impossible by self-contradictory, what can be said?

Carla. The problem is not identical to that of contradiction, since what should be here represented is ‘always holds’. It is an affirmative argument for which it seems that a constructive argument could be in order. Trying to prove it through a reduction to absurd reasoning is another possibility that, notwithstanding, has the trouble of representing the ‘always’.

In the case of the unit interval before mentioned, notice that the numbers that are not self-contradictory are those in $(0.5, 1)$, coming from the inequality $p' \leq p \Leftrightarrow$

$1 - p \leq p \Leftrightarrow 0.5 \leq p$, equivalent to $p' \leq (p')'$. This last expression shows that the numbers p not self-contradictory are those whose negation is self-contradictory; these numbers, being not self-contradictory, and after being understood those in $[0, 0.5]$ as ‘impossible’, can be seen as ‘possible’. Thus, and since in presence of duality it is $p + p' = (p \cdot p')'$, it appears the possibility of identifying the excluded middle with the inequality $(p + p')' \leq ((p + p')')'$. Can it be proven?

Karl. I see. Under duality and double negation, last inequality reduces to $p' \cdot p \leq p + p'$, whose validity is obvious. I suppose, by analogy with the proof of contradiction, that it should come from accepting the inferences $p \leq p + q$, and $q \leq p + q$. Is not it?

Carla. Yes, it seems so. They imply $(p + q)' \leq p'$, $(p + q)' \leq q'$, and then the question is how to continue for arriving at a proof.

Karl. Very easy once accepted that the negation verifies $p \leq (p')'$. The proof is:

$$p \leq p + p' \Rightarrow (p + p')' \leq p' \leq p + p' \leq ((p + p')')'.$$

Carla. Yes, but there is a problem in it. The step $p' \leq p + p'$, is allowed?

Karl. You are right ... we did not yet accept the commutative property of the operations. It is $p' \leq p' + p$.

Carla. There is the exit's way of supposing not only $p \leq p + q$, but also $p \leq q + p$, that does not seem anything bizarre. What do you think?

Karl. Certainly it is not a bizarre supposition that can be also accepted for conjunction: $p \cdot q \leq p$, and $q \cdot p \leq p$. Hence, with it the proof is ok.

Carla. In the case of the unit interval, the inequalities translating the two principles should be verified by all the numbers between 0 and 1. Since there is duality and the negation is strong it suffices to see it for just one of them, for instance, $p \cdot p' \leq (p \cdot p')' \Leftrightarrow \min(p, 1 - p) \leq 1 - \min(p, 1 - p) = \max(1 - p, p)$, obviously true.

Karl. What turns in my mind is that intuitionistic logic not accepts excluded middle, but that in the form we established the principle, it should hold. There seems to be something rare.

Carla. I don't think so since no constructive procedure asserts here that $p + p'$ is true. We just prove the principle as a theorem by saying that “not (p or not q)” is self-contradictory that, in some way, means that “ p or not p ” is not self-contradictory. Passing from this to truth, it lacks a form of asserting what is true, and we don't entered in such a question that, I think, is in the center of intuitionism. We left constructive truth aside. What we add is just that in all those cases in which the inference relation, the negation, the conjunction, and the disjunction, satisfy a few and very soft properties, the principles hold as theorems on the basis of considering self-contradiction as a death ‘sin’. We escaped from truth and the ways for asserting it. But nothing else!

Karl. May be ours is merely an idealistic way of posing the principles . . .

Carla. Why? I am not sure. In the classical case $p \cdot p' \leq ((p \cdot p')')$ just means $p \cdot p' = 0$, and $(p + p')' \leq ((p + p')')'$ means $p + p' = 1$. That is, our way of posing the principles reduces to the common way in which they are classically posed, with the difference that there the principles are axioms and here theorems. I see the classical axiomatic way more idealistic than is ours. We find minimal conditions supporting the principles, and these conditions seem to be of a so great generality that allow the validity of both principles in many settings.

Karl. The Aristotle's idea that the principle of contradiction cannot be submitted to proof seems to be challenged by an interpretation of the word impossible he did use, as the concept self-contradictory, and in a framework of great amplitude, where the two principles of contradiction and excluded middle, appear as theorems. These theorems reduce, in the classical case, to the set interpretation of the principles, and shows that both the algebras of fuzzy sets and also De Morgan algebras satisfy these principles: It cannot be properly said that De Morgan algebras and fuzzy sets violate the two principles.

Carla. There are indeed some multiple-valued logics also verifying them, although some others did not because of the properties their negations 'enjoy'. Nevertheless, what still is not fully clear is which is the relationship between what has been presented with both the concept of truth, and, specially, with the constructivism for asserting truth and that intuitionists consider essential for assuming mathematical results. In fact, from what we escape is from constructivism.

Karl. This, jointly with the analysis of the two principles for consequences operators whatsoever, that is, for any kind of deductive inference, remains open to scrutiny.

Carla. Anyway, all that on which we jointly debated here cannot be seen as something belonging to just the province of logic, but to the larger domains of natural language and commonsense reasoning. And, at this respect the words of Russell and Whitehead, that mathematics and logic are merely concerned with 'the correct use of a certain small number of words', should be specifically taken into account. Common reasoning concerns all words, and in all contexts.

Karl. All right. May be, and provided our discussion could have any value at all, it could lie in a change of perspective for the study of the reasoning's phenomena by placing at its center the ideas of a natural linking between statements, and contradiction, instead of truth and the representation of conditional, If/Then, statements by any sort of material conditional.

Carla. Also, and from my point of view, to abandon the expression of the conditional 'If p , then q ' coming from, for instance in Boolean algebras, the logical equivalence between $p' + q = 1$ and $p \leq q$, by substituting the partial order by a just transitive 'natural' relation that links statements, could arrive to be fruitful. Who knows!



Fig. 2.3 Claudio Moraga, at the “1. International Symposium Fuzziness, Philosophy and Medicine” at the *European Centre for Soft Computing*, Mieres (Asturias), Spain, March 23–24, 2011, in the background, Clara Barroso; photograph by Rudolf Seising

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