Chapter 2
Oscillations of Systems with Several Degrees of Freedom

Abstract In the first part of this chapter, we study oscillating systems with two (and subsequently with *n*) degrees of freedom. We learn the existence of particular motions, the normal modes, in which all parts of the system oscillate together harmonically. The number of modes, and the number of resonances, is equal to the number of degrees of freedom. We then study the modes of a vibrating sting. In the second part of the chapter, we study Fourier analysis, in regard to both periodic and non-periodic functions and for functions of both time and space.

The oscillators we studied in the previous chapter had one degree of freedom, namely their state was defined by a single variable. In the first part of this chapter, we study oscillating systems with two and then more degrees of freedom. We shall deal, as an example, with the system of two pendulums linked together by a spring. In general, the motions of the system are not harmonic oscillations, but can be quite complicated. We shall find, however, that special, very simple motions exist in which both pendulums, or, generally speaking, both parts of the system, oscillate in a harmonic motion with the same frequency and in the same initial phase. These stationary motions, called the normal modes, are two in number for a system with two degrees of freedom; indeed, in general, the number of modes is the same as the number of degrees of freedom. The oscillation frequencies of the modes, called proper frequencies, are characteristic of the system.

Subsequently, in Sect. 2.3, we shall study the vibrations of a continuous system through the example of an elastic string with fixed extremes. We shall see that normal modes also exist for continuous systems. In these motions, all the points of the system vibrate harmonically in phase with the same frequency. The number of modes is infinite, but numerable. Once again, the frequencies of the succession of modes are characteristic of the system. For an elastic string, the proper frequencies form the arithmetic succession, which is the succession of the natural numbers. The lowest frequency is called the fundamental, while those subsequent are its harmonics. The name is a consequence of the fact that a sound composed of simple sounds is pleasant if the frequencies of the components are in the ratio of small natural numbers.

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The systems we study are linear, namely the differential equation ruling their motion is linear and the superposition principle holds. A consequence of this is that every motion of the system, as complicated as it can be, can always be expressed as a linear combination of its normal modes. The functions (which are cosines for an elastic string) giving the time dependence of the modes constitute, as we say, a complete ortho-normal system of functions. Every function representing a motion of the system can be developed as a linear combination of these functions. This property allows for important simplifications in the study of the vibrations.

In the second part of the chapter, we study the harmonic (Fourier) analysis. This is the chapter of mathematics that originates from the physical phenomena we have just mentioned. We shall give only the mathematical statements without rigor and without proof. We are interested in the way in which harmonic analysis is very useful in physics. We shall see in Sects. 2.4 and 2.5 how a periodic function of time can be expressed in its Fourier series, which is a linear combination of an infinite sequence of harmonic functions, whose frequencies are the integer multiples of the lowest of them.

In Sect. 2.6, we shall extend the result to non-periodic functions. For them, an integral over the angular frequency, called the Fourier transform, takes the place of the Fourier series. We shall discuss two relevant examples.

Finally, in Sect. 2.7, we shall consider the Fourier analysis for a function of a space coordinate, rather than one of time. The function might be, for example, the gray level of a picture. The problems are substantially equal from the mathematical point of view, but their physical aspects are different. We shall need these concepts when studying optical phenomena.

2.1 Free Oscillators with Several Degrees of Freedom

After having studied the oscillations of systems with one degree of freedom in the previous chapter, we study here oscillations of systems with a discrete number of degrees of freedom. The motions, or more generally the evolution, which we shall discuss will always be about a stable equilibrium configuration under the action of a restoring force, or, more generally, of an agent, of magnitude proportional to the displacement from equilibrium. In other words, we shall deal with *linear systems*.

We start with the free oscillations of systems with two degrees of freedom in the absence of dissipative agents. To be specific, we consider a system of two *coupled oscillators*, as shown in Fig. 2.1. We shall subsequently generalize our findings. The coupled oscillators in the example of Fig. 2.1 are two identical pendulums, a and b, having length l and mass m, joined by a spring, having a rest length equal to the distance between the equilibrium positions of the pendulums. In order to have the oscillators coupled only weakly, we must choose the spring constant k as being small enough to have an elastic force substantially smaller than the weight.

We shall consider motions of the pendulums in the direction joining their equilibrium positions. Let x_a and x_b be the coordinates of the pendulums *each*

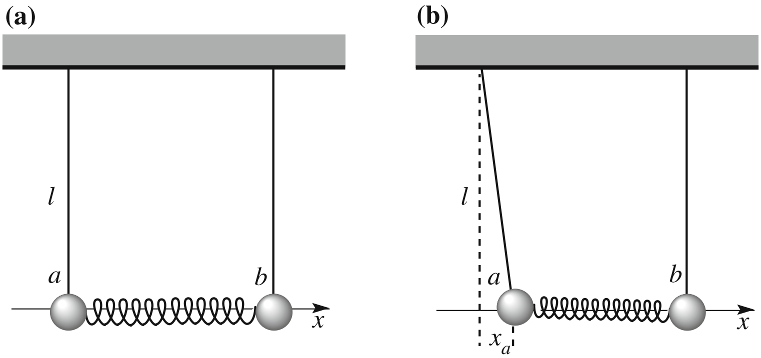


Fig. 2.1 Two coupled pendulums. **a** Equilibrium condition. **b** Initial condition

measured from their respective equilibrium position. We move pendulum a to the distance A from equilibrium, keeping b in its equilibrium position and letting both go with zero speeds. The initial conditions of the motions are the positions and velocities of the two pendulums, namely $x_a(0) = A$, $x_b(0) = 0$, $dx_a/dt(0) = 0$, $dx_b/dt(0) = 0$ (Fig. 2.1b).

We observe the following phenomenon. Initially, a oscillates with an amplitude equal to A. In so doing, it exerts on b a periodic force through the spring. The force is quite feeble (our having chosen to make k small) but it is at the resonance frequency of b (the pendulums being equal). As a consequence, b starts oscillating. The oscillations of b grow in amplitude with time, while those of a decrease, up to the point when a stops, or almost does, for a moment. The amplitude of b is now equal, or almost so, to A. The configuration is like that of the initial one, with the roles exchanged. The amplitude of b starts decreasing with that of a increasing until the system is back in its initial configuration, and so on. In this motion, energy goes back and forth from one pendulum to the other. This motion would continue forever in the absence of dissipative forces.

The motion we just described is more complex than the harmonic motion of a single pendulum. However, a system with two degrees of freedom can perform harmonic motions. More precisely, it is always possible to choose the *initial conditions* in such a way that *all the parts of the system* (namely the two pendulums, in this case) *perform a harmonic motion*. As a matter of fact, two different motions of this type exist. The initial conditions are shown in Fig. 2.2.

The first motion is obtained by taking both pendulums out of equilibrium by the same distance, say A, and letting them go at the same instant with zero speed. Namely, the initial conditions are $x_a(0) = A$, $x_b(0) = A$, $dx_a/dt(0) = 0$, $dx_b/dt(0) = 0$. Clearly, under these conditions, the spring is not deformed, namely its length initially and always is the rest length. Consequently, the spring does not exert any force (we assume its weight to be negligible) and each pendulum oscillates as if it was free (not coupled). Namely, we have

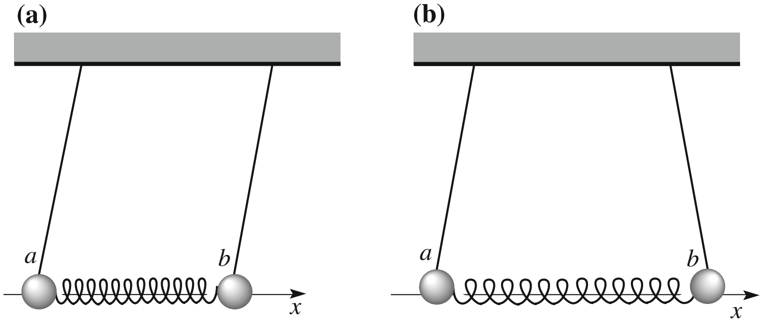


Fig. 2.2 The normal modes of two coupled pendulums

$$
\begin{aligned}
x_a(t) &= A\cos(\omega_1 t + \phi) \\
x_b(t) &= A\cos(\omega_1 t + \phi),
\end{aligned}
\tag{2.1}
$$

where

$$\omega_1^2 = g/l,$$

which, we must remember, is the restoring force per unit displacement and per unit mass. Note that the amplitude A is arbitrary, as long as we do not stray from the condition of having the restoring force proportional to the displacement. What does matter for having a harmonic motion is the value of the *ratio* between the initial amplitudes of the two pendulums, which must be equal to 1. In Eq. (2.1), we have explicitly written the initial phase ϕ in the arguments of the cosines to be complete. In this case, it is $\phi = 0$. As a matter of fact, the initial phase is arbitrary too, but *must be the same* for both pendulums.

The second way to obtain harmonic oscillations for both pendulums (Fig. 2.2b) is to take them out of equilibrium by the same distance in opposite directions and let them go with zero initial speeds. The initial conditions are $x_a(0) = A$, $x_b(0) = -A$, $dx_a/dt(0) = 0$, $dx_b/dt(0) = 0$. In this case, the spring acts. The forces that it exerts on the pendulums, internal to the system, are equal and opposite of one another in any instant. The center of mass, which was initially at rest, remains at rest. Consequently, in every moment, we have $x_a(t) = -x_b(t)$.

Let us start analyzing the motion of a. Three forces act on the pendulum. Two forces are exactly the same as for a non-coupled pendulum, namely the weight and the tension of the suspension wire. Together, they give a contribution proportional to the displacement equal to $-x_a g/l$. The third force is due to the spring. Taking signs into account, the stretch of the spring is $x_a - x_b$. Consequently, the force is $-k(x_a - x_b) = -2kx_a$. We see that, in the present motion, it is proportional to the displacement. In conclusion, the resultant force on a is a restoring force proportional to its displacement from equilibrium. This is the condition necessary for harmonic motion. The square of its angular frequency, which we call ω_2, is, as always, the restoring force per unit displacement per unit mass, namely

$$\omega_2^2 = \frac{g}{l} + 2\frac{k}{m}.$$

The situation for b is completely analogous. It moves under the action of its weight, the tension and the spring. The force of the latter is equal to and opposite of that of a and we can write it in terms of x_b only as $k(x_a - x_b) = -2kx_b$. The restoring force is the same as that for a when the displacement is the same. Consequently, being that the masses are also equal, b oscillates in harmonic motion with the same angular frequency ω_2. The motions of the two pendulums are harmonic motions *with the same angular frequency* and in phase opposition, or, to put it more properly, with *the same phase* and equal and opposite amplitudes, namely

$$\begin{aligned} x_a(t) &= A\cos(\omega_2 t + \phi) \\ x_b(t) &= -A\cos(\omega_2 t + \phi), \end{aligned} \tag{2.2}$$

In this case too, the initial amplitude is arbitrary. What matters is the *ratio* between the initial amplitudes, which must be equal to -1. Also, one of the initial phases is arbitrary, just as before. What matters is that the two initial phases must be equal (both $\phi = 0$, in this case).

The motions we described, namely those given by Eqs. (2.1) and (2.2), are called *normal modes* of the system. In general, the motion of a freely oscillating system with several degrees of freedom is said to be a normal mode when *all the parts of the system move in harmonic motion with the same frequency and the same phase.* "With the same phase" means that all the parts pass through their equilibrium positions in the same instant. We can also say that the normal modes are the *stationary motions* of the system, namely the motions whose characteristics are constant over time. Contrastingly, the characteristics of a generic motion, such as the one considered at the beginning of the section, evolve over time.

The system we considered is particularly simple and symmetric, the two pendulums being equal to one another. Its symmetry substantially allowed us to find the normal modes through intuition. To solve the problem in general, we need a bit of mathematics. We shall now show that the number of normal modes (or simply modes) of a system is equal to the number of its degrees of freedom. For linear systems, such as the ones we will consider, the superposition principle holds. Consequently, a linear combination of its normal modes is a possible motion of the system as well. In addition, we shall now show that *every motion* of the system can be expressed as a linear combination of its modes.

Let us start with the formal treatment of the example just discussed. The differential equations of its motion are

$$\begin{aligned} \frac{d^2x_a}{dt^2} + \omega_1^2 x_a &= -\frac{k}{m}(x_a - x_b) \\ \frac{d^2x_b}{dt^2} + \omega_1^2 x_b &= -\frac{k}{m}(x_b - x_a), \end{aligned} \tag{2.3}$$

where $\omega_1^2 = g/l$ is a constant. We re-write the system in the form

$$\begin{aligned} \frac{d^2x_a}{dt^2} &= -\left(\omega_1^2 + \frac{k}{m}\right)x_a + \frac{k}{m}x_b \\ \frac{d^2x_b}{dt^2} &= +\frac{k}{m}x_a - \left(\omega_1^2 + \frac{k}{m}\right)x_b. \end{aligned}$$

This form can be made general. Indeed, we define a linear, not damped, freely oscillating system with two degree of freedom as a system obeying the following system of differential equations

$$\begin{aligned} \frac{d^2x_a}{dt^2} &= -a_{11}x_a - a_{12}x_b \\ \frac{d^2x_b}{dt^2} &= -a_{21}x_a - a_{22}x_b, \end{aligned} \tag{2.4}$$

where the coefficients a_{ij} are constants characteristic of the system (in the example, they are combinations of the masses, the elastic constant of the spring and the gravity acceleration), which are independent of the initial conditions.

Let us search for normal modes, namely motions in which all the parts of the system oscillate with the same frequency and in the same initial phase. The question is: does any value of ω exist such that the functions of time

$$\begin{aligned} x_a(t) &= A\cos(\omega t + \phi) \\ x_b(t) &= B\cos(\omega t + \phi), \end{aligned} \tag{2.5}$$

are solutions to the system in Eq. (2.4)?

To find an answer, we substitute these functions in Eq. (2.4), obtaining

$$\begin{aligned} \left(a_{11} - \omega^2\right)x_a + a_{12}x_b &= 0 \\ a_{21}x_a + \left(a_{22} - \omega^2\right)x_b &= 0. \end{aligned} \tag{2.6}$$

This is a homogeneous algebraic system. The condition for obtaining non-trivial solutions is that the determinant must be zero, namely that

$$\begin{vmatrix} a_{11} - \omega^2, & a_{12} \\ a_{21}, & a_{22} - \omega^2 \end{vmatrix} = 0. \tag{2.7}$$

This is a second-degree algebraic equation in the unknown ω^2. The equation is so important that it has a name, the *secular equation*. The equation has two roots (corresponding to the two degrees of freedom), say ω_1 and ω_2, which are called *proper angular frequencies* of the system. For each solution, there is a normal mode, say mode 1 and mode 2. Once the secular equation is satisfied, Eq. (2.6) give, for each mode, the *ratios* between the oscillation amplitudes of x_a and x_b.

For mode 1, we have

$$\left(\frac{x_b}{x_a}\right)_{\text{mode 1}} = \frac{B_1}{A_1} = \frac{\omega_1^2 - a_{11}}{a_{12}}, \tag{2.8}$$

where A_1 and B_1 are the amplitudes of the harmonic motions of x_a and x_b, respectively. The equations of the motions, for mode 1, are then

$$\begin{aligned} x_a(t) &= A_1 \cos(\omega_1 t + \phi_1) \\ x_b(t) &= B_1 \cos(\omega_1 t + \phi_1). \end{aligned} \tag{2.9}$$

Analogously, for mode 2, Eq. (2.6) give us

$$\left(\frac{x_b}{x_a}\right)_{\text{mode 2}} = \frac{B_2}{A_2} = \frac{\omega_2^2 - a_{11}}{a_{12}}. \tag{2.10}$$

where A_1 and B_1 are the amplitudes of the harmonic motions of x_a and x_b, respectively. The equations of motions for mode 2 are

$$\begin{aligned} x_a(t) &= A_2 \cos(\omega_2 t + \phi_2) \\ x_b(t) &= B_2 \cos(\omega_2 t + \phi_2). \end{aligned} \tag{2.11}$$

Note that the physical characteristics of the system, which are encoded in the a_{ij} constants, determine the proper frequency and the ratio between the amplitudes (called the *mode shape*) for each mode. Contrastingly, the values of the amplitude and the initial phase are determined by the initial conditions of the motion.

The most general solution to a system of two linear differential equations is given by a linear combination of two independent solutions. It is then evident that the most general solution of the system in Eq. (2.4) is

$$\begin{aligned} x_a(t) &= A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2) \\ x_b(t) &= B_1 \cos(\omega_1 t + \phi_1) + B_2 \cos(\omega_2 t + \phi_2). \end{aligned} \tag{2.12}$$

The initial conditions determine four quantities. As a matter of fact, only four of the six constants (A_1, A_2, B_1, B_2, ϕ_1 and ϕ_2) in Eq. (2.12) are independent. Indeed, Eqs. (2.8) and (2.10) determine the mode shapes, namely B_1/A and B_2/A_2, independently of the initial conditions, and we can rewrite the general solution in the form

$$\begin{aligned} x_a(t) &= A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2) \\ x_b(t) &= \left(\frac{B_1}{A_1}\right) A_1 \cos(\omega_1 t + \phi_1) + \left(\frac{B_2}{A_2}\right) A_2 \cos(\omega_2 t + \phi_2). \end{aligned} \tag{2.13}$$

The four constants A_1, A_2, ϕ_1 and ϕ_2 are determined by the initial conditions ($x_a(0)$, $x_b(0)$, $dx_a/dt(0)$ and $dx_b/dt(0)$).

A generic motion of a system with two degrees of freedom may be quite complicated. The motions of its parts may not be harmonic, but all these motions can be expressed as linear combinations of two simple harmonic motions.

Coming back to our initial example, let us find the combination of normal modes that expresses it. The shapes of the two modes of the system are given by the ratios $B_1/A_1 = 1$ and $B_2/A_2 = -1$. Thus, we write

$$\begin{aligned} x_a(t) &= A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2) \\ x_b(t) &= A_1 \cos(\omega_1 t + \phi_1) - A_2 \cos(\omega_2 t + \phi_2). \end{aligned}$$

We must find A_1, A_2, ϕ_1 and ϕ_2 in order to have the initial conditions $x_a(0) = A$, $x_b(0) = 0$, $dx_a/dt(0) = 0$ and $dx_b/dt(0) = 0$ satisfied. The solution is clearly $A_1 = A_2 = A/2$, $\phi_1 = \phi_2 = 0$. The equations of the motions are then

$$\begin{aligned} x_a(t) &= \frac{A}{2}(\cos \omega_1 t + \cos \omega_2 t) \\ x_b(t) &= \frac{A}{2}(\cos \omega_1 t - \cos \omega_2 t), \end{aligned} \tag{2.14}$$

which can also be written in the more transparent form

$$\begin{aligned} x_a(t) &= A \cos\left(\frac{\omega_2 - \omega_1}{2} t\right) \cos\left(\frac{\omega_2 + \omega_1}{2} t\right) \\ x_b(t) &= A \sin\left(\frac{\omega_2 - \omega_1}{2} t\right) \sin\left(\frac{\omega_2 + \omega_1}{2} t\right). \end{aligned} \tag{2.15}$$

The two pendulums are weakly coupled, namely we have

$$2\frac{k}{m} \ll \frac{g}{l}. \tag{2.16}$$

The difference between ω_1 and ω_2 is small compared to their values. The motion of each pendulum can be thought of as an almost harmonic motion at the mean angular frequency $(\omega_2 + \omega_1)/2$ with an amplitude $A \cos[(\omega_2 - \omega_1)t/2]$, which is not constant in time, but varies periodically, as a sine function, at low frequency (half the difference between the proper frequencies), as shown in Fig. 2.3. When the amplitude of a is large, that of b is small, and vice versa.

The energy of each pendulum is proportional to the square of its amplitude. The total energy of the system is proportional to the sum

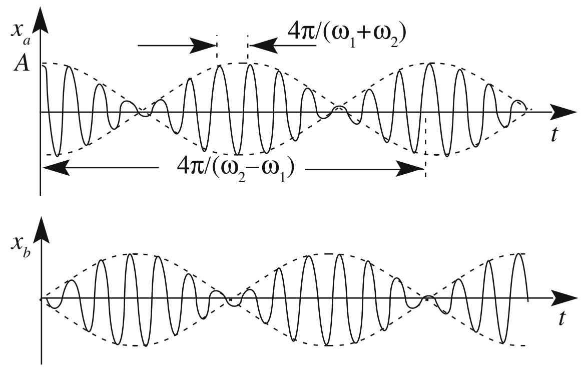


Fig. 2.3 The motions of two equal, weakly-coupled pendulums starting from the initial conditions $x_a(0) = A$, $x_b(0) = 0$, $dx_a/\mathrm{d}t(0) = 0$ and $dx_b/\mathrm{d}t(0) = 0$

$$A^2\left[\cos^2\left(\frac{\omega_2-\omega_1}{2}t\right)+\sin^2\left(\frac{\omega_2-\omega_1}{2}t\right)\right]=A^2,$$

which is constant, as expected.

Let us again use the example of the two equal, weakly-coupled pendulums to introduce the concept of *normal coordinates*. Let us substitute in the system of Eq. (2.3) the two linear combinations of x_a and x_b, which, remember, are the displacement for each pendulum of its equilibrium position

$$\begin{aligned} x_1(t) &= x_a(t)+x_b(t) \\ x_2(t) &= x_a(t)-x_b(t). \end{aligned} \tag{2.17}$$

Let us add and subtract the two equations in Eq. (2.3). We obtain

$$\begin{aligned} \frac{d^2x_1}{dt^2}+\omega_1^2x_1(t)=0 \\ \frac{d^2x_2}{dt^2}+\omega_2^2x_2(t)=0. \end{aligned} \tag{2.18}$$

The equations are now independent of one another. The coordinates enjoying such a property, like x_1 and x_2 in the example, are called *normal coordinates*. Note that each normal coordinate corresponds to one of the modes. Indeed, for mode 1, in which $x_a(t)=x_b(t)$, it is $x_2(t)=0$ identically. We see that only x_1 is excited. Similarly, in mode 2, only x_2 is non-zero. Note also that the equation of each normal coordinate is the harmonic motion equation.

Analogous considerations hold for any linear, freely oscillating system with two degrees of freedom, neglecting dissipative forces. However, the simple expressions of the normal coordinates and normal modes we have found hold only in the simple and symmetric example we have considered. They are not even valid for a system similar to that shown in Fig. 2.1, but with two pendulums of different lengths, for example, with *a* shorter than *b*. The system is still simple, but not symmetric. The configurations analogous to those shown in Fig. 2.2 are not the normal modes. In addition, if we observe the evolution of the system starting from an initial state analogous to that shown in Fig. 2.1a, we still see that, initially, the oscillation amplitude of a decreases over time and that of b increases. However, the amplitude of a does not go down to zero, but rather reaches a non-zero minimum and then goes back up again. In other words, energy is never transferred completely from the pendulum that is initially moving to the one that starts from rest.

The problem of finding the normal modes in general must be treated with the proper mathematics. We shall not do this here.

There exist freely oscillating systems with two degrees of freedom of very different nature. Figure 2.4 shows some examples, three mechanical systems and one electric. Very important cases of systems of two degrees of freedom exist in molecules, atoms, nuclei and elementary particles as well. Just to quote a few examples, we mention the ammonia molecule (and the MASER based on it), the hydrogen molecule, a number of dye molecules (and the colors they produce), the electrons in a magnetic field (and the electron-spin resonance) and the oscillation phenomena of elementary particles. These systems are properly described by quantum, rather than classical, physics. However, a proper classical oscillation frequency corresponds in quantum physics to the energy, or to the mass, of a state. Consequently, the analogy with classical physics is strong and can help us in intuiting an understanding of these quantum phenomena.

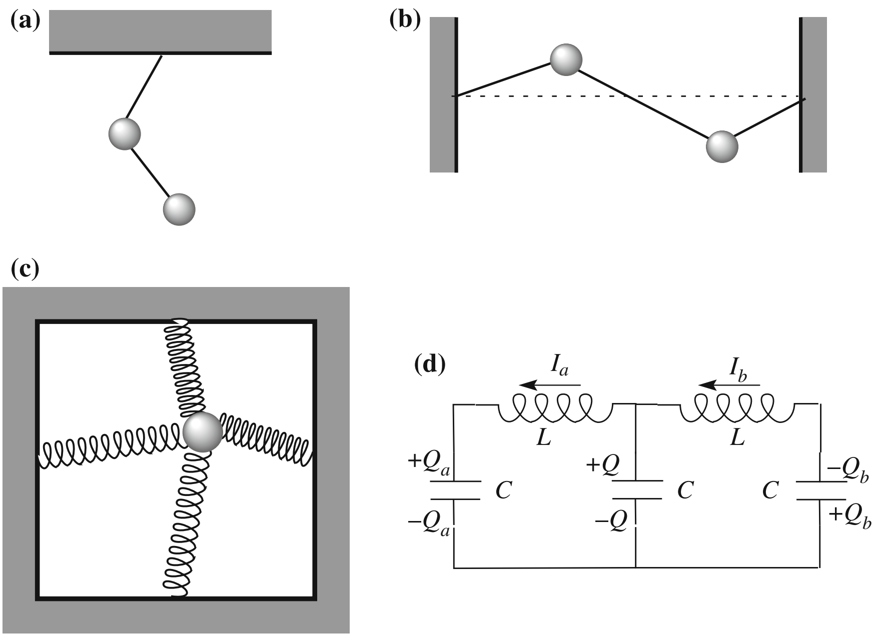


Fig. 2.4 Examples of systems with two degrees of freedom. **a** Double pendulum. **b** Two-mass loaded string, **c** Two-dimensional oscillator. **d** Capacitance coupled oscillating system

Let us consider, as another example, the two oscillating circuits in Fig. 2.4d. They have a capacitor in common and are consequently said to be capacity-coupled (an alternative is to have an inductor in common; the reader can analyze this as an exercise).

With the positive signs for charges and currents shown in the figure, the differential equations of the system are

$$-L\frac{dI_a}{dt} = \frac{Q_a(t)}{C} - \frac{Q(t)}{C}$$
$$-L\frac{dI_b}{dt} = \frac{Q_b(t)}{C} + \frac{Q(t)}{C}.$$

With these sign conventions, we have $dQ_a/dt = +I_a$ and $dQ_b/dt = +I_b$, meaning an I_a running in the positive direction positively charges the capacitor a, and the same goes for I_b and capacitor b. Considering that there are instants in which all the charges on the capacitors are zero and that charge is conserved, we conclude that $Q(t) + Q_a(t) + Q_b(t) = 0$. We can then write the above differential equations as

$$\begin{aligned} \frac{d^2Q_a}{dt^2} &= -\frac{1}{LC}(2Q_a - Q_b) \\ \frac{d^2Q_b}{dt^2} &= -\frac{1}{LC}(2Q_b - Q_a). \end{aligned} \tag{2.19}$$

These equations are also equal to those of the two coupled pendulums with $1/LC$ in place of ω_1^2 and $1/LC$ in place of k/m. We can conclude that the proper frequencies of the two modes are $1/\sqrt{LC}$ and $\sqrt{3}/\sqrt{LC}$.

QUESTION Q 2.1. Analyze an inductance-coupled oscillating circuit, namely a circuit similar to that shown in Fig. 2.4, with capacitances where there are inductances and inductances where there are capacitances. □

That which we have discussed can be easily generalized to systems of n degrees of freedom. We call the coordinate measuring the displacement from its own equilibrium position of each part of the system with x_1, x_2, ..., x_n, respectively. The differential equations of the motion of the system, neglecting dissipative forces, are

$$\begin{aligned} &\tfrac{d^2x_a}{dt^2} = -a_{11}x_1 - a_{12}x_2 - \ldots - a_{1n}x_n \\ &\tfrac{d^2x_2}{dt^2} = -a_{21}x_1 - a_{22}x_2 - \ldots - a_{2n}x_n. \\ &\ldots \\ &\tfrac{d^2x_n}{dt^2} = -a_{n1}x_1 - a_{n2}x_2 - \ldots - a_{nn}x_n \end{aligned} \tag{2.20}$$

We look for solutions that are normal modes, namely motions in which all the coordinates move in harmonic motion with the same frequency and with the same initial phase. In other words, we look to see if we can find one or more values of ω such that $x_1 = A^{(1)}\cos(\omega t + \phi), \ldots, x_n = A^{(n)}\cos(\omega t + \phi)$ is a solution. We substitute these functions in Eq. (2.20) and obtain the algebraic system

$$
\begin{aligned}
&(a_{11}-\omega^2)x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0 \\
&a_{21}x_1 + (a_{22}-\omega^2)x_2 + \ldots + a_{2n}x_n = 0 \\
&\ldots \\
&a_{n1}x_1 + a_{n2}x_2 + \ldots + (a_{nn}-\omega^2)x_n = 0.
\end{aligned}
\tag{2.21}
$$

In this case too, we must impose that the determinant be zero, in order to obtain non-trivial solutions. We have an algebraic equation of n degrees in ω^2. Its n solutions, say $\omega_1^2, \omega_2^2, \ldots \omega_n^2$, are the proper angular frequencies of the n normal modes. While it can be shown that all the solutions are positive, and consequently physically meaningful, it can happen, depending on the system, that the values of some of them coincide. In this case, the corresponding normal modes are said to be *degenerate*.

The system in Eq. (2.21) gives the ratios between the n amplitudes and one that is arbitrary, namely the mode shape. The initial conditions determine the other constants.

2.2 Forced Oscillators with Several Degrees of Freedom

We shall now consider a linear oscillating system with two degrees of freedom, forced by an external force and in the presence of a drag force proportional to the velocity. We shall start with the example of the two equal pendulums from the previous section, applying a force to one of them, namely pendulum a, as shown in Fig. 2.5. We consider an external force sinusoidally dependent on time, say $F(t) = F_0 \cos \omega t$.

The differential equations of motions of the system are

$$
\begin{aligned}
&\frac{d^2x_a}{dt^2} + \gamma\frac{dx_a}{dt} + \frac{g}{l}x_a + \frac{k}{m}(x_a - x_b) = \frac{F_0}{m}\cos\omega t \\
&\frac{d^2x_b}{dt^2} + \gamma\frac{dx_b}{dt} + \frac{g}{l}x_b + \frac{k}{m}(x_b - x_a) = 0.
\end{aligned}
\tag{2.22}
$$

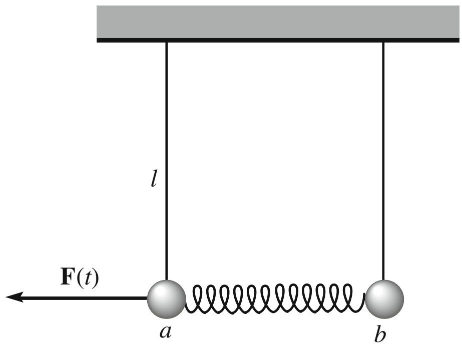


Fig. 2.5 Two equal forced coupled pendulums

We know that the two normal modes of the system when it is free and in the absence of drag, namely if $\gamma = F_0 = 0$, are

$$\begin{array}{lll} \text{mode 1} & x_a(t) = x_b(t); & \omega_1^2 = \frac{g}{l} \\ \text{mode 2} & x_a(t) = -x_b(t); & \omega_2^2 = \frac{g}{l} + 2\frac{k}{m}. \end{array}$$

Let us try and see if the normal coordinates that worked for the free system, namely

$$x_1(t) = x_a(t) + x_b(t)$$
$$x_2(t) = x_a(t) - x_b(t)$$

still work now. Let us add and subtract the two sides of Eq. (2.22), obtaining

$$\begin{aligned} &\frac{d^2x_1}{dt^2} + \gamma\frac{dx_1}{dt} + \frac{g}{l}x_1 = \frac{F_0}{m}\cos\omega t \\ &\frac{d^2x_2}{dt^2} + \gamma\frac{dx_2}{dt} + \left(\frac{g}{l} + \frac{2k}{m}\right)x_2 = \frac{F_0}{m}\cos\omega t. \end{aligned} \tag{2.23}$$

We find that we have been lucky; the two equations are independent of one another. Consequently, x_1 and x_2 are the normal coordinates for the present system as well.

We see that each of the equations in Eq. (2.23) is the differential equation of a damped and forced oscillation. The normal coordinate x_1 behaves like the coordinate of such an oscillator with proper square angular frequency $\omega_1^2 = g/l$ and damping γ forced by the force $F(t) = F_0 \cos wt$. The normal coordinate x_2 behaves similarly with proper square angular frequency $\omega_2^2 = g/l + 2k/m$. The two oscillations are independent. Each of them behaves like a one degree of freedom oscillator. Remember, however, that x_1 and x_2 are not physical displacements, but rather combinations of these.

The system has two resonances, one for each of its proper frequencies. If the frequency of the external force is close to one of the resonance frequencies, the system, after the transient phase has finished, reaches its steady regime. It moves in the normal mode corresponding to that proper frequency. The other normal coordinate is zero.

The resonances of the two modes, in general, not only have different frequencies but also different widths. This is not the case in the simple example we have just discussed, but suppose, for example, that the spring joining the pendulums dissipates a certain amount of energy when it is stretched back and forth. In this case, the energy loss rate will be larger for mode 2 (in which the spring is stretched) than for mode 1 (in which the length of the spring does not vary). As a consequence, the width of the second resonance will be larger than that of the first.

The discussion we developed around a simple example is valid in general, provided that the difference between the proper frequencies is substantially larger than the widths of both of them. In these cases, normal motions exist, even if finding the normal coordinates is not as simple as in the above example. This is not the case in the presence of damping if the resonances are too close to one another.

All the arguments can be extended to systems with n degrees of freedom.

2.3 Transverse Oscillations of a String

Up to now, we have considered systems with a discrete number of degrees of freedom. We shall now discuss the normal modes of a system with a continuous, infinite number of degrees of freedom. We shall analyze the important case of an elastic string with fixed extremes. More precisely, our string is an ideal one, namely it is perfectly elastic (its tension is proportional to the stretching), perfectly flexible (it does not oppose to folding) and is homogenous (its linear mass density ρ is uniform). The extremes are fixed at a distance L. At equilibrium, the tension of the rope, T_0, is independent of the position. The motions we shall study are the small oscillations.

Let x, y, z be a reference frame, with the z-axis on the equilibrium position of the string and the origin in its left extreme, as in Fig. 2.6. We identify each element of the string by the z coordinate of the *equilibrium position* of the element. The position of that element when the string is moving at the instant in time t is a vector function of z and t, which we call $\boldsymbol{\psi}(z, t)$. This vector has a component along the string, corresponding to the *longitudinal oscillation*, and two components normal to the string, corresponding to the *transverse oscillations*. We shall limit the discussion here to the transverse oscillations. The z-component of $\boldsymbol{\psi}(z, t)$ is identically zero.

In general, in a transverse oscillation, the direction of $\boldsymbol{\psi}(z, t)$ on the xy plane is different both for different z and for different times. A transverse oscillation is said to be *plane polarized* or, alternatively, *linearly polarized* if the direction of $\boldsymbol{\psi}(z, t)$ is independent of both z and t. In this case, if we take a picture at any instant in time, we see the string shape as a plane curve, always in the same plane. We shall limit the discussion, for the sake of simplicity, to a linearly polarized motion, in which the displacement is represented by a single function $\psi(z, t)$, rather than by a vector.

Let us consider a small segment of the string, of length at rest Δz. Its mass is $\Delta m = \rho \Delta z$. In the generic non-equilibrium configuration, the element we are considering is removed from the z-axis, as shown in Fig. 2.7. The displacements of both extremes are normal to the z-axis, because we assumed a purely transverse

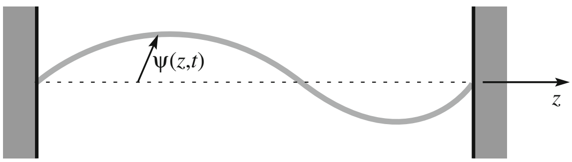


Fig. 2.6 A configuration of an elastic string with fixed extremes

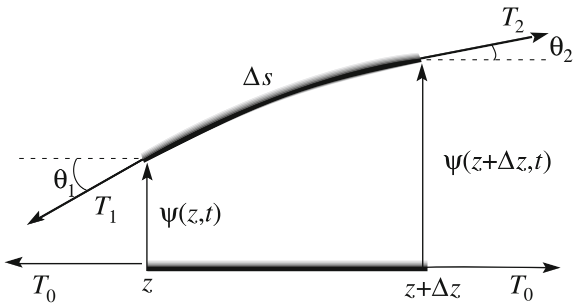


Fig. 2.7 A segment of the elastic string in a generic position

oscillation and they are, in general, different from one another. In general, the element is not straight, namely the angles with the z-axis at the two extremes may be different ($\theta_1 \neq \theta_2$ in Fig. 2.7). The component of the resultant force in the direction opposite to that of the displacement is

$$F(t) = T_2 \sin\theta_2 - T_1 \sin\theta_1.$$

Two important simplifications are possible if the displacements from equilibrium of all the elements of the string are small. First, we can approximate the tangent of the angles with the angle or with its sine. Second, we can use the Pythagorean theorem to find an approximate relation between the length Δs of the element and its length at rest Δz. The theorem gives us

$$\Delta s^2 = \Delta z^2 + [\psi(z+\Delta z,t) - \psi(z,t)]^2 = \Delta z^2 + \left(\frac{\partial\psi}{\partial z}\Delta z\right)^2,$$

namely

$$\Delta s = \left[1 + \left(\frac{\partial\psi}{\partial z}\right)^2\right]^{1/2} \Delta z$$

Now, $\partial\psi/\partial z = \tan\theta$ is small, say infinitesimal, and we can expand the square root on the right-hand side of this expression, stopping at the first term. We have

$$\Delta s = \left[1 + \frac{1}{2}\left(\frac{\partial\psi}{\partial z}\right)^2\right] \Delta z.$$

This expression shows, finally, that the difference between Δs and Δz is infinitesimal of the second order. Under these conditions, the tensions T_1 and T_2 at the two extremes differ from the tension at rest T_0 by infinitesimals of the second order, which we neglect. Under these approximations, for the restoring force, we can write

$$F(t) = T_0 \sin\theta_2 - T_0 \sin\theta_1 = T_0\left(\frac{\partial\psi}{\partial z}\right)_2 - T_0\left(\frac{\partial\psi}{\partial z}\right)_1 = T_0\frac{\partial^2\psi}{\partial z^2}\Delta z.$$

The acceleration of the element is $\partial^2\psi/\partial t^2$ and the second Newton law gives us

$$T_0\frac{\partial^2\psi}{\partial z^2}\Delta z = \rho\Delta z\frac{\partial^2\psi}{\partial t^2}.$$

Finally, simplifying out Δz, we obtain

$$\frac{\partial^2\psi}{\partial t^2} - \upsilon^2\frac{\partial^2\psi}{\partial z^2} = 0, \tag{2.24}$$

where we have introduced the constant

$$\upsilon = \sqrt{T_0/\rho}, \tag{2.25}$$

which has the physical dimension of a velocity, as it is easy to check.

Equation (2.24) is a very famous partial differential equation known as the *vibrating string equation* and, more commonly, the *wave equation*, for reasons that will become clear in the subsequent chapter. From now on, through the entire book, we shall discuss the properties of its solutions in a number of sectors of physics.

Let us now search for the *normal modes* of the system. Namely, we look for solutions in which all the parts of the system move sinusoidally with the same frequency and with the same phase, meaning that both ω and ϕ are independent of z. The solution should have the form

$$\psi(z,t) = A(z)\cos(\omega t + \phi). \tag{2.26}$$

Let us substitute this in Eq. (2.24). We obtain

$$\frac{d^2A(z)}{dz^2} = -\frac{\omega^2}{\upsilon^2}A(z). \tag{2.27}$$

The solution to this differential equation $A(z)$ gives the *shape of the mode*. Different modes have different shapes due to the factor ω^2 in the equation, which differs from one mode to another.

Equation (2.27) is formally equal to the equation of the harmonic oscillator, with the space coordinate z at the place of the time t. Hence, we know how to solve it. In the present case, the most useful form is

$$A(z) = A\sin kz + B\cos kz, \tag{2.28}$$

where A and B are the integration constant that we shall soon find and

$$k^2 = \frac{\omega^2}{\upsilon^2}. \tag{2.29}$$

The quantity k is called the *wave number*, which is inversely proportional to the *wave length* λ as

$$k = \frac{2\pi}{\lambda}. \tag{2.30}$$

The function $A(z)$ is the oscillation amplitude of the element at z, namely the shape of the mode. We see that it is a sinusoidal function of z. The wavelength is the period in length, which is equivalent to the period in time T in a periodic function of time. The unit of the wavelength is the meter. Similarly, the wave number k is the equivalent in length of the angular frequency ω ($\omega = 2\pi/T$) in time. Consequently, it is also called *spatial frequency*. Its measurement units are the inverse of a meter (m^{-1}). Note that we shall use the same symbol k for spatial frequency as for the spring constant, but this should not generate confusion.

Note also that ω and k are not at all independent. Rather, when one of the two is known, the other is known as well. Indeed, for every system, a relation between ω and k exists, called the *dispersion relation* . The dispersion relation of the ideal elastic string we are discussing is given by Eq. (2.29). Other systems, in general, have more complicated dispersion relations. The dispersion relation is independent of the boundary conditions.

Let us go back to the normal modes we have found, putting together Eqs. (2.26) and (2.28). The solution we have found is

$$\psi(z,t) = (A\sin kz + B\cos kz)\cos(\omega t + \phi). \tag{2.31}$$

We now determine the integration constant A and B by imposing the boundary conditions. These are the equivalent of the initial conditions we have always used when working in the time domain. The boundary conditions are that the extremes do not move, namely $\psi(0, t) = 0$ and $\psi(L, t) = 0$ for any t. The first of these gives $B = 0$ and we can consequently rewrite Eq. (2.31) as

$$\psi(z,t) = A\sin(kz)\cos(\omega t + \phi) = A\sin\left(\frac{2\pi}{\lambda}z\right)\cos(\omega t + \phi). \tag{2.32}$$

The second condition, and $\psi(L, t) = 0$, imposes that $\sin(2\pi L/\lambda) = 0$, trivial solution $A = 0$ apart. The unique quantity we can adjust is the wavelength λ. This means that the modes exist only for definite wavelengths. There is an infinite sequence of them, namely

$$\lambda_1 = 2L,\ \lambda_2 = \lambda_1/2,\ \lambda_3 = \lambda_1/3,\ \lambda_4 = \lambda_1/4, \ldots. \tag{2.33}$$

Understanding the reason for that is easy if we recognize that the wavelengths of the modes are those that have one half of the string length as an integer multiple.

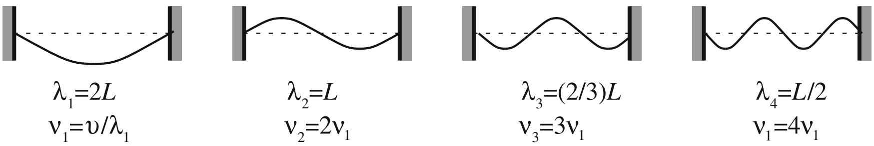


Fig. 2.8 The first four oscillation modes of an elastic string with fixed extremes

Namely, an integer number of half wavelengths must fit exactly between the extremes, as shown in Fig. 2.8.

Summarizing, when the string vibrates in one of its normal modes, each of its elements oscillates in a sinusoidal motion, all of them at the same frequency and in the same phase. There are instants in which the string is straight, passing through its equilibrium configuration. The vibration amplitude is different in the different positions. There are points that are always at rest, which are called the *nodes*, and points that vibrate with maximum amplitude, called the *anti-nodes*. Two contiguous nodes and two contiguous anti-nodes are separated by half a wavelength.

The dispersion relation in Eq. (2.29) can be written in terms of the wavelength λ and of the frequency $\nu = 1/T$ as

$$\nu\lambda = \upsilon. \tag{2.34}$$

It follows that each mode vibrates with a definite frequency. The sequence (or progression) of these frequencies is

$$\nu_1 = \upsilon/\lambda_1,\ \nu_2 = 2\nu_1,\ \nu_3 = 3\nu_1,\ \nu_4 = 4\nu_1, \ldots \tag{2.35}$$

We see that the proper frequencies are the integer multiples of the smallest one, called the *fundamental frequency*. This is a sequence (or progression) well known from mathematics courses, called the harmonic sequence. The frequencies above the fundamental are called harmonics. The reason for these names is in the physical phenomenon we are discussing. Indeed, Pythagoras of Samos (Greece, −570 to −495) discovered early on that two vibrating strings of a musical instrument give a pleasant sound, we say a harmonic sound, if their lengths are multiples of one another, and consequently if the fundamentals are multiples of one another as well. The sound is also pleasant if the lengths are in the ratio of two small integer numbers, when some of the harmonics of the two strings coincide.

Note that what we have just observed is a consequence of the linear relation between ω and k. Namely, the vibration frequency is directly proportional to the wave number, hence inversely proportional to the wavelength ($\nu = \upsilon/\lambda$, with υ being a constant). We also note that the proportionality constant

$$\upsilon = \sqrt{T_0/\rho}. \tag{2.36}$$

is directly proportional to the square root of the tension (namely of the restoring force) and inversely to the square root of the density (namely the inertia). In other words, for a given length of the string, the vibration frequency is higher if the tension is larger and if the density is smaller. As always, the proper frequency square is proportional to the restoring force per unit mass.

As we already stated, a dispersion relation exists for every system, however, such a relation is often more complicated than Eq. (2.29) or Eq. (2.34) $\lambda v = \omega/k = \upsilon$ = constant. Piano strings, for example, are not perfectly flexible. They present a small degree of stiffness, which opposes flexion. Being that flexion is larger for larger curvatures, namely for shorter wavelengths or larger wavenumbers k, the contribution of stiffness to the restoring force is larger for larger wave numbers. Let us develop the unknown dispersion relation $\omega^2(k^2)$ in series of powers of k^2 and stop at the first term. This is proportional $(k^2)^2$ by a certain constant α. We write

$$\omega^2 = \frac{T_0}{\rho}k^2 + \alpha k^4.$$

We do not really know the constant α, but, remembering that ω^2 is the restoring force per unit displacement and per unit mass, we know that it is positive for the argument given above.

The boundary conditions for our piano string are the same as those we discussed for an ideal string, and consequently the shapes of the modes are the same. Their wavelengths are still $\lambda_1 = 2L$, $\lambda_2 = \lambda_1/2$, etc., but now the proper frequencies of the modes are not exactly integer multiples of the fundamental. Being that $\alpha > 0$, the frequencies of the higher modes are a bit higher than those of the harmonic sequence. The sound is still (or even more) pleasant if the differences are not too large.

We have seen in Sects. 2.1 and 2.2 that every motion of a linear system with n degrees of freedom (where n is an integer number) can be expressed as a linear combination of its n normal modes. The vibrating string we are now studying has a continuous infinite number of degrees of freedom. As we have seen, its normal modes are numerable infinite. Well, even in this case, it can be shown (although we shall not do that) that every possible motion of the string (with fixed extremes) can be expressed as a linear combination of its normal modes. This means that, given an arbitrary motion represented by the function $\psi(z, t)$, we can find two infinite sequences of numbers, F_0, F_1, F_2… (amplitudes) and ϕ_1, ϕ_2,… (initial phases) such that

$$\psi(z,t) = \sum_{m=1}^{\infty} F_m \cos(\omega_m t + \phi_m) \sin k_m z, \tag{2.37}$$

where ω_m are the proper angular frequencies and $k_m = \omega_m/\upsilon$.

The normal modes of the vibrating string that are solutions to the wave equation are also called *stationary waves*.

Let us finally consider the energy of a vibrating string. Every element of the string has kinetic and potential energy in every moment. As we have just seen, every motion can be expressed as a superposition of the modes. Consider now the energy the string would have if it were vibrating in the generic mode m with the same amplitude F_m appearing in that superposition. Let us call it the *energy of the mode*. Well, it is easy to show (although we shall not do that) that the energy of the string in the considered motion is equal to the sum of the energies of the modes of the superposition taken separately.

2.4 The Harmonic Analysis

We have already stated that the functions that we usually encounter in the description of physical phenomena can be expressed as linear combinations of sines or cosines. The mathematical process for finding such combinations is called the *harmonic analysis* or *Fourier analysis* of the function under examination. This represents an important chapter in mathematics initiated in modern terms by Joseph Fourier (France; 1768–1830) in 1809 (after important contributions by several predecessors). In both this section and the one that follows it, we shall present the elements of the harmonic analysis that we shall need in the subsequent study, having in mind physics rather than mathematical rigor. We shall not provide the mathematical demonstrations of the statements. However, we shall discuss several examples of interest for physics.

We preliminarily note that harmonic analysis is extremely useful, both for functions of time, such as the displacement of a point of a string or the evolution of a force, and for functions of the position in space, such as the surface of the sea with its waves at a certain instant or the level of gray in a photograph.

This section has an introductory character, intended to provide a sense of the issue as it applies to physics. As we are dealing with an analysis that is called harmonic, let us start from harmony, namely from music and musical tones.

To be specific, let us consider a guitar string tuned to an A at 440 Hz. We assume the string to be perfectly flexible. Consequently, the frequencies of its normal nodes are in harmonic sequence, namely

$$\nu_1 = 440\,\text{Hz},\ \nu_2 = 880\,\text{Hz},\ \nu_3 = 1320\,\text{Hz},\ldots, \tag{2.38}$$

If we pluck the string, we move it out of equilibrium. Initially, the string is not in a normal mode. The disturbance propagates in both directions, reaches the extremes, is reflected there, and then comes back, and so on. After a short moment of transition, the motion of the string becomes stationary. The stationary vibration continues for a duration that is very long compared to the period of the fundamental, which is $1/\nu_1 = 22.7$ ms.

In its vibrations, the string produces periodic variations of the air pressure that propagates in space. This is the sound wave we shall be studying in Sect. 3.4. The sound wave, in turn, sets our eardrum into vibration, which is a motion completely similar to the vibrations of the points of the string; and thus we perceive the sound.

Let $f(t)$ be the displacement from equilibrium of the eardrum at the instant t. If we were to measure it, we would obtain something like that represented in Fig. 2.9.

As a comparison, let us think about recording the motion of our eardrum when we perceive noise, for example, the clapping of our hands. We would find something similar to Fig. 2.10. Comparing the two cases, we see that a musical sound corresponds to a *periodic* function of time, and a noise, a non-periodic one. The function shown by Fig. 2.9 repeats identically when the time is incremented by a well-defined quantity T, which is the *period* of the function. Mathematically expressed, the function has the property that $f(t + T) = f(t)$ for every t. This is clearly an idealization. Indeed, rigorously speaking, a periodic function should exist for an infinitely long time from minus infinite in the past to plus infinite in the future. No physical phenomenon is represented by a truly periodic function. However, if the duration of the phenomenon is long compared to the period, we can consider the function as being approximately periodic.

The periodic function $f(t)$ representing the sound of the string in general is not, however, a simple sinusoidal function. This is because the string does not vibrate in a normal mode, as a consequence of its initial state not having had the shape of any mode. If the motion was the first normal motion (the fundamental), the vibration of the eardrum (and of any point on the string) would be a cosine of period $T = 1/\nu_1 = 22.7$ ms. The second normal motion is a cosine function as well, with period $T/2$. This means that the function repeats itself every $T/2$ s. But it repeats itself every T seconds as well. Similarly, the third mode repeats itself every $T/3$ s,

Fig. 2.9 A musical sound

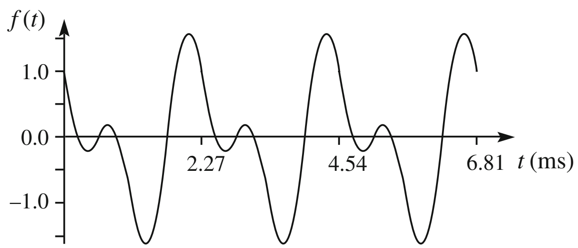
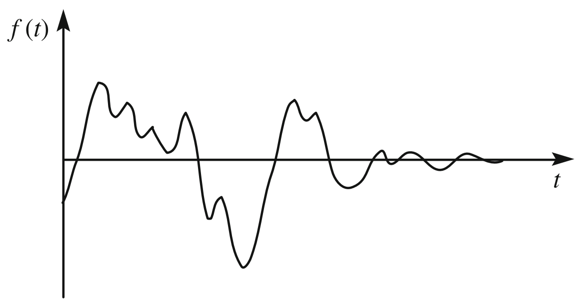


Fig. 2.10 A noise



and hence also every T seconds, etc. In conclusion, all the functions cos $\omega_0 t$, cos $2\omega_0 t$, cos $3\omega_0 t$, … (and sin $\omega_0 t$, sin $2\omega_0 t$, sin $3\omega_0 t$, … as well) are periodic functions with period T. Hence, every linear combination of these functions is periodic with period T as well.

Hence, the motion of our eardrum, which is a linear combination of harmonic motions as given by Eq. (2.38), is a periodic motion at the frequency $T = 1/v_1$. The term corresponding to each mode enters into the linear combination with a certain amplitude (we shall call F_m the amplitude of the mode at the frequency v_m) and with a certain initial phase (which we shall indicate with ϕ_m). Both quantities depend on the initial configuration of the string. If the angular frequency of the fundamental is $\omega_0 = 2\pi/T$, the general motion of the eardrum can be expressed as

$$f(t) = F_1\cos(\omega_0 t + \phi_1) + F_2\cos(2\omega_0 t + \phi_2) + F_3\cos(3\omega_0 t + \phi_3) + \ldots, \quad (2.39)$$

We obtained the example shown in Fig. 2.9 by adding together only two terms, namely as

$$f(t) = \cos(2\pi \cdot 440 \cdot t + 0) + 0.8 \cdot \cos(2\pi \cdot 880 \cdot t + \pi/2),$$

with t in seconds. Figure 2.11 graphically represents the two components and the resulting combination. Note, in particular, how the two components start, at $t = 0$, at a different point of their period. This is because the initial phases are different: one is 0 and the other is $\pi/2$.

We shall now use this simple example to discuss some general features.

Let us first consider doubling the amplitudes of both components. The result is simply twice what we had. The new function has the same shape as the old one. Hence, generally speaking, the shape of the resulting function depends on the ratios between the component amplitudes, and not on their absolute value.

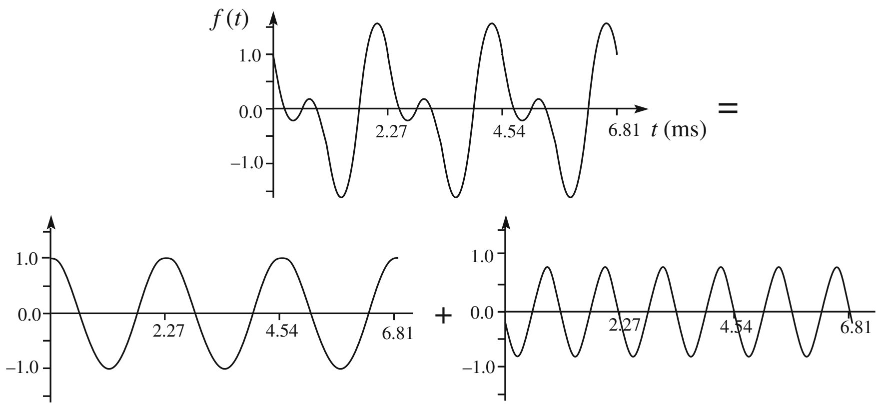


Fig. 2.11 The components of the function in Fig. 2.9 having periods T and $T/2$

Let us now change the initial phases by adding the same quantity to each of them, for example, $\pi/2$. We have the function

$$f(t) = \cos(2\pi \cdot 440 \cdot t + \pi) + 0.8 \cdot \cos(2\pi \cdot 880 \cdot t + 3\pi/2),$$

The result is shown as a dotted curve in Fig. 2.12. The new function has the same shape and magnitude as the old one. It is simply translated forward in time by half a period.

Let us now change the difference between the initial phases. Let us have, for example, $\pi/4$ in the second term instead of $\pi/2$ without changing the argument of the first, obtaining

$$f(t) = F_1\cos(2\pi \cdot 440 \cdot t) + 0.8 \cdot F_1\cos(2\pi \cdot 880 \cdot t + \pi/4).$$

Figure 2.13b shows the result, while Fig. 2.13a shows the original function for comparison. Now, the shape has changed. Generalizing the result, the shape of the combination depends on the differences between the initial phases, but is independent of their absolute values.

All the functions we have considered in the above examples have mean values over a period equal to zero. Indeed, this is the case for any function representing the displacement from a fixed equilibrium position in an oscillation about it. More generally, the mean value of a function might be different from zero. Let us consider, for example, the function

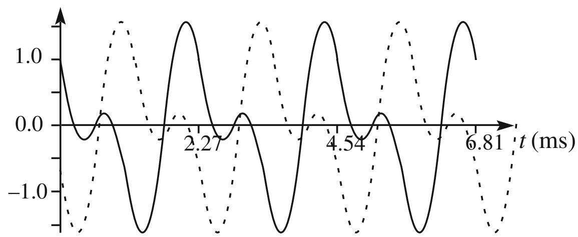


Fig. 2.12 Changing by the same amount both phases of the components

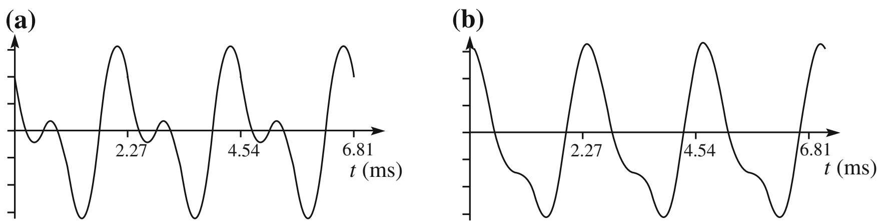


Fig. 2.13 Changing the phase difference between the components

$$f(t) = F_0 + \cos(2\pi \cdot 440 \cdot t) + 0.8 \cdot \cos(2\pi \cdot 880 \cdot t + \pi/2).$$

Being that the mean values over a period of all the cosines are zero, the mean value of f is F_0. Figure 2.14 shows this function as a dotted line compared with the original one, which is a continuous line. Clearly, the shapes of both are equal. We have only a vertical shift up or down, depending on the sign of F_0.

We have concluded the observations for our simple example. They have a general character. We are ready to state the Fourier theorem, which is valid for a very large class of periodic function, as precisely defined by mathematics. We shall not demonstrate the theorem, but shall simply state that practically all the functions encountered in physics obey the theorem. We state that, given any "reasonable" function of time $f(t)$ periodic of period T, one can always find two infinite sequences of real numbers F_0, F_1, F_2, … and ϕ_1, ϕ_2, …, which, with $\omega_0 = 2\pi/T$, are such that

$$f(t) = F_0 + F_1 \cos(\omega_0 t + \phi_1) + F_2 \cos(2\omega_0 t + \phi_2) + \ldots, \tag{2.40}$$

Let us go back to our example of the musical tone. In general, the motion of our eardrum can be expressed as the series in Eq. (2.40) (with $F_0 = 0$). The term in $\nu_1 = \omega_0/2\pi$ is the fundamental, the subsequent ones are the harmonics. Each of them is the same note in a higher octave. Two notes differ by an octave when the frequency of the second is twice the frequency of the first. The amplitudes F_m define the relative importance of the subsequent harmonics. Our ear is capable of appreciating the relative weights of the harmonics, namely the F_m. In other words, our ear performs a Fourier analysis. On the other hand, we are not sensitive to the phases.

The proportions of the fundamental and of the different harmonics determine what is called the *timbre* (and also *color*) of the sound. A sound is said to be *pure* if it contains the fundamental alone, *rich* if, on the contrary, several harmonics are important. The same note, the A we have been considering, for example, is different if it is played by a piano, an oboe, a violin or another instrument, because the different instruments produce harmonics in different proportions. Similarly, the timbre distinguishes the same note sung once as a-a-a, and then again as o-o-o.

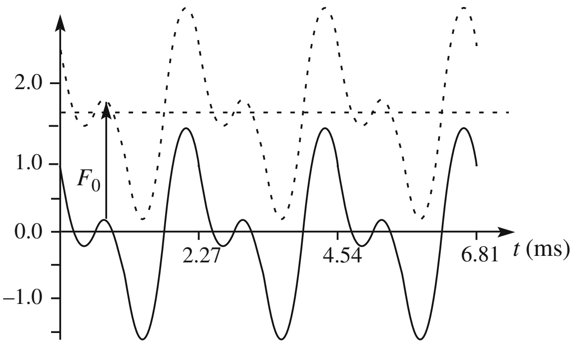


Fig. 2.14 Including a constant term

We recall now that, at the end of Sect. 2.3, we stated that every motion of the string can be expressed as a linear combination of its normal modes writing Eq. (2.37). Let us now consider an arbitrary point of the string, for example, the point at $z = z^*$, and let us indicate its displacement from equilibrium with $g(t)$. This function of time is given by Eq. (2.37), which is a function of z and t, valuated for $z = z^*$, namely

$$g(t) = \psi(z^*, t) = \sum_{m=1}^{\infty} [F_m sink_m z^*] \cos(m\omega_0 t + \phi_m).$$

This expression is just Eq. (2.40) with $F_m sink_m z^*$ in place of F_m, which is just another way to write the constants. This consideration shows that the Fourier series is also the series of the normal modes of a physical system, the flexible string.

The development in Eq. (2.40) can be written in an equivalent form, which will be useful and that we will find immediately. We start from the trigonometric identity $\cos(\omega t + \phi) = \cos\phi \cdot \cos\omega t - \sin\phi \cdot \sin\omega t$. Considering that $\cos\phi$ and $\sin\phi$ are two constants, we can absorb them into the amplitudes of the terms and express the function as the sum of a linear combination of sines $(\sin m\omega_0 t)$ and one of cosines $(\cos m\omega_0 t)$ with initial phase zero. The equivalent form of Eq. (2.40) is

$$f(t) = A_0 + A_1 \cos\omega_0 t + A_2 \cos 2\omega_0 t + \ldots + B_1 \sin\omega_0 t + B_2 \sin 2\omega_0 t + \ldots, \tag{2.41}$$

2.5 Harmonic Analysis of a Periodic Phenomena

In this section, we generalize the conclusions we have reached, giving, without any demonstration, the proper mathematical expressions for the Fourier series. We shall see how to calculate the coefficients in three equivalent, but all useful, expressions.

Consider the *periodic* function of time $f(t)$ with period T, and, correspondingly, angular frequency $\omega_0 = 2\pi/T$. As we know, all the functions $\cos\omega_0 t$, $\cos 2\omega_0 t, \ldots$, $\cos m\omega_0 t\ldots$ and $\sin\omega_0 t$, $\sin 2\omega_0 t, \ldots$, $\sin m\omega_0 t\ldots$are periodic with period T. In addition, these functions constitute a complete set of normal orthogonal functions over the interval 2π. This means that the functions enjoy the following properties:

$$\begin{aligned}
&\frac{1}{\pi}\int_0^{2\pi} \cos mx \cdot \cos nx \cdot dx = \begin{matrix} 0 & \text{for } m \neq n \\ 1 & \text{for } m = n \end{matrix} \\
&\frac{1}{\pi}\int_0^{2\pi} \sin mx \cdot \sin nx \cdot dx = \begin{matrix} 0 & \text{for } m \neq n \\ 1 & \text{for } m = n \end{matrix} \\
&\frac{1}{\pi}\int_0^{2\pi} \cos mx \cdot \sin nx \cdot dx = 0 \quad \text{for any } m \text{ and } n.
\end{aligned} \tag{2.42}$$

In other words, the properties are as follows. Firstly, the integral over a period of the product of two *different* functions is zero. Such functions are said to be orthogonal in analogy to the fact that the scalar product of two orthogonal vectors is zero. Here, we have the integration in place of the scalar product. Secondly, the integral of the square of each function is equal to 1. Such functions are said to be normal. This has been obtained by including the "normalization factor" $1/\pi$.

The Fourier theorem states (as we already discussed in Sect. 2.4) that, given any "reasonable" function of time $f(t)$ periodic of period T, it is always possible to find two infinite successions of real numbers A_0, A_1, A_2, … and B_1, B_2, … such that

$$f(t) = A_0 + \sum_{m=1}^{\infty} A_m \cos(m\omega_0 t) + \sum_{m=1}^{\infty} B_m \sin(m\omega_0 t), \tag{2.43}$$

where $\omega_0 = 2\pi/T$. The constant term A_0 is the mean value of the function in *any* time interval T, namely it is

$$A_0 = \langle f \rangle = \frac{1}{T} \int_{\tau}^{\tau+T} f(t)dt \tag{2.44}$$

Note that the cosine is an even function of its argument (its values in opposite values of the argument are equal). Consequently, in the development of an even function $f(t)$ of t, (namely such that $f(-t) = f(t)$), $B_m = 0$ for all m. Similarly, in the development of an odd function ($f(-t) = -f(t)$), $A_m = 0$ for all m. In the general case, expression (2.43) shows the even and odd parts of the function separately.

The ortho-normality property of the set of sine and cosine functions allows us to find the expressions of the coefficients of the series immediately. To find the generic A_n, we multiply both sides of Eq. (2.43) by $\cos n\omega_0 t$, obtaining

$$\begin{aligned} f(t)\cos(n\omega_0 t) = A_0 \cos(n\omega_0 t) &+ \sum_{m=1}^{\infty} A_m \cos(m\omega_0 t)\cos(n\omega_0 t) \\ &+ \sum_{m=1}^{\infty} B_m \sin(m\omega_0 t)\cos(n\omega_0 t) \end{aligned}$$

and integrate over a period, say from τ to $\tau + T$. For the orthogonality property, the only integral different from zero on the right-hand side is the term in the first sum with $m = n$, so that we have

$$\int_{\tau}^{\tau+T} f(t)\cos(n\omega_0 t)dt = A_n \int_{\tau}^{\tau+T} \cos^2(n\omega_0 t)dt = A_n \frac{T}{2},$$

and hence, finally,

$$A_n = \frac{2}{T}\int_{\tau}^{\tau+T} f(t)\cos(n\omega_0 t)dt.$$

Similarly, we obtain B_n by multiplying by sin $n\omega_0 t$ and integrating over a period. In conclusion, the coefficients of the Fourier series of the periodic function $f(t)$ are given by

$$A_n = \frac{2}{T}\int_{\tau}^{\tau+T} f(t)\cos(n\omega_0 t)dt; \quad B_n = \frac{2}{T}\int_{\tau}^{\tau+T} f(t)\sin(n\omega_0 t)dt \tag{2.45}$$

The second equivalent expression of the Fourier series is

$$f(t) = F_0 + \sum_{m=1}^{\infty} F_m \cos(m\omega_0 t + \phi_m), \tag{2.46}$$

where the coefficients are now the F_m (which are non-negative) and the $_m$. We immediately state these quantities in terms of the A_m and B_m as

$$F_0 = A_0,\ F_m = \sqrt{A_m^2 + B_m^2},\ \phi_m = -\arctan\frac{B_m}{A_m}. \tag{2.47}$$

The third equivalent form of the series, which we shall use very often, is obtained from Eq. (2.46), using the identity

$$F_m \cos(m\omega_0 t + \phi_m) = \frac{F_m}{2}\mathrm{e}^{im\phi}\mathrm{e}^{im\omega_0 t} + \frac{F_m}{2}\mathrm{e}^{-im\phi}\mathrm{e}^{-im\omega_0 t}$$

and defining the coefficients as

$$C_0 = F_0,\ C_m = \frac{F_m}{2}\mathrm{e}^{i\phi_m},\ C_{-m} = \frac{F_m}{2}\mathrm{e}^{-i\phi_m}. \tag{2.48}$$

Equation (2.46) becomes

$$f(t) = \sum_{m=-\infty}^{\infty} C_m \mathrm{e}^{im\omega_0 t}, \tag{2.49}$$

Note that the sum now runs on all the integer numbers, not just the positive ones, and that the coefficients are complex numbers. Note also that the two coefficients corresponding to the opposite values of the index are the complex conjugates of one another. As immediately seen from their definitions, we have

$$C_{-m} = C_m^*. \tag{2.50}$$

We may also immediately verify that the complex coefficients C_m are given by

$$C_m = \frac{1}{T}\int_{\tau}^{\tau+T} f(t)\mathrm{e}^{-in\omega_0 t}dt. \tag{2.51}$$

We shall call the coefficients F_m the *Fourier amplitudes* and ϕ_m the *Fourier phases* of the Fourier series. They are, respectively, the modulus and the argument of the *complex Fourier amplitudes* C_m.

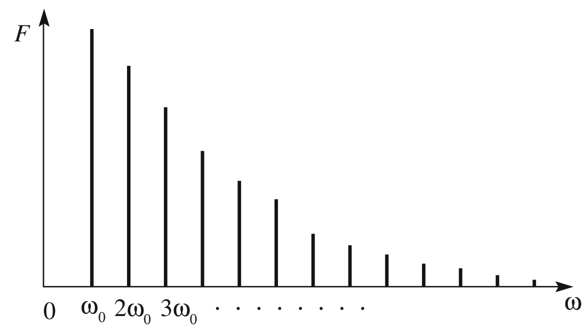
Finding the coefficient of the Fourier series of a given function is called Fourier analysis or harmonic analysis.

The sequence of Fourier amplitudes F_m of a time-dependent phenomenon is called the *amplitude spectrum* or simply the spectrum of the phenomenon. The periodic phenomena we are considering have a discrete spectrum. An example is shown in Fig. 2.15. We can appreciate the importance of the concept of an amplitude spectrum if we remember that the energy of the oscillating system is equal to the sum of the energies of its normal modes, namely of the energies of its Fourier components. The latter are proportional to the squares of the amplitudes F_m and do not depend on the phases. The energies of the Fourier components are completely determined by the amplitude spectrum of the function.

Let us now consider an example, which is important for our study in the subsequent chapters. Consider the function of time shown in Fig. 2.16a. It is a periodic sequence, with period T, of equal rectangular pulses of height L and length Δt ($\Delta t < T$). We assume the function to be exactly periodic, namely that the sequence of pulses extends through infinite time. Clearly, this is an idealized condition. Real situations will approach it the longer the duration of the sequence is compared to the period.

Taking advantage of the symmetry of the problem, we choose the origin of time in the center of a pulse. The function is an even one. The angular frequency is $\omega_0 = 2\pi/T$. We want to find the Fourier coefficients. The choice of the integration

Fig. 2.15 Example of a spectrum of a periodic phenomenon



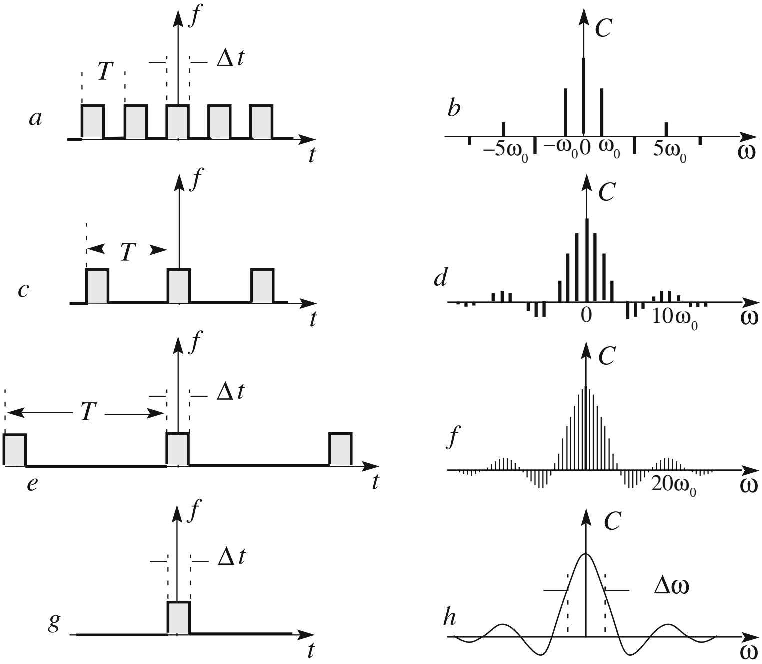


Fig. 2.16 An example of a periodic function and of its spectrum, for different values of the period

period being arbitrary, we choose the easier one, which is cantered on zero, namely $-T/2 \leq t \leq T/2$. In this interval, our function is different from 0 only in the interval $-\Delta t/2 \leq t \leq +\Delta t/2$, in which it is $f(t) = L$. The complex amplitudes are given by

$$C_m = \frac{1}{T}\int_{-\Delta T/2}^{\Delta T/2} L\mathrm{e}^{-im\omega_0 t}dt = -\frac{L}{T}\frac{\mathrm{e}^{-im\omega_0\Delta t/2} - \mathrm{e}^{+im\omega_0\Delta t/2}}{im\omega_0} = \frac{L\Delta t}{T}\frac{\sin(m\omega_0\Delta t/2)}{m\omega_0\Delta t/2}.$$

Let us analyze the result. The mth coefficient of the series is the product of a constant times a function. The constant is the product of the height of the pulse (L) and the ratio between its length (Δt) and the period (T). The function is (sin x)/x, a function that we shall often encounter. To analyze its behavior, we start by observing that the quantity $m\omega_0$ in its argument is the mth angular frequency in the series. Let us call it $\omega_m = m\omega_0$. We then write

$$C_m = \frac{L\Delta t}{T}\frac{\sin(\omega_m\Delta t/2)}{\omega_m\Delta t/2}. \tag{2.52}$$

We note that, as a consequence of $f(t)$ being an even function, the coefficients are real and, for $m \neq 0$, $C_m = C_{-m}$. The coefficients are shown in Fig. 2.16b. We shall use the other parts of the figure in the next section. We recall that these coefficients are one half of the Fourier amplitudes F_m.

2.6 Harmonic Analysis of a Non-periodic Phenomena

The Fourier analysis can be extended to non-periodic functions. These functions can represent phenomena that are "periodic" only for a certain duration or that are not periodic at all, like the noise represented in Fig. 2.2.

Let us consider a function $f(t)$, which is different from zero within a definite time interval Δt (as are all the functions describing physical phenomena). Let us consider another function of time, which is equal to $f(t)$ during Δt and that repeats itself periodically with the same form in an arbitrary period $T > \Delta t$. Being that this new function is periodic, we can calculate its Fourier coefficients, write down its Fourier series and then look to see if we can find its limit for T going to the infinite.

Let us consider a very simple, non-periodic function, namely a rectangular pulse of height L and duration Δt. The corresponding periodic auxiliary function is represented in Fig. 2.16a. We have already calculated its Fourier coefficients. Let T now grow, keeping Δt fixed. Figure 2.16a, c, e represent the result for $T = 2\Delta t$, $T = 4\Delta t$ and $T = 8\Delta t$, respectively. Note that when T increases, the fundamental angular frequency $\omega_0 = 2\pi/T$ decreases in an inverse proportion. The abscissa of the diagram is the angular frequency ω. The m-th Fourier amplitude C_m is at the abscissa $\omega_m = m\omega_0$. Consequently, when ω_0 decreases, the amplitudes C_m get closer to one another, while Eq. (2.52) continues to hold, describing their envelope. In other words, when T varies, the values of the ω_ms vary as well, namely the positions on the abscissa at which the amplitudes are evaluated vary, but the dependence of the amplitudes on ω does not change. This is shown in Fig. 2.16b, d, f. It is then convenient to think of the Fourier spectrum, namely of the amplitudes C_m, as the values of a continuous function $C(\omega)$ evaluated in ω_m, namely as $C_m = C(\omega_m)$. In the limit $T \to \infty$ (Fig. 2.16g), the spectrum becomes a continuum function, namely the domain of the function $C(\omega)$ becomes defined on the entire real axis ω and not only at the discrete values ω_m (Fig. 2.16h).

In conclusion, for a non-periodic function, in place of an infinite discrete sequence of Fourier amplitudes, we have a continuous function of the angular frequency. Correspondingly, in place of the Fourier series, we have an integral. The integral is called the *Fourier transform*. We shall now give, without demonstration, the three equivalent expressions of the Fourier transform (analogous to the three expressions of the Fourier series in the previous section).

Let us start with the analogy of Eq. (2.43). In place of the discrete sequences of A_m and B_m, we now have two functions of the continuous variable ω, which we call $A(\omega)$ and $B(\omega)$, respectively. In place of the sums, we have integrals on ω on the entire domain that is $\omega \geq 0$,

$$f(t) = \int_{0}^{\infty} A(\omega) \cos \omega t \, d\omega + \int_{0}^{\infty} B(\omega) \sin \omega t \, d\omega. \tag{2.53}$$

The expressions for the two functions ω, $A(\omega)$ and $B(\omega)$ (analogous to Eq. (2.45)), are

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos \omega t \, dt, \quad B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin \omega t \, dt. \tag{2.54}$$

Note that the domain is on the entire axis of time, from $-\infty$ to $+\infty$.

The second equivalent form, analogous to Eq. (2.46) in the periodic case, is

$$f(t) = \int_{0}^{\infty} F(\omega) \cos(\omega t + \phi(\omega)) d\omega. \tag{2.55}$$

The functions $F(\omega)$ and (ω) are given in terms of the $A(\omega)$ and $B(\omega)$ by the expressions

$$F(\omega) = \sqrt{A^2(\omega) + B^2(\omega)}, \quad \phi(\omega) = -\arctan \frac{B(\omega)}{A(\omega)}. \tag{2.56}$$

The third expression, analogous to Eq. (2.49), is

$$f(t) = \int_{-\infty}^{+\infty} C(\omega) \mathrm{e}^{i\omega t} d\omega. \tag{2.57}$$

The function $C(\omega)$ is complex and so equivalent to two real functions. Its expression, analogous to Eq. (2.51), is

$$C(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \mathrm{e}^{-i\omega t} dt. \tag{2.58}$$

The continuous complex function of the angular frequency ω, $C(\omega)$, is called the *Fourier transform* of the function of time $f(t)$. Twice its absolute value, namely $2|C(\omega)|$, is the *frequency spectrum*, which is a continuous rather than a discrete function, as in the case of the periodic functions. The function $f(t)$ itself, as given by Eq. (2.57), is called the *Fourier antitransform* of the function $C(\omega)$.

Note that the Fourier integral in the third form of Eq. (2.57) extends on both the positive and negative values of the angular frequency. This was already the case for periodic functions in the sum of Eq. (2.49). As we know, the angular frequency is inversely proportional to the period, and is consequently a physically positive quantity. The negative values of appear as a consequence of the mathematical rearrangements we have made, but do not have a direct physical meaning. However, we shall see in the next section that the corresponding quantity for functions of space, which is the spatial frequency, does have a physical meaning both for positive and negative values.

Let us now go back now to the function in Fig. 2.16g, which is not only simple, but very important as well. Its Fourier transform is found using Eq. (2.58). The calculation is simple and completely analogous to what we did for the C_m. We shall not develop it, but rather go directly to the result, which is

$$C(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t)\mathrm{e}^{-i\omega t}dt = \frac{L}{2\pi}\int_{-\Delta t/2}^{+\Delta t/2} \mathrm{e}^{-i\omega t}dt = \frac{L\Delta t}{2\pi}\frac{\sin(\omega\Delta t/2)}{\omega\Delta t/2}. \tag{2.59}$$

Looking at Fig. 2.16h, we see that this function has its absolute maximum at $\omega = 0$. Two minima, where the function is null, are located symmetrically on the sides of the maximum. Going further, we encounter a succession of maxima and minima. The heights of the maxima decrease monotonically. We note that the most important components of the spectrum are located at low frequencies.

This is, indeed, a general characteristic of all the functions of time that have a beginning and an end, namely a finite duration. The most important part of their spectrum is situated in a region of lower frequencies, which we call *bandwidth*. The concept of bandwidth has a certain degree of arbitrariness, but the main argument for defining it is as follows. Consider a certain function of time $f(t)$ and calculate its Fourier transform $C(\omega)$. Let us then reasonably define a bandwidth $\Delta\omega$ and consider the function $C'(\omega)$, which is equal to $C(\omega)$ inside the bandwidth and 0 outside it. Let us antitransform $C'(\omega)$, obtaining, say, the function of time $f'(t)$. The result will not be exactly equal to $f(t)$, but the differences may be small enough for our purposes. If this is not the case, we need to define a somewhat wider bandwidth.

Coming back to the case under discussion, we now define the bandwidth $\Delta\omega$ as one half of the interval between the first two zeroes of $C(\omega)$, which is about the full width of the peak at half maximum (FWHM). The latter are at the values of ω for which $\omega\Delta t = \pm\pi$. We see that the bandwidth $\Delta\omega$ and the duration Δt are linked by the expression

$$\Delta\omega\Delta t = 2\pi. \tag{2.60}$$

This is an extremely important equation. Indeed, it is a particular case of a theorem of general validity, which we call the bandwidth theorem. We shall not demonstrate the theorem, but just give its statement, which is: *the bandwidth $\Delta\omega$ of the Fourier spectrum of a function limited in time to a duration Δt is inversely proportional to that duration.* The result is general, even if the specific coefficient (2π, in our example) depends both on the shape of the function of time and by our specific definitions of the bandwidth and of the duration (which is somewhat arbitrary as well for functions that are not rectangular pulses). Equation (2.60) has very relevant consequences for optics, as we shall see in Sect. 5.3. Even more important are the consequences for quantum physics, where its equivalent is one of the fundamental laws, namely the uncertainty relation between energy and time.

In conclusion, the spectrum of a non-periodic function of time is a continuous function of the angular frequency; the spectrum of a periodic function of time is a function defined for discrete values of the frequency alone. In particular, the spectrum of a sinusoidal function of time has one component only, at the frequency of the sine.

As we shall see in the subsequent chapters, light is an electromagnetic wave. When the wave has a sinusoidal dependence on time, light has a definite color to our eyes. Contrastingly, in white light, there are waves of different frequency, continuously distributed over a wide range. As a consequence, an electromagnetic wave with sinusoidal dependence on time is said to be *monochromatic*, meaning a single color in Greek. By extensions, all the sinusoidal functions of time are often called monochromatic as well.

We shall now discuss two important examples.

Damped oscillation. Elastic oscillations in the presence of resistive forces proportional to velocity are quite common phenomena in different fields of physics. We have studied their equations in Sect. 1.2. We would now like to work with an acoustic oscillator vibrating on a single mode. The tuning forks used to tune the musical instruments have this property. They are U-shaped metal objects properly shaped to the purpose. Let ω_1 be the proper angular frequency of our tuning fork. We excite its vibrations by hitting one of its prongs with a small wooden hammer. Let us consider the displacement from equilibrium of one of its points as a function of time $\psi(t)$ and let ψ_0 be the initial displacement. We assume the air drag to be proportional to the velocity as

$$F_r = -m\gamma\frac{d\psi}{dt}.$$

As we know, the equation of motion for $t > 0$, namely after the oscillation has started, is

$$\psi(t) = \psi_0 \mathrm{e}^{-\frac{\gamma}{2}t}\cos\omega_1 t = \psi_0 \mathrm{e}^{-\frac{t}{2\tau}}\cos\omega_1 t, \tag{2.61}$$

where we have taken, as in Sect. 1.2.,

$$\tau = 1/\gamma$$

We recall that the time τ is the time interval in which the energy of the oscillator (which is proportional to the square of the vibration amplitude) decreases to $1/e$ of the initial value. The oscillation is not exactly sinusoidal, at angular frequency ω_1, as a consequence of damping. Consequently, its spectrum contains components at angular frequencies different from ω_1. The spectrum is obtained by performing the integral in Eq. (2.58) with $f(t) = \psi(t)$. The limits of the integral are 0 and $+\infty$ because $\psi(t) = 0$ for $t < 0$. We also recall that $\omega_1^2 = \omega_0^2 - (\gamma/2)^2$, where ω_0 is the proper frequency in the absence of damping. Performing the Fourier transform integral and taking the square of the result, one finds for the square of the frequency spectrum the expression

$$F^2(\omega) = \frac{1}{(2\pi)^2}\frac{4\omega^2 + \gamma^2}{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}. \tag{2.62}$$

We see that the denominator is the same as that of the resonance curves. If the damping is small, as is often the case, namely if it is $\gamma \ll \omega$, we can neglect the second term in the numerator and write

$$F^2(\omega) = \frac{1}{\pi^2}\frac{\omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}, \tag{2.63}$$

which, constant apart, is exactly the response function, or the resonance curve, of the oscillator in Eq. (1.64). The function $F(\omega)$ is shown in Fig. 2.17. We can then state that *the square of the Fourier transform of a damped oscillation is proportional to the response curve $R(\omega)$ of the resonance of the same oscillator when it is forced.*

Rigorously speaking, the spectrum extends over an infinite frequency range. However, the important contributions are those that are not very different from ω_0 within a certain bandwidth. We here define the bandwidth to be the full width at half maximum (FWHM) of the resonance curve $\Delta\omega_F$. Under this definition, recalling what we stated in Sects. 1.2 and 1.3, we can conclude that:

(a) the bandwidth of the Fourier transform of a weakly damped oscillator is equal to the width of the same oscillator when forced by a periodic external force;
(b) both widths are inversely proportional to the decay time τ of the free oscillations, namely the time in which the stored energy decreases to a value equal to $1/e$ of the initial value.

Namely, we have

$$\Delta\omega_{\text{ris}} = \Delta\omega_F = 1/\tau. \tag{2.64}$$

The bandwidth theorem requires defining the duration Δt of the phenomenon. Rigorously speaking, the duration would be infinite, but, in practice, the vibration is finished after several τ. Somewhat arbitrarily taking $\Delta t = 2\pi\tau$, we can write Eq. (2.64) as

$$\Delta\omega_F\Delta t = 2\pi. \tag{2.65}$$

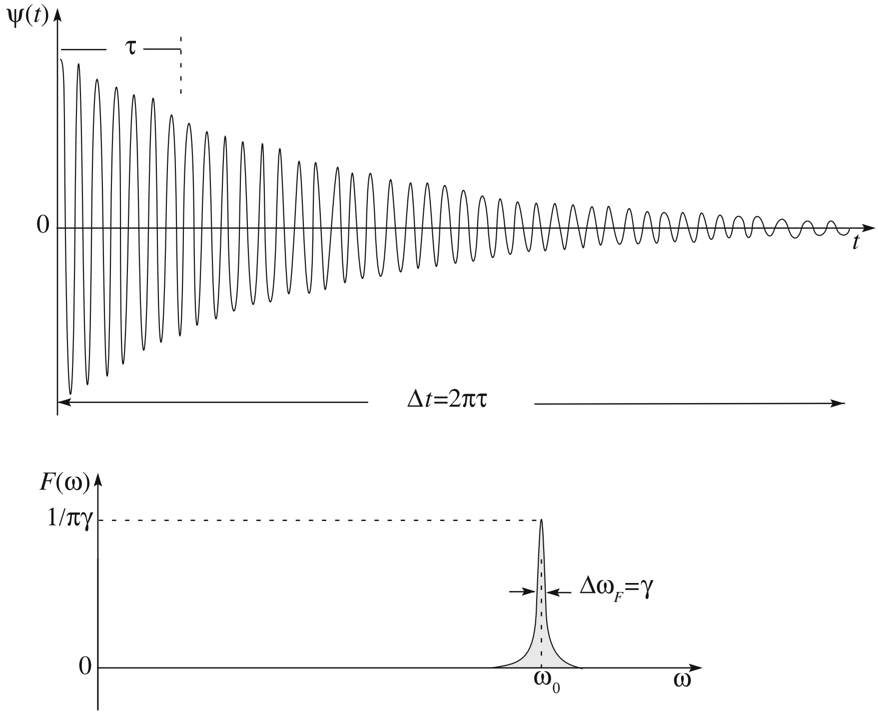


Fig. 2.17 A weakly damped oscillation and its Fourier transform

Even if the oscillators encountered in nature and technology are often damped to some degree, their damping is often small and their behavior approaches that of an ideal oscillator. The *quality factor*, or simply the *Q-factor*, is defined as the ratio between the resonance angular frequency and the FWHM of the resonance curve. The same quantity can be expressed in the following equivalent forms:

$$Q \equiv \frac{\omega_0}{\Delta\omega_F} = \frac{\omega_0}{\gamma} = 2\pi\frac{\tau}{T}. \tag{2.66}$$

Q-factors of mechanical oscillators are typically between several hundreds and several thousands (as in the case of a piano string), and up to millions or more for electronic oscillators.

The just found expressions have important experimental consequences. Indeed, while it is usually easy to measure the decay time of a macroscopic oscillator, for example, making a film of its vibrations, the same cannot be done for an atomic, or subatomic, oscillator. We can, however, enclose a number of atoms, say a gas, in a container and excite them, for example, with an electric discharge that force the ions, a few of which are always present, to violently collide with atoms. The electron cloud of each excited atom will then oscillate with a frequency and decay time that are characteristic of the atomic species under study (as a matter of fact, the atom has several normal modes, each with a frequency and a decay time). The excited atoms emit the energy they have acquired as electromagnetic radiation. The decay time τ, called the lifetime of the excited atomic state, is an important quantity in atomic physics, but is usually too short to be directly measurable. However, if we measure the Fourier spectrum of the intensity of the emitted radiation, we obtain a resonance curve, whose width $\Delta\omega_F$ we can measure. We then have the lifetime from the relation $\tau = 1/\Delta\omega_F$.

QUESTION Q 2.2. Calculate the Fourier transform of Eq. (2.59). □

QUESTION Q 2.3. You hit the prong of a tuning fork of 440 Hz and hear a sound lasting about 30″. How much is its γ? How much is its Q-factor? □

QUESTION Q 2.4. Consider an excited atom having a resonance angular frequency of $2 \times 10^{16}\ \mathrm{s}^{-1}$ and a lifetime of 2 ns. How much is the Q-factor? □

QUESTION Q 2.5. The Q-factor of a harmonic oscillator of frequency 850 Hz is 7000. What is the time in which the amplitude reduces by a factor $1/e$? How many oscillations happen in this time? What is the difference between two consequent oscillation amplitudes relative to the amplitude itself? □

QUESTION Q 2.6. With reference to Eq. (1.40), prove that the Q-factor is equal to 2π times $\langle U\rangle/(d\langle U\rangle/dt)$. □

Beats. A beat is a sound resulting from the interference between two sounds of slightly different frequencies. We perceive it as a periodic variation in volume whose rate of change is the difference between the two frequencies, say ω_1 and ω_2. The sound can be easily heard using two equal tuning forks, and altering the frequency of one of them by fixing a small weight to one of its prongs. If the weight is near the bottom of the prong, the change in frequency is quite small. If we now hit both forks, we hear the beat. If we fix the weight a little higher and repeat the experiment, we hear the volume of the sound periodically varying at a higher frequency. If we further increase the difference, we reach a limit in which the system of ear-plus-brain perceives the two sounds as separate. The limit depends on the person, usually being at about $\Delta\omega/\omega = 6\ \%$. The ear of a musician is substantially more sensitive.

Let us analyze the phenomenon. Assume, for simplicity, that the two vibrations have equal initial amplitudes and phases. The two displacements from equilibrium are then

$$\psi_1(t) = A\cos\omega_1 t, \quad \psi_2(t) = A\cos\omega_2 t.$$

The motion of our eardrum is proportional to the sum of these two functions, namely it is given by

$$\psi = \psi_1 + \psi_2 = A\cos\omega_1 t + A\cos\omega_2 t = 2A\cos\left(\frac{\omega_1-\omega_2}{2}t\right)\cos\left(\frac{\omega_1+\omega_2}{2}t\right).$$

As we stated, the beat happens when the difference between the two frequencies $\Delta\omega = \omega_1 - \omega_2$ is small in absolute value. Under these conditions, the average of the two, namely $\omega_0 = (\omega_1 + \omega_2)/2$, is very close to each of them and we can write within a good approximation

$$\psi(t) = 2A\cos\left(\frac{\Delta\omega}{2}t\right)\cos(\omega_0 t). \tag{2.67}$$

We can think of this expression as describing an almost harmonic motion taking place at the mean frequency ω_0, whose amplitude varies slowly (and harmonically as well) in time with angular frequency $\Delta\omega/2$. The function is shown in Fig. 2.18.

Note that the perceived angular frequency of the modulation is $\Delta\omega$ and not $\Delta\omega/2$. The reason for this is that the ear is sensitive to the *intensity* of sound. The intensity is proportional to the square of the amplitude, namely to $4A^2\cos^2(\Delta\omega t/2) = 2A^2(1 + \cos\Delta\omega t)$. The constant term on the right-hand side is irrelevant; the angular frequency of the varying term is $\Delta\omega$.

Let us now consider the more general case in which the amplitudes of the two motions are different. Let us call them A_1 and A_2, and represent the motions as rotating vectors (Fig. 2.19). The vector sum of the two represents the resultant motion. If the oscillations have the same angular frequency ω, then all vectors rotate as a rigid structure with angular velocity ω. As a consequence, the magnitude of the resultant is constant over time and its x component makes a harmonic motion. Contrastingly, if the two frequencies are a little different, the angle between the vectors A_1 and A_2 slowly varies, and consequently the magnitude of their resultant slowly varies as well. From the figure, we see that the magnitude of the resultant is

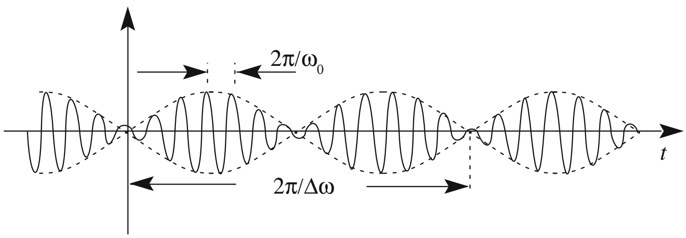
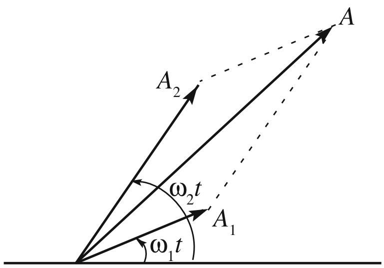


Fig. 2.18 The amplitude of a beat sound

Fig. 2.19 Two rotating vectors and their resultant



$$A = \left[A_1^2 + A_2^2 + 2A_1A_2\cos(\omega_1 - \omega_2)t\right]^{1/2}. \tag{2.68}$$

2.7 Harmonic Analysis in Space

In this section, we shall consider the Fourier analysis as a function of space rather than of time. A function of space is, in general, a function of three variables, which are the coordinates. Such is the case with, for example, the temperature of a fluid. Examples of functions of two spatial coordinates exist as well, like the height of the waves on the surface of a lake or the sea and the gray levels of a photograph. For the sake of simplicity, we shall limit the discussion to functions of one space variable only. Such is the case with, for example, the configuration of a rope at a certain instant in time. Under these conditions, the mathematics of the spatial Fourier analysis is exactly the same as in the time domain we have considered in the previous sections.

Let us start by considering a periodic function of the coordinate x, which we call $f(x)$. As in the case of time, no physical system is described by a strictly periodic function of space, because no system exists of infinite dimensions. However, approximately periodic spatial structures are often encountered.

The period in space is the wavelength λ. The quantity corresponding to the frequency is the wave number, namely the number of wavelengths in one meter

$$\nu_s = 1/\lambda \tag{2.69}$$

and that, corresponding to the angular frequency, is the spatial frequency

$$k = 2\pi/\lambda = 2\pi\nu_s. \tag{2.70}$$

Let $f(x)$ now be a periodic function of x of period λ, and let $k_0 = 2\pi/\lambda$ be its fundamental spatial frequency. Clearly, we can express the Fourier series of $f(x)$ in any of the three forms we have seen for a function of time. We shall write down only the third one, which is, in complete analogy with Eq. (2.49),

$$f(x) = \sum_{m=-\infty}^{\infty} C_m e^{imk_0x}. \tag{2.71}$$

The complex coefficients are given by the integral over a period, analogous to Eq. (2.51), as

$$C_m = \frac{1}{T}\int_{\xi}^{\xi+\lambda} f(x)e^{-ink_0x}dx, \tag{2.72}$$

starting from any ξ.

If the function $f(x)$ is not periodic, then instead of a Fourier series, we have a Fourier integral, or a spatial Fourier transform. Analogous to Eq. (2.57), the transformation is

$$f(x) = \int_{-\infty}^{+\infty} C(k)\mathrm{e}^{ikx} dk. \tag{2.73}$$

The function $C(k)$, analogous to Eq. (2.58), is given by the Fourier spatial antitransform

$$C(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-ikx} dx. \tag{2.74}$$

We shall now come back to the gray level of a (black and white) picture, as an example of a space function. These are, in general, functions of two variables, say the coordinates x and y, but, for the sake of simplicity, we shall limit the discussion to functions of one coordinate only, say on x.

In the time domain, the Fourier analysis is the simplest for a sine function of time. The same is obviously true in the space domain as well. If the physical quantity described by the function is a gray level of a picture, it cannot have physically negative values. Let us consider the simplest case, namely

$$f(x) = \frac{1}{2} + \frac{1}{2}\cos kx. \tag{2.75}$$

The two constants on the right-hand side might have different values. We have chosen them so as to have the function vary between 0 and 1. The gray level and the function in Eq. (2.75) are shown in Fig. 2.20. Note that the gray level does not depend on y. This is obviously an idealization.

As shown in the figure, the wavelength is the distance between two homologous points, for example, between two consecutive maxima or minima. The wave number is the number of periods in one meter. The space frequency k is the wave number times 2π. The larger the spatial frequency, the closer the clear and dark bands are to one another. This structure is called sine grating in optics.

In the following example, we consider an infinite succession of completely black rectangular bands of width D repeating with a period in x equal to λ. Such successions are called Fraunhofer gratings in optics. This situation, as shown in Fig. 2.21, is again idealized, because neither the succession nor the length of the bars in y can be infinite. Being that the function is periodic, it admits a Fourier series, which is given by Eq. (2.71) with $k_0 = 2\pi/\lambda$. This function of x is identical to the function of time we considered in Sect. 2.5. Consequently, we already know the

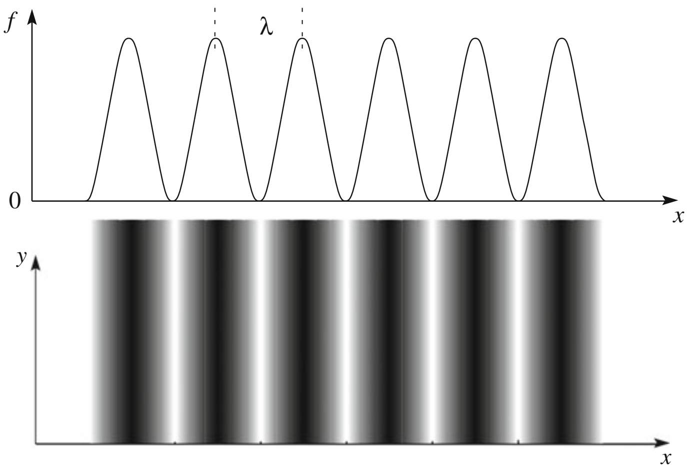
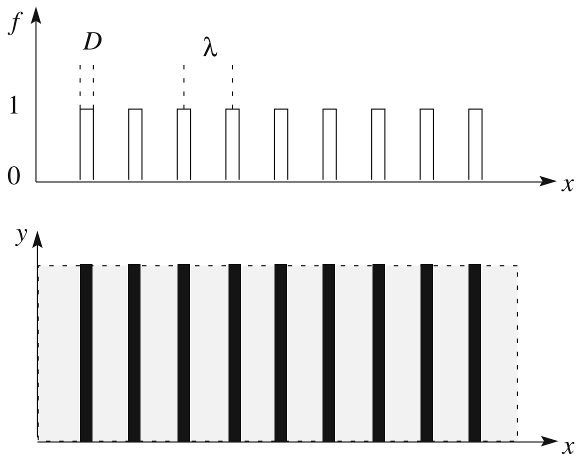


Fig. 2.20 A sinusoidal function of one space coordinate

Fig. 2.21 A periodic spatial grating



coefficients of the series, which are those of Eq. (2.52), with the obvious changes in the variables, namely

$$C_m = \frac{D}{\lambda}\frac{\sin(k_m D/2)}{k_m D/2}. \tag{2.76}$$

Let us now move to a non-periodic function of *x*, namely a single black band of width *D*, as shown in Fig. 2.22. Being that the function is not periodic, we must consider the integral of Eq. (2.73). In this case too, this is the spatial analogy of the rectangular function of time that we considered in Sect. 2.6. Consequently, we

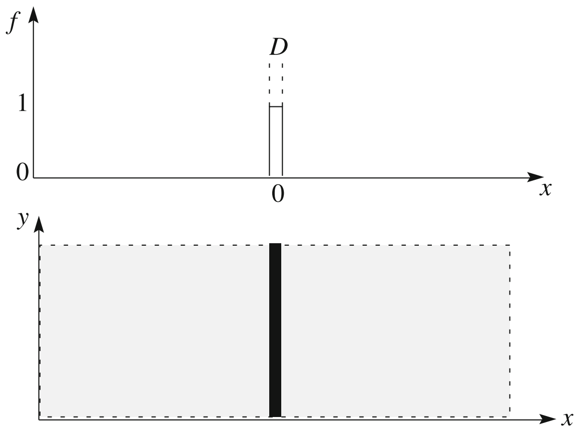


Fig. 2.22 A single black band

already know the Fourier transform in Eq. (2.74) of our function, which we obtain from Eq. (2.59). Changing the variables as needed, we obtain

$$C(k) = \frac{D}{2\pi}\frac{\sin(Dk/2)}{Dk/2}. \tag{2.77}$$

We shall return to this equation when we study the diffraction phenomenon of a grating in optics in Sect. 5.9.

The spatial Fourier transform in Eq. (2.78) is shown in Fig. 2.23, which is obviously the analogy in space of Fig. 2.16h in the time domain. The abscissa is now the spatial frequency k in place of the angular frequency ω.

Clearly, the bandwidth theorem holds in the space domain, as it does in the time domain. In this example, the function $f(x)$ is different from zero in a limited interval, which is $\Delta x = D$. The most important part of the Fourier transform has a certain width Δk, which we define similarly to what we did for the angular frequency (see Fig. 2.23) The relation between them is

$$\Delta k \cdot \Delta x = 2\pi. \tag{2.78}$$

Namely, the narrower the space function, the larger the frequency interval needed to represent it in the Fourier transform. This relation has important

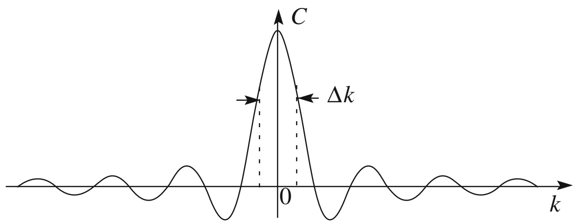


Fig. 2.23 The spatial Fourier transform of the function in Fig. 2.22

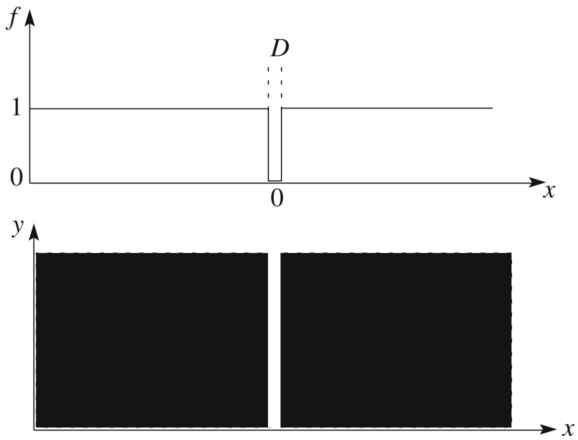


Fig. 2.24 A slit

consequences in optics, which we shall discuss in Chap. 5. Even more important are the consequences for quantum physics, where its equivalent is one of the fundamental laws, namely the uncertainty relation between momentum and position.

Let us finally note that Eq. (2.78) is, a sign apart, the Fourier transform of the "gray level" function shown in Fig. 2.24, which is the same as Fig. 2.22 with inverted blacks and whites. But we can also think of Fig. 2.24 as a screen that completely absorbs an incident light beam, except for the part in the white band, which can go through. This can be obtained, for example, by opening a slit in an absorbing screen. We shall discuss this phenomenon in Chap. 5, where we shall see, in particular, that the light intensity beyond the slit depends on the coordinates as the square of the function shown in Fig. 2.23. What we shall learn to be the (Fraunhofer) diffraction pattern of the slit is the physical materialization of the mathematical concept of the spatial Fourier transform.

Summary

The most important concepts studied in this chapter are the following:

1. Two coupled harmonic oscillators can move in two particular motions, the normal modes, in which all the parts of the system oscillate with the same frequency and in the same initial phase. The oscillation frequencies of the modes are the proper frequencies of the system. The proper frequencies and the shapes of the modes depend on the structure of the system, but not on the initial conditions.
2. Particular coordinates can be found, called the normal coordinates, in which the equations of motions are decoupled.
3. The equations of motion of two forced coupled oscillators are independent when written in normal coordinates. Two resonances exist at the two proper frequencies.

4. Coupled oscillating electrical and mechanical systems obey the same differential equations and behave in analogous manners.
5. The above properties can be generalized to systems with n degrees of freedom.
6. An oscillating string with fixed extremes is a system with an infinite continuous number of degrees of freedom. Its motion is described by an important partial differential equation.
7. The normal modes of the vibrating elastic string have the following characteristics: (a) all the points of the string oscillate with the same amplitude and in the same phase, (b) each mode has its own (proper) oscillation frequency, dependent on the structure of the system and independent of the initial state, (c) the shape of the modes are sine functions, whose period is the wave length.
8. The relation between the angular frequency and the wave number of the modes is the dispersion relation. The dispersion relation is linear for a perfectly elastic string. In this case, the proper frequencies are in the arithmetic succession.
9. A periodic function, both of time and of a space coordinate, can be expressed as a sum of cosine functions and in other equivalent forms.
10. A non-periodic function can be expressed as an integral, performing a Fourier transform. The Fourier transform of a function of time (space coordinate) is a function of the angular frequency (space frequency).
11. The bandwidth of the Fourier transform of a function of time of limited duration is inversely proportional to that duration. The bandwidth of the Fourier transform of a function of a space coordinate extending throughout a limited interval is inversely proportional to that interval.
12. The width of the resonance curve of a weakly damped forced oscillator is equal to the width of the Fourier transform of the displacement as a function of time in the free oscillations of the same oscillator. Both widths are inversely proportional to the decay time of the free oscillations.

Problems

2.1 Consider two coupled pendulums, such as those discussed in Sect. 2.1. Suppose that the squares of their proper angular frequencies differ by 4 s^{-2}. If we change the spring with another one having a spring constant 10 times smaller, how much is the new difference?

2.2 Consider the two coupled pendulums again. Does the proper frequency of the lower frequency mode depend on the spring constant? Does it depend on the masses? Answer the same questions for the higher frequency mode.

2.3 A string of a viola is 0.5 m long and is tuned 440 Hz. You can play it at 550 Hz, pressing it with your finger to make it shorter. How much shorter should we make it?

2.4 A string of an instrument should be tuned to 440 Hz, but instead is at 435 Hz. How should we change its tension to tune it?

2.5 The width of a forced oscillator is $\Delta\omega_{\mathrm{ris}} = 35\ \mathrm{s}^{-1}$. We let it freely oscillate and we measure its displacement as a function of time. We then Fourier transform the function we have found. What is the bandwidth of this transform?

2.6 Consider the motion $\psi(t) = (10\,\text{mm})\cos\left[(6.28\text{s}^{-1})t + 32^{\circ}\right] + (15\,\text{mm})\sin\left[(6.21\text{s}^{-1})t - 72^{\circ}\right]$. Calculate the mean frequency and the beat frequency

2.7 The bandwidth of the Fourier transform of a function of time is 120 s^{-1}. How much is the duration of the function?

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