Chapter 2
Motion and Strain (Rate)

The tensor analysis providing the mathematical foundation for the continuum mechanics is described in Chap. 1. Basic concepts and quantities for continuum mechanics will be studied in the three chapters up to Chap. 4. The description of motion and deformation of a material body constitutes the basic introductory part of the continuum mechanics. Various expressions of motion and a variety of strain and strain rate measures are employed for the description of reversible and irreversible deformations of materials. Some selected basic expressions and measures will be explained in this chapter.

2.1 Motion of Material Point

A material body is assembly of material particles (or material elements). The map of positions of material particles in a space is referred to as the *configuration*. Here, the configurations in the initial time $t = t_0$ and the current time t are called the *initial* (or *Lagrangian*) *configuration* and the *current* (or *Eulerian*) *configuration*, respectively. Deformation is described by the change of configuration from a particular configuration which is called the *reference configuration.* Here, the reference configuration can be chosen at arbitrary intermediate time $\tau(t_0 \le \tau \le t)$ is called the *reference time*.

The position vectors of material particle in the initial and the current configurations are designated by $\mathbf{X}$ and $\mathbf{x}(t)$, respectively. Here, $\mathbf{X}$ is fixed and thus it can be regarded as a label of each material particle. The motion of material point during the time $t_0 \to t$ is described as

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X},t),\ \mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x},t) \tag{2.1}$$

Besides, the motion of material point during the time $t_0 \to \tau$ is described as

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$$\mathbf{x}(\tau) = \boldsymbol{\chi}(\mathbf{X},\tau), \quad \mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}(\tau),\tau) \tag{2.2}$$

The fact that a material does not overlap or separate by the motion of material requires the existence of the one-to-one correspondence between **X** and **x** (**x** is uniquely determined by **x** and vice versa) so that $x_1(X_1,X_2,X_3)$, $x_2(X_1,X_2,X_3)$ and $x_3(X_1,X_2,X_3)$ must be mutually independent. Now, let the mathematical requirement for this fact be derived by the reductive absurdity. x_1,x_2,x_3 are not mutually independent if they satisfy the constraint

$$f(x_1(X_1,X_2,X_3),x_2(X_1,X_2,X_3),x_3(X_1,X_2,X_3)) = 0 \tag{2.3}$$

from which it follows that

$$\left.\begin{array}{l} \dfrac{\partial f}{\partial x_1}\dfrac{\partial x_1}{\partial X_1} + \dfrac{\partial f}{\partial x_2}\dfrac{\partial x_2}{\partial X_1} + \dfrac{\partial f}{\partial x_3}\dfrac{\partial x_3}{\partial X_1} = 0 \\ \dfrac{\partial f}{\partial x_1}\dfrac{\partial x_1}{\partial X_2} + \dfrac{\partial f}{\partial x_2}\dfrac{\partial x_2}{\partial X_2} + \dfrac{\partial f}{\partial x_3}\dfrac{\partial x_3}{\partial X_2} = 0 \\ \dfrac{\partial f}{\partial x_1}\dfrac{\partial x_1}{\partial X_3} + \dfrac{\partial f}{\partial x_2}\dfrac{\partial x_2}{\partial X_3} + \dfrac{\partial f}{\partial x_3}\dfrac{\partial x_3}{\partial X_3} = 0 \end{array}\right\}, \text{ i.e. } \begin{bmatrix} \dfrac{\partial x_1}{\partial X_1} & \dfrac{\partial x_2}{\partial X_1} & \dfrac{\partial x_3}{\partial X_1} \\ \dfrac{\partial x_1}{\partial X_2} & \dfrac{\partial x_2}{\partial X_2} & \dfrac{\partial x_3}{\partial X_2} \\ \dfrac{\partial x_1}{\partial X_3} & \dfrac{\partial x_2}{\partial X_3} & \dfrac{\partial x_3}{\partial X_3} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \\ \dfrac{\partial f}{\partial x_3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{2.4}$$

The equation

$$J = 0 \tag{2.5}$$

must hold in order that $\partial f/\partial x_1$, $\partial f/\partial x_2$, $\partial f/\partial x_3$ are determined uniquely on account of Eq. (1.125) regarding $T_{iJ} = \partial x_i/\partial X_J$ and $v_i = \partial f/\partial x_i$, where J is defined by

$$\boxed{J \equiv \det\left(\frac{\partial x_i}{\partial X_J}\right) = \varepsilon_{IJK}\frac{\partial x_1}{\partial X_I}\frac{\partial x_2}{\partial X_J}\frac{\partial x_3}{\partial X_K}} \tag{2.6}$$

and is called the *functional determinant* or *Jacobian*. In contrast, in order that they are mutually independent, it must hold that

$$J \neq 0 \tag{2.7}$$

being free from the constraint in Eq. (2.3). The transformation between **x** and **X** is called the *admissible transformation*, if f_1, f_2, f_3 in $x_1 = f_1(X_1,X_2,X_3)\, x_2 = f_2(X_1,X_2,X_3)$, $x_3 = f_3(X_1X_2X_3)$ are single-valued and continuous functions, so that the Jacobian is not zero as shown in Eq. (2.7). Further, if the Jacobian is positive, a right-hand coordinate system is transformed to other right-hand one, and it is called the *positive transformation*. Inversely, if the Jacobian is negative, a right-hand coordinate system is transformed to a left-hand one, and it is called the *negative transformation*. Admissible and positive transformation with $J > 0$ is assumed throughout this book.

Physical quantity, say $\mathbf{T}$, in the body changes generally with the position and the time. Physical quantity at current time is described $\mathbf{T}(\mathbf{X},t)$ in terms of the current configuration $\mathbf{X}$ and the current time t. This type of description of mechanical state is called the *Lagrangian* (*or material*) *description*. On the other hand, the physical quantity at current time is described $\mathbf{T}(\mathbf{x},t)$ in terms of the current configuration $\mathbf{x}$ and the current time t. This type of description of physical quantity is called the *Eulerian description* or *spatial description*. Further, the physical quantity at current time can be described in terms of the current configuration $\mathbf{x}(\tau)$ at arbitrary reference time τ and the current time t as

$$\mathbf{T}(\chi^{-1}(\mathbf{x}(\tau),\tau),t) = {}_{\tau}\mathbf{T}(\mathbf{x}(\tau),\ t) \tag{2.8}$$

${}_{\tau}(\ )$ designating to choose the reference time τ as $\tau > t_0$. Equation (2.8) is called the *relative description*. Specifically, ${}_{t}\mathbf{T}(\mathbf{x}(t),\ t)$ is called the *updated Lagrangian description*, where the reference configuration is taken as the current configuration, choosing the reference time as the current time, i.e. $\tau = t$. In contrast, the description $\mathbf{T}(\mathbf{X},t)$ will be called the *total Lagrangian description.*

2.2 Time-Derivatives

The time derivative of the tensor in the spatial description

$$\frac{\partial \mathbf{T}(\mathbf{x},t)}{\partial t} \tag{2.9}$$

describes the rate of the physical quantity at a certain spatial point and thus it is called the *spatial-time* (or *local*) derivative. In many cases of fluid mechanics, a motion and its history of individual particle from the initial state is immaterial and thus the spatial-time derivative is often adopted. In contrast, the time-derivative of the tensor in the material description

$$\frac{\partial \mathbf{T}(\mathbf{X},t)}{\partial t} \tag{2.10}$$

describes the rate of the physical quantity in a certain material particle and thus is called the *material-time derivative*. It is denoted by the symbol

$$\dot{\mathbf{T}} \equiv \frac{\partial \mathbf{T}(\mathbf{X},t)}{\partial t} \ \text{or}\ \frac{D\mathbf{T}}{Dt} \equiv \frac{\partial \mathbf{T}(\mathbf{X},t)}{\partial t} \tag{2.11}$$

In solid mechanics, the rate of deformation and its history of individual material particle is required and thus the material-time derivative is used usually.

The material-time derivative in Eq. (2.11) and the spatial-time derivative in Eq. (2.9) are related by

$$\boxed{\dot{\mathbf{T}} \equiv \frac{\partial \mathbf{T}(\mathbf{x},t)}{\partial t} + \frac{\partial \mathbf{T}(\mathbf{x},t)}{\partial \mathbf{x}}\mathbf{v}}, \; \dot{T}_{ij} \equiv \frac{\partial T_{ij}(\mathbf{x},t)}{\partial t} + \frac{\partial T_{ij}(\mathbf{x},t)}{\partial x_k} v_k \tag{2.12}$$

where $\mathbf{v} \equiv \partial \mathbf{x}/\partial t$ is the velocity vector of material particle. The first term in the right-hand side signifies the *non-steady* (or *local time derivative*) *term* describing the change with time at fixed spatial point and the second term signifies the *steady* (or *convective*) *term* describing the change due to the movement of material, which results from the existence of a spatial gradient of the physical quantity $\mathbf{T}$.

Rate-type constitutive equations for the irreversible deformation of solids, e.g. the viscoelastic, the elastoplastic and the viscoplastic deformation, must be described by the material-time derivative pursuing a material particle because they must describe the relation of physical quantities in each material particle. Here, it should be noticed that the material-time derivative of physical quantity describes the rate observed by moving in parallel with material particle as known from Eq. (2.11) which concerns only with the position vector of material particle and the time. Then, the objective time-derivative based on the rate of physical quantity observed by the coordinate system deforming/rotating with a material must be used for constitutive equations of solids as will be described in Chap. 4.

2.3 Deformation Gradient and Deformation Tensors

At the initial state of deformation ($t = 0$), consider a material particle, the position vector of which is $\mathbf{X}$, and the adjacent material point, the position vector of which is $\mathbf{X} + d\mathbf{X}$. Furthermore, consider the current state ($t = t$) in which these points move to the points with position vectors $\mathbf{x}$ and $\mathbf{x} + d\mathbf{x}$, respectively. The infinitesimal line elements before and after the deformation are described as

$$d\mathbf{X} = dX_A \mathbf{e}_A, \quad d\mathbf{x}(t) = dx_i(t)\mathbf{e}_i(t) \tag{2.13}$$

where the current base $\{\mathbf{e}_i(t)\}$ rotates with the elapse of time so that it changes different from the fixed reference base $\{\mathbf{e}_A\}$, i.e. $\{\mathbf{e}_i(t)\} \neq \{\mathbf{e}_A\}$ for $t > 0$ in general. However, the same base is often used for the reference and the current bases for the sake of simplicity.

Here, based on the relation $d\mathbf{x}(t) = (\partial \mathbf{x}(t)/\partial \mathbf{X})d\mathbf{X}$, we define the deformation gradient tensor

$$\boxed{\mathbf{F}(t) \equiv \frac{\partial \mathbf{x}(t)}{\partial \mathbf{X}}} = F_{iA}(t)\mathbf{e}_i(t) \otimes \mathbf{e}_A \equiv \frac{\partial x_i(t)}{\partial X_A}\mathbf{e}_i(t) \otimes \mathbf{e}_A = x_{i,A}(t)\mathbf{e}_i(t) \otimes \mathbf{e}_A \tag{2.14}$$

$\mathbf{F}(t)$ is based in the current and the reference bases which can be chosen different to each other and thus it is called the *Eulerian-Lagrangian two-point tensor*. The infinitesimal line-element $d\mathbf{x}(t)$ is described by $d\mathbf{X}$ from Eq. (2.14) as follows:

$$d\mathbf{x}(t) = dx_i(t)\mathbf{e}_i(t) = \frac{\partial x_i(t)}{\partial X_A} dX_A \mathbf{e}_i(t) = \frac{\partial x_i(t)}{\partial X_A} \mathbf{e}_i(t) \otimes \mathbf{e}_A dX_B \mathbf{e}_B = \mathbf{F}(t) d\mathbf{X} \quad (2.15)$$

The deformation gradient tensor $\mathbf{F}(t)$ transforms the reference infinitesimal line element to the current infinitesimal line element and thus it is the most fundamental variable for the description of deformation of materials. Equation (2.6) is written in terms of the deformation gradient as

$$J = \det \mathbf{F} \quad (2.16)$$

Therefore, if $J = \det \mathbf{F} \neq 0$ holds, the inverse tensor $\mathbf{F}^{-1}$ exists by virtue of Eq. (1.120), and it is derived from $\mathbf{F}\mathbf{F}^{-1} = (\partial\mathbf{x}/\partial\mathbf{X})(\partial\mathbf{X}/\partial\mathbf{x}) = \mathbf{I}$ as

$$\mathbf{F}^{-1} = \frac{\partial \mathbf{X}}{\partial \mathbf{x}}, \quad (\mathbf{F}^{-1})_{Ai} \mathbf{e}_A \otimes \mathbf{e}_i(t) = \frac{\partial X_A}{\partial x_i} \mathbf{e}_A \otimes \mathbf{e}_i(t) = X_{A,i}(t) \mathbf{e}_A \otimes \mathbf{e}_i(t) \quad (2.17)$$

noting

$$d\mathbf{X} = dX_A \mathbf{e}_A = \frac{\partial X_A}{\partial x_i(t)} dx_i(t) \mathbf{e}_A = \frac{\partial X_A}{\partial x_i(t)} \mathbf{e}_A \otimes \mathbf{e}(t) dx_j(t) \mathbf{e}_j(t) = \mathbf{F}^{-1}(t) d\mathbf{x}(t)$$

As described above, the deformation gradient tensor $\mathbf{F}$ plays the most basic role to describe the deformation of materials. Any exact deformation (rate) measure must be represented by it. In addition, the transformation of the infinitesimal current line-element to its rate is described by the velocity gradient tensor $\boldsymbol{l}$ which is the most basic measure for deformation rate as will be described in Sect. 2.5.

Besides, consider the unit cubic cell (a parallelepiped) whose sides at the initial (reference) configuration are given by the triad $\{\mathbf{e}_I\}$ and then assume that it deforms to the cell whose sides are formed by the triad $\{\bar{\mathbf{e}}_i\}$. They are related by Eq. $(2.15)_1$ regarding $d\mathbf{x}$ and $d\mathbf{X}$ as $\bar{\mathbf{e}}_i$ and $\mathbf{e}_I$, respectively, as follows:

$$\bar{\mathbf{e}}_i = \delta_{iI} \mathbf{F} \mathbf{e}_I \quad (2.18)$$

The vectors $\bar{\mathbf{e}}_i$ are neither unit vectors nor orthonormal except for the rigid-body rotation. The curvilinear coordinate system with the base $\{\bar{\mathbf{e}}_i\}$ is referred to as the *convected coordinate system*. It is indispensable for general interpretation of deformation and rotation of materials, the detailed explanation of which can be referred to Sect. 4.4 and **Appendix B** briefly or Hashiguchi and Yamakawa (2012) in detail.

Applying the polar decomposition in Sect. 1.11 to the deformation gradient $\mathbf{F}$, we have

$$\boxed{\mathbf{F} = \mathbf{RU} = \mathbf{VR}}, \quad F_{iA} = R_{iR}U_{RA} = V_{ir}R_{rA} \tag{2.19}$$

where

$$\mathbf{U} = \mathbf{R}^T\mathbf{F} = (\mathbf{F}^T\mathbf{F})^{1/2}(= \mathbf{U}^T)(\mathbf{U}^2 = \mathbf{F}^T\mathbf{F}) \tag{2.20}$$

$$\mathbf{V} = \mathbf{FR}^T = (\mathbf{FF}^T)^{1/2}(= \mathbf{V}^T)(\mathbf{V}^2 = \mathbf{FF}^T) \tag{2.21}$$

$$\mathbf{R} = \mathbf{FU}^{-1} = \mathbf{F}(\mathbf{F}^T\mathbf{F})^{-1/2}, \quad \mathbf{R} = \mathbf{V}^{-1}\mathbf{F} = (\mathbf{FF}^T)^{-1/2}\mathbf{F} \quad (\det\mathbf{R} = 1) \tag{2.22}$$

$$\mathbf{V} = \mathbf{RUR}^T, \quad \mathbf{U} = \mathbf{R}^T\mathbf{VR} \tag{2.23}$$

$\mathbf{U}$ and $\mathbf{V}$ are the symmetric tensors so that there exist the two principal direction triads in which the deformation is described by the three-dimensional stretching resulting in the volume change and the shape change (shear deformation). Further, they are the similar tensors to each other, since Eq. (2.23) holds for the orthogonal tensor $\mathbf{R}$ as was described in Sect. 1.6. Therefore, they possess the same principal values, say $\lambda_\alpha(>0)(\alpha = 1, 2, 3)$. Denoting the bases for the principal directions of $\mathbf{U}$ and $\mathbf{V}$ by $\{\mathbf{N}^\alpha(t)\}$ and $\{\mathbf{n}^{(\alpha)}(t)\}$, respectively, they can be written as

$$\boxed{\mathbf{U} = \sum_{\alpha=1}^{3} \lambda_\alpha \mathbf{N}^{(\alpha)}(t) \otimes \mathbf{N}^{(\alpha)}(t), \quad \mathbf{V} = \sum_{\alpha=1}^{3} \lambda_\alpha \mathbf{n}^{(\alpha)}(t) \otimes \mathbf{n}^{(\alpha)}(t)} \tag{2.24}$$

where the relation of $\mathbf{N}^{(\alpha)}(t)$ and $\mathbf{n}^{(\alpha)}(t)$ is given from Eq. (1.228) as follows:

$$\boxed{\mathbf{n}^{(\alpha)}(t) = \mathbf{R}(t)\mathbf{N}^{(\alpha)}(t), \mathbf{N}^{(\alpha)}(t) = \mathbf{R}^T(t)\mathbf{n}^{(\alpha)}(t)} \tag{2.25}$$

with

$$\boxed{\mathbf{R}(t) = \sum_{\alpha=1}^{3} \mathbf{n}^{(\alpha)}(t) \otimes \mathbf{N}^{(\alpha)}(t)} \tag{2.26}$$

$\mathbf{N}^{(\alpha)}(t)$ and $\mathbf{n}^{(\alpha)}(t)$ are called the *Lagrangian triad* and the *Eulerian triad*, respectively.

Substituting Eqs. (2.24) and (2.26) into Eq. (2.19), $\mathbf{F}$ and its inverse tensor are described by

$$\boxed{\mathbf{F}(t) = \sum_{\alpha=1}^{3} \lambda_\alpha(t)\mathbf{n}^{(\alpha)}(t) \otimes \mathbf{N}^{(\alpha)}(t), \quad \mathbf{F}^{-1}(t) = \sum_{\alpha=1}^{3} \frac{1}{\lambda_\alpha}(t)\mathbf{N}^{(\alpha)}(t) \otimes \mathbf{n}^{(\alpha)}(t)} \tag{2.27}$$

Let the mechanical meanings of $\mathbf{U}, \mathbf{V}$ and $\mathbf{R}$ be examined below.

The variation of infinitesimal line-element is given by the polar decomposition $\mathbf{F} = \mathbf{RU}$ as follows:

$$d\mathbf{x} = \mathbf{F}d\mathbf{X} = \mathbf{RU}d\mathbf{X} = \mathbf{R}\sum_{\beta=1}^{3}\lambda_\beta \mathbf{N}^{(\beta)} \otimes \mathbf{N}^{(\beta)} \sum_{\alpha=1}^{3} dX_\alpha \mathbf{N}^{(\alpha)} = \mathbf{R}\sum_{\alpha=1}^{3}\lambda_\alpha dX_\alpha \mathbf{N}^{(\alpha)} \quad (2.28)$$

Equation (2.28) means that the infinitesimal line-elements $dX_\alpha \mathbf{N}^{(\alpha)}$ (no sum) in the principal directions $\mathbf{N}^{(\alpha)}$ are first stretched by λ_α times to $\lambda_\alpha dX_\alpha \mathbf{N}^{(\alpha)}$ (no sum) and then undergoes the rotation $\mathbf{R}$ as shown in Fig. 2.1.

On the other hand, the change of the infinitesimal line-element by the polar decomposition $\mathbf{VR}$ is described as

$$\begin{aligned} d\mathbf{x} = \mathbf{VR}d\mathbf{X} &= \sum_{\beta=1}^{3}\lambda_\beta \mathbf{n}^{(\beta)} \otimes \mathbf{n}^{(\beta)}\mathbf{R}\sum_{\alpha=1}^{3} dX_\alpha \mathbf{N}^{(\alpha)} = \sum_{\beta=1}^{3}\lambda_\beta \mathbf{n}^{(\beta)} \otimes \mathbf{n}^{(\beta)}\sum_{\alpha=1}^{3} dX_\alpha \mathbf{n}^{(\alpha)} \\ &= \sum_{\alpha=1}^{3}\lambda_\alpha dX_\alpha \mathbf{n}^{(\alpha)} = \sum_{\alpha=1}^{3}\lambda_\alpha \mathbf{R} dX_\alpha \mathbf{N}^{(\alpha)} \end{aligned} \quad (2.29)$$

Equation (2.29) means that the infinitesimal line-elements $dX_\alpha \mathbf{N}^{(\alpha)}$ (no sum) in the principal directions $\mathbf{N}^{(\alpha)}$ first becomes $dX_\alpha \mathbf{n}^{(\alpha)}$ (no sum) by rotation $\mathbf{R}$ and then are stretched by λ_α times to $\lambda_\alpha dX_\alpha \mathbf{n}^{(\alpha)}$ (no sum) (see Fig. 2.1).

As described above, $\mathbf{U}$ and $\mathbf{V}$ designates the deformation and $\mathbf{R}$ the rotation. λ_α is called the *principal stretch*, and $\mathbf{U}$ and $\mathbf{V}$ are called the *right* and *left stretch tensor*, respectively.

Letting $\mathbf{R}^L$ and $\mathbf{R}^E$ designate the rotations of the Lagrangian triad $\{\mathbf{N}^{(\alpha)}\}$ and the Eulerian triad $\{\mathbf{n}^{(\alpha)}\}$, respectively, from the fixed base $\{\mathbf{e}_\alpha\}(\alpha = 1, 2, 3)$, they are given by

$$\boxed{\mathbf{R}^L \equiv \sum_{\alpha=1}^{3} \mathbf{N}^{(\alpha)} \otimes \mathbf{e}_\alpha, \quad \mathbf{R}^E \equiv \sum_{\alpha=1}^{3} \mathbf{n}^{(\alpha)} \otimes \mathbf{e}_\alpha} \quad (2.30)$$

where the following relations hold.

$$\boxed{\mathbf{N}^{(\alpha)} = \mathbf{R}^L \mathbf{e}_\alpha, \quad \mathbf{n}^{(\alpha)} = \mathbf{R}^E \mathbf{e}_\alpha} \quad (2.31)$$

$$\boxed{\mathbf{R}^E = \mathbf{R}\mathbf{R}^L} \quad (2.32)$$

Considering the particle P and the adjacent particles P′ and P″, we designate their position vectors before and after the deformation by $\mathbf{X}$, $\mathbf{X} + d\mathbf{X}$, $\mathbf{X} + \delta\mathbf{X}$ and $\mathbf{x}$, $\mathbf{x} + d\mathbf{x}$, $\mathbf{x} + \delta\mathbf{x}$, respectively. Then, noting (1.116), one has

$$d\mathbf{x} \cdot \delta\mathbf{x} = \mathbf{F}d\mathbf{X} \cdot \mathbf{F}\delta\mathbf{X} = \mathbf{F}^T\mathbf{F}d\mathbf{X} \cdot \delta\mathbf{X} = \mathbf{C}d\mathbf{X} \cdot \delta\mathbf{X} \quad (2.33)$$

$$d\mathbf{X} \cdot \delta\mathbf{X} = \mathbf{F}^{-1}d\mathbf{x} \cdot \mathbf{F}^{-1}\delta\mathbf{x} = \mathbf{F}^{-T}\mathbf{F}^{-1}d\mathbf{x} \cdot \delta\mathbf{x} = (\mathbf{F}\mathbf{F}^T)^{-1}d\mathbf{x} \cdot \delta\mathbf{x} = \mathbf{b}^{-1}d\mathbf{x} \cdot \delta\mathbf{x} \quad (2.34)$$

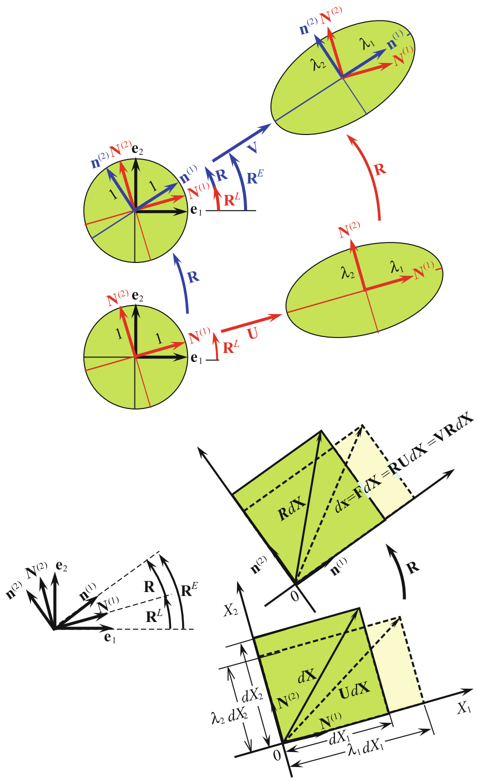

Fig. 2.1 Polar decomposition of deformation gradient

where

$$\boxed{\mathbf{C} \equiv \mathbf{F}^T\mathbf{F} = \mathbf{U}^2 (= \mathbf{C}^T)}, \quad C_{AB} = F_{kA}F_{kB} \tag{2.35}$$

and

$$\boxed{\mathbf{b} \equiv \mathbf{F}\mathbf{F}^T = \mathbf{V}^2\,(= \mathbf{R}\mathbf{C}\mathbf{R}^T)(= \mathbf{b}^T)}, \; b_{ij} = F_{iA}F_{jA} \tag{2.36}$$

are the tensors which describe how the scalar product of two line-element vectors passing through a material point is influenced by a deformation. **C** and **b** are called the *right* and *left Cauchy-Green deformation tensor*, respectively. In accordance with Eq. (2.24) they are described by

$$\boxed{\mathbf{C} = \sum_{\alpha=1}^{3} \lambda_\alpha^2 \mathbf{N}^{(\alpha)} \otimes \mathbf{N}^{(\alpha)}}, \quad \boxed{\mathbf{b} = \sum_{\alpha=1}^{3} \lambda_\alpha^2 \mathbf{n}^{(\alpha)} \otimes \mathbf{n}^{(\alpha)}} \tag{2.37}$$

The principal values λ_α are obtained by the solutions of the characteristic equation

$$\lambda^3 - \mathrm{I}_c\lambda^2 - \mathrm{II}_c\lambda + \mathrm{III}_c = 0 \tag{2.38}$$

based on Eq. (1.177), where

$$\mathrm{I_c} \equiv \mathrm{tr}\mathbf{C},\ \mathrm{II_c} \equiv \frac{1}{2}(\mathrm{tr}\mathbf{C} - \mathrm{tr}\mathbf{C}^2),\ \ \mathrm{III_c} \equiv \frac{1}{6}\mathrm{tr}^3\mathbf{C} - \frac{1}{2}\mathrm{tr}\mathbf{C}\mathrm{tr}\mathbf{C}^2 + \frac{1}{3}\mathrm{tr}\mathbf{C}^3 \tag{2.39}$$

The principal values and directions are calculated by the method described in 1.6. The similar equations hold for $\mathbf{b}$ instead of $\mathbf{C}$. Using the relative description (Eq. 2.8), the *relative deformation gradient tensor* in the reference configuration $\mathbf{x}(\tau)$ is defined as

$$\boxed{{}_\tau\mathbf{F}(t) = \frac{\partial \mathbf{x}(t)}{\partial \mathbf{x}(\tau)}} \tag{2.40}$$

which is related to the deformation gradient $\mathbf{F}(t)$ $(\equiv {}_0\mathbf{F}(t))$ as

$$\mathbf{F}(t) = \frac{\partial \mathbf{x}(t)}{\partial \mathbf{X}} = \frac{\partial \mathbf{x}(t)}{\partial \mathbf{x}(\tau)}\frac{\partial \mathbf{x}(\tau)}{\partial \mathbf{X}} = {}_\tau\mathbf{F}(t)\mathbf{F}(\tau) \tag{2.41}$$

and is further expressed in the polar decomposition as

$${}_\tau\mathbf{F}(t) = {}_\tau\mathbf{R}(t){}_\tau\mathbf{U}(t) = {}_\tau\mathbf{V}(t){}_\tau\mathbf{R}(t) \tag{2.42}$$

where ${}_\tau\mathbf{C}(t)$, ${}_\tau\mathbf{b}(t)$ defined by

$$\left.\begin{aligned} {}_\tau\mathbf{C}(t) &= ({}_\tau\mathbf{F}(t))^T\, {}_\tau\mathbf{F}(t) = {}_\tau\mathbf{U}^2(t) \\ {}_\tau\mathbf{b}(t) &= {}_\tau\mathbf{F}(t)\,({}_\tau\mathbf{F}(t))^T = {}_\tau\mathbf{V}^2(t) \end{aligned}\right\} \tag{2.43}$$

which are called the *relative right* and the *left Cauchy-Green deformation tensors*.

2.4 Strain Tensors

Consider the scalar quantity which changes only by the pure deformation but is independent of the rotation. Subtracting Eq. (2.34) from Eq. (2.33), one has

$$d\mathbf{x}\bullet\delta\mathbf{x} - d\mathbf{X}\bullet\delta\mathbf{X} = \begin{cases} 2\boldsymbol{E}d\mathbf{X}\bullet\delta\mathbf{X}(= 2E_{AB}dX_A\delta X_B) \\ 2\boldsymbol{e}d\mathbf{x}\bullet\delta\mathbf{x}(= 2e_{ij}dx_i\delta x_j) \end{cases} \tag{2.44}$$

where

$$\left.\begin{array}{l}
\boxed{\boldsymbol{E} \equiv \frac{1}{2}(\mathbf{C}-\mathbf{I}) = \frac{1}{2}(\mathbf{U}^2-\mathbf{I}) = \frac{1}{2}(\mathbf{F}^T\mathbf{F}-\mathbf{I}) = \frac{1}{2}\left[\left(\frac{\partial\mathbf{x}}{\partial\mathbf{X}}\right)^T\left(\frac{\partial\mathbf{x}}{\partial\mathbf{X}}\right)-\mathbf{I}\right]} \\
E_{AB} \equiv \frac{1}{2}(F_{kA}F_{kB}-\delta_{AB}) = \frac{1}{2}\left(\frac{\partial x_k}{\partial X_A}\frac{\partial x_k}{\partial X_B}-\delta_{AB}\right) \\
\boxed{\boldsymbol{e} \equiv \frac{1}{2}(\mathbf{I}-\mathbf{b}^{-1}) = \frac{1}{2}(\mathbf{I}-\mathbf{V}^{-2}) = \frac{1}{2}(\mathbf{I}-\mathbf{F}^{-T}\mathbf{F}^{-1}) = \frac{1}{2}\left[\mathbf{I}-\left(\frac{\partial\mathbf{X}}{\partial\mathbf{x}}\right)^T\frac{\partial\mathbf{X}}{\partial\mathbf{x}}\right]} \\
e_{ij} \equiv \frac{1}{2}[\delta_{ij}-(\mathbf{F}^{-1})_{Ki}(\mathbf{F}^{-1})_{Kj}] = \frac{1}{2}\left(\delta_{ij}-\frac{\partial X_K}{\partial x_i}\frac{\partial X_K}{\partial x_j}\right)
\end{array}\right\} \tag{2.45}$$

which are defined by **C** and **b** describing the pure deformations. Applying the quotient law described in Sect. 1.3.2 to Eq. (2.44), it is confirmed that $\boldsymbol{E}$ and $\boldsymbol{e}$ are the second-order tensors.

If a deformation is not induced, the triangle PP′P″ keeps the same shape as in the initial state and thus the left-hand side in Eq. (2.44) is zero so that $\boldsymbol{E}$ and $\boldsymbol{e}$ are zero independent of rotation. Conversely, if $\boldsymbol{E} \neq \mathbf{O}$, $\boldsymbol{e} \neq \mathbf{O}$, the left-hand side in Eq. (2.44) is not zero so that the shape of the triangle is not same as in the initial state, resulting in a deformation. Therefore, $\boldsymbol{E}$ and $\boldsymbol{e}$ are the quantities describing the deformation independent of rigid-body rotation and called the *Green* (or *Lagrangian*) *strain tensor* and the *Almansi* (or *Eulerian*) *strain tensor*, respectively. Using the displacement vector

$$\boxed{\mathbf{u} = \mathbf{x}-\mathbf{X} = u_i\mathbf{e}_i}, \quad \frac{\partial\mathbf{u}}{\partial\mathbf{X}} = \mathbf{F}-\mathbf{I} \tag{2.46}$$

they are expressed by

$$\left.\begin{array}{ll}
\boxed{\boldsymbol{E} = \frac{1}{2}\left[\frac{\partial\mathbf{u}}{\partial\mathbf{X}}+\left(\frac{\partial\mathbf{u}}{\partial\mathbf{X}}\right)^T+\left(\frac{\partial\mathbf{u}}{\partial\mathbf{X}}\right)^T\left(\frac{\partial\mathbf{u}}{\partial\mathbf{X}}\right)\right]}, & E_{AB} = \frac{1}{2}\left(\frac{\partial u_A}{\partial X_B}+\frac{\partial u_B}{\partial X_A}+\frac{\partial u_K}{\partial X_A}\frac{\partial u_K}{\partial X_B}\right) \\
\boxed{\boldsymbol{e} = \frac{1}{2}\left[\frac{\partial\mathbf{u}}{\partial\mathbf{x}}+\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)^T-\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)^T\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right)\right]}, & e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)
\end{array}\right\} \tag{2.47}$$

The following relation exists between them,

$$\boxed{\boldsymbol{E} = \mathbf{F}^T\boldsymbol{e}\mathbf{F}}, \quad E_{AB} = F_{iA}F_{jB}e_{ij} \tag{2.48}$$

Now, consider the symmetric part of the displacement gradient which is the eliminations of the third terms in the brackets $\boldsymbol{E}$ and $\boldsymbol{e}$ in Eq. (2.47), i.e

$$\boxed{\boldsymbol{\varepsilon} \equiv \left(\frac{\partial \mathbf{u}}{\partial \mathbf{X}}\right)^S = \frac{1}{2}\left[\frac{\partial \mathbf{u}}{\partial \mathbf{X}} + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{X}}\right)^T\right] = \frac{1}{2}(\mathbf{F} + \mathbf{F}^T) - \mathbf{I}}, \quad \varepsilon_{AB} \equiv \frac{1}{2}\left(\frac{\partial u_A}{\partial X_B} + \frac{\partial u_B}{\partial X_A}\right) \tag{2.49}$$

or

$$\boxed{\boldsymbol{\varepsilon} \equiv \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^S = \frac{1}{2}\left[\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^T\right] = \mathbf{I} - \frac{1}{2}(\mathbf{F}^{-1} + \mathbf{F}^{-T})}, \quad \varepsilon_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.50}$$

which describe roughly deformation, depending not only on $\mathbf{U}$ or $\mathbf{V}$ but also on the rotation tensor $\mathbf{R}$. Besides, the Lagrangian and the Eulerian tensors explained in Sect. 4.4 are mixed without the distinction of $\mathbf{X}$ and $\mathbf{x}$. Then, $\boldsymbol{\varepsilon}$ is called the *infinitesimal strain tensor*. The difference of Eqs. (2.49) and (2.50) vanishes in an infinitesimal deformation leading to $d\mathbf{x} \cong d\mathbf{X}$. It possesses various impertinence as will be described in Sect. 2.6.

In what follows, the geometrical interpretation of $\boldsymbol{E}$ and $\boldsymbol{e}$ will be given.

Considering the case that the two infinitesimal line-elements PP′ and PP″ coincide to each other, i.e. $d\mathbf{X} = \delta\mathbf{X}, d\mathbf{x} = \delta\mathbf{x}$ and denoting its direction vector in the initial configuration by $\mathbf{N}(\|\mathbf{N}\| = 1)$, it follows from Eq. (2.44) that

$$||d\mathbf{x}||^2 - ||d\mathbf{X}||^2 = \begin{cases} 2\boldsymbol{E}\mathbf{N}\cdot\mathbf{N}||d\mathbf{X}||^2 \\ 2\boldsymbol{e}\mathbf{N}\cdot\mathbf{N}||d\mathbf{x}||^2 \end{cases} \tag{2.51}$$

Selecting the X_1-axis for this line-element, $(N_1, N_2, N_3) = (1, 0, 0)$ holds and thus we have

$$\left.\begin{aligned} E_{11} &= \frac{1}{2}\frac{||d\mathbf{x}||^2 - ||d\mathbf{X}||^2}{||d\mathbf{X}||^2}\left(= \frac{1}{2}\left[\left(\frac{||d\mathbf{x}||}{||d\mathbf{X}||}\right)^2 - 1\right]\right) \\ e_{11} &= \frac{1}{2}\frac{||d\mathbf{x}||^2 - ||d\mathbf{X}||^2}{||d\mathbf{x}||^2}\left(= \frac{1}{2}\left[1 - \left(\frac{||d\mathbf{X}||}{||d\mathbf{x}||}\right)^2\right]\right) \end{aligned}\right\} \tag{2.52}$$

from which the ratio of the line-elements before and after the deformation is given by

$$\frac{||d\mathbf{x}||}{||d\mathbf{X}||} = \begin{cases} \sqrt{1 + 2E_{11}} \\ 1/\sqrt{1 - 2e_{11}} \end{cases} \tag{2.53}$$

In the case that the variation of the length of the line-element is infinitesimal $(||d\mathbf{x}||/||d\mathbf{X}|| \cong 1)$, Eq. (2.52) becomes

$$\left.\begin{aligned} E_{11} &\cong \frac{||d\mathbf{x}|| - ||d\mathbf{X}||}{||d\mathbf{X}||} \\ e_{11} &\cong \frac{||d\mathbf{x}|| - ||d\mathbf{X}||}{||d\mathbf{x}||} \end{aligned}\right\} \cong \varepsilon_{11} \tag{2.54}$$

so that E_{11} and e_{11} describe the rate of elongation coinciding with the normal strain in the infinitesimal strain $\boldsymbol{\varepsilon}$.

On the other hand, denoting the direction vectors of the two distinct infinitesimal line-element PP$'$ and PP$''$ as $\mathbf{N}'$ and $\mathbf{N}''$, respectively, in the initial state and the angles contained by them as θ, it holds from Eq. (2.44) that

$$||d\mathbf{x}||||\delta\mathbf{x}|| \cos\theta - ||d\mathbf{X}||||\delta\mathbf{X}|| \cos\theta_0 = 2\boldsymbol{E}\mathbf{N}' \bullet \mathbf{N}''||d\mathbf{X}||||\delta\mathbf{X}|| \tag{2.55}$$

i.e.

$$\frac{||d\mathbf{x}||}{||d\mathbf{X}||}\frac{||\delta\mathbf{x}||}{||\delta\mathbf{X}||}\cos\theta - \cos\theta_0 = 2\boldsymbol{E}\mathbf{N}' \bullet \mathbf{N}'' = 2E_{ij}N'_iN''_j \tag{2.56}$$

where θ_0 is the initial value of θ. Here, assuming that the infinitesimal line-elements PP$'$ and PP$''$ were mutually perpendicular before a deformation, i.e. $\theta_0 = \pi/2$ leading to $\cos\theta_0 = 0$, and making their directions coincide to the X_1- and X_2-axes, i.e. $(N'_1, N'_2, N'_3) = (1, 0, 0)$, $(N''_1, N''_2, N''_3) = (0, 1, 0)$ leading to $E_{ij}N'_iN''_j = E_{12}$, it follows that

$$E_{12} = \frac{1}{2}\frac{||d\mathbf{x}||}{||d\mathbf{X}||}\frac{||\delta\mathbf{x}||}{||\delta\mathbf{X}||}\cos\theta = \frac{1}{2}\frac{||d\mathbf{x}||}{||d\mathbf{X}||}\frac{||\delta\mathbf{x}||}{||\delta\mathbf{X}||}\sin(\pi/2-\theta) \tag{2.57}$$

which describes the half of decrease in the sine of angle contained by the two line-elements which were mutually perpendicular before deformation when the changes in lengths of these line-elements are infinitesimal ($||d\mathbf{x}||/||d\mathbf{X}|| \cong 1$, $||\delta\mathbf{x}||/||\delta\mathbf{X}|| \cong 1$). Furthermore, when the change in the angle formed by these line-elements is infinitesimal ($\theta \cong \pi/2$), one has

$$E_{12} \cong (\pi/2 - \theta)/2 \tag{2.58}$$

Consequently, E_{12} describes half of the decrease in the angle contained by the two line-elements which were perpendicular before deformation.

In addition to the Lagrangian and Eulerian strain tensors defined above, we can define various strain tensors in terms of $\mathbf{U}$ or $\mathbf{V}$, fulfilling the condition that they are zero when $\mathbf{U} = \mathbf{V} = \mathbf{I}$ as follows (Seth 1964; Hill 1968):

$$\boxed{\boldsymbol{E}^{(m)} = \mathbf{f}(\mathbf{U}) = \begin{cases} \dfrac{1}{2m}(\mathbf{U}^{2m} - \mathbf{I}) & \text{for } m \neq 0 \\ \ln\mathbf{U} & \text{for } m = 0 \end{cases}} \tag{2.59}$$

$$\boxed{\boldsymbol{e}^{(m)} = \mathbf{f}(\mathbf{V}) = \begin{cases} \dfrac{1}{2m}(\mathbf{V}^{2m} - \mathbf{I}) & \text{for } \quad m \neq 0 \\ \ln\mathbf{V} & \text{for } \quad m = 0 \end{cases}} \tag{2.60}$$

where $2m$ is the integer (positive or negative). The Green strain tensor is obtained by choosing $m = 1$ in Eq. (2.59) and the Almansi strain tensor is obtained by choosing $m = -1$ in Eq. (2.60). The *Biot strain tensor* (Biot 1965) is given by choosing $m = 1/2$ in Eq. (2.59), i.e.

$$\boxed{\mathbf{B} \equiv \mathbf{U} - \mathbf{I}} \tag{2.61}$$

The generalized strain tensors in Eqs. (2.59) and (2.60) are mutually related by virtue of Eq. (2.23) as follows.

$$\boxed{\boldsymbol{E}^{(m)} = \mathbf{R}^T \boldsymbol{e}^{(m)} \mathbf{R}} \tag{2.62}$$

The strain tensors in Eqs. (2.59) and (2.60) are coaxial with $\mathbf{U}$ and $\mathbf{V}$, respectively, and their principal values are given by

$$\boxed{f(\lambda_\alpha) = \begin{cases} \dfrac{1}{2m}(\lambda_\alpha^{2m} - 1) & \text{for} \quad m \neq 0 \\ \ln \lambda_\alpha & \text{for} \quad m = 0 \end{cases}} \tag{2.63}$$

for $\alpha = 1, 2, 3$. The function $f(\lambda_\alpha)$ fulfills

$$f(1) = 0, \; f'(1) = 1 \tag{2.64}$$

and

$$f'(s) > 0 \tag{2.65}$$

where s is an arbitrary positive scalar quantity. The function $f(\lambda_\alpha)$ for several values of m is shown in Fig. 2.2.

(Note) Eq. $(2.63)_2$ for $m = 0$ is derived as follows:

$$\lim_{m\to 0} \frac{1}{m}(\lambda_\alpha^m - 1) = \lim_{m\to 0} \frac{\exp(m \ln \lambda_\alpha) - 1}{m} = \lim_{m\to 0} \frac{\exp(m \ln \lambda_\alpha) \ln \lambda_\alpha}{1} = \ln \lambda_\alpha \; (\text{no sum})$$

by the aid of l'Hôpital's.

Further, adopt the second-order tensor function $\mathbf{f}(\mathbf{U})$ which is coaxial with the right stretch tensor $\mathbf{U}$ and has the principal values $f(\lambda_\alpha)$. Therefore, we can define the general strain tensor in the spectral decomposition as follows:

$$\boxed{\mathbf{f}(\mathbf{U}) = \sum_{\alpha=1}^{3} f(\lambda_\alpha) \mathbf{N}^{(\alpha)} \otimes \mathbf{N}^{(\alpha)}} \tag{2.66}$$

In addition, for the left stretch tensor $\mathbf{V}$, we can define the following strain tensor.

$$\boxed{\mathbf{f}(\mathbf{V}) = \sum_{\alpha=1}^{3} f(\lambda_\alpha) \mathbf{n}^{(\alpha)} \otimes \mathbf{n}^{(\alpha)}} = \sum_{\alpha=1}^{3} f(\lambda_\alpha) \mathbf{R}\mathbf{N}^{(\alpha)} \otimes \mathbf{R}\mathbf{N}^{(\alpha)} = \mathbf{R}\mathbf{f}(\mathbf{U})\mathbf{R}^T \tag{2.67}$$

noting Eq. (1.106).

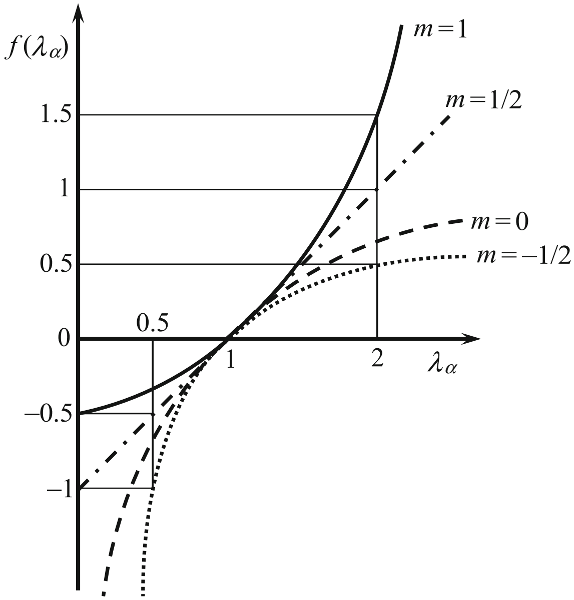

Fig. 2.2 Function of general principal strain measures

In the particular case of $m = 0$, noting $\lambda_\alpha > 0$, the strains defined by the following equation are called the *logarithmic* or *Hencky strain tensor*.

$$\begin{array}{|ll|} \hline \text{Lagrangian-logarithmic strain tensor:} & \boldsymbol{E}^{(0)} = \sum_{\alpha=1}^{3} \ln \lambda_\alpha \mathbf{N}^{(\alpha)} \otimes \mathbf{N}^{(\alpha)} \equiv \ln\mathbf{U} = \frac{1}{2}\ln\mathbf{C} \\ \text{Eulerian-logarithmic strain tensor:} & \boldsymbol{e}^{(0)} = \sum_{\alpha=1}^{3} \ln \lambda_\alpha \mathbf{n}^{(\alpha)} \otimes \mathbf{n}^{(\alpha)} \equiv \ln\mathbf{V} = \frac{1}{2}\ln\mathbf{b} \\ \hline \end{array} \tag{2.68}$$

where

$$\lambda_\alpha = U_\alpha = V_\alpha = \sqrt{C_\alpha} = \sqrt{b_\alpha} \tag{2.69}$$

$U_\alpha, C_\alpha, V_\alpha$ and b_α being the principal values of $\mathbf{U}, \mathbf{C}, \mathbf{V}$ and $\mathbf{b}$, respectively.

When the principal directions of $\mathbf{U}$ and $\mathbf{V}$ are fixed, the following equations hold in these directions.

$$\lambda_\alpha = \frac{\partial x_\alpha}{\partial X_\alpha} \text{ (no sum)} \tag{2.70}$$

$$(\dot{\boldsymbol{E}}^{(0)})_\alpha = (\dot{\boldsymbol{e}}^{(0)})_\alpha = \left(\frac{\partial x_\alpha}{\partial X_\alpha}\right)^{\bullet} \Big/ \left(\frac{\partial x_\alpha}{\partial X_\alpha}\right) = \frac{\partial \dot{\boldsymbol{x}}_\alpha}{\partial x_\alpha} = (\ln\lambda_\alpha)^{\bullet} = d_{\alpha\alpha} \text{ (no sum)} \tag{2.71}$$

where $\ln \lambda_\alpha$ in Eq. (2.70) is the *logarithmic strain* and $d_{\alpha\alpha}$ (no sum) is the principal component of the strain rate tensor defined in the next section. It follows from Eq. (2.68) that

$$\left.\begin{aligned}\mathrm{tr}\boldsymbol{E}^{(0)} &= \sum_{\alpha=1}^{3}\ln\lambda_\alpha = \frac{1}{2}\sum_{\alpha=1}^{3}\ln C_\alpha = \sum_{\alpha=1}^{3}\ln U_\alpha = \ln(U_1U_2U_3)\\ \mathrm{tr}\boldsymbol{e}^{(0)} &= \sum_{\alpha=1}^{3}\ln\lambda_\alpha = \frac{1}{2}\sum_{\alpha=1}^{3}\ln b_\alpha = \sum_{\alpha=1}^{3}\ln V_\alpha = \ln(V_1V_2V_3)\end{aligned}\right\} \tag{2.72}$$

which is nothing but the *logarithmic volumetric strain*

$$\boxed{\mathrm{tr}\boldsymbol{E}^{(0)} = \mathrm{tr}\boldsymbol{e}^{(0)} = \sum_{\alpha=1}^{3}\ln\frac{\partial x_\alpha}{\partial X_\alpha} = \ln J = \ln\frac{v}{V} = \varepsilon_v} \tag{2.73}$$

The Hencky strain is relevant to the strain rate defined by the symmetric part of the velocity gradient, as the rates of principal components and volumetric part in the former coincides to the latter as will be described in Sect. 2.6.

2.5 Strain Rate and Spin Tensors

The idealized deformation process in which the deformation is uniquely determined by the state of stress independent of the loading path is called the *elastic deformation process*. To describe it, it is required to introduce the strain tensor describing the deformation from the initial state and relate it to the stress. Here, since the superposition rule does not hold in the strain tensor, the null stress state is chosen usually as the reference state of strain.

On the other hand, the deformation is not determined uniquely by the state of stress depending on the loading path and thus it cannot be related to the stress in the *irreversible deformation process*, e.g. the viscoelastic, the plastic and the viscoplastic loading processes. Therefore, it is obligatory to relate the infinitesimal changes of stress and deformation and to integrate them along the loading path in order to capture the current states of stress and deformation.

Here, introduce the *velocity gradient tensor* defined as

$$\boxed{\boldsymbol{l} \equiv \frac{\partial \mathbf{v}}{\partial \mathbf{x}}}, \quad l_{ij} \equiv \frac{\partial v_i}{\partial x_j} \equiv \partial_j v_i \tag{2.74}$$

Noting $\dot{\mathbf{F}} = \partial\dot{\mathbf{x}}/\partial\mathbf{X} = \partial\mathbf{v}/\partial\mathbf{X}$ $(d\mathbf{v} = \dot{\mathbf{F}}d\mathbf{X})$ and the chain rule of derivative, Eq. (2.74) can be rewritten as

$$\boxed{\boldsymbol{l} = \frac{\partial \mathbf{v}}{\partial \mathbf{X}}\frac{\partial \mathbf{X}}{\partial \mathbf{x}} = \dot{\mathbf{F}}\mathbf{F}^{-1}}(\dot{\mathbf{F}} = \boldsymbol{l}\mathbf{F}),\, l_{ij} = \frac{\partial v_i}{\partial X_A}\frac{\partial X_A}{\partial x_j} \tag{2.75}$$

$$\boxed{(d\mathbf{x})^{\bullet} = \boldsymbol{l}d\mathbf{x}} \tag{2.76}$$

noting

$$(d\mathbf{x})^{\bullet} = d\mathbf{v} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} d\mathbf{x}$$

$(d\mathbf{x})^{\bullet}$, i.e. $d\mathbf{v}$ describes the rate of the infinitesimal line element, i.e. the relative velocity between the velocities of material particles in both sides of infinitesimal line element $d\mathbf{x}$. Here, we can choose the time $\tau(\leq t)$ to be arbitrary, resulting in $\boldsymbol{l} = {}_{\tau}\dot{\mathbf{F}}(t)_{\tau}\mathbf{F}^{-1}(t)$ because the velocity gradient tensor $\boldsymbol{l}$ is substantially independent of the reference infinitesimal line element $d\mathbf{X}$ but dependent only on rates of deformation and rotation. Now, choosing the current state for the reference state leading to ${}_{t}\mathbf{F}^{-1}(t) = \mathbf{I}$, the velocity gradient tensor $\boldsymbol{l}$ can be expressed in the updated Lagrangian description as follows:

$$\boldsymbol{l} = {}_{t}\dot{\mathbf{F}}(t) \tag{2.77}$$

Further, taking the time-derivative of Eq. (2.42) and noting ${}_{t}\mathbf{R}(t) = {}_{t}\mathbf{U}(t) = {}_{t}\mathbf{V}(t) = \mathbf{I}$, it follows that

$${}_{t}\dot{\mathbf{F}}(t) = {}_{t}\dot{\mathbf{U}}(t) + {}_{t}\dot{\mathbf{R}}(t) = {}_{t}\dot{\mathbf{V}}(t) + {}_{t}\dot{\mathbf{R}}(t) \tag{2.78}$$

Decomposing $\boldsymbol{l}$ additively into the symmetric and the skew-symmetric parts and noting Eqs. (2.75)–(2.78), it is obtained that

$$\boxed{\boldsymbol{l} = \mathbf{d} + \mathbf{w}} \tag{2.79}$$

where

$$\left.\begin{array}{l} \boxed{\mathbf{d} \equiv \frac{1}{2}(\boldsymbol{l} + \boldsymbol{l}^T) = \frac{1}{2}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{x}} + \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)^T\right]} = {}_{t}\dot{\mathbf{U}}(t) = {}_{t}\dot{\mathbf{V}}(t) \\ d_{ij} \equiv \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \end{array}\right\} \tag{2.80}$$

$$\left.\begin{array}{l} \boxed{\mathbf{w} \equiv \frac{1}{2}(\boldsymbol{l} - \boldsymbol{l}^T) = \frac{1}{2}\left[\frac{\partial \mathbf{v}}{\partial \mathbf{x}} - \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)^T\right]} = {}_{t}\dot{\mathbf{R}}(t) \\ w_{ij} \equiv \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) \end{array}\right\} \tag{2.81}$$

where $\mathbf{d}$ is called the *strain rate tensor* or *deformation rate tensor* or *stretching* and $\mathbf{w}$ is called the (*continuum*) *rotation rate tensor* or *continuum spin tensor*. Here, note that $\mathbf{d}$ is not a time-derivative of any strain tensor but is defined independently as the rate variable although it is called the strain rate tensor. In addition, note that the time-integration of $\mathbf{d}$ cannot play any deformation measure in general, because it concerns with different material line-elements which rotate with material. Only the time-integration of axial component of **d** coincides with the axial component of the Hencky strain in Eq. (2.68) if the axial direction is fixed.

Substituting Eqs. (2.19) and (2.75) into Eqs. (2.80) and (2.81), **d** and **w** are described by $\mathbf{U}, \mathbf{R}$ as follows:

$$\left.\begin{aligned}\mathbf{d} &= \frac{1}{2}[\dot{\mathbf{F}}\mathbf{F}^{-1} + (\dot{\mathbf{F}}\mathbf{F}^{-1})^T] = \frac{1}{2}\left\{(\mathbf{RU})^{\bullet}(\mathbf{RU})^{-1} + (\mathbf{RU})^{-T}[(\mathbf{RU})^{\bullet}]^T\right\}\\ &= \frac{1}{2}\mathbf{R}(\dot{\mathbf{U}}\mathbf{U}^{-1} + \mathbf{U}^{-1}\dot{\mathbf{U}})\mathbf{R}^T\\ \mathbf{w} &= \frac{1}{2}[\dot{\mathbf{F}}\mathbf{F}^{-1} - (\dot{\mathbf{F}}\mathbf{F}^{-1})^T] = \frac{1}{2}\left\{(\mathbf{RU})^{\bullet}(\mathbf{RU})^{-1} - (\mathbf{RU})^{-T}[(\mathbf{RU})^{\bullet}]^T\right\}\\ &= \dot{\mathbf{R}}\mathbf{R}^T + \frac{1}{2}\mathbf{R}(\dot{\mathbf{U}}\mathbf{U}^{-1} - \mathbf{U}^{-1}\dot{\mathbf{U}})\mathbf{R}^T\end{aligned}\right\} \tag{2.82}$$

Consequently, we obtain

$$\left.\begin{aligned}\mathbf{d} &= \frac{1}{2}\mathbf{R}(\dot{\tilde{\mathbf{U}}} + \dot{\tilde{\mathbf{U}}}^T)\mathbf{R}^T\\ \mathbf{w} &= \mathbf{\Omega}^R + \frac{1}{2}\mathbf{R}(\dot{\tilde{\mathbf{U}}} - \dot{\tilde{\mathbf{U}}}^T)\mathbf{R}^T\end{aligned}\right\} \tag{2.83}$$

where

$$\dot{\tilde{\mathbf{U}}} \equiv \dot{\mathbf{U}}\mathbf{U}^{-1} \tag{2.84}$$

$$\boxed{\mathbf{\Omega}^R \equiv \dot{\mathbf{R}}\mathbf{R}^T} \tag{2.85}$$

$\mathbf{\Omega}^R$ is called the *relative* (or *polar*) *spin tensor*. Further, **d** and **w** are described by $\mathbf{V}, \mathbf{R}$ as follows:

$$\left.\begin{aligned}\mathbf{d} &= \frac{1}{2}\left\{(\mathbf{VR})^{\bullet}(\mathbf{VR})^{-1} + (\mathbf{VR})^{-T}[(\mathbf{VR})^{\bullet}]^T\right\}\\ &= \frac{1}{2}(\dot{\mathbf{V}}\mathbf{V}^{-1} + \mathbf{V}^{-T}\dot{\mathbf{V}}^T) + \frac{1}{2}(\mathbf{V}\dot{\mathbf{R}}\mathbf{R}^T\mathbf{V}^{-T} - \mathbf{V}^{-1}\dot{\mathbf{R}}\mathbf{R}^T\mathbf{V}^T)\\ \mathbf{w} &= \frac{1}{2}\left\{(\mathbf{VR})^{\bullet}(\mathbf{VR})^{-1} - (\mathbf{VR})^{-T}[(\mathbf{VR})^{\bullet}]^T\right\}\\ &= \frac{1}{2}(\mathbf{V}\dot{\mathbf{R}}\mathbf{R}^T\mathbf{V}^{-1} + \mathbf{V}^{-T}\dot{\mathbf{R}}\mathbf{R}^T\mathbf{V}^T) + \frac{1}{2}(\dot{\mathbf{V}}\mathbf{V}^{-1} - \mathbf{V}^{-T}\dot{\mathbf{V}}^T)\end{aligned}\right\} \tag{2.86}$$

and thus

$$\left.\begin{aligned}\mathbf{d} &= \frac{1}{2}(\dot{\tilde{\mathbf{V}}} + \dot{\tilde{\mathbf{V}}}^T) + \frac{1}{2}(\tilde{\mathbf{\Omega}}^R + \tilde{\mathbf{\Omega}}^{RT})\\ \mathbf{w} &= \frac{1}{2}(\tilde{\mathbf{\Omega}}^R - \tilde{\mathbf{\Omega}}^{RT}) + \frac{1}{2}(\dot{\tilde{\mathbf{V}}} - \dot{\tilde{\mathbf{V}}}^T)\end{aligned}\right\} \tag{2.87}$$

where

$$\dot{\tilde{\mathbf{V}}} \equiv \dot{\mathbf{V}}\mathbf{V}^{-1} \tag{2.88}$$

$$\widetilde{\boldsymbol{\Omega}}^R \equiv \mathbf{V}\boldsymbol{\Omega}^R\mathbf{V}^{-1} \tag{2.89}$$

It follows from Eq. (2.30) that

$$\begin{aligned}\boxed{\boldsymbol{\Omega}^L \equiv \dot{\mathbf{R}}^L\mathbf{R}^{LT}} &= \sum_{\alpha,\beta=1}^{3}(\mathbf{N}^{(\alpha)} \otimes \mathbf{e}_\alpha)^{\bullet}(\mathbf{N}^{(\beta)} \otimes \mathbf{e}_\beta)^T \\ &= \sum_{\alpha,\beta=1}^{3}(\dot{\mathbf{N}}^{(\alpha)} \otimes \mathbf{e}^{(\alpha)} + \mathbf{N}^{(\alpha)} \otimes \dot{\mathbf{e}}^{(\alpha)})\mathbf{e}^{(\beta)} \otimes \mathbf{N}^{(\beta)} \\ &= \sum_{\alpha=1}^{3}\dot{\mathbf{N}}^{(\alpha)} \otimes \mathbf{N}^{(\alpha)}\end{aligned} \tag{2.90}$$

and thus one has

$$\boxed{\dot{\mathbf{N}}^{(\alpha)} = \boldsymbol{\Omega}^L\mathbf{N}^{(\alpha)}} \tag{2.91}$$

noting $\dot{\mathbf{e}}^{(\alpha)} = \mathbf{0}$ since $\{\mathbf{e}^{(\alpha)}\}$ is the fixed base. Therefore, $\boldsymbol{\Omega}^L$ describes the spin of the Lagrangian principal triad $\{\mathbf{N}^{(\alpha)}\}$ of the right stretch tensor $\mathbf{U}$ and is called the *Lagrangian spin tensor*. The components of $\boldsymbol{\Omega}^L$ in the Lagrangian principal triad $\{\mathbf{N}^{(\alpha)}\}$ are described as

$$\Omega^L_{\alpha\beta} = \mathbf{N}^{(\alpha)} \bullet \boldsymbol{\Omega}^L\mathbf{N}^{(\beta)} = \mathbf{N}^{(\alpha)} \bullet \dot{\mathbf{N}}^{(\beta)} \tag{2.92}$$

On the other hand, it follows from Eq. (2.30) that

$$\begin{aligned}\boxed{\boldsymbol{\Omega}^E \equiv \dot{\mathbf{R}}^E\mathbf{R}^{ET}} &= \sum_{\alpha,\beta=1}^{3}(\mathbf{n}^{(\alpha)} \otimes \mathbf{e}^{(\alpha)})^{\bullet}(\mathbf{n}^{(\beta)} \otimes \mathbf{e}^{(\beta)})^T \\ &= \sum_{\alpha=1}^{3}\dot{\mathbf{n}}^{(\alpha)} \otimes \mathbf{n}^{(\alpha)}\end{aligned} \tag{2.93}$$

and thus one has

$$\boxed{\dot{\mathbf{n}}^{(\alpha)} = \boldsymbol{\Omega}^E\mathbf{n}^{(\alpha)}} \tag{2.94}$$

Therefore, $\mathbf{\Omega}^E$ describes the spin of the Eulerian principal triad $\{\mathbf{n}^{(\alpha)}\}$ of the right stretch tensor $\mathbf{V}$ and is called the *Eulerian spin tensor*. The components of $\mathbf{\Omega}^E$ in the Eulerian triad $\{\mathbf{n}^{(\alpha)}\}$ are described as

$$\Omega^E_{\alpha\beta} = \mathbf{n}^{(\alpha)} \cdot \mathbf{\Omega}^E \mathbf{n}^{(\beta)} = \mathbf{n}^{(\alpha)} \cdot \dot{\mathbf{n}}^{(\beta)} \tag{2.95}$$

It follows from Eq. (2.27) that

$$\dot{\mathbf{F}} = \sum_{\alpha=1}^{3} \left[\dot{\lambda}_\alpha \mathbf{n}^\alpha \otimes \mathbf{N}^\alpha + \lambda_\alpha (\dot{\mathbf{n}}^\alpha \otimes \mathbf{N}^\alpha + \mathbf{n}^{(\alpha)} \otimes \dot{\mathbf{N}}^\alpha) \right] \tag{2.96}$$

which is rewritten by Eqs. (2.91) and (2.94) as

$$\dot{\mathbf{F}} = \sum_{\alpha=1}^{3} \dot{\lambda}_\alpha \mathbf{n}^{(\alpha)} \otimes \mathbf{N}^{(\alpha)} + \mathbf{\Omega}^E \mathbf{F} - \mathbf{F}\mathbf{\Omega}^L \tag{2.97}$$

or by Eqs. (2.92) and (2.95), Eq. (2.96) leads to

$$\dot{\mathbf{F}} = \sum_{\alpha=1}^{3} \dot{\lambda}_\alpha \mathbf{n}^{(\alpha)} \otimes \mathbf{N}^{(\alpha)} + \sum_{\alpha,\beta=1}^{3} (\lambda_\beta \Omega^E_{\alpha\beta} - \lambda_\alpha \Omega^L_{\alpha\beta}) \mathbf{n}^{(\alpha)} \otimes \mathbf{N}^{(\beta)} \tag{2.98}$$

Here, it holds that

$$\begin{aligned}
\dot{\mathbf{R}}\mathbf{R}^T &= \sum_{\alpha,\beta=1}^{3} (\mathbf{n}^{(\alpha)} \otimes \mathbf{N}^{(\alpha)})^{\bullet} (\mathbf{n}^{(\beta)} \otimes \mathbf{N}^{(\beta)})^T \\
&= \sum_{\alpha,\beta=1}^{3} (\dot{\mathbf{n}}^{(\alpha)} \otimes \mathbf{N}^{(\alpha)} + \mathbf{n}^{(\alpha)} \otimes \dot{\mathbf{N}}^{(\alpha)}) \mathbf{N}^{(\beta)} \otimes \mathbf{n}^{(\beta)} \\
&= \sum_{\alpha,\beta,\gamma=1}^{3} \left[\dot{\mathbf{n}}^{(\alpha)} \otimes \mathbf{n}^{(\alpha)} + \mathbf{n}^{(\alpha)} \otimes \mathbf{N}^{(\alpha)} (\dot{\mathbf{N}}^{(\gamma)} \otimes \dot{\mathbf{N}}^{(\gamma)}) \mathbf{N}^{(\beta)} \otimes \mathbf{n}^{(\beta)} \right] \\
&= \sum_{\alpha,\beta,\gamma=1}^{3} \left[\dot{\mathbf{n}}^{(\alpha)} \otimes \mathbf{n}^{(\alpha)} - \mathbf{n}^{(\alpha)} \otimes \mathbf{N}^{(\alpha)} (\dot{\mathbf{N}}^{(\gamma)} \otimes \mathbf{N}^{(\gamma)}) \mathbf{N}^{(\beta)} \otimes \mathbf{n}^{(\beta)} \right]
\end{aligned}$$

and thus the following relations hold.

$$\boxed{\mathbf{\Omega}^R = \mathbf{\Omega}^E - \mathbf{R}\mathbf{\Omega}^L\mathbf{R}^T,\ \mathbf{\Omega}^E = \mathbf{\Omega}^R + \mathbf{R}\mathbf{\Omega}^L\mathbf{R}^T, \mathbf{\Omega}^L = \mathbf{R}^T(\mathbf{\Omega}^E - \mathbf{\Omega}^R)\mathbf{R}} \tag{2.99}$$

The velocity gradient is described noting Eq. (2.27) as

$$\begin{aligned}\boldsymbol{l} &= \left(\sum_{\alpha=1}^{3}\lambda_\alpha\mathbf{n}^{(\alpha)}\otimes\mathbf{N}^{(\alpha)}\right)^{\bullet}\sum_{\beta=1}^{3}\frac{1}{\lambda_\beta}\mathbf{N}^{(\beta)}\otimes\mathbf{n}^{(\beta)}\\ &= \sum_{\alpha=1}^{3}\left(\dot{\lambda}_\alpha\mathbf{n}^{(\alpha)}\otimes\mathbf{N}^{(\alpha)}+\lambda_\alpha\dot{\mathbf{n}}^{(\alpha)}\otimes\mathbf{N}^{(\alpha)}+\lambda_\alpha\mathbf{n}^{(\alpha)}\otimes\dot{\mathbf{N}}^{(\alpha)}\right)\sum_{\beta=1}^{3}\frac{1}{\lambda_\beta}\mathbf{N}^{(\beta)}\otimes\mathbf{n}^{(\beta)}\\ &= \sum_{\alpha=1}^{3}\left(\frac{\dot{\lambda}_\alpha}{\lambda_\alpha}\mathbf{n}^{(\alpha)}\otimes\mathbf{n}^{(\alpha)}+\dot{\mathbf{n}}^{(\alpha)}\otimes\mathbf{n}^{(\alpha)}\right)+\sum_{\alpha,\beta=1}^{3}\frac{\lambda_\alpha}{\lambda_\beta}\left(\dot{\mathbf{N}}^{(\alpha)}\boldsymbol{\cdot}\mathbf{N}^{(\beta)}\right)\mathbf{n}^{(\alpha)}\otimes\mathbf{n}^{(\beta)}\end{aligned} \tag{2.100}$$

which is rewritten using Eqs. (2.92) and (2.95) as

$$\boldsymbol{l} = \sum_{\alpha=1}^{3}\frac{\dot{\lambda}_\alpha}{\lambda_\alpha}\mathbf{n}^{(\alpha)}\otimes\mathbf{n}^{(\alpha)}+\sum_{\alpha,\beta=1}^{3}\left(\Omega^E_{\alpha\beta}-\frac{\lambda_\alpha}{\lambda_\beta}\Omega^L_{\alpha\beta}\right)\mathbf{n}^{(\alpha)}\otimes\mathbf{n}^{(\beta)} \tag{2.101}$$

from which the strain rate and the continuum spin are represented as

$$\mathbf{d} = \sum_{\alpha=1}^{3}\frac{\dot{\lambda}_\alpha}{\lambda_\alpha}\mathbf{n}^{(\alpha)}\otimes\mathbf{n}^{(\alpha)}+\sum_{\alpha,\beta=1}^{3}\frac{\lambda_\alpha^2-\lambda_\beta^2}{2\lambda_\alpha\lambda_\beta}\Omega^L_{\alpha\beta}\mathbf{n}^{(\alpha)}\otimes\mathbf{n}^{(\beta)} \tag{2.102}$$

$$\mathbf{w} = \sum_{\alpha,\beta=1}^{3}\left(\Omega^E_{\alpha\beta}-\frac{\lambda_\alpha^2+\lambda_\beta^2}{2\lambda_\alpha\lambda_\beta}\Omega^L_{\alpha\beta}\right)\mathbf{n}^{(\alpha)}\otimes\mathbf{n}^{(\beta)}\ (\alpha\neq\beta) \tag{2.103}$$

noting

$$\mathbf{d} = \sum_{\alpha=1}^{3}\frac{\dot{\lambda}_\alpha}{\lambda_\alpha}\mathbf{n}^{(\alpha)}\otimes\mathbf{n}^{(\alpha)}+\frac{1}{2}\sum_{\alpha,\beta=1}^{3}\left[\frac{\lambda_\alpha}{\lambda_\beta}(\dot{\mathbf{N}}^{(\alpha)}\boldsymbol{\cdot}\mathbf{N}^{(\beta)})+\frac{\lambda_\beta}{\lambda_\alpha}(\dot{\mathbf{N}}^{(\beta)}\boldsymbol{\cdot}\mathbf{N}^{(\alpha)})\right]\mathbf{n}^{(\alpha)}\otimes\mathbf{n}^{(\beta)} \tag{2.104}$$

$$\begin{aligned}\mathbf{w} &= \frac{1}{2}\sum_{\alpha=1}^{3}(\dot{\mathbf{n}}^{(\alpha)}\otimes\mathbf{n}^{(\alpha)}-\mathbf{n}^{(\alpha)}\otimes\dot{\mathbf{n}}^{(\alpha)})\\ &\quad+\frac{1}{2}\sum_{\alpha,\beta=1}^{3}\left[\frac{\lambda_\alpha}{\lambda_\beta}(\dot{\mathbf{N}}^{(\alpha)}\boldsymbol{\cdot}\mathbf{N}^{(\beta)})-\frac{\lambda_\beta}{\lambda_\alpha}(\dot{\mathbf{N}}^{(\beta)}\boldsymbol{\cdot}\mathbf{N}^{(\alpha)})\right]\mathbf{n}^{(\alpha)}\otimes\mathbf{n}^{(\beta)}\end{aligned} \tag{2.105}$$

$$\begin{aligned}2\mathrm{sym}\left[\sum_{\alpha=1}^{3}\dot{\mathbf{n}}^{(\alpha)}\otimes\mathbf{n}^{(\alpha)}\right] &= \sum_{\alpha=1}^{3}(\dot{\mathbf{n}}^{(\alpha)}\otimes\mathbf{n}^{(\alpha)}+\mathbf{n}^{(\alpha)}\otimes\dot{\mathbf{n}}^{(\alpha)})\\ &= \left(\sum_{\alpha=1}^{3}\mathbf{n}^{(\alpha)}\otimes\mathbf{n}^{(\alpha)}\right)^{\bullet} = \dot{\mathbf{I}} = \mathbf{O}\end{aligned} \tag{2.106}$$

From the relation

$$d_{\alpha\beta} = \frac{\lambda_\alpha^2 - \lambda_\beta^2}{2\lambda_\alpha\lambda_\beta}\Omega_{\alpha\beta}^L \tag{2.107}$$

obtained from the anti-symmetric part in Eq. (2.102), the Lagrangian spin is written as

$$\Omega_{\alpha\beta}^L = \frac{2\lambda_\alpha\lambda_\beta}{\lambda_\beta^2 - \lambda_\alpha^2} d_{\alpha\beta}\ (\alpha \neq \beta) \tag{2.108}$$

The Eulerian spin is given from Eq. (2.103) as follows:

$$\Omega_{\alpha\beta}^E = w_{\alpha\beta} - \frac{\lambda_\alpha^2 + \lambda_\beta^2}{\lambda_\alpha^2 - \lambda_\beta^2} d_{\alpha\beta}\ (\alpha \neq \beta) \tag{2.109}$$

In the rigid-body rotation ($\dot{\mathbf{U}} = \dot{\bar{\mathbf{U}}} = \dot{\mathbf{V}} = \dot{\bar{\mathbf{V}}} = \mathbf{O}$, $\dot{\mathbf{N}}^{(\alpha)} = \mathbf{O}$), it follows from Eqs. (2.79), (2.83), (2.90) and (2.99) that

$$\boldsymbol{l} = \mathbf{w} = \boldsymbol{\Omega}^R = \boldsymbol{\Omega}^E,\ \boldsymbol{\Omega}^L = \mathbf{O} \tag{2.110}$$

In what follows, we consider the physical meanings of **d** and **w**.

The relative velocity of the particle points P and P′, the current position vectors of which are $\mathbf{x}$ and $\mathbf{x} + d\mathbf{x}$, respectively, is given by

$$d\mathbf{v} = \boldsymbol{l} d\mathbf{x} \tag{2.111}$$

from Eq. (2.76) and it is additively decomposed as

$$d\mathbf{v} = d\mathbf{v}^d + d\mathbf{v}^w \tag{2.112}$$

where

$$d\mathbf{v}^d \equiv \mathbf{d} d\mathbf{x} \tag{2.113}$$

$$d\mathbf{v}^w \equiv \mathbf{w} d\mathbf{x} \tag{2.114}$$

The following equation is obtained for the infinitesimal line-element $d\mathbf{x} = dx_i\mathbf{e}_i$ (no sum).

$$d_{ji} = \frac{dv_j^d}{dx_i}\ \text{(no sum)} \tag{2.115}$$

noting

$$d_{ji} = \mathbf{e}_j \cdot \mathbf{d}\mathbf{e}_i = \mathbf{e}_j \cdot \mathbf{d}\frac{d\mathbf{x}}{dx_i} = \mathbf{e}_j \cdot \frac{d\mathbf{v}^d}{dx_i} = \frac{dv_j^d}{dx_i} \tag{2.116}$$

with the aid of Eq. (2.113). Therefore, d_{ji} is the $\mathbf{e}_j-$component of the relative velocity $d\mathbf{v}^d$ of the unit line element $(dx_i = 1)$ in the $\mathbf{e}_i-$direction. Consequently, the infinitesimal line-element $d\mathbf{x} = dx_i\mathbf{e}_i$ (no sum) rotates in the velocity given by the tangential component $d_{ji}(j \neq i)$ of the strain rate $\mathbf{d}$.

On the other hand, denoting the axial vector described in Eq. (1.139) for the skew-symmetric tensor $\mathbf{w}$ by $\widehat{\boldsymbol{\omega}}$, it holds that

$$\widehat{\omega}_i = -\frac{1}{2}\varepsilon_{irs}w_{rs}, \quad w_{ij} = -\varepsilon_{ijr}\,\widehat{\omega}_r \tag{2.117}$$

and thus Eq. (2.114) is rewritten from Eq. (1.142) as

$$d\mathbf{v}^w = \widehat{\boldsymbol{\omega}} \times d\mathbf{x}, dv_i^w (= w_{is}dx_s = -\varepsilon_{isr}\widehat{\omega}_r dx_s) = \varepsilon_{irs}\widehat{\omega}_r dx_s \tag{2.118}$$

Therefore, the arbitrary line-element $d\mathbf{x}$ rotates in the peripheral velocity $d\mathbf{v}^w$ and angular velocity $\widehat{\boldsymbol{\omega}}$, termed often the *spin vector*, by the continuum spin $\mathbf{w}$, whereas $2\widehat{\boldsymbol{\omega}}$ is called the *vorticity*.

Noting Eqs. (2.112)–(2.118), it follows for the material line element $dx_1\mathbf{e}_1$ that

$$\begin{aligned} d\mathbf{v} &= l_{ij}\mathbf{e}_i \otimes \mathbf{e}_j dx_1\mathbf{e}_1 = l_{i1}dx_1\mathbf{e}_i = (d_{21} + w_{21})dx_1\mathbf{e}_2 + d_{11}dx_1\mathbf{e}_1 \\ &= (d_{12} - w_{12})dx_1\mathbf{e}_2 + d_{11}dx_1\mathbf{e}_1 = \varpi_3 dx_1\mathbf{e}_2 + d_{11}dx_1\mathbf{e}_1 \end{aligned} \tag{2.119}$$

as shown in Fig. 2.3, where we set

$$\varpi_3 \equiv -w_{12} + d_{12} = \widehat{\omega}_3 + d_{12} \tag{2.120}$$

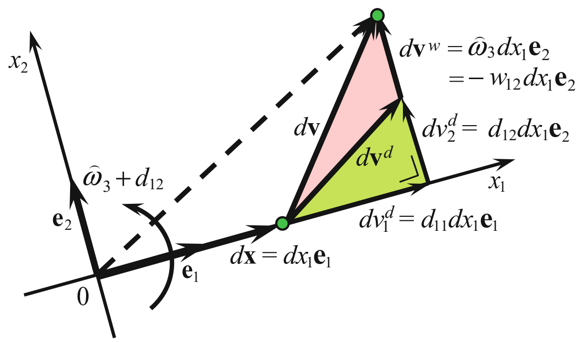

Fig. 2.3 Extension and rotation of the line-element

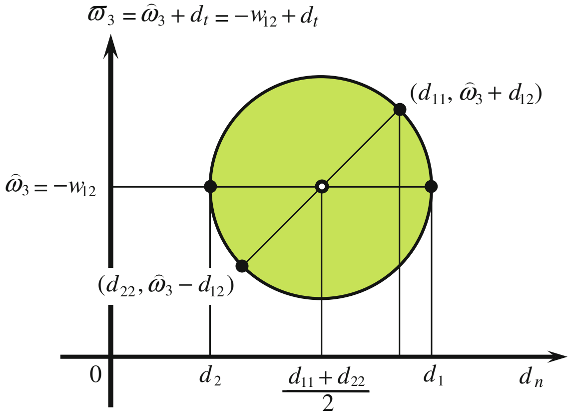

Fig. 2.4 Circle of relative velocity

ϖ_3 designates the clock-wise angular velocity of the line-element. $\widehat{\omega}_3$ denotes the average angular velocity of the line-elements in the plane, which coincides with the angular velocity of the line element in the principal directions of strain rate fulfilling $d_{12} = 0$. By choosing d_n, d_t for T_n, T_t described in Sect. 1.14, the relation of the rate of elongation and the rate of rotation is shown in Fig. 2.4. It is depicted by the *circle of relative velocity* with the radius $\sqrt{[(d_{11} + d_{22})/2]^2 + d_{12}^2}$ centering in $((d_{11} + d_{22})/2,\ \widehat{\omega}_3)$ in the two-dimensional plane $(d_n,\ \varpi_3)$.

The parallelepiped in the principal directions of the strain rate $\mathbf{d}$ rotates by the angular velocity $\widehat{\boldsymbol{\omega}}$ as shown in Fig. 2.5.

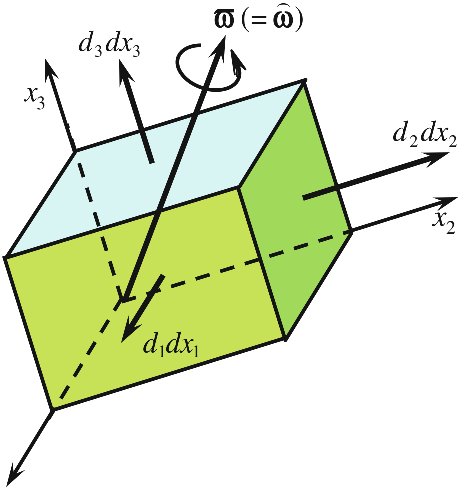

Fig. 2.5 Deformation and rotation for principal directions of strain rate

The rate of the scalar product of the vectors $d\mathbf{x}$ and $\delta\mathbf{x}$ of the infinitesimal elements connecting the three points P, P', P'' with the position vectors $\mathbf{x}, \mathbf{x}+d\mathbf{x},\ \mathbf{x}+\delta\mathbf{x}$, respectively, is given noting Eq. (1.116) as follows:

$$(d\mathbf{x}\cdot\delta\mathbf{x})^{\bullet} = d\mathbf{v}\cdot\delta\mathbf{x} + d\mathbf{x}\cdot\delta\mathbf{v}$$
$$= \frac{\partial\mathbf{v}}{\partial\mathbf{x}}d\mathbf{x}\cdot\delta\mathbf{x} + d\mathbf{x}\cdot\frac{\partial\mathbf{v}}{\partial\mathbf{x}}\delta\mathbf{x} = \left\{\left[\frac{\partial\mathbf{v}}{\partial\mathbf{x}} + \left(\frac{\partial\mathbf{v}}{\partial\mathbf{x}}\right)^{T}\right]d\mathbf{x}\right\}\cdot\delta\mathbf{x}$$

leading to

$$\boxed{(d\mathbf{x}\cdot\delta\mathbf{x})^{\bullet} = 2\mathbf{d}d\mathbf{x}\cdot\delta\mathbf{x}} \tag{2.121}$$

If the vicinity of the particle P undergoes the rigid-body rotation, the quantity in Eq. (2.121) for an arbitrary scalar quantity $d\mathbf{x}\cdot\delta\mathbf{x}$ is zero and thus $\mathbf{d}=\mathbf{O}$ has to hold. Inversely, if $\mathbf{d}=\mathbf{O}$, the quantity in Eq. (2.121) for the scalar quantity $d\mathbf{x}\cdot\delta\mathbf{x}$ of arbitrary line-element vectors becomes zero and thus it can be stated that the vicinity of the particle P does not undergo a deformation. Then, $\mathbf{d}=\mathbf{O}$ is the necessary and the sufficient condition for the situation that a deformation is not induced, allowing only a rigid-body rotation.

Denoting the lengths of the line-elements PP′ and PP″ as ds and δs, respectively, and the angle contained by them as θ, it holds that

$$(d\mathbf{x}\cdot\delta\mathbf{x})^{\bullet} = (ds\delta s\cos\theta)^{\bullet}$$
$$= \left\{\left[\frac{(ds)^{\bullet}}{ds} + \frac{(\delta s)^{\bullet}}{\delta s}\right]\cos\theta - \dot{\theta}\sin\theta\right\}ds\delta s \tag{2.122}$$

Further, denoting the unit vectors in the directions of the line-elements PP′ and PP″ as $\boldsymbol{\mu}$ and $\mathbf{v}$, respectively, and noting $d\mathbf{x}=\boldsymbol{\mu}ds$, $\delta\mathbf{x} = \mathbf{v}\delta s$, it holds from Eqs. (2.121) and (2.122) that

$$\left[\frac{(ds)^{\bullet}}{ds} + \frac{(\delta s)^{\bullet}}{\delta s}\right]\cos\theta - \dot{\theta}\sin\theta = 2\mathbf{d}\boldsymbol{\mu}\cdot\mathbf{v}\,(=2d_{ij}\mu_i v_j) \tag{2.123}$$

If the particles P′ and P″ chosen in same direction $(\theta=0)$, it follows from Eq. (2.123) that

$$\frac{(ds)^{\bullet}}{ds} = \mathbf{d}\boldsymbol{\mu}\cdot\boldsymbol{\mu} \tag{2.124}$$

The left-hand side of Eq. (2.124) designates the *rate of elongation* of the line-element. Therefore, the rate of elongation is given by the normal component of $\mathbf{d}$ in the relevant direction, noting Eq. (1.102).

On the other hand, choosing the line-element PP″ to be perpendicular to the line element PP′ $(\theta = \pi/2)$, it follows from Eq. (2.123) that

$$-\dot{\theta}\, 2\mathbf{d}\boldsymbol{\mu}\cdot\mathbf{v} \quad (\boldsymbol{\mu}\cdot\mathbf{v} = 0) \tag{2.125}$$

The left-hand side of Eq. (2.125) designates the decreasing rate of the angle contained by the two line-elements mutually perpendicular instantaneously and is called the *shear strain rate*.

Next, the relations of the rate $\dot{\boldsymbol{E}}$ of Green strain tensor $\boldsymbol{E}$ and the rate $\dot{\boldsymbol{e}}$ of the Almansi strain tensor $\boldsymbol{e}$ to the strain rate tensor $\mathbf{d}$ are formulated below.

The material-time derivative of Eq. (2.44) is given by

$$(d\mathbf{x}\cdot\delta\mathbf{x})^{\bullet} = 2\dot{\boldsymbol{E}}\, d\mathbf{X}\cdot\delta\mathbf{X} \tag{2.126}$$

It follows from Eqs. (2.121) and (2.126) that

$$\mathbf{d}d\mathbf{x}\cdot\delta\mathbf{x} = \mathbf{dF}d\mathbf{X}\cdot\mathbf{F}\delta\mathbf{X} = \dot{\boldsymbol{E}}\; d\mathbf{X}\cdot\delta\mathbf{X} \tag{2.127}$$

from which, noting Eq. (1.116), we have the relation of Green strain tensor to the strain rate tensor as follows:

$$\boxed{\dot{\boldsymbol{E}} = \mathbf{F}^T\mathbf{dF},\ \mathbf{d} = \mathbf{F}^{-T}\dot{\boldsymbol{E}}\,\mathbf{F}^{-1}} \tag{2.128}$$

which is obtained also from

$$\begin{aligned}
\dot{\boldsymbol{E}} &= \frac{1}{2}(\mathbf{F}^T\mathbf{F}-\mathbf{I})^{\bullet} = \frac{1}{2}(\dot{\mathbf{F}}^T\mathbf{F}+\mathbf{F}^T\dot{\mathbf{F}})\\
&= \frac{1}{2}\ [\mathbf{F}^T(\mathbf{F}^{-T}\dot{\mathbf{F}}^T)\mathbf{F}+\mathbf{F}^T(\dot{\mathbf{F}}\mathbf{F}^{-1})\mathbf{F}]\\
&= \frac{1}{2}(\mathbf{F}^T\boldsymbol{l}^T\mathbf{F}+\mathbf{F}^T\boldsymbol{l}\mathbf{F}) = \frac{1}{2}\mathbf{F}^T(\boldsymbol{l}^T+\boldsymbol{l})\mathbf{F}\\
&= \mathbf{F}^T\mathbf{dF}
\end{aligned} \tag{2.129}$$

Next, the time-differentiation of Eq. $(2.45)_2$ leads to

$$\dot{\boldsymbol{e}} = -\frac{1}{2}(\dot{\mathbf{F}}^{-T}\mathbf{F}^{-1}+\mathbf{F}^{-T}\dot{\mathbf{F}}^{-1}) \tag{2.130}$$

Here, it follows from $(\mathbf{F}\mathbf{F}^{-1})^{\bullet} = \mathbf{F}\dot{\mathbf{F}}^{-1}+\dot{\mathbf{F}}\mathbf{F}^{-1} = \mathbf{O}$ with Eq. (2.75) that

$$\boxed{\dot{\mathbf{F}}^{-1} = -\mathbf{F}^{-1}\boldsymbol{l}} \tag{2.131}$$

which is derived also by

$$
\begin{aligned}
\dot{\mathbf{F}}^{-1} &= \frac{\partial \mathbf{F}^{-1}}{\partial t} + \frac{\partial \mathbf{F}^{-1}}{\partial \mathbf{x}}\mathbf{v} = \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{X}}{\partial \mathbf{x}}\right) + \frac{\partial}{\partial \mathbf{x}}\left(\frac{\partial \mathbf{X}}{\partial \mathbf{x}}\right)\mathbf{v} \\
&= \frac{\partial}{\partial \mathbf{x}}\left(\frac{\partial \mathbf{X}}{\partial t}\right) + \frac{\partial}{\partial \mathbf{x}}\left(\frac{\partial \mathbf{X}}{\partial \mathbf{x}}\mathbf{v}\right) - \frac{\partial \mathbf{X}}{\partial \mathbf{x}}\frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{x}}\left(\frac{\partial \mathbf{X}}{\partial t} + \frac{\partial \mathbf{X}}{\partial \mathbf{x}}\mathbf{v}\right) - \frac{\partial \mathbf{X}}{\partial \mathbf{x}}\frac{\partial \mathbf{v}}{\partial \mathbf{x}} \\
&= \frac{\partial \dot{\mathbf{X}}}{\partial \mathbf{x}} - \frac{\partial \mathbf{X}}{\partial \mathbf{x}}\frac{\partial \mathbf{v}}{\partial \mathbf{x}} = -\frac{\partial \mathbf{X}}{\partial \mathbf{x}}\frac{\partial \mathbf{v}}{\partial \mathbf{x}}
\end{aligned}
\tag{2.132}
$$

noting that the inside of the bracket ( ) in the last side of Eq. (2.132) is the material-time derivative of the initial position vector $\mathbf{X}$ and thus it is zero. Substituting Eq. (2.131) into Eq. (2.130), one has

$$
\begin{aligned}
\dot{\boldsymbol{e}} &= \frac{1}{2}[(\mathbf{F}^{-1}\boldsymbol{l})^T\mathbf{F}^{-1} + \mathbf{F}^{-T}\mathbf{F}^{-1}\boldsymbol{l}] \\
&= \boldsymbol{l}^T\left[\frac{1}{2}\mathbf{I} - \frac{1}{2}(\mathbf{I} - \mathbf{F}^{-T}\mathbf{F}^{-1})\right] + \frac{1}{2}\left[\mathbf{I} - \left(\frac{1}{2}\mathbf{I} - \mathbf{F}^{-T}\mathbf{F}^{-1}\right)\right]\boldsymbol{l} \\
&= \boldsymbol{l}^T\left(\frac{1}{2}\mathbf{I} - \boldsymbol{e}\right) + \left(\frac{1}{2}\mathbf{I} - \boldsymbol{e}\right)\boldsymbol{l}
\end{aligned}
\tag{2.133}
$$

from which one has the relation of the rate of the Almansi strain tensor to the strain rate tensor:

$$
\boxed{\dot{\boldsymbol{e}} = \mathbf{d} - \boldsymbol{l}^T\boldsymbol{e} - \boldsymbol{e}\boldsymbol{l}}
\tag{2.134}
$$

Equation (2.134) is rewritten as

$$
\dot{\boldsymbol{e}} = \mathbf{d} - \frac{1}{2}[(\boldsymbol{l} + \boldsymbol{l}^T) - (\boldsymbol{l} - \boldsymbol{l}^T)]\boldsymbol{e} - \frac{1}{2}\boldsymbol{e}[(\boldsymbol{l} + \boldsymbol{l}^T) + (\boldsymbol{l} - \boldsymbol{l}^T)]
$$

and thus we obtain

$$
\dot{\boldsymbol{e}} - \mathbf{w}\boldsymbol{e} + \boldsymbol{e}\mathbf{w} = \mathbf{d} - \mathbf{d}\boldsymbol{e} - \boldsymbol{e}\mathbf{d}
\tag{2.135}
$$

where

$$
\boxed{\overset{\circ}{\boldsymbol{e}}{}^{w} \equiv \dot{\boldsymbol{e}} - \mathbf{w}\boldsymbol{e} + \boldsymbol{e}\mathbf{w}}
\tag{2.136}
$$

is called the *Zaremba-Jaumann rate of Almansi strain tensor*, while the Zaremba-Jaumann rate will be explained in Sect. 4.4. $\dot{\boldsymbol{E}} = \dot{\boldsymbol{e}} = \overset{\circ}{\boldsymbol{e}}{}^{w} = \mathbf{d}$ holds in the initial state ($\mathbf{F} = \mathbf{I}$, $\boldsymbol{E} = \boldsymbol{e} = \mathbf{O}$) and thus all the strain rates mutually coincide by Eqs. (2.128), (2.134) and (2.136).

2.6 Logarithmic and Nominal Strains

In what follows, let **d** and **d**dt be designated as $\dot{\boldsymbol{\varepsilon}}$ and $d\boldsymbol{\varepsilon}$, respectively.

If the direction of the material line-element always coincides with the x_i-axis, the principal strain rate in this direction is given by

$$d_i = \dot{\varepsilon}_i = \frac{\partial \dot{u}_i}{\partial x_i} = \frac{\partial \dot{u}_i}{\partial (X_i + u_i)} = \frac{\partial \dot{u}_i}{\partial X_i} \Big/ \left(1 + \frac{\partial u_i}{\partial X_i}\right) \text{ (no sum)} \tag{2.137}$$

The time-integration of Eq. (2.137) leads to

$$\boxed{\varepsilon_i = \ln\left(1 + \frac{\partial u_i}{\partial X_i}\right) = 1\mathrm{n}\frac{\partial x_i}{\partial X_i} = 1\mathrm{n}\lambda_i = E_i^{(0)} = e_i^{(0)} = 1\mathrm{n}(1 + \varepsilon_i) \text{ (no sum)}} \tag{2.138}$$

Therefore, the time-integration ε_i of principal strain rate d_i does not coincide with the principal infinitesimal strain ε_i in Eq. (2.50). Setting $\partial X_i \to l^0$, $\partial x_i \to l$, $\partial u_i \to l - l^0$, where l^0 and l are the lengths of the line-element in the initial and the current states, respectively, it follows that

$$\left.\begin{aligned} \varepsilon_i &= 1\mathrm{n}\frac{l}{l^0} = \ln(1 + \varepsilon_i) \\ \varepsilon_i &= \frac{l - l^0}{l^0} \end{aligned}\right\} \tag{2.139}$$

Consequently, the time-integration of principal component of strain rate tensor **d** and the Hencky strain tensor $\boldsymbol{E}^{(0)}$ and $\boldsymbol{e}^{(0)}$ coincide with ε_i which is called the *logarithmic* (or *natural*) *strain*, provided that their principal directions are fixed. On the other hand, the principal value of infinitesimal strain tensor $\boldsymbol{\varepsilon}$ does not coincide with them and it is called the *nominal strain*.

It follows from Eq. (2.139) that

$$\left.\begin{aligned} &\boxed{\varepsilon_i = \ln_e 2(\cong 0.693) \text{ for } l = 2l^0 \text{ and } \varepsilon_i = -\infty \text{ for } l = 0} \\ &\varepsilon_i = +1 \text{ for } l = 2l^0 \text{ and } \varepsilon_i = -1 \text{ for } l = 0 \end{aligned}\right\} \tag{2.140}$$

Therefore, the magnitude of nominal strain in the deformation that the material length becomes zero, i.e. the material diminishes is identical with that in the deformation that the material length becomes only twice. As a practical example, about 5 % difference is induced in the nominal strain for 10 % elongation as known from $\varepsilon_i/\varepsilon_i = 0.1/\ln(1.1) = 1.049$ for $l = 1.1 \times l_0$. This property would cause the inconvenience for the adoption in constitutive equation for the wide range of deformation.

Further, one has

$$\int_{l^0}^{l^n}\frac{dl}{l}=\int_{l^0}^{l^1}\frac{dl}{l}+\int_{l^1}^{l^2}\frac{dl}{l}+\cdots+\int_{l^{n-1}}^{l^n}\frac{dl}{l}$$

i.e.

$$1n\frac{l^n}{l^0}=1n\frac{l^1}{l^0}+1\mathrm{n}\frac{l^2}{l^1}+\cdots+1\mathrm{n}\frac{l^n}{l^{n-1}}$$

On the other hand, one sees

$$\frac{l^n-l^0}{l^0}\neq\frac{l^1-l^0}{l^0}+\frac{l^2-l^1}{l^1}+\cdots+\frac{l^n-l^{n-1}}{l^{n-1}}$$

Thus, it follows for the superposition of strains that

$$\left.\begin{array}{l}\boxed{\varepsilon_i^{0\sim n}=\varepsilon_i^{0\sim 1}+\varepsilon_i^{0\sim 2}+\cdots+\varepsilon_i^{n-1\sim n}}\\ \varepsilon_i^{0\sim n}\neq\varepsilon_i^{0\sim 1}+\varepsilon_i^{0\sim 2}+\cdots+\varepsilon_i^{n-1\sim n}\end{array}\right\}\tag{2.141}$$

while $\varepsilon_i^{a\sim b}$ designates the longitudinal strain in the x_i-direction when the length of the line-element changes from l^a to l^b, provided that the principal direction of strains are fixed. Consequently, the superposition rule holds in the logarithmic strain but it does not hold in the nominal strain.

Furthermore, one has

$$1\mathrm{n}\frac{l_1l_2l_3}{l_1^0l_2^0l_3^0}=1\mathrm{n}\frac{l_1}{l_1^0}+1\mathrm{n}\frac{l_2}{l_2^0}+1\mathrm{n}\frac{l_3}{l_3^0}$$

$$\frac{l_1l_2l_3-l_1^0l_2^0l_3^0}{l_1^0l_2^0l_3^0}\neq\frac{l_1-l_1^0}{l_1^0}+\frac{l_2-l_2^0}{l_2^0}+\frac{l_3-l_3^0}{l_3^0}$$

where l_1, l_2, l_3 are the lengths of line-elements in the directions of three fixed principal strains. Thus, it follows for the sum of the principal strains that

$$\left.\begin{array}{l}\boxed{\varepsilon_v=\ln\frac{v}{V}=\sum_{i=1}^{3}\varepsilon_i}=\ln(1+\varepsilon_v)\\ \varepsilon_v=\frac{v-V}{V}\neq\sum_{i=1}^{3}\varepsilon_i\end{array}\right\}\tag{2.142}$$

where V and v are the initial and the current volumes, respectively, of material. Therefore, the sum of logarithmic strains in orthogonal directions coincides with the logarithmic volumetric strain but the sum of nominal strains in orthogonal directions does not coincide with the nominal volumetric strain. Further, it follows from Eqs. (2.137) and (2.142) that

$$d_v \equiv \overset{\bullet}{\varepsilon}_v = \mathrm{tr}\mathbf{d} = \frac{\overset{\bullet}{v}}{v} \neq \overset{\bullet}{\varepsilon}_v = \frac{\overset{\bullet}{v}}{V} \tag{2.143}$$

Therefore, the volumetric strain rate tr**d** coincides with the material-time derivative of the logarithmic volumetric strain ε_v but it does not coincide with that of the nominal volumetric strain ε_v.

Consequently, the nominal strain is applicable only to the description of infinitesimal deformation, so that the logarithmic strain should be adopted to constitutive equations for finite deformation.

2.7 Surface Element, Volume Element and Their Rates

Presuming that the line-elements $d\mathbf{X}^a$, $d\mathbf{X}^b$, $d\mathbf{X}^c$ change to $d\mathbf{x}^a$, $d\mathbf{x}^b$, $d\mathbf{x}^c$ by the deformation, the following relation holds for the volume element before and after the deformation by exploiting Eq. (1.150)$_3$.

$$dv = [d\mathbf{x}^a\ d\mathbf{x}^b\ d\mathbf{x}^c] = [\mathbf{F}d\mathbf{X}^a\ \mathbf{F}d\mathbf{X}^b\ \mathbf{F}d\mathbf{X}^c] = \det\mathbf{F}[d\mathbf{X}^a\ d\mathbf{X}^b\ d\mathbf{X}^c] = \det\mathbf{F}dV \tag{2.144}$$

which can be also derived from Eqs. (1.16), (1.55), (1.56), (2.6) and (2.16).

$$\begin{aligned} dv &= (d\mathbf{x}^a \times d\mathbf{x}^b) \cdot d\mathbf{x}^c = \varepsilon_{ijk} dx_i^a dx_j^b dx_k^c \\ &= \begin{vmatrix} dx_1^a & dx_1^b & dx_1^c \\ dx_2^a & dx_2^b & dx_2^c \\ dx_3^a & dx_3^b & dx_3^c \end{vmatrix} = \begin{vmatrix} F_{1R}dX_R^a & F_{1R}dX_R^b & F_{1R}dX_R^c \\ F_{1R}dX_R^a & F_{2R}dX_R^b & F_{2R}dX_R^c \\ F_{1R}dX_R^c & F_{3R}dX_R^b & F_{3R}dX_R^c \end{vmatrix} \\ &= \left| \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} dX_1^a & dX_1^b & dX_1^c \\ dX_2^a & dX_2^b & dX_2^c \\ dX_3^a & dX_3^b & dX_3^c \end{bmatrix} \right| \\ &= \begin{vmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{vmatrix} \begin{vmatrix} dX_1^a & dX_1^b & dX_1^c \\ dX_2^a & dX_2^b & dX_2^c \\ dX_3^a & dX_3^b & dX_3^c \end{vmatrix} = JdV \end{aligned}$$

from which one has

$$\boxed{J = \det\mathbf{F} = \frac{dv}{dV} = \frac{\rho_0}{\rho}} \tag{2.145}$$

where ρ_0 and ρ are the initial and the current mass densities. On the other hand, denoting the areas and the unit normal vectors of the surface elements formed by the two line-elements $d\mathbf{X}^a, d\mathbf{X}^b$ and $d\mathbf{x}^a, d\mathbf{x}^b$ as dA , $\mathbf{N}$ and da, $\mathbf{n}$, respectively, we have

$$\left.\begin{array}{l} dV = [d\mathbf{X}^a\, d\mathbf{X}^b d\mathbf{X}^c] = (d\mathbf{X}^a \times d\mathbf{X}^b)\cdot d\mathbf{X}^c = \mathbf{N}dA\cdot d\mathbf{X}^c \\ dv = [d\mathbf{x}^a\, d\mathbf{x}^b\, d\mathbf{x}^c] = (d\mathbf{x}^a \times d\mathbf{x}^b)\cdot d\mathbf{x}^c = d\mathbf{x}^c\cdot\mathbf{n}da = \mathbf{F}d\mathbf{X}^c\cdot\mathbf{n}da = \mathbf{F}^T\mathbf{n}da\cdot d\mathbf{X}^c \end{array}\right\} \tag{2.146}$$

noting Eq. (1.116). The following *Nanson's formula* is derived from Eqs. (2.145) and (2.146).

$$\boxed{\mathbf{n}da = J\mathbf{F}^{-T}\mathbf{N}dA,\ \mathbf{N}dA = \frac{1}{J}\mathbf{F}^T\mathbf{n}da} \tag{2.147}$$

or

$$d\mathbf{a} = J\mathbf{F}^{-T}d\mathbf{A}, \quad d\mathbf{A} = \frac{1}{J}\mathbf{F}^T d\mathbf{a} \tag{2.148}$$

where

$$d\mathbf{a} \equiv \mathbf{n}da, \quad d\mathbf{A} \equiv \mathbf{N}dA \tag{2.149}$$

It follows from Eq. (2.145) noting Eq. (1.121) that

$$\dot{J} = (\det\mathbf{F})^{\bullet} = \mathrm{tr}\left(\frac{\partial\det\mathbf{F}}{\partial\mathbf{F}}\dot{\mathbf{F}}^T\right) = \mathrm{tr}[(\det\mathbf{F})\mathbf{F}^{-T}\dot{\mathbf{F}}^T] = \mathrm{tr}[J(\dot{\mathbf{F}}\mathbf{F}^{-1})^T] = J\mathrm{tr}\boldsymbol{l}$$

Then, the following relation holds for the rate of volume element, noting $\mathrm{tr}\boldsymbol{l} = \mathrm{tr}\mathbf{d}$.

$$\boxed{\dot{\varepsilon}_v = d_v = \mathrm{tr}\mathbf{d} = \frac{\partial v_r}{\partial x_r} = \frac{(dv)^{\bullet}}{dv} = \frac{\dot{J}}{J}}, \quad \text{i.e.} \quad (dv)^{\bullet} = \dot{J}dV = dv\mathrm{tr}\mathbf{d} \tag{2.150}$$

which was derived already in Eq. (2.143) exploiting the principal values under the fixed principal directions. Then, the time-integration of the volumetric strain rate leads to the logarithmic volumetric strain.

$$\varepsilon_v = \int \mathrm{tr}\mathbf{d}dt = \ln\frac{dv}{dV} \tag{2.151}$$

which was shown already in Eq. (2.142).

Moreover, it follows from Eq. (2.75) and the Nanson's formula (2.147) that

$$\begin{aligned} (\mathbf{n}da)^{\bullet} &= (J\mathbf{F}^{-T}\mathbf{N}dA)^{\bullet} = (\dot{J}\,\mathbf{F}^{-T} + J\dot{\mathbf{F}}^{-T})\mathbf{N}dA \\ &= [(\mathrm{tr}\mathbf{d})\mathbf{I} + \dot{\mathbf{F}}^{-T}\mathbf{F}^T]\mathbf{F}^{-T}J\mathbf{N}dA \\ &= [(\mathrm{tr}\mathbf{d})\mathbf{I} - \mathbf{F}^{-T}\dot{\mathbf{F}}^T]\mathbf{n}da \\ &= [(\mathrm{tr}\mathbf{d})\mathbf{I} - \boldsymbol{l}^T]\mathbf{n}da \end{aligned} \tag{2.152}$$

On the other hand, noting $\dot{\mathbf{n}} \cdot \mathbf{n} = 0$ because of $\mathbf{n} \cdot \mathbf{n} = 1$ for the unit vector $\mathbf{n}$, it follows that

$$(da)^{\bullet} = \mathbf{n} \cdot \mathbf{n}(da)^{\bullet} = \mathbf{n} \cdot [(\mathbf{n}da)^{\bullet} - \dot{\mathbf{n}}\, da] = \mathbf{n} \cdot (\mathbf{n}da)^{\bullet} \tag{2.153}$$

Substituting Eq. (2.152) into Eq. (2.153), one obtains the rate of the current infinitesimal area as follows:

$$(da)^{\bullet} = \mathbf{n} \cdot [(\mathrm{tr}\mathbf{d})\mathbf{I} - \boldsymbol{l}^{T}]\ \mathbf{n}da \tag{2.154}$$

or

$$\boxed{(da)^{\bullet} = (\mathrm{tr}\mathbf{d} - \mathbf{n} \cdot \mathbf{dn})da} \tag{2.155}$$

Further, it holds from Eqs. (2.152) and (2.155) that

$$\begin{aligned}\dot{\mathbf{n}}\, da &= (\mathbf{n}da)^{\bullet} - \mathbf{n}(da)^{\bullet} \\ &= [(\mathrm{tr}\mathbf{d})\mathbf{I} - \boldsymbol{l}^{T}]\mathbf{n}da - \mathbf{n}[(\mathrm{tr}\mathbf{d}) - \mathbf{n} \cdot \mathbf{dn}]da\end{aligned} \tag{2.156}$$

Then, the rate of the unit normal vector of the current surface element is given by

$$\boxed{\dot{\mathbf{n}} = [(\mathbf{n} \cdot \mathbf{dn})\mathbf{I} - \boldsymbol{l}^{T}]\mathbf{n}} \tag{2.157}$$

2.8 Material-Time Derivative of Volume Integration

Supposing that the zone of material occupying the volume v at the current moment $(t = t)$ changes to occupy the volume $v + \delta v$ after the infinitesimal time $(t = t + \delta t)$, the material-time-derivative of the volume integration $\int_v T(\mathbf{x}, t)dv$ of the physical quantity $T(\mathbf{x}, t)$ involved in the volume is given by the following equation.

$$\begin{aligned}\left(\int_v T(\mathbf{x}, t)dv\right)^{\bullet} &= \lim_{\delta t \to 0} \frac{1}{\delta t}\left[\int_{v+\delta v} T(\mathbf{x}, t+\delta t)dv - \int_v T(\mathbf{x}, t)dv\right] \\ &= \lim_{\delta t \to 0} \frac{1}{\delta t}\left[\int_v \{T(\mathbf{x}, t+\delta t) - T(\mathbf{x}, t)\}dv + \int_{\delta v} T(\mathbf{x}, t+\delta t)dv\right]\end{aligned} \tag{2.158}$$

The integration of the first term in the right-hand side in Eq. (2.158) is transformed as

$$\lim_{\delta t \to 0} \frac{1}{\delta t}\int_v [T(\mathbf{x}, t+\delta t) - T(\mathbf{x}, t)]dv = \int_v \frac{\partial T(\mathbf{x}, t)}{\partial t}dv \tag{2.159}$$

On the other hand, the second term in Eq. (2.158) describes the influence caused by the change of volume during the infinitesimal time increment. Here, the volume increment δv is given by subtracting the volume flowing out from the boundary of the zone from the volume flowing into the boundary, which is the sum of $dv (= \mathbf{v} \cdot \mathbf{n} da \delta t)$ over the whole boundary surface (Fig. 2.6). Therefore, substituting the Gauss' divergence theorem in Eq. (1.324) and ignoring the second-order infinitesimal quantity, the integration of the second term in the right-hand side of Eq. (2.158) is given by

$$\begin{aligned}\lim_{\delta t \to 0} \frac{1}{\delta t} \int_{\delta v} T(\mathbf{x}, t + \delta t) dv &\cong \lim_{\delta t \to 0} \frac{1}{\delta t} \int_{\delta v} T(\mathbf{x}, t) dv \\ &= \lim_{\delta t \to 0} \frac{1}{\delta t} \int_a T(\mathbf{x}, t) v_r n_r da \delta t = \int_a T(\mathbf{x}, t) v_r n_r da \\ &= \int_v \frac{\partial T(\mathbf{x}, t) v_r}{\partial x_r} dv = \int_v \frac{\partial T(\mathbf{x}, t)}{\partial x_r} v_r dv + \int_v T(\mathbf{x}, t) \frac{\partial v_r}{\partial x_r} dv\end{aligned}$$

The sum of the first term in the right-hand side in this equation and the Eq. (2.159) is equal to the material-time derivative of $T(\mathbf{x}, t)$ by virtue of Eq. (2.12). Then, Eq. (2.158) is given by

$$\boxed{\left(\int_v T(\mathbf{x}, t) dv \right)^{\bullet} = \int_v \left[\dot{T}(\mathbf{x}, t) + T(\mathbf{x}, t) \frac{\partial v_r}{\partial x_r} \right] dv = \int_v [\dot{T}(\mathbf{x}, t) + T(\mathbf{x}, t) \operatorname{div} \mathbf{v}] dv} \tag{2.160}$$

which is called the *Reynolds' transportation theorem*.

Equation (2.160) can be obtained also by the following simple manner.

$$\begin{aligned}\left(\int_v T(\mathbf{x}, t) dv \right)^{\bullet} &= \left(\int_V T(\mathbf{X}, t) J dV \right)^{\bullet} = \int_V (\dot{T}(\mathbf{X}, t) J + T(\mathbf{X}, t) \dot{J}) dV \\ &= \int_v \left(\dot{T}(\mathbf{x}, t) + T(\mathbf{x}, t) \frac{\partial v_r}{\partial x_r} \right) dv\end{aligned}$$

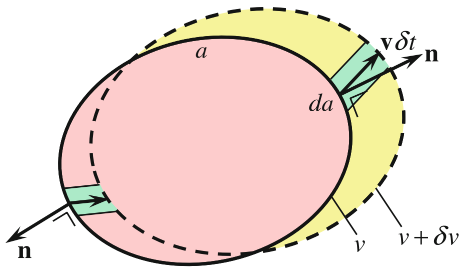

Fig. 2.6 Translation of a material in zone

where V is the initial volume. Here, Eq. (2.150), i.e. $J = dv/dV$ and $\dot{J}/J = \partial v_r/\partial x_r$ hold.

For the physical quantity T kept constant in a volume element, say a mass, Eq. (2.160) leads to

$$\int_v (\dot{T}(\mathbf{x}, t) + T(\mathbf{x}, t)div\mathbf{v})dv = 0 \tag{2.161}$$

The local (weak) form of Eq. (2.161) is given as

$$\boxed{\dot{T}(\mathbf{x}, t) + T(\mathbf{x}, t)div\mathbf{v} = 0} \tag{2.162}$$

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