Chapter 2
Vector-matrix Differential Equation and Numerical Inversion of Laplace Transform

2.1 Vector-matrix Differential Equation

A differential equation and a set of differential (simultaneous linear ordinary differential equations or partial differential equations) equations are written in the form of a Vector-matrix differential equation which is then solved by eigenvalue approach methodology.

Examples

1.

$$\frac{dv_1}{dt} = a_{11}v_1 + a_{12}v_2$$
$$\frac{dv_2}{dt} = a_{21}v_1 + a_{22}v_2 \tag{2.1}$$

Or, equivalently written as:

$$\frac{dv}{dt} = Av \tag{2.2}$$

where $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, $A = [a_{ij}]_{i,j=1,2}$ = Constant coefficients of the differential Eq. (2.1).

Similarly, we can take $v = [v_1 \ v_2 \ \ldots v_n]^T$, $A = [a_{ij}]_{i,j=1,2,\ldots,n}$ for n linear differential equations, whereas the elements a_{ij} of the matrix A are not all simultaneously zero. Equation (2.1) can be modified as

2.

$$\frac{dv_1}{dt} = a_{11}v_1 + a_{12}v_2 + f_1$$
$$\frac{dv_2}{dt} = a_{21}v_1 + a_{22}v_2 + f_2 \tag{2.3}$$

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where f_i's ($i = 1, 2$) are any scalars.
Then, Eq. (2.2) becomes

$$\frac{dv}{dt} = Av + f \tag{2.4}$$

where $f = [f_1\ f_2]^T$.
In a similar way, the Eqs. (2.2) and (2.4) can be generalized for n equations, where $f = [f_1\ f_2\ \ldots f_n]^T$.
Another type of linear differential equations are

3.

$$Lt = x^2 A\ t \tag{2.5}$$

where L is the Bessel operator and $L = x^2 \frac{d^2}{dx^2} + xp(x)\frac{d}{dx} + q(x)$, $t = [t_1\ t_2\ \ldots t_n]^T$, and $A = [a_{ij}]_{i,j=1,2,\ldots,n}$, a_{ij} is constant for all i and j, and $p(x)$ and $q(x)$ are real-valued continuous functions in [0, 1].
Expanded form of equation (2.5) is

$$L\begin{bmatrix} t_1 \\ t_2 \\ \ldots \\ t_n \end{bmatrix} = x^2 \begin{bmatrix} a_{11}\ a_{12}\ \ldots\ a_{1n} \\ a_{21}\ a_{22}\ \ldots\ a_{2n} \\ \ldots\ \ldots\ \ldots\ \ldots \\ a_{n1}\ a_{n2}\ \ldots\ a_{nn} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ \ldots \\ t_n \end{bmatrix} \tag{2.6}$$

which also gives the n linear differential equations.
This Eq. (2.6) can be restricted for one, two, three,… equations, putting the value of $n = 1, 2, 3, \ldots$.
Henceforth, the Eqs. (2.2), (2.4), and (2.5) are defined as **vector–matrix differential equations**.

2.2 Solution of Vector-matrix Differential Equation

The problem of thermoelasticity and magnetoelasticity should be solved. In this field, governing equation in laplace and/or Fourier transformed domain should be written in the form of vector–matrix differential equation and solved them by eignvalue approach. So, the above-discussed vector–matrix differential equations are given below:

- (i) $\frac{dv}{dt} = A\ v$; $\frac{dv}{dx} = A\ v\ +\ f$
- (ii) $Lt = x^2 A\ t$

(i) Taking the Eqs. (2.2) and (2.4) for n-differential equations, the solution of the vector–matrix differential equation of the form of Eqs. (2.2) and (2.4) with initial condition $v(0) = c$, and we make a substitution $v = Xe^{\lambda t}$ such that X is a nonzero independent vector and obviously λ is a scalar.

This implies that

$$(A - \lambda I)X = 0 \tag{2.7}$$

this equation interprets that $\lambda(\lambda_i; i = 1(1)n)$ are nothing, but the eigenvalues of the matrix A and corresponding eigenvectors are $X(X_i; i = 1(1)n)$. These n-linearly independent vectors form a basis of complex n-dimensional Euclidean space E^n. If we take into consideration any vector c which is belong to E^n, then for any scalars $(c_1, c_2, \ldots, c_n)$.

c can be expressed as $c = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n = \sum_{i=1}^{n} c_i X_i$.

If $u(x) = \sum_{i=1}^{n} c_i X_i e^{\lambda_i x}$ is the solution of equation (2.2).

Then $v(0) = \sum_{i=1}^{n} c_i X_i e^{\lambda_i x_o} = \sum c_i x_i = c$.

Which also satisfies the initial condition.

For the uniqueness of the solution of equation (2.2), we can express the solution as $v(t) = c_1 X_1 e^{\lambda_1 t} + c_2 X_1 e^{\lambda_2 t} + \cdots + c_n X_n e^{\lambda_n t}$ $v(o) = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n = \sum_{i=1}^{n} c_i X_i = c$

Hence, $v(t)$ is the unique solution of the Vector-matrix differential equation (2.2) satisfying the initial condition. We now show that the uniqueness of the solution of Vector-matrix differential equation (2.4). For any n-scalar functions $b_1(x), b_2(x), \ldots, b_n(x)$, we can take

$$v(x) = \sum_{i=1}^{n} b_i(x) X_i e^{\lambda_i x} \; such \; that \; b_i(x_o) = 0 \tag{2.8}$$

Differentiating both sides of Eqs. (2.8) with respect to x, then we get $v'(x) = \sum_{i=1}^{n} b_i'(x) X_i e^{\lambda_i x} + \sum_{i=1}^{n} b_i(x) \lambda_i e^{\lambda_i x}$

Substituting the values of $v(x)$ in Eq. (2.8), we have

$$\begin{aligned} \sum_{i=1}^{n} b_i'(x) X_i \, e^{\lambda_i x} + \sum_{i=1}^{n} b_i(x) \lambda_i X_i \, e^{\lambda_i x} \\ = \sum_{i=1}^{n} b_i(x) A \; X_i \, e^{\lambda_i x} + f(x) \end{aligned} \tag{2.9}$$

or,

$$\begin{aligned} \sum_{i=1}^{n} b_i'(x) X_i \, e^{\lambda_i x} = \sum_{i=1}^{n} b_i(x) [A \; X_i - \lambda_i X_i] \, e^{\lambda_i x} \\ + f(x) = f(x) \end{aligned} \tag{2.10}$$

Multiplying Eq. (2.10) by $Z_j \; e^{-\lambda_j x}$ (where $Z_1, Z_2, Z_3, \ldots, Z_n$ are left eigenvector corresponding to the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$), we get

$$\sum_{i=1}^{n} b_i'(x) X_i Z_j \; e^{(\lambda_i - \lambda_j)x} = Z_j \; f(x) \; e^{-\lambda_j x} \tag{2.11}$$

or,

$$b_j'(x) Z_j \; X_j = Z_j \; f(x) \; e^{-\lambda_j x}, \; [Z_j \; X_j = 0 \; for \; i \neq j]$$

$$b_j'(x) = \frac{1}{Z_j \; X_j} Z_j \; f(x) \; e^{-\lambda_j x}$$

$$b_j(x) = \int_{x_0}^{x} (Z_j \; X_j)^{-1} Z_j \; f(x) \; e^{-\lambda_j s} ds, \tag{2.12}$$

taking $b_j(x_0) = 0, \; for \; j = 1(1)n$ Now take

$$v(x) = v_1(x) + v_2(x) \tag{2.13}$$

By differentiating, we get
$v'(x) = {v_1}'(x) + {v_2}'(x) = A \; v_1(x) + A \; v_2(x) + f(x) = A[v_1(x) + v_2(x)] + f(x) = A \; v(x) + f(x)$ i.e., $v'(x_0) = {v_1}'(x_0) + {v_2}'(x_0) = c$

Hence, $v(x) = v_1(x) + v_2(x)$ is the unique solution of the Vector-matrix differential equation (2.4), satisfying the condition $v(x_0) = c$.

(ii) Now, we consider another type of Vector-matrix differential equation of the form

$$Lt = x^2 A \; t \tag{2.14}$$

where $A = [a_{ij}], \; (i, j) = 1(1)n$, all a_{ij}'s are constant, not all simultaneously zero, and $p(x)$ and $q(x)$ are two real-valued continuous function in [0, 1].
The initial conditions are

$$t(1) = c \; and \; t'(1) = d \tag{2.15}$$

where t, c, and d are vectors with n-components.

Assume that $t(x) = X(\alpha)\omega(x, \alpha)$ be a solution of the equation (2.14), X is n-vector independent of x, and $\omega(x, \alpha)$ is a non-trivial solution of second-order linear differential equation

$$Ly = \alpha x^2 y \tag{2.16}$$

taking α as a scalar.
We now get using the operator L on t

$$Lt = L(X, \omega) = XL\omega = X(x^2 \alpha \omega) = \alpha x^2 X \omega \tag{2.17}$$

Hence, Eq. (2.14) becomes

$$x^2(\alpha X - AX)\omega = 0 \tag{2.18}$$

Since, $t(x)$ is the non-trivial solution of equation (2.14), $\omega(x, \alpha) \neq 0$. So, it follows that

$$\alpha\ X = A\ X \tag{2.19}$$

Equation (2.19) is an algebraic eigenvalue problem where α is the eigenvalue and X is the corresponding eigenvector of the matrix A; also, Let $\alpha_i, i = 1(1)n$ be the distinct eigenvalues and let $X_i, i = 1(1)n$ be the corresponding eigenvectors of the matrix A.

Then, $X_i, i = 1(1)n$ are linearly independent, it forms a complex space $\mathcal{C}^n$, and $\mathcal{C}$ is the field of the complex numbers.

We can find the scalars $c_i, i = 1(1)n$ and $d_i, i = 1(1)n$, for any two vectors C and D, such that $C = c_1X_1+a_2X_2+\cdots+c_nX_n, and\ D = d_1X_1+d_2X_2+\cdots+dnX_n$

Taking $f(x, \alpha_i)$ and $g(x, \alpha_i)$ as two linearly independent solutions of the differential equations

$$Ly = \alpha_i x^2 y$$

with the initial conditions $f(1, \alpha_i) = 1,\ f^{'}(1, \alpha_i) = 0\, and\ g(1, \alpha i) = 1,\ g^{'}(1, \alpha_i) = 1$,we now get

$$t(x) = \sum_{i=1}^{n} X_i[c_i f(x, \alpha_i) + d_i g(x, \alpha_i)] \tag{2.20}$$

So, $t(x)$ also satisfies the Eq. (2.14) also

$$t(1) = \sum_{i=1}^{n} X_i[c_i f(1, \alpha_i) + d_i g(1, \alpha_i)] = \sum_{i=1}^{n} c_i X_i = C \tag{2.21}$$

$$t^{'}(1) = \sum_{i=1}^{n} X_i[c_i f^{'}(1, \alpha_i) + d_i g^{'}(1, \alpha_i)] = \sum_{i=1}^{n} d_i X_i = D \tag{2.22}$$

where prime (') denotes the differentiation, and it satisfies the prescribed initial conditions. Hence, $t(x)$, which is given by Eq. (2.20), which is also the unique of the system of linear differential equations (2.14) satisfying the initial conditions (2.15).

2.3 Applications

In this section, we shall show that the results obtained by the applications of the present theory are in complete agreement with those in the existing literature.

AI

We can solve an ordinary differential equation with the help of above theory. Consider the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{4x} \tag{2.23}$$

The Eq. (2.23) can be written as

$$\frac{d}{dx}\begin{bmatrix} \frac{dy}{dx} \\ y \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{dy}{dx} \\ y \end{bmatrix} + \begin{bmatrix} e^{4x} \\ 0 \end{bmatrix} \Longrightarrow \frac{d\underline{v}}{dx} = \underline{A}\ \underline{v} + \underline{f} \tag{2.24}$$

Therefore $\underline{v} = \begin{bmatrix} \frac{dy}{dx} \\ y \end{bmatrix}$

$\underline{A} = \begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix}$; $\underline{f} = \begin{bmatrix} e^{4x} \\ 0 \end{bmatrix}$

The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$, and eigenvectors are

$V_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ when $\lambda_1 = 2$ and $V_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ when $\lambda_2 = 3$

Therefore,

$V = \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$

and also $V^{-1} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$

The r-th equation of the Vector-matrix differential equation (2.24) is

$$\frac{dy_r}{dx} = \lambda_r y_r + Q_r \tag{2.25}$$

where, $Q_r = V_r^{-1}\underline{f}$; $V^{-1} = [w_{ij}]$;

$$Q_r = \sum_{i=1}^{n} W_{ri} f_i; \quad r = 1, 2 \tag{2.26}$$

$V^{-1}\underline{f} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} e^{4x} \\ 0 \end{bmatrix} = \begin{bmatrix} -e^{4x} \\ e^{4x} \end{bmatrix}$

From Eq. (2.25), we get

$$y_r = c_r e^{\lambda_r x} + e^{\lambda_r x}\int Q_r e^{-\lambda_r x} dx \tag{2.27}$$

Taking $r = 1$, we get

$$y_1 = c_1 e^{\lambda_1 x} + e^{\lambda_1 x}\int Q_1 e^{-\lambda_1 x} dx \tag{2.28}$$

Putting $\lambda_1 = 2$, we get

$$y_1 = c_1 e^{2x} + e^{2x} \int -e^{4x} e^{-2x} dx$$
$$= c_1 e^{2x} - \frac{1}{2} e^{4x} \tag{2.29}$$

Again from Eq. (2.25),we get

$$y_2 = c_2 e^{\lambda_2 x} + e^{\lambda_2 x} \int Q_2 e^{-\lambda_2 x} dx \tag{2.30}$$

Putting $\lambda_2 = 3$, we get

$$y_2 = c_2 e^{3x} + e^{3x} \int -e^{4x} e^{-3x} dx$$
$$= c_2 e^{3x} + e^{4x} \tag{2.31}$$

Now we get, $\underline{v} = V_1\ y_1 + V_2\ y_2$

Combining Eqs. (2.29) and (2.31), we get

$$\begin{bmatrix} \frac{dy}{dx} \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} (c_1 e^{2x} - \frac{1}{2} e^{4x})$$
$$+ \begin{bmatrix} 3 \\ 1 \end{bmatrix} c_2 e^{3x} + e^{4x} \tag{2.32}$$

Then, the general solution of equation (2.23) is

$$y = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{2} e^{4x} \tag{2.33}$$

AII

We can solve a set of differential equations with the help of above theory. Consider the set of differential equations

$$\frac{d^2 e}{dy^2} = b_1 e + b_2 \theta + b_3 \tag{2.34}$$
$$\frac{d^2 \theta}{dy^2} = c_1 e + c_2 \theta + c_3 \tag{2.35}$$

where b_i's and c_i's are arbitrary parameters which can be determined from the initial conditions.

As in the theory stated above, Eqs. (2.34) and (2.35) are written in the form of vector–matrix differential equation

$$\frac{d\underline{V}}{dy} = \underline{A}\ \underline{V} + \underline{F} \tag{2.36}$$

where

$$\underline{V} = \begin{bmatrix} e & \theta & \frac{de}{dy} & \frac{d\theta}{dy} \end{bmatrix}^T$$
$$and \quad \underline{F} = [0 \quad 0 \quad b_3 \quad c_3]^T \tag{2.37}$$

The matrix $\underline{A}$ is given by

$$\underline{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ b_1 & b_2 & 0 & 0 \\ c_1 & c_2 & 0 & 0 \end{bmatrix} \tag{2.38}$$

For the solution of the vector–matrix differential equation (2.36), we now apply the method of eigenvalue approach methodology.

The characteristic equation of the matrix $\underline{A}$ is given by

$$\lambda^4 - (b_1 + c_2)\lambda^2 + (b_1c_2 - b_2c_1) = 0 \tag{2.39}$$

The roots of the characteristic equation (2.39) are $\lambda = \lambda_i$; $(i = 1(1)4)$, and these are of the form $\lambda = +\lambda_1$, $\lambda = -\lambda_1$, $\lambda = +\lambda_2$, and $\lambda = -\lambda_2$, which are also the eigenvalues of the matrix $\underline{A}$.

The eigenvector X corresponding to the eigenvalue λ can be calculated as

$$X = \left[(c_2 - \lambda^2) \ \ -c_1 \ \lambda(c_2 - \lambda^2) \ \ -\lambda c_1\right]^T \tag{2.40}$$

Let V_i be the eigenvectors of the matrix $\underline{A}$ corresponding to the eigenvalues λ_i respectively, where

$$V_1 = [X]_{\lambda=\lambda_1}, \quad V_2 = [X]_{\lambda=-\lambda_1}, \quad V_3 = [X]_{\lambda=\lambda_2}, \quad V_4 = [X]_{\lambda=-\lambda_2} \tag{2.41}$$

The general solution of equation (2.36) can be written as:

$$\underline{V} = \sum_{i=1}^{4} V_i x_i$$
$$where\ x_i = A_i e^{\lambda_i y} + e^{\lambda_i y} \int q_i e^{-\lambda_i y} dy$$
$$and\ q_i = V^{-1}\ \underline{F}$$
$$where\ V = [V_i],\ i = 1(1)4 \tag{2.42}$$

A_i's are the arbitrary parameters which are determined from the boundary conditions.

2.4 Numerical Inversion of Laplace Transform

Numerical inversion of Laplace transform is carried out by two different methods:

(i) Bellman method and
(ii) Zakian method.

(i) Bellman Method [8]
The definition[1] of Laplace transform is

$$f(p) = \int_0^\infty f(t)e^{-pt}dt \tag{2.43}$$

It is assumed that $f(t)$ is integrable and also is of exponential order $\sigma > 0$. For the approximation of the integral of Eq. (2.43), we substitute, $u = e^{-t}$

$$f(p) = \int_0^\infty u^{p-1}h(u)du \tag{2.44}$$

by taking $f(-logu) = h(u)$

Using the Gaussian quadrature formula, we get from Eq. (2.44)

$$\sum_{i=1}^{n} X_i u_i^{p-1} h(u_i) = f(p) \tag{2.45}$$

where X_i's are coefficients, and X_i's are the corresponding roots of the Legendre equation $P_n(u) = 0$.

Putting the values $p = 1, 2, 3, \ldots, N$ in Eq. (2.45), we get

$$\begin{aligned}
X_1h(u_1) + X_2h(u_2) + \cdots + X_nh(u_n) &= f(1)\\
X_1u_1h(u_1) + X_2u_2h(u_2) + \cdots + X_nu_nh(u_n) &= f(2)\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
X_1u_1^{N-1}h(u_1) + X_2u_2^{N-1}h(u_2) + \cdots + X_nu_n^{N-1}h(u_n) &= f(N)
\end{aligned} \tag{2.46}$$

From Eq. (1.46), we get the values of $h(u_i$'s, $i = 1(1)$n, where $f(-logu_1) = h(u_1)$, $f(-logu_2) = h(u_2), \ldots, f(-logu_n) = h(u_n)$.

From equation (2.46), we get the numerical inversion of Laplace transform according to the numeric values of p.

(ii) Zakian Method [30]
The definition[2] of Laplace transform of the piecewise continuous function $f(t)$ of exponential order $\sigma > 0$ which is given by

[1]"Numerical Inversion of Laplace Transform," Amer. Elsevier Pub. Com., New York, 1966.
[2]"Electronics Letters," 5(6), 120–121, 1969.

$$f(p) = \int_0^\infty f(t)e^{-pt}dt \tag{2.47}$$

Now, we define the scaled delta function as

$$\begin{aligned}\int_0^T \delta(\frac{\alpha}{t} - 1)d\alpha = t;\ such\ that\ 0 < t < T,\\ where\ as\ \delta(\frac{\alpha}{t} - 1) = 0\ when\ \ t \neq \alpha\end{aligned} \tag{2.48}$$

We can define the integral as

$$I_1 = \frac{1}{t}\int_0^T g(\alpha)\delta(\frac{\alpha}{t} - 1)d\alpha;\ t \in (0, T) \tag{2.49}$$

where $\delta(\frac{\alpha}{t} - 1)$ is the delta function. So, using the property of the delta function, we have from Eq. (2.48)

$$I_1 = \frac{g(t)}{t}\int_0^T \delta(\frac{\alpha}{t} - 1)d\alpha,\ \ where\ as\ t \in (0, T) \tag{2.50}$$

We also have from Eqs. (2.48) and (2.49)

$$g(t) = \frac{1}{t}\int_0^T g(\alpha)\delta(\frac{\alpha}{t} - 1)d\alpha;\ \ where\ as,\ t \in (0, T) \tag{2.51}$$

The function g is the discontinuous function and has the jump discontinuity from $g(t-)$ to $g(t+)$ is $\frac{1}{2}\{s_1 g(t-) + s_2 g(t+)\}$, where s_1 and s_2 are two nonnegative real parameters such that $s_1 + s_2 = 2$.

So, the delta function $\delta(\frac{\alpha}{t} - 1)$ may be expanded as

$$\delta(\frac{\alpha}{t} - 1) = \delta_n(\frac{\alpha}{t} - 1) = \sum_{j=1}^{n} s_j e^{(-\beta_j \frac{\alpha}{t})} \tag{2.52}$$

and for every point of continuity at t, we get

$$g(t) = \lim_{n\to\infty} g_n(t);\ \ where\ as,\ t \in (0, T) \tag{2.53}$$

From Eqs. (2.51) and (2.52), we get

$$\begin{aligned}g_n(t) &= \frac{1}{t}\int_0^T g(\alpha)\delta_n(\frac{\alpha}{t} - 1)d\alpha;\ \ where\ as,\ t \in (0, T)\\ &= \frac{1}{t}\int_0^T g(\alpha)\sum_{j=1}^{n} s_j e^{(-\beta_j \frac{\alpha}{t})}d\alpha = \frac{1}{t}\sum_{j=1}^{n} s_j \int_0^T g(\alpha)e^{(-\beta_j \frac{\alpha}{t})}d\alpha\end{aligned} \tag{2.54}$$

Taking $T \to \infty$ and the definition of Laplace transform, i.e., Eq. (2.47), we get

$$g_n(t) = \frac{1}{t}\sum_{j=1}^{n} s_j G(\frac{\beta_j}{t});\ \ for\ any\ region\ \ 0 < t < t_c \tag{2.55}$$

where,

$$t_c = \min_{j=1,2,\ldots,n}\{Re(\frac{\beta_j}{\sigma})\};\ \ \sigma > 0 \tag{2.56}$$

So, making as $n \to \infty$, $Re(\beta_j) \to \infty$, we get $t_c \to \infty$.
We get the explicit expression for the inversion of Laplace transform as

$$\begin{aligned} g(t) &= \lim_{n\to\infty}\frac{1}{t}\sum_{j=1}^{n} s_j G(\frac{\beta_j}{t});\ \ 0 < t < \infty \\ where\ G(\frac{\beta_j}{t}) &= \int_0^T g(\alpha)e^{(-\beta_j\frac{\alpha}{t})}d\alpha \end{aligned} \tag{2.57}$$

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