Chapter 2
Mathematical Background

This chapter presents some fundamental mathematical knowledge and basic results which facilitate the analysis and design in the subsequent chapters. The motivation is to help readers understand the theoretical work presented in this book.

2.1 Lipschitz Function

This section will present the well-known Lipschitz condition and the generalised Lipschitz condition.

2.1.1 Lipschitz Condition

Definition 2.1 A function $f(x) : \mathbb{R}^n \mapsto \mathbb{R}^m$ is said to satisfy the *Lipschitz condition* in the domain $\Omega \subset \mathbb{R}^n$ if there exists a nonnegative constant L such that the inequality

$$\| f(x) - f(\hat{x})\| \le L\|x - \hat{x}\| \tag{2.1}$$

holds for any $x \in \Omega$ and $\hat{x} \in \Omega$. Then L is called the *Lipschitz constant* and $f(x)$ is called a *Lipschitz function* in Ω. If $\Omega = \mathbb{R}^n$, then $f(x)$ is said to satisfy the *global Lipschitz condition.*

From Definition 2.1, it is clear that a Lipschitz function must be continuous. However, the converse is not true and a typical example is the scalar function

$$f(x) = x^{1/3}$$

X.-G. Yan et al., *Variable Structure Control of Complex Systems*,
Communications and Control Engineering, DOI 10.1007/978-3-319-48962-9_2

in a neighbourhood of the origin $x = 0$. A Lipschitz function may not be differentiable and a simple example is the scalar function

$$f(x) = |x|$$

at the origin $x = 0$ in $x \in \mathbb{R}$. Moreover, a differentiable function may not be Lipschitz on a compact set, for example the function

$$f(x) = \begin{cases} x^{\alpha} \sin \dfrac{1}{x}, & 0 < x \leq 1 \\ 0, & x = 0 \end{cases} \tag{2.2}$$

is not Lipschitz in the compact set $x \in [0,\ 1]$ for any constant α satisfying $1 < \alpha < 2$. The reason is that the derivative of the function $f(x)$ defined in (2.2) is not bounded in the interval $[0,\ 1]$.

Lemma 2.1 ([91]) *Consider a function $f(x) : \mathbb{R}^n \mapsto \mathbb{R}^m$ which is differentiable in the domain Ω. If its Jacobian matrix is bounded in Ω, that is, there exists a constant L such that*

$$\|J_f\| \leq L$$

for any $x \in \Omega$, then $f(x)$ satisfies the Lipschitz condition, and the inequality (2.1) holds.

2.1.2 Generalised Lipschitz Condition

The well-known Lipschitz condition in Sect. 2.1.1 will be extended to a more general case which will be used later in the analysis.

Definition 2.2 A function $f(x_1, x_2, x_3) : \Omega_1 \times \Omega_2 \times \Omega_3 \mapsto \mathbb{R}^n$ is said to satisfy a generalised Lipschitz condition with respect to (w.r.t.) the variables $x_1 \in \Omega_1 \subset \mathbb{R}^{n_1}$ and $x_2 \in \Omega_2 \subset \mathbb{R}^{n_2}$ uniformly for x_3 in $\Omega_3 \subset \mathbb{R}^{n_3}$ if there exist nonnegative continuous functions $\mathscr{L}_{f1}(\cdot)$ and $\mathscr{L}_{f2}(\cdot)$ defined in Ω_3 such that for any $\hat{x}_1, x_1 \in \Omega_1$ and $\hat{x}_2, x_2 \in \Omega_2$, the inequality

$$\left\| f(x_1, x_2, x_3) - f(\hat{x}_1, \hat{x}_2, x_3) \right\| \leq \mathscr{L}_{f1}(x_3) \left\| x_1 - \hat{x}_1 \right\| + \mathscr{L}_{f2}(x_3) \left\| x_2 - \hat{x}_2 \right\|$$

holds for any $x_3 \in \Omega_3$. Then, $f(\cdot)$ is called a generalised Lipschitz function, and $\mathscr{L}_{f1}(\cdot)$ and $\mathscr{L}_{f2}(\cdot)$ are called generalised Lipschitz bounds. Further, if $\Omega_1 = \mathbb{R}^{n_1}$ and $\Omega_2 = \mathbb{R}^{n_2}$, then, it is said that $f(\cdot)$ satisfies a global generalised Lipschitz condition w.r.t. x_1 and x_2 uniformly for x_3 in Ω_3.

Remark 2.1 The symbols $\mathscr{L}_{f1}(\cdot)$ and $\mathscr{L}_{f2}(\cdot)$ introduced above are usually nonnegative functions instead of constants. This is different from the Lipschitz condition.

Thus, the nonnegative continuous functions $\mathscr{L}_{f1}(x_3)$ and $\mathscr{L}_{f2}(x_3)$ are called generalised Lipschitz bounds which correspond to the Lipschitz constant for the Lipschitz condition.

Clearly, the generalised Lipschitz condition is more relaxed than the Lipschitz condition. For example, the function

$$f(x_1, x_2, x_3) := x_1 x_3^2 + x_2 x_3$$

with $x_1, x_2, x_3 \in \mathbb{R}$ does not satisfy the global Lipschitz condition. However, from the inequality that for any $\mathrm{col}(x_1, x_2, x_3) \in \mathbb{R}^3$ and $\mathrm{col}(\hat{x}_1, \hat{x}_2, x_3) \in \mathbb{R}^3$

$$|f(x_1, x_2, x_3) - f\left(\hat{x}_1, \hat{x}_2, x_3\right)| \le |x_1 - \hat{x}_1| x_3^2 + |x_2 - \hat{x}_2|\,|x_3|$$

it is clear to see that $f(\cdot)$ satisfies the global generalised Lipschitz condition w.r.t. x_1 and x_2, uniformly for $x_3 \in \mathbb{R}$.

2.2 Comparison Functions

This section will present the definitions and properties of the class $\mathscr{K}$ function and related functions.

Definition 2.3 (*see [91]*) A continuous function $\alpha : [0, a) \mapsto \mathbb{R}^+$ is said to belong to class $\mathscr{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathscr{K}_\infty$ if $a = \infty$ and $\lim_{r\to\infty} \alpha(r) = \infty$.

Definition 2.4 (*see [91]*) A continuous function $\beta : [0, a) \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ is said to belong to class $\mathscr{K}\mathscr{L}$ if, for any given $s \in \mathbb{R}^+$, the mapping $\beta(r, s)$ belongs to class $\mathscr{K}$ with respect to the variable r, and for any given $r \in [0, a)$, the mapping $\beta(r, s)$ is decreasing with respect to the variable s and $\lim_{s\to\infty} \beta(r, s) = 0$.

Definition 2.5 If a class $\mathscr{K}$ function is a C^1 function, then it is said to belong to class $\mathscr{K}C^1$. A continuous function $\beta : \mathbb{R}^n \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ is said to be a class $\mathscr{K}I$ function if for any given $x \in \mathbb{R}^n$ the function $\beta(x, s)$ is increasing with respect to the variable s in $\mathbb{R}^+$, that is, $\beta(x, s_1) \le \beta(x, s_2)$ for any $0 \le s_1 \le s_2$.

The functions defined in Definitions 2.3 and 2.4 above are directly from [91]. The new concepts of class $\mathscr{K}C^1$ functions and class $\mathscr{K}I$ functions introduced in Definition 2.5 will be used for later analysis.

The following new concept is introduced, which will be termed as a class $\mathscr{W}\mathscr{S}$ function and will be used in Sect. 7.3.

Definition 2.6 A continuous function $\beta(t, x_1, x_2) : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ with $\beta(t, 0, 0) = 0$ is said to be weak w.r.t the variable x_1 and strong w.r.t. the variable x_2 if there exist functions $\chi_1(t, x_1, x_2)$ and $\chi_2(t, x_1, x_2)$ such that

$$\beta(t, x_1, x_2) = \chi_1(t, x_1, x_2)x_1 + \chi_2(t, x_1, x_2)x_2, \tag{2.3}$$

where both $\chi_1(\cdot, \cdot, x_2)$ and $\chi_2(\cdot, \cdot, x_2)$ are continuous and nondecreasing w.r.t. the variable x_2. Further, the function $\beta(t, x_1, x_2)$ is said to be a class $\mathscr{W}\mathscr{S}$ function w.r.t. the variables x_1 and x_2.

Remark 2.2 It should be noted that if a function $\beta(t, x_1, x_2) : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ with $\beta(t, 0, 0) = 0$ is smooth enough, then it follows from [3] that there exist continuous functions $\beta_1(\cdot)$ and $\beta_2(\cdot)$ such that the expression

$$\beta(t, x_1, x_2) = \beta_1(t, x_1, x_2)x_1 + \beta_2(t, x_1, x_2)x_2$$

holds. Moreover, if $\beta_1(t, x_1, x_2)$ and $\beta_2(t, x_1, x_2)$ are nondecreasing w.r.t. x_2, then $\beta(t, x_1, x_2)$ is a class $\mathscr{W}\mathscr{S}$ function w.r.t. x_1 and x_2.

Lemma 2.2 (see [91]) *Assume that $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are class $\mathscr{K}$ functions in* $[0,\ a)$, *$\alpha_3(\cdot)$ and $\alpha_4(\cdot)$ are class $\mathscr{K}_\infty$ functions, and $\beta(\cdot)$ is a class $\mathscr{K}\mathscr{L}$ function defined in* $[0,\ a) \times \mathbb{R}^+$. *Then, the following results hold:*

- *the inverse function $\alpha_1^{-1}(\cdot)$ is a class $\mathscr{K}$ function defined in* $[0,\ \alpha_1(a))$.
- *the inverse function $\alpha_3^{-1}(\cdot)$ is a class $\mathscr{K}_\infty$ function defined in* $[0,\ \infty)$.
- *the composite function $\alpha_1 \circ \alpha_2$ is a class $\mathscr{K}$ function.*
- *the composite function $\alpha_3 \circ \alpha_4$ is a class $\mathscr{K}_\infty$ function.*
- *the function $\sigma(r, s) = \alpha_1\left(\beta(\alpha_2(r), s)\right)$ is a class $\mathscr{K}\mathscr{L}$ function.*

Lemma 2.3 *The following results hold:*

(i) If $\beta(x, s) : \mathbb{R}^n \times R^+ \mapsto \mathbb{R}^+$ is a class $\mathscr{K}I$ function, then $\beta^2(x, s)$ is a class $\mathscr{K}I$ function.

(ii) Suppose that a function $\phi_1 : [0, a) \mapsto \mathbb{R}^+$ is a C^1 function with $\phi_1(0) = 0$. Then there exists a continuous function $\phi_2(\cdot)$ in $[0, a)$ such that

$$\phi_1(s) = \phi_2(s)s, \qquad s \in [0, a)$$

Proof (i) Suppose that $\beta(x, s) : \mathbb{R}^n \times R^+ \mapsto \mathbb{R}^+$ is a class $\mathscr{K}I$ function. Then for any $0 \le s_1 \le s_2$ and $x \in \mathbb{R}^n$,

$$\beta(x, s_1) \le \beta(x, s_2)$$

Since $\beta(x, s) \ge 0$ for any $(x, s) \in \mathbb{R}^n \times \mathbb{R}^+$

$$\begin{aligned} &\beta^2(x, s_1) - \beta^2(x, s_2) \\ &= (\beta(x, s_1) + \beta(x, s_2))(\beta(x, s_1) - \beta(x, s_2)) \\ &\le 0 \end{aligned}$$

This shows that $\beta^2(x, s)$ is a class $\mathscr{K}I$ function

(ii) Since the function $\phi_1(\cdot)$ is a C^1 function in $[0, a)$, its derivative $\frac{d\phi_1(s)}{ds}$ is continuous in $[0, a)$. For any $s \in [0, a)$, construct a function

$$\phi_2(s) := \begin{cases} \dfrac{\phi_1(s)}{s}, & s \neq 0 \\ \dfrac{d\phi_1(s)}{ds}\Big|_{s=0}, & s = 0 \end{cases} \tag{2.4}$$

From the definition of $\phi_2(\cdot)$, it is clear to see that

(1) if $s \neq 0$, then $\phi_1(s) = \phi_2(s)s$;
(2) if $s = 0$, then from $\phi_1(0) = 0$, $\phi_1(s) = \phi_2(s)s$.

Therefore, the expression

$$\phi_1(s) = \phi_2(s)s$$

holds for $s \in [0, a)$. It remains to prove that the function $\phi_2(\cdot)$ defined in (2.4) is continuous in $[0, a)$.

It is clear that $\phi_2(s)$ is continuous in $(0, a)$. Since ϕ_1 is a C^1 function in $[0, a)$, from the continuity of $\frac{d\phi_1(s)}{ds}$ at $s = 0$,

$$\lim_{s \to 0^+} \phi_2(s) = \lim_{s \to 0^+} \frac{\phi_1(s)}{s} = \frac{d\phi_1(s)}{ds}\Big|_{s=0} = \phi_2(0)$$

which implies that $\phi_2(\cdot)$ is continuous at $s = 0$. Therefore, $\phi_2(\cdot)$ is continuous in $[0, a)$.

Hence the conclusion follows. ∇

2.3 Lyapunov Stability Theorems

The results given in this section are available in [91].

Consider the nonlinear system

$$\dot{x}(t) = f(t, x(t)), \tag{2.5}$$

where the function $f : \mathbb{R}^+ \times D \mapsto \mathbb{R}^n$ is continuous and $D \subset \mathbb{R}^n$ is a domain which contains the origin $x = 0$. It is assumed that

$$f(t, 0) = 0, \qquad t \in \mathbb{R}^+$$

which implies that the origin is an equilibrium point of the system.

Definition 2.7 The equilibrium point $x = 0$ of System (2.5) is called exponentially stable if there exist positive constants c_i for $i = 1, 2, 3$ such that for any $x(t_0)$ satisfying $\|x(t_0)\| \leq c_1$,

$$\|x(t)\| \leq c_2 \|x(t_0)\| e^{-c_3(t-t_0)} \tag{2.6}$$

If Inequality (2.6) holds for any $x(t_0) \in \mathbb{R}^n$, then, the equilibrium point $x = 0$ of System (2.5) is called globally exponentially stable.

2.3.1 Asymptotic Stability

Theorem 2.1 *Consider System (2.5). Let $V : \mathbb{R}^+ \times D \mapsto \mathbb{R}^+$ be a continuously differentiable function such that*

$$\begin{aligned} W_1(x) \leq V(t, x) &\leq W_2(x) \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) &\leq -W_3(x) \end{aligned}$$

for any $t \in \mathbb{R}^+$ and $x \in D$, where $W_i(x)$ for $i = 1, 2, 3$ are continuous positive definite functions in D. Then $x = 0$ is uniformly asymptotically stable. Further if $D = \mathbb{R}^n$, and $w(x)$ is radially unbounded, then $x = 0$ is globally uniformly asymptotically stable.

2.3.2 Exponential Stability

Theorem 2.2 *Consider System (2.5). Let $V : \mathbb{R}^+ \times D \mapsto \mathbb{R}^+$ be a continuously differentiable function such that for $t \in \mathbb{R}^+$ and $x \in D$,*

$$\begin{aligned} k_1 \|x\|^a \leq V(t, x) &\leq k_2 \|x\|^a \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) &\leq -k_3 \|x\|^a, \end{aligned}$$

where k_i for $i = 1, 2, 3$ and a are positive constants. Then $x = 0$ is exponentially stable. Further if $D = \mathbb{R}^n$, then $x = 0$ is global exponentially stable.

Comparing Theorems 2.1 and 2.2 above, it is straightforward to see that exponential stability implies uniform asymptotic stability.

2.3.3 Converse Lyapunov Theorem

The following result is the well-known converse Lyapunov theorem.

Theorem 2.3 *Consider System (2.5) in domain $D := \mathscr{B}_r = \{x \in \mathbb{R}^n \mid \|x\| < r\}$. Let $\beta(\cdot)$ be a class $\mathscr{K}\mathscr{L}$ function and r_0 be a positive constant such that*

$$\beta(r_0, 0) < r \quad \text{and} \quad \mathscr{B}_{r_0} := \{x \mid \|x\| < r_0\}$$

Assume that the Jacobian matrix $\frac{\partial f}{\partial x}$ is bounded[1] *in domain D uniformly for $t \in \mathbb{R}^+$, and that the trajectory of System (2.5) satisfies*

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \qquad x(t_0) \in \mathscr{B}_{r_0}, \quad t \geq t_0 \geq 0$$

Then, there exists a continuously differentiable function $V : \mathbb{R}^+ \times \mathscr{B}_{r_0} \mapsto \mathbb{R}^+$ such that

$$\begin{aligned} \alpha_1(\|x\|) \leq V(t, x) &\leq \alpha_2(\|x\|) \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) &\leq -\alpha_3(\|x\|) \\ \left\| \frac{\partial V}{\partial x} \right\| &\leq \alpha_4(\|x\|), \end{aligned}$$

where α_i for $i = 1, 2, 3, 4$ are class $\mathscr{K}$ functions defined on the interval $[0, r_0]$. The function $V(\cdot)$ can be chosen independent of time t if $f(\cdot)$ in System (2.5) is independent of the time t.

2.4 Uniformly Ultimate Boundedness

For a given System (2.5), if asymptotic stability is not possible, uniform ultimate bounded stability can be considered. This is very useful in practical cases.

Theorem 2.4 *Consider System (2.5). Let $V : \mathbb{R}^+ \times D \mapsto \mathbb{R}^+$ be a continuously differentiable function such that in $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^n$,*

$$\begin{aligned} \alpha_1(\|x\|) \leq V(t, x) &\leq \alpha_2(\|x\|) \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) &\leq -W_3(x), \quad \text{for any} \quad \|x\| \geq \mu > 0, \end{aligned}$$

where $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are class $\mathscr{K}$ functions and $W_3(\cdot)$ is a continuous positive definite function in domain D. Then $x = 0$ is uniformly ultimately bounded.[2] *Further if $D = \mathbb{R}^n$, and $\alpha_1(\cdot)$ belongs to class $\mathscr{K}_\infty$, then $x = 0$ is globally uniformly ultimately bounded.*

Proof See the reference [91] (Theorem 4.18, p. 172). #

From Theorem 2.4, the following result is ready to be presented:

[1] If the function $f(\cdot)$ in (2.5) is continuously differentiable in the ball $\overline{\mathscr{B}}_r$, then $\frac{\partial f}{\partial x}$ is bounded in the domain $D = \mathscr{B}_r$.

[2] The ultimate bound depends on the parameters μ, which can be estimated using the result given in Theorem 4.18 in [91].

Lemma 2.4 *Consider the nonlinear system*

$$\dot{x} = \omega(x), \tag{2.7}$$

where $x \in \mathbb{R}^n$ is the system state, and the function $\omega(\cdot)$ is continuous in $\mathbb{R}^n$. Let $\mathscr{V} : \mathbb{R}^n \mapsto \mathbb{R}^+$ be a continuously differentiable class $\mathscr{K}_\infty$ function of $\|x\|$ such that the inequality

$$\frac{\partial \mathscr{V}}{\partial x}\omega(x) \leq -\vartheta(\|x\|), \qquad x \in \mathbb{R}^n \setminus \mathscr{B}_\mu \tag{2.8}$$

holds for some domain $\mathscr{B}_\mu$, where ϑ is a class $\mathscr{K}$ function, and μ is a positive constant. Then, the trajectory of System (2.7) enters into the domain $\mathscr{B}_\mu$ in finite time.

Proof From the condition of Lemma 2.4, there exists a class $\mathscr{K}_\infty$ function $\vartheta_1(\cdot)$ such that

$$\mathscr{V}(x) = \vartheta_1(\|x\|). \tag{2.9}$$

Then, from (2.8), (2.9) and using Theorem 2.4, the trajectory of System (2.7) is driven to the domain $\overline{\mathscr{B}}_\mu$ in a finite time, and remains there. That means there exists t_1 such that $x \in \overline{\mathscr{B}}_\mu$ for $t \geq t_1$.

The aim now is to prove that the trajectory of System (2.7) enters into $\mathscr{B}_\mu$ in a finite time. Suppose for a contradiction that this is not the case, then there exists some time t_2 such that the solution $x(x_0, t)$ of System (2.7) starting from some point x_0 satisfies $x(x_0, t) \in \partial\overline{\mathscr{B}}_\mu$ after t_2. This is equivalent to

$$\|x(x_0, t)\| = \mu, \qquad\qquad t \geq t_2. \tag{2.10}$$

By (2.9) and (2.10), it follows that

$$\mathscr{V}(x(x_0, t)) = \vartheta_1(\|x(x_0, t)\|) = \vartheta_1(\mu), \qquad\qquad t \geq t_2, \tag{2.11}$$

where μ is a positive constant. This shows that $\dot{\mathscr{V}}\,|_{(2.7)} \equiv 0$ after t_2, and it contradicts (2.8). Hence, the conclusion follows. #

Remark 2.3 Lemma 2.4 demonstrates that the solution enters the open set $\mathscr{B}_\mu$ in finite time and remains on $\overline{\mathscr{B}}_\mu$. It does not claim that the solution subsequently remains in $\mathscr{B}_\mu$.

2.5 Razumikhin Theorem

Consider a time-delay system

$$\dot{x}(t) = f(t, x(t - d(t)) \tag{2.12}$$

with an initial condition

$$x(t) = \phi(t), \qquad t \in [-\overline{d}, 0],$$

where the function vector $f : \mathbb{R}^+ \times \mathscr{C}_{[-\overline{d},0]} \mapsto \mathbb{R}^n$ takes $\mathbb{R} \times$ (bounded sets of $\mathscr{C}_{[-\overline{d},0]}$) into bounded sets in $\mathbb{R}^n$; $d(t) > 0$ is the time-delay and

$$\overline{d} := \sup_{t \in \mathbb{R}^+} \{d(t)\} < \infty$$

which implies that the time-delay $d(t)$ has a finite upper bound in $t \in \mathbb{R}^+$.

Theorem 2.5 (Razumikhin Theorem) *If there exist class $\mathscr{K}_\infty$ functions $\zeta_1(\cdot)$ and $\zeta_2(\cdot)$, a class $\mathscr{K}$ function $\zeta_3(\cdot)$ and a continuous function $V_1(\cdot) : [-\overline{d}, \infty] \times \mathbb{R}^n \mapsto \mathbb{R}^+$ satisfying*

$$\zeta_1(\|x\|) \le V_1(t, x) \le \zeta_2(\|x\|), \qquad t \in \mathbb{R}^+, \quad x \in \mathbb{R}^n$$

such that the time derivative of V_1 along the solution of System (2.12) satisfies

$$\dot{V}_1(t, x) \le -\zeta_3(\|x\|) \quad \text{if} \quad V_1(t - d, x(t - d)) \le V_1(t, x(t)) \tag{2.13}$$

for any $d \in [0, \overline{d}]$, then the System (2.12) is uniformly stable. If in addition, $\zeta_3(\tau) > 0$ for $\tau > 0$ and there exists a continuous nondecreasing function $\xi(\tau) > \tau$ for $\tau > 0$ such that (2.13) is strengthened to

$$\dot{V}_1(t, x) \le -\zeta_3(\|x\|) \quad \text{if} \quad V_1(t - d, x(t - d)) \le \xi(V_1(t, x(t))) \tag{2.14}$$

for $d \in [0, \overline{d}]$, then the System (2.12) is uniformly asymptotic stable.

Proof See pages 14–15 in [65]. ∇

From the Razumikhin Theorem 2.5, the following conclusion can be obtained directly:

Lemma 2.5 *Consider the time-delay system (2.12). If there exist constants $\gamma > 0$ and $\zeta > 1$ and a function*

$$V_2(x(t)) = x^T \tilde{P} x$$

with $\tilde{P} > 0$ such that the time derivative of $V_2(\cdot)$ along the solution of System (2.12) satisfies

$$\dot{V}_2 \,|_{(2.12)} \le -\gamma \left\| \tilde{P}^{\frac{1}{2}} x(t) \right\|^2 \tag{2.15}$$

whenever

$$\|\tilde{P}^{\frac{1}{2}} x(t + \theta)\| \le \zeta \|\tilde{P}^{\frac{1}{2}} x(t)\|$$

for any $\theta \in [-\overline{d}, 0]$, then, System (2.12) is uniformly asymptotic stable.

Proof From the definition of $V_2(\cdot)$ it follows that

$$\lambda_{\min}(\tilde{P})\|x\|^2 \leq V_2(t, x(t)) \leq \lambda_{\max}(\tilde{P})\|x\|^2$$

and from (2.15)

$$\dot{V}_2 \mid_{(2.12)} \leq -\gamma x(t)^T \tilde{P} x(t) \leq -\gamma \lambda_{\max}(\tilde{P})\|x\|^2.$$

It is clear that the inequality

$$V_2(x(t+\theta)) \leq \zeta^2 V_2(x(t))$$

is equivalent to the inequality

$$\|\tilde{P}^{\frac{1}{2}} x(t+\theta)\| \leq \zeta \|\tilde{P}^{\frac{1}{2}} x(t)\|$$

Then, from Razumikhin Theorem 2.5 and $\tilde{P} > 0$, the conclusion follows by letting

$$\begin{array}{ll} \gamma_1(\tau) = \lambda_{\min}(\tilde{P})\tau^2, & \gamma_2(\tau) = \lambda_{\max}(\tilde{P})\tau^2 \\ \gamma_3(\tau) = \gamma\lambda_{\min}(\tilde{P})\tau^2, \ \gamma_4(\tau) = \zeta^2\tau & \end{array}$$

in Theorem 2.5. #

2.6 Output Sliding Surface Design

In order to form an output feedback sliding mode control scheme, it is usually required that the designed switching function is a function of the system outputs. The corresponding sliding surface is called *an output sliding surface* in this book. The output sliding surface algorithm proposed in [37, 38] is outlined here, and this will be frequently used in the sequel.

Consider initially a linear system

$$\dot{x} = Ax + Bu \tag{2.16}$$

$$y = Cx, \tag{2.17}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ are the states, inputs and outputs, respectively, and assume $m \leq p < n$. The triple (A, B, C) comprises constant matrices of appropriate dimensions with B and C both being of full rank.

For System (2.16) and (2.17), it is assumed that

$$\text{rank}(CB) = m$$

Then, from [37] it can be shown that a coordinate transformation $\widetilde{x} = \widetilde{T}x$ exists such that the system triple (A, B, C) with respect to the new coordinate $\widetilde{x}$ has the following structure

$$\begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix}, \quad \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad \begin{bmatrix} 0 & \breve{T} \end{bmatrix}, \tag{2.18}$$

where $\widetilde{A}_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$, $B_2 \in \mathbb{R}^{m\times m}$ and $\breve{T} \in \mathbb{R}^{p\times p}$ is orthogonal. Further, it is assumed that the system $(\widetilde{A}_{11}, \widetilde{A}_{12}, \widetilde{C}_1)$ with $\widetilde{C}_1$ defined by

$$\widetilde{C}_1 = \begin{bmatrix} 0_{(p-m)\times(n-m)} & I_{p-m} \end{bmatrix} \tag{2.19}$$

is output feedback stabilisable, i.e., there exists a matrix $K \in \mathbb{R}^{m\times(p-m)}$ such that

$$\widetilde{A}_{11} - \widetilde{A}_{12}K\widetilde{C}_1$$

is stable. It is shown in [37, 38] that a necessary condition for $(\widetilde{A}_{11}, \widetilde{A}_{12}, \widetilde{C}_1)$ to be stabilisable is that the invariant zeros of (A, B, C) lie in the open left-half plane. In [37, 38] a sliding surface of the form

$$FCx = 0 \tag{2.20}$$

is proposed, where

$$F = F_2 \begin{bmatrix} K & I_m \end{bmatrix} \breve{T}^\tau \tag{2.21}$$

and $F_2 \in \mathbb{R}^{m\times m}$ is any nonsingular matrix.

If a further coordinate change is introduced based on the nonsingular transformation $z = \widehat{T}\widetilde{x}$ with $\widehat{T}$ defined by

$$\widehat{T} = \begin{bmatrix} I_{n-m} & 0 \\ K\widetilde{C}_1 & I_m \end{bmatrix}$$

then in the new coordinates z, System (2.16) and (2.17) has the following form

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \qquad \widehat{C},$$

where $A_{11} = \widetilde{A}_{11} - \widetilde{A}_{12}K\widetilde{C}_1$ is stable and $\widehat{C}$ satisfies

$$F\widehat{C} = \begin{bmatrix} 0 & F_2 \end{bmatrix}$$

with F_2 nonsingular. From the analysis above, the following conclusion is obtained directly:

Lemma 2.6 *Consider System (2.16) and (2.17). Suppose that*

(i) $\mathrm{rank}(CB) = m$;

(ii) *the invariant zeros of* (A, B, C) *lie in the open left-half plane;*
(iii) *the matrix triple* $(\widetilde{A}_{11}, \widetilde{A}_{12}, \widetilde{C}_1)$ *is output feedback stabilisable, where* $(\widetilde{A}_{11}, \widetilde{A}_{12})$ *and* $\widetilde{C}_1$ *are defined, respectively, by (2.18) and (2.19).*

Then,

(i) *there exists a transformation* $z = Tx$ *such that the new coordinate* z *system (2.16) and (2.17) has the following form*

$$\dot{z} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} z + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u \tag{2.22}$$

$$y = \begin{bmatrix} 0 & C_2 \end{bmatrix} z, \tag{2.23}$$

where $A_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$ *is stable. Both matrices* $B_2 \in \mathbb{R}^{m\times m}$ *and* $C_2 \in \mathbb{R}^{p\times p}$ *are nonsingular;*
(ii) *there exists a matrix* F *such that* $FCx = 0$ *provides a stable sliding motion for System (2.16) and (2.17) and* $F\begin{bmatrix} 0 & C_2 \end{bmatrix} = \begin{bmatrix} 0 & F_2 \end{bmatrix}$*, where* $F_2 \in \mathbb{R}^{m\times m}$ *is nonsingular.*

Proof All that remains to be shown is that the output distribution matrix has the form given in (2.23) and that C_2 is nonsingular. The output distribution matrix in the new coordinates is given by

$$\begin{aligned}
\begin{bmatrix} 0 & \breve{T} \end{bmatrix} \widehat{T}^{-1} &= \begin{bmatrix} 0 & \breve{T} \end{bmatrix} \begin{bmatrix} I_{n-m} & 0 \\ -K\widetilde{C}_1 & I_m \end{bmatrix} \\
&= \begin{bmatrix} 0 & \breve{T} \end{bmatrix} \begin{bmatrix} I_{n-p} & 0 & 0 \\ 0 & I_{p-m} & 0 \\ 0 & -K & I_m \end{bmatrix} \\
&= \begin{bmatrix} 0 & \breve{T} \end{bmatrix} \begin{bmatrix} I_{n-m} & 0 \\ 0 & \begin{bmatrix} I_{p-m} & 0 \\ -K & I_m \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} 0 & \breve{T} \begin{bmatrix} I_{p-m} & 0 \\ -K & I_m \end{bmatrix} \end{bmatrix}.
\end{aligned}$$

and so by inspection,

$$C_2 = \breve{T} \begin{bmatrix} I_{p-m} & 0 \\ -K & I_m \end{bmatrix}$$

which is nonsingular. Hence the result follows. #

From the analysis above, it is clear to see that the coordinate transformation

$$z = Tx,$$

where $T := \widehat{T}\tilde{T}$, transfers the system (2.16) and (2.17) to the regular form (2.22) and (2.23). Choose the sliding surface

$$\mathscr{S} = \{x \mid FCx = 0,\ x \in \mathbb{R}^n\} \tag{2.24}$$

Then, the analysis above shows that the sliding motion of System (2.16) and (2.17) corresponding to the sliding surface (2.24) is asymptotically stable. The sliding surface (2.24) can be described by

$$\mathscr{S} = \{y \mid Fy = 0,\ y \in \mathbb{R}^p\} \tag{2.25}$$

which is a subspace of the *output space*. Therefore, $\mathscr{S}$ in (2.24) or (2.25) denote *output sliding surfaces*.

Remark 2.4 Lemma 2.6 gives a condition for the existence of the output switching surface (2.20) on which system (2.16) is stable. It should be emphasised that the sliding surface given by Lemma 2.6 can be obtained from a systematic algorithm together with any output feedback pole placement algorithm of choice. Details of appropriate algorithms and how to determine the switching surface (2.20) are described in [37, 38], where the necessary and sufficient condition to guarantee the existence of the matrix F is available in Proposition 5.2 of [38]. If $p = m$ then there is no design freedom and the sliding motion is governed by the invariant zeros of (A, B, C).

2.7 Geometric Structure of Nonlinear System

Consider the nonlinear system

$$\dot{x}(t) = F(x(t), u(t)) \tag{2.26}$$

$$y(t) = h(x(t)), \qquad x_0 = x(0), \tag{2.27}$$

where $x \in \Omega \subset \mathscr{R}^n$ (and Ω is a neighbourhood of x_0), $u = \mathrm{col}(u_1, u_2, \ldots, u_m) \in \mathscr{U} \in \mathscr{R}^m$, and $y = \mathrm{col}(y_1, y_2, \ldots, y_p) \in \mathscr{R}^p$ are the state variables, inputs and outputs, respectively, where $\mathscr{U}$ is an admissible control set. $F(x, u)$ is a known smooth vector field in $\Omega \times \mathscr{U}$ and the known function $h : \Omega \mapsto \mathscr{R}^p$ is smooth. For convenience, the system (2.26) and (2.27) is also denoted by the pair $(F(x, u), h(x))$.

Definition 2.8 (*See, e.g., [58]*) System (2.26) and (2.27) is said to be *observable* at $(x_0, u_0) \in \Omega \times \mathscr{U}$ if there exists a neighbourhood $\mathscr{N}$ of (x_0, u_0) in $\Omega \times \mathscr{U}$ and a set of nonnegative integer numbers $\{r_1, r_2, \cdots, r_p\}$ with $\sum_{i=1}^{p} r_i = n$ such that

(1) for all $(x, u) \in \mathscr{N}$

$$\frac{\partial}{\partial u_j} L^k_{F(x,u)} h_i(x) = 0 \tag{2.28}$$

for indices $i = 1, 2, \ldots, p$, $k = 0, 1, 2, \ldots, r_i - 1$ and $j = 1, 2, \ldots, m$;

(2) the $p \times m$ matrix $M(x, u) := \{\frac{\partial}{\partial u_j} L^{r_i}_{F(x,u)} h_i(x)\}$ has rank p in (x_0, u_0)

Then, $\{r_1, r_2, \cdots, r_p\}$ is called the *observability index* of System (2.26) and (2.27) at (x_0, u_0). Further, System (2.26) and (2.27) is said to be *uniformly observable* in $\Omega \times \mathscr{U}$ if for any $(x_0, u_0) \in \Omega \times \mathscr{U}$, the system is observable and the observability indices are fixed.

Assume the pair $(F(x, u), h(x))$ has uniform observability index $\{r_1, r_2, \cdots, r_p\}$ with $\sum_{i=1}^{p} r_i = n$ in the domain $\Omega \times \mathscr{U}$. Construct a nonlinear transformation $T : x \mapsto z$ as follows:

$$z_{i1} = h_i(x) \tag{2.29}$$

$$z_{i2} = L_{F(x,u)} h_i(x) \tag{2.30}$$

$$\vdots$$

$$z_{ir_i} = L_{F(x,u)}^{r_i - 1} h_i(x), \tag{2.31}$$

where $z_i := \mathrm{col}(z_{i1}, z_{i2}, \cdots, z_{ir_i})$ for $i = 1, 2, \ldots, p$ and $z := \mathrm{col}(z_1, z_2, \cdots, z_p)$.

It follows from Definition 2.8 that $M(x, u)$ has rank p in $\Omega \times \mathscr{U}$, implying that all z_i are independent of the control u, which combined with the restriction $\sum_{i=1}^{p} r_i = n$ means that the corresponding Jacobian matrix of $T(x)$, $\frac{\partial T}{\partial x}$, is nonsingular. Therefore, (2.29) and (2.31) is a diffeomorphism in the domain Ω, and $z = \mathrm{col}(z_1, z_2, \ldots, z_p)$ forms a new coordinate system which can be obtained by direct computation from (2.29) to (2.31).

Since $L_{F(x,u)}^{j} h_i(x)$ is independent of u for all $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, r_i - 1$, it follows by direct computation that for $i = 1, 2, \ldots, p$

$$\begin{aligned}
\dot{z}_{i1} &= \tfrac{\partial h_i}{\partial x} F(x, u) = L_{F(x,u)} h_i(x) = z_{i2} \\
\dot{z}_{i2} &= \frac{\partial \left(L_{F(x,u)} h_i(x)\right)}{\partial x} F(x, u) = L_{F(x,u)}^{2} h_i(x) = z_{i3} \\
&\ \vdots \\
\dot{z}_{ir_{i-1}} &= L_{F(x,u)}^{r_i - 1} h_i(x) = z_{ir_i} \\
\dot{z}_{ir_i} &= L_{F(x,u)}^{r_i} h_i(x)
\end{aligned}$$

Therefore, in the new coordinates z defined by (2.29) and (2.31), System (2.26) and (2.27) has the following form:

$$\begin{aligned}
\dot{z} &= Az + B\Phi(z, u) \\
y &= Cz,
\end{aligned}$$

where

$$A = \mathrm{diag}\{A_1, \ldots, A_p\}, \quad B = \mathrm{diag}\{B_1, \ldots, B_p\} \quad \text{and} \quad C = \mathrm{diag}\{C_1, \ldots, C_p\},$$

where $A_i \in \mathscr{R}^{r_i \times r_i}$, $B_i \in \mathscr{R}^{r_i \times 1}$ and $C_i \in \mathscr{R}^{1 \times r_i}$ for $i = 1, 2, \ldots, p$ are defined by

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad B_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \qquad C_i = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \tag{2.32}$$

and

$$\Phi(z,u) := \begin{bmatrix} \phi_1(z,u) \\ \phi_2(z,u) \\ \vdots \\ \phi_p(z,u) \end{bmatrix} := \begin{bmatrix} L_{F(x,u)}^{r_1} h_1(x) \\ L_{F(x,u)}^{r_2} h_2(x) \\ \vdots \\ L_{F(x,u)}^{r_p} h_p(x) \end{bmatrix}_{x=T^{-1}(z)}, \tag{2.33}$$

where $\phi_i : T(\Omega) \times \mathscr{U} \mapsto \mathscr{R}$ for $i = 1, 2, \ldots, p$.

2.8 Summary

This chapter has presented the fundamental concepts and results which underpin the theoretical analysis in this book. Some of the results are taken from the existing literature and others are developed by the authors, but with rigorous proofs provided. The content covers Lipschitz conditions, comparison functions, stability of nonlinear systems, the converse Lyapunov theorem and uniform ultimate boundedness. The well-known Razumikhin theorem has been presented, for the readers' convenience, and will be employed to deal with time-delay systems throughout the book. Section 2.5 summarises the output sliding surface design approach proposed in [38] which will be frequently used in the sequel. Finally, the geometric structure of nonlinear systems with uniform observability index has been provided.

http://www.springer.com/978-3-319-48961-2

Variable Structure Control of Complex Systems
Analysis and Design
Yan, X.-G.; Spurgeon, S.K.; Edwards, C.
2017, XXII, 350 p. 41 illus., 23 illus. in color., Hardcover
ISBN: 978-3-319-48961-2