Efficiency in Contests Between Groups

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1 Introduction

In recent years a number of articles have investigated contests between groups of players. Most of this literature is characterized by a structure in which players choose a level of input and the inputs of all members of a group determines (positively) the lobbying effort for the group through an impact or production function. The probability that a group wins the contest is the ratio of its lobbying effort to aggregate lobbying effort. The prize awarded to the winning group might be regarded by that group as a within-group public good or a private good which is to be divided according to a sharing rule to which members of the group commit before the contest is run. It might even exhibit a mixture of these characteristics, but in all cases the fundamental difference from a simple contest is the strategic tension between a contestant's incentive to increase input in order to increase the probability of winning the inter-group contest and the incentive to free ride on the input of other members of the same group. The positive externalities implicit in the production function may lead one to anticipate that the production of lobbying effort in equilibrium is inefficient. Indeed, such inefficiency, often labelled as free riding, is typically invoked to explain apparently counter-intuitive comparative statics. However, one of our main observations is that, in equilibrium, each group's lobbying effort is efficiently produced. This does not preclude a member of a group choosing zero input but this will occur if and only if equilibrium lobbying effort can be efficiently produced with no input from that member. As well as challenging conventional explanations of the properties of equilibria, this observation also leads to a two-stage procedure for studying collective contests that is considerably simpler and perhaps more insightful than working directly from first-order conditions.

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Some of the earliest literature on contests played by groups arose from an attempt to construct a formal framework in which to study Olsen's (1965) analysis of the relative lobbying effectiveness of small and large groups. For example, Esteban and Ray (2001) conclude that, in certain circumstances, members of large groups may fare less will than those in smaller groups and ascribe this to free-riding. However, we argue instead that, in a situation where large groups perform less well than small groups, this is not due to inefficient production of lobbying effort, but rather to the fact that the prize is divided amongst more members of the group.

An early formal analysis of existence and uniqueness of equilibrium was carried out by Katz et al. (1990) who considered two groups in which the group's production function is just a sum of the inputs of members of the group and costs are linear. Baik (1993) generalized this to more than two groups but still supposed that the probability of winning was influenced by the inputs of members of groups only through aggregate input. Baik later gave a more complete development (Baik 2008), and considered the impact of budget restrictions on players. A similar model was analyzed by Riaz et al. (1995) and Dijkstra (1998) and, recently, Ryvkin (2011) has used the model to examine how lobbying effort is affected by how players are sorted into groups. However, the assumption that production functions are a simple sum of inputs and costs are linear leads to equilibria in which, typically,[1] at most one player in each group is active (chooses positive input). Recent articles by Epstein and Mealam (2009) and by Kolmar and Rommeswinkel (2013) avoid this perhaps implausible outcome by considering production functions which do not simply aggregate inputs within groups. Our decomposition procedure considerably simplifies the analysis of equilibria in such cases and allows us to demonstrate that, under plausible and standard conditions, an equilibrium exists and the profile of equilibrium lobbying effort is unique. Three recent contributions which do not satisfy these condition are by Chowdhury et al. (2013) who study group contests in which only the highest effort in a group affects the probability of that group winning and by Baik et al. (2001) and Topolyan (2014) who assume the group with the highest output wins.

When the prize is wholly or partially a private good as opposed to a pure within-group public good, adding players to a group may reduce payoffs. The clearest case is when the prize is shared equally between members of the winning group. An early study of the implications of this by Nitzan (1991) looks at the effects of different sharing rules and Ueda (2002) examined oligopolization (groups becoming inactive). Esteban and Ray (2001), draw on insights from these models in their formal examination of Olsen's famous group size paradox. Once again, the assumption made in these articles that production functions are a sum of inputs and costs are linear lead, typically, to equilibria with at most one active player in each group. In recent contributions, Nitzan and Ueda (2011, 2014) use techniques developed by Cornes and Hartley (2005, 2007) to study collective contests in which

[1]When more than one member of a group is active, the equilibrium profile of inputs within the group is not unique.

many of these restrictions are relaxed. In both of these articles, Nitzan and Ueda assume the prize is a mixture of a public and a private good and, in Nitzan and Ueda (2014) use this approach to study the relative effectiveness of more or less homogeneous groups as measured by the values members ascribe to winning. In Nitzan and Ueda (2011), they assume that sharing rules are exogenously determined but that members of one group cannot observe the sharing rule chosen by other groups. Such incomplete information takes us outside the scope of the analysis below.

In Sects. 2 and 3, we set out the model and describe the decomposition theorem. Its use allows us to avoid a direct attack on the study of equilibria of collective contests. Such an approach typically involves studying first order conditions for best responses and this leads to a system of non-linear inequalities whose complicated nature has led many scholars to simplify the analysis by imposing symmetry and/or restricting the number of groups to two. However, this may be an unwelcome restriction as analysis of contests played by individuals shows that new features may emerge in general asymmetric contests with many players (see, for example, Cornes and Hartley 2005). Our aim in this article is to show how the decomposition theorem permits us to study contests played by groups with minimal restrictions on contest success and cost functions and an arbitrary number of players. The decomposition theorem entails defining a collective cost function for each group. This is the minimal normalized aggregate cost of producing a given level of group effort and its calculation is a straightforward optimization problem. The overall contest can then be studied by finding equilibria of a conventional contest played by groups with a simple lottery-type contest success function and the collective cost functions for each group. The equilibrium efforts of each group in this reduced contest can then be used to determine full equilibrium strategy profiles of the original contest.

In Sect. 4, we study the implications of this result for existence and uniqueness of equilibria and discuss properties of equilibria, in particular efficiency issues within groups. In Sect. 5, we present a number of examples of group cost functions. In particular, we look first at linear impact and cost functions and then at homogeneous impact functions and constant elasticity cost functions. These latter assumptions include all cases of the general model we have encountered in the literature. In Sect. 6, we study rent dissipation and note that in a fully symmetric case, rent dissipation is independent of the size of groups. Section 7 applies the decomposition theorem to comparative statics and, in particular the effect on winning probabilities and payoffs of adding a new group and of adding members to an existing group. A final section concludes.

In this preamble, we have assumed that the prize is an indivisible good. However, since we assume that contestants are risk neutral, all of our results apply equally well to contests in which the prize is a divisible good and the contest success function determines the proportion of the prize received by each group. As in the case of an indivisible prize, the prize can be a public good within groups, or a private good provided we assume that members of groups have pre-committed to a rule for sharing any winnings. We can even imagine that sharing is determined in a second stage game, once the contest has been won. In that case, our analysis applies to the first stage of a subgame perfect equilibrium. For example, Katz and Tokatlidu (1996) examine the case where the sharing is determined in a second intra-group rent-seeking contest, though to simplify the analysis they assume only two groups and linear production functions.[2] The model is formally the same in all these cases, but for expositional simplicity, we refer to the indivisible, public-good case except where explicitly stated otherwise.

2 Notation and Payoffs

We study a contest played by n groups in which group i contains $N_i \geq 1$ contestants. Groups compete to win an indivisible prize which is won by one group. Each member of each group provides input into this process and the probability that group i wins is affected positively by the inputs of members of that group and negatively by members of other groups. Specifically, we suppose that the input of the kth contestant in group i is a non-negative real number x_{ik},[3] where $k = 1, \ldots, N_i$. For each group i, it is convenient to write $\mathbf{x}_i$ for the vector $(x_{i1}, \ldots, x_{iN_i})$ and $\mathbf{x}$ for the strategy profile $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$. For any profile $\mathbf{x} \neq \mathbf{0}$, we assume that the probability that group i wins is given by the generalized logistic contest success function:

$$\rho_i(\mathbf{x}) = f_i(\mathbf{x}_i) \Big/ \sum_{j=1}^{n} f_j(\mathbf{x}_j),$$

where the real-valued functions $f_1, \ldots, f_n$ satisfy the following assumptions.

A1 For $i = 1, \ldots, n$, the function f_i is continuously differentiable,[4] concave, strictly increasing ($\mathbf{x}_i \geq \mathbf{x}_i', \mathbf{x}_i \neq \mathbf{x}_i' \implies f_i(\mathbf{x}_i) > f_i(\mathbf{x}_i')$), satisfies $f_i(\mathbf{0}) = 0$ and is unbounded above.

We can view f_i as a production function for group i, which maps the input vector $\mathbf{x}_i$ of members of the members of group i into lobbying effort $y_i = f_i(\mathbf{x}_i)$ of the group. When no contestant in any group supplies input, we suppose the prize is not awarded: $\rho_i(\mathbf{0}) = 0$ for $i = 1, \ldots, n$.

We assume that the kth contestant in group i values winning the prize at $v_{ik}(> 0)$ and incurs a cost $d_{ik}(x_{ik})$ for supplying input x_{ik}. We suppose that contestants are

[2] Choi et al. (2016) study a similar model, but with the internal and external contests running simultaneously.

[3] Our analysis extends readily, but at the expense of notational complexity, to vector inputs, possibly of different dimensions. We will omit the details for reasons of clarity.

[4] We interpret derivatives for functions not defined for negative arguments as one-sided on the boundary.

risk neutral, so the payoff of the kth contestant in group i is

$$v_{ik} f_i(\mathbf{x}_i) / \sum_{j=1}^{n} f_j(\mathbf{x}_j) - d_{ik}(x_{ik}). \tag{1}$$

This defines a simultaneous-move game, which we denote $\mathcal{C}$. We make the following assumptions about the cost functions.

A2 For $i = 1, \ldots, n$ and $k = 1, \ldots, N_i$, the function d_{ik} is continuously differentiable, convex, strictly increasing and satisfies $d_{ik}(0) = 0$.

The value v_{ik}, which the kth member of group i assigns to winning the prize could be interpreted either as her personal evaluation of the benefit of winning when the prize is a within-group public good or her share of the prize if it is a private good. Most of our analysis applies to both cases. However, when considering comparative statics of group size it is important to distinguish between these cases. Adding players to a group in the public-good case has no effect on the payoffs of incumbent players but this is not so in the private-good case. We might typically expect v_{ik} to fall if extra players join group i and, indeed, we ascribe apparently paradoxical results on group size to the consequence of sharing the prize amongst more players rather than inefficiency.

3 Decomposition

In this and subsequent sections it will prove convenient to normalize payoffs by dividing (1) by the positive number v_{ik}. Thus, the kth contestant in group i seeks to maximize

$$\pi_{ik}(\mathbf{x}) = f_i(\mathbf{x}_i) / \sum_{j=1}^{n} f_j(\mathbf{x}_j) - c_{ik}(x_{ik}),$$

where $c_{ik}(x_{ik}) = d_{ik}(x_{ik}) / v_{ik}$, for all $i = 1, \ldots, n$ and $k = 1, \ldots, N_i$. Note that Assumption **A2** holds for c_{ik}.

To motivate our results it is helpful to look at the first-order conditions for a best response by the kth member of group i in an equilibrium profile $\widetilde{\mathbf{x}}$. These can be written as

$$\widetilde{\lambda}_i \frac{\partial f_i(\widetilde{\mathbf{x}}_i)}{\partial x_{ik}} \leq c'_{ik}(\widetilde{x}_{ik}), \tag{2}$$

with equality if $\widetilde{x}_{ik} > 0$, where

$$\widetilde{\lambda}_i = \sum_{j \neq i} f_j\left(\widetilde{\mathbf{x}}_j\right) / \left[\sum_{j=1}^{n} f_j\left(\widetilde{\mathbf{x}}_j\right)\right]^2 .$$

Interpreting $\widetilde{\lambda}_i$ as a Lagrange multiplier, these are also the first-order conditions for minimizing $\sum_{\ell} c_{i\ell}\left(x_{i\ell}\right)$, the total normalized cost of group i subject to $f_i\left(\mathbf{x}_i\right) \geq f_i\left(\widetilde{\mathbf{x}}_i\right)$. This shows that the equilibrium strategy profile for group i produces the lobbying effort for that group at minimum aggregate cost; production of lobbying effort is constrained efficient. It also suggests that it is helpful to define a *collective cost function* for group i for all $y \geq 0$ as

$$C_i\left(y\right) = \min_{\mathbf{x}_i \geq \mathbf{0}} \left\{\sum_{\ell=1}^{N_i} c_{i\ell}\left(x_{i\ell}\right) : f_i\left(\mathbf{x}_i\right) \geq y\right\} . \tag{3}$$

We note, in the following lemma, proved in the Appendix, that C_i inherits many of the properties of individual cost functions.

Lemma 1 *Under Assumptions **A1** and **A2**, C_i is a convex, strictly increasing function for any i and satisfies $C_i\left(0\right) = 0$.*

We can now define a game played by the groups. The game has n players in which player (group) $i\,(= 1, \ldots, n)$ chooses $y_i \geq 0$ and receives payoff

$$\frac{y_i}{Y} - C_i\left(y_i\right),$$

where $Y = \sum_{j=1}^{n} y_j$, provided $Y > 0$ and payoff 0 if $Y = 0$. We can view this simultaneous-move game as a contest in which the ith contestant supplies effort y_i at cost $C_i\left(y_i\right)$ and the contest success function takes a simple lottery form. We refer to this game as the *reduced contest* and denote it $\mathcal{D}$.

The following theorem is our central result. The proof, in the Appendix, is an elaboration of the argument following (2).

Theorem 2 (Decomposition Theorem) *Suppose Assumptions **A1** and **A2** hold. Then, $\widetilde{\mathbf{x}}$ is a Nash equilibrium of $\mathcal{C}$ if and only if*

1. *$\left(f_1\left(\widetilde{\mathbf{x}}_1\right), \ldots, f_n\left(\widetilde{\mathbf{x}}_n\right)\right)$ is a Nash equilibrium of $\mathcal{D}$, and*
2. *$\widetilde{\mathbf{x}}_i$ achieves the minimum in the definition of $C_i\left(f_i\left(\widetilde{\mathbf{x}}_i\right)\right)$ for $i = 1, \ldots, n$.*

The constrained efficiency of production implicit in the decomposition theorem raises questions about the interpretation of comparative statics of group sizes when the prize is a private good and groups commit to sharing rules before entering the contest. In particular, the group size paradox, (larger groups being less likely to win) is often ascribed partly to free riding. However, Theorem 2 suggests that such "free-riding" is not an efficiency issue. Whilst it is certainly possible that an equilibrium entails zero input by one or more members of a group, this can occur only if it is inefficient for those members to choose a positive input level.

To study this further, suppose each group delegates the choice of strategies $\mathbf{x}_i$ for each member of the group to a manager who is tasked with looking after the interests of the group as a whole. Specifically, the manager of group i is charged with maximizing the value of winning net of the total cost incurred. This entails the manager choosing $\mathbf{x}_i$ to maximize:

$$\pi_i^* (\mathbf{x}) = f_i (\mathbf{x}_i) / \sum_{j=1}^{n} f_j \left(\mathbf{x}_j\right) - \sum_{\ell=1}^{N_i} c_{i\ell} (x_{i\ell}) .$$

Observe that, if $\widetilde{\mathbf{x}}_i$ is the manager's best response to her rivals, then $\widetilde{\mathbf{x}}_i$ must solve (3) with $y = f_i (\widetilde{\mathbf{x}}_i)$, for otherwise there would be another strategy $\mathbf{x}_i$ with a greater or equal value of $f_i (\mathbf{x}_i)$ and a smaller value of $\sum_\ell c_{i\ell} (x_{i\ell})$ and this would increase π_{ik}^* contradicting the supposition that $\widetilde{\mathbf{x}}_i$ is a best response. So the set of equilibria of the game played by the managers is the same as for the original game $\mathcal{C}$. In particular, no member of a group has an incentive to deviate unilaterally from the strategy prescribed for them by the manager.

It follows that appeals to free-riding as an explanation of the nature of equilibria need care. A recent article by Kolmar and Wagener (2013) discusses costless incentive schemes for avoiding such free-riding and conclude that groups may or may not choose to use such a scheme. (The choice is made in an initial stage of the game.) However, an alternative explanation of their results arises from noting that the incentive schemes can also be viewed as increasing the value of winning or, equivalently, decreasing costs in the second (contest) stage of the game. It is not hard to verify that, even in a simple two-player contest, the option to reduce costs in a second stage of the game, may or may not be in the interests of a contestant and, since "groups" in this case consist of a single member, this cannot be explained by appeal to free-riding.

In the next section, we apply the decomposition theorem to the existence and uniqueness of equilibria and, in the following section analyze some specific cost and production functions.

4 Existence, Uniqueness and Properties of Equilibrium

It is well-known that the reduced contest $\mathcal{D}$ has a unique Nash equilibrium (see, e.g. Cornes and Hartley 2005). It follows from the decomposition theorem that the profile of group lobbying efforts in any equilibrium is unique. The following corollary provides a formal statement.

Corollary 3 *If Assumptions **A1** and **A2** hold, then $\mathcal{C}$ has a Nash equilibrium. If $\widetilde{\mathbf{x}}$ and $\widetilde{\mathbf{x}}'$ are both equilibria of $\mathcal{C}$, then $f_i(\widetilde{\mathbf{x}}_i) = f_i(\widetilde{\mathbf{x}}'_i)$ for $i = 1, \ldots, n$.*

If, furthermore, each c_{ik} is strictly convex, then $\sum_\ell c_{i\ell}(x_{i\ell})$ is a strictly convex function of $\mathbf{x}_i$. This means that the optimization problem in (3) has a unique solution. Combined with the decomposition theorem we have the following result.[5]

Corollary 4 *Suppose Assumption **A1** and holds and d_{ik} is continuously differentiable, strictly convex, increasing and satisfies $d_{ik}(0) = 0$ for all $i = 1, \ldots, n$ and $k = 1, \ldots, N_i$. Then $\mathcal{C}$ has a unique equilibrium.*

5 Examples

The most commonly encountered production function in the literature is the aggregative form:

$$f_i(\mathbf{x}_i) = \sum_{\ell=1}^{N_i} x_{i\ell},$$

where it is often combined with the assumption that all cost functions are linear. It is instructive to apply the decomposition theorem to that case. Without loss of generality, we can assume $d_{ik}(x_{ik}) = x_{ik}$ for all k, so that

$$C_i(y) = \min_{\mathbf{x}_i \geq \mathbf{0}} \left\{ \sum_{\ell=1}^{N_i} \frac{1}{v_{ik}} x_{i\ell} : \sum_{\ell=1}^{N_i} x_{i\ell} \geq y \right\} = \frac{y}{\overline{v}_i},$$

where $\overline{v}_i = \max_k v_{ik}$. If there is a unique member of group i who places the highest value on winning (say the first member), the minimum is achieved at $x_{i1} = y$ and $x_{ik} = 0$ for all $k \neq 1$. If group i makes a positive effort to win the contest, only the first member supplies input.

When several players have the same maximum value of v_{ik}, any $\mathbf{x}_i \geq \mathbf{0}$ with $x_{i\ell} = 0$ for the remaining players and whose components add up to y will achieve the minimum in the definition of C_i. If group i is active in equilibrium, the equilibrium strategy profile will not be unique, although equilibrium group efforts will be. To have a unique equilibrium with several members of a group contributing positive equilibrium input, we need production or cost functions, or both, to be nonlinear and we now turn to such a case in which collective cost functions can still be calculated.

[5] A careful study of (3) shows that this conclusion remains true if for each i, in addition to Assumption **A2**, c_{ik} is strictly convex for all but one of $k = 1, \ldots, N_i$.

We will suppose that production functions are homogeneous and cost functions have constant elasticity, which is the same for all group members. This means that, for any group i, there exist $\mu_i, \omega_i > 0$, such that for any strategy profile $\mathbf{x}_i$ and $\lambda > 0$, we have (without further loss of generality)

$$f_i(\lambda \mathbf{x}_i) = \lambda^{\mu_i} f_i(\mathbf{x}_i),$$
$$d_{ik}(x_{ik}) = x_{ik}^{\omega_i}$$

for any $k = 1, \ldots, N_i$. These suppositions have implications for the form of C_i. To investigate these we will write $\widehat{\mathbf{x}}_i(y)$ for a profile that achieves the minimum in the definition of $C_i(y)$. Then

$$\sum_{\ell=1}^{N_i} c_{i\ell}\left(\widehat{x}_{il}(y)\right) \leq \sum_{\ell=1}^{N_i} c_{i\ell}(x_{i\ell})$$

for all $\mathbf{x}_i$ satisfying $f_i(\mathbf{x}_i) = y$. Multiplying this inequality by λ^{ω_i/μ_i}, we have

$$\lambda^{\omega_i/\mu_i} \sum_{\ell=1}^{N_i} c_{i\ell}\left(\widehat{x}_{il}(y)\right) \leq \lambda^{\omega_i/\mu_i} \sum_{\ell=1}^{N_i} c_{i\ell}(x_{i\ell}),$$

or

$$\sum_{\ell=1}^{N_i} c_{i\ell}\left(\lambda^{1/\mu_i}\widehat{x}_{il}(y)\right) \leq \sum_{\ell=1}^{N_i} c_{i\ell}\left(\lambda^{1/\mu_i} x_{i\ell}\right),$$

for all $\mathbf{x}_i$ satisfying $f_i(\mathbf{x}_i) = y$ and therefore also for all $\lambda^{1/\mu_i}\mathbf{x}_i$ satisfying $f_i\left(\lambda^{1/\mu_i}\mathbf{x}_i\right) = \lambda y$. It follows that $\lambda^{1/\mu_i}\widehat{\mathbf{x}}_i(y)$ achieves the minimum in the definition of $C_i(\lambda y)$. Hence,

$$\begin{aligned} C_i(\lambda y) &= \sum_{\ell=1}^{N_i} c_{i\ell}\left(\widehat{x}_{il}(\lambda y)\right) = \sum_{\ell=1}^{N_i} c_{i\ell}\left(\lambda^{1/\mu_i}\widehat{x}_{il}(y)\right) \\ &= \lambda^{\omega_i/\mu_i} \sum_{\ell=1}^{N_i} c_{i\ell}\left(\widehat{x}_{il}(y)\right) = \lambda^{\omega_i/\mu_i} C_i(y). \end{aligned}$$

It follows that group cost functions are characterized by a single parameter $C_i(1)$ (apart from μ_i and ω_i) and they take the power form: $C_i(y) = y^{\omega_i/\mu_i} C_i(1)$. When **A1** and **A2** are satisfied, f_i is concave which implies $\mu_i \leq 1$ and c_{ik} is convex, which implies $\omega_i \geq 1$. It follows that $\omega_i/\mu_i \geq 1$, confirming that C_i is convex.

As an example, consider linear cost function $c_{ik}(x) = x$ for all i and k and a Cobb-Douglas production function:

$$f_i(\mathbf{x}) = g_i \prod_{l=1}^{N_i} x_{il}^{\alpha_{il}},$$

where $g_i, \alpha_{ik} > 0$ for all k are positive and $A_i = \sum_{l=1}^{N_i} \alpha_{il} \leq 1$ to ensure concavity. In this case, $\omega_i/\mu_i = 1/A_i$ and

$$C_i(1) = A_i g_i^{-1/A_i} / \prod_{l=1}^{N_i} (\alpha_{il} v_{il})^{\alpha_{il}/A_i}.$$

With the same cost functions and the CES production function considered by Kolmar and Rommeswinkel (2013):

$$f_i(\mathbf{x}) = g_i \left[\sum_{l=1}^{N_i} \alpha_{il} x_{il}^{\gamma_i} \right]^{1/\gamma_i},$$

where all $g_i, \alpha_{ik} > 0$ for all k and $\gamma_i \leq 1$ and $\gamma_i \neq 0$, we have $\omega_i/\mu_i = 1$ and

$$C_i(1) = g_i^{-1} / \left[\sum_{l=1}^{N_i} \alpha_{il} (\alpha_{il} v_{il})^{\rho_i} \right]^{1/\rho_i},$$

where $\rho_i = \gamma_i / (1 - \gamma_i)$.

In the CES case, ω_i/μ_i is independent of i (actually equal to 1 for all i) and this is also true for Cobb-Douglas productions functions if A_i is independent of i. In such an instance we can order the marginal values of the collective cost functions and therefore also the equilibrium probabilities of groups winning the contest. If we arrange the groups so that $C_i(1) < C_j(1)$ if $i < j$, the probability that group i wins is increasing in i. If the common value of ω_i/μ_i exceeds 1, all groups are active ($y_i > 0$). If the common value is unity (for example, with linear costs and CRS production functions), it is possible that some groups may be inactive. In this case, there will be an integer $\overline{n} \geq 2$ such that groups $i = 1, \ldots, \overline{n}$ are active and any group i for which $i > \overline{n}$ is inactive. Kolmar and Rommeswinkel write down a procedure to determine $\overline{n}$ and equilibrium group efforts and probabilities.

6 Rent Dissipation

One major theme in the literature on contests is: what proportion of the value of the prize is expended in the effort, often assumed to have no other economic worth, to win the prize? In a collective contest, intra-group inefficiency would be expected to reduce rent-dissipation through a reduction in rent-seeking activity. However, since we have shown that groups provide effort efficiently, there should be no effect on rent dissipation. It is useful to note that, since we have normalized the prize to 1, rent dissipation as a proportion of the rent takes the form

$$\rho = \sum_{i=1}^{n} \sum_{\ell=1}^{N_i} c_{i\ell}\left(\widetilde{x}_{i\ell}\right) = \sum_{j=1}^{n} C_j\left(\widetilde{y}_j\right),$$

where $\widetilde{\mathbf{x}}$ is an equilibrium strategy profile, $\widetilde{\mathbf{y}}$ is an equilibrium vector of group efforts and we refer to ρ as the *dissipation ratio*. If there are multiple equilibria, the sum is the same for all equilibria. The second sum shows that we can analyze dissipation ratios just using the reduced contest $\mathcal{D}$.

When contests are played by groups, an important issue is how group size affects the dissipation ratio. In the case where production and cost functions are linear and cost functions are the same for all members of a group, there is always an equilibrium in which only one player is active in each active group. This implies that collective costs and therefore the dissipation ratio is independent of the number of players in each group.

To study the case where costs are non-linear, suppose production functions are additive: $f_i\left(\mathbf{x}_i\right) = \sum_{\ell=1}^{N_i} x_{i\ell}$ for each group i and all contestants have the same cost functions and assign the same value to the prize: $d_{ik}\left(x\right) = x^{\alpha}$ and $v_{ik} = R$ for all i and k, where $\alpha \geq 1$. Then, from (3),

$$C_i\left(y\right) = \min_{\mathbf{x}_i \geq \mathbf{0}} \left\{ \sum_{\ell=1}^{N_i} x_{i\ell}^{\alpha}/R : \sum_{\ell=1}^{N_i} x_{i\ell} \geq y \right\}.$$

If $\alpha > 1$, the minimizer in the definition of C_i must be unique and symmetric and the constraint binding, which gives $\widehat{x}_{ik} = y/N_i$ for all k. Hence, $C_i\left(y\right) = y^{\alpha}/RN_i^{\alpha-1}$. If $\alpha = 1$, there are multiple optimal solutions but the formula for C_i remains valid.

The simplest case to consider is where $N_i = N$ for all i in which case the reduced game $\mathcal{D}$ is symmetric. It is then straightforward to see that the first-order conditions for best responses imply that the equilibrium value of Y is $\widetilde{Y}$, where

$$1 - \frac{1}{n} = \frac{\alpha \widetilde{Y}^{\alpha}}{Rn^{\alpha-1}N^{\alpha-1}}.$$

Since the equilibrium is symmetric, $\widetilde{y}_i = \widetilde{Y}/n$ for all i and therefore

$$C_i\left(\widetilde{y}_i\right) = C_i\left(\frac{\widetilde{Y}}{n}\right) = \left(1 - \frac{1}{n}\right)\frac{1}{n\alpha}.$$

It follows that the dissipation ratio is $\rho = \left(n-1\right)/n\alpha$. Intriguingly, this is independent of the number of members in each group. Furthermore, as $n \longrightarrow \infty$, the dissipation ratio approaches $1/\alpha$ (from below). This is the same dissipation ratio as in a conventional contest in which all contestants have cost function x^{α}. It is not hard to see that this limiting result remains true even if groups differ in size, provided that the number of groups of each size approaches infinity.

7 Comparative Statics

In this section, we show how the decomposition theorem can be used to study comparative statics. Collective contests are rich enough to allow for many possible studies, but here we confine ourselves to two. What are the effects of (1) adding a new group to the contest and (2) of adding members to an existing group? The general approach is to start by examining the effect of the change on collective cost functions and then investigate the effect of this change on the reduced contest. For example, adding an active group reduces the probability of an incumbent group winning the reduced contest and does not increase payoffs in equilibrium, at least under plausible assumptions on cost and production functions. To investigate the effects of such results on individual members of groups it is useful to know how an increase of group lobbying effort is reflected in the inputs of members of that group. It turns out that, under the following strengthening of Assumption **A1** and with convex costs, individual inputs rise or remain the same in equilibrium.

A1* For $i = 1, \ldots, n$,

$$f_i(\mathbf{x}_i) = \phi_i\left(\sum_{\ell=1}^{N_i} g_{i\ell}(x_{i\ell})\right), \tag{4}$$

where each ϕ_i and each g_{ik} is continuously differentiable, concave, strictly increasing, satisfies $\phi_i(0) = g_{ik}(0) = 0$ for $k = 1, \ldots, N_i$ and f_i is unbounded above.

Under this assumption, we have the following result proved in the Appendix.

Lemma 5 *Suppose Assumptions **A1*** and **A2** hold and, in addition, all cost functions in group i are strictly convex. If $y < y^*$, then the optimal solution in (3) satisfies $\widehat{\mathbf{x}}_i(y) \leq \widehat{\mathbf{x}}_i(y^*)$.*

Note that the lemma implies that, if $\widehat{x}_{ik}(y^*) = 0$, then $\widehat{x}_{ik}(y) = 0$. Active contestants do not become inactive, when the lobbying effort of the group of which they are a member increases.

7.1 Adding a Group

We first consider the effect on incumbent groups of new groups joining the contest. So, consider a collective contest in which the equilibrium value of Y in the reduced contest $\mathcal{D}$ is $\widetilde{Y}$ and suppose a new group $n+1$ joins the contest. If at least one member of the new group is active in the new equilibrium, then the equilibrium value of Y rises to $\widetilde{Y}^{*}$, say. This was shown in Cornes and Hartley (2005), where it was also proved that the probabilities of incumbent groups winning the prize fall as do payoffs. The latter result can be written:

$$\frac{\widetilde{y}_i^*}{\widetilde{Y}^*} - C_i\left(\widetilde{y}_i^*\right) < \frac{\widetilde{y}_i}{\widetilde{Y}} - C_i\left(\widetilde{y}_i\right), \tag{5}$$

where $\widetilde{y}_i$ $(\widetilde{y}_i^*)$ is the equilibrium effort of group $i \le n$ before (after) entry. However, without further information on cost functions, we cannot say whether $\widetilde{y}_i^*$ is bigger or smaller than $\widetilde{y}_i$.

To examine the implications for members of groups, we need to consider the two cases separately. If we have $\widetilde{y}_i^* \ge \widetilde{y}_i$, then for any member k of group i the preceding lemma gives $\widetilde{x}_{ik}^* \ge \widetilde{x}_{ik}$, where $\widetilde{x}_{ik}$ $(\widetilde{x}_{ik}^*)$ is the equilibrium input of k before (after) entry. Since cost functions are increasing, this means that costs do not fall and the probability of winning does fall with entry. Hence, individual payoffs must also fall.

In the alternative case in which $\widetilde{y}_i^* < \widetilde{y}_i$, then $c_{ik}\left(\widetilde{x}_{ik}^*\right) \le c_{ik}\left(\widetilde{x}_{ik}\right)$ for all k and (5) gives

$$\begin{aligned}\sum_{\ell=1}^{N_i}\left[c_{i\ell}\left(\widetilde{x}_{i\ell}\right) - c_{i\ell}\left(\widetilde{x}_{i\ell}^*\right)\right] &= C_i\left(\widetilde{y}_i\right) - C_i\left(\widetilde{y}_i^*\right) \\ &< \frac{\widetilde{y}_i}{\widetilde{Y}} - \frac{\widetilde{y}_i^*}{\widetilde{Y}^*}.\end{aligned}$$

Since the left-hand side of this inequality is a sum of non-negative terms, we deduce that each term in this sum is less than the right-hand side. This implies

$$\frac{\widetilde{y}_i^*}{\widetilde{Y}^*} - c_{ik}\left(\widetilde{x}_{ik}^*\right) < \frac{\widetilde{y}_i}{\widetilde{Y}} - c_{ik}\left(\widetilde{x}_{ik}\right)$$

for all k. Since $\widetilde{y}_i/\widetilde{Y}$ is the probability that group i wins and the value of the prize is normalized to 1, this says that payoffs fall in this case also. That is, although the cost of the input of the kth member of group i decreases, this is more than offset by the fall in the expected value of the prize.

The following proposition summarizes these results.

Proposition 6 *Suppose additional groups join a collective contest $\mathcal{C}$ and at least one is active in the equilibrium of the enlarged contest in which Assumptions **A1*** and **A2** hold. Then, inactive incumbent groups remain inactive and the equilibrium* *payoff of members of incumbent groups in which cost functions are strictly convex falls.*

7.2 Adding Contestants

It is also interesting to consider the implications of new members entering existing groups when entry does not affect valuations (the public-good case). Under the following further strengthening of **A1**, we can show that the group's marginal cost reduces.

A1** For $i = 1, \ldots, n$,

$$f_i(\mathbf{x}_i) = \sum_{\ell=1}^{N_i} g_{i\ell}(x_{i\ell}),$$

where each g_{ik} is continuously differentiable, concave, strictly increasing, satisfies $g_{ik}(0) = 0$ for $k = 1, \ldots, N_i$ and f_i is unbounded above.

Under this assumption (and **A2**), marginal group costs fall as new members join the group. The following lemma, proved in the Appendix, gives a formal statement.

Lemma 7 *Suppose C_i is defined as in (3) with $N_i \geq 2$ and Assumptions **A2** and **A1**** hold. If*

$$D_i(y) = \min_{\mathbf{x}_i} \left\{ \sum_{\ell=2}^{N_i} c_{i\ell}(x_{i\ell}) : \sum_{\ell=2}^{N_i} g_{i\ell}(x_{i\ell}) \geq y \right\},$$

then $C_i'(y) \leq D_i'(y)$.

Adding players to group i reduces the marginal group cost of group i and therefore (because $C_i(0) = 0$) actual costs as well. It follows that group i has an increased probability of winning the reduced contest $\mathcal{D}$ whilst all other groups are less likely to win and face reduced equilibrium payoffs. For further details of such comparative statics exercises, see Cornes and Hartley (2005).

We can apply these observations in the manner of the previous section to get the following result.

Proposition 8 *Suppose additional contestants join a group in a collective contest $\mathcal{C}$ and Assumptions **A2** and **A1**** hold in the enlarged contest. Then, the probability of that group winning does not fall and original members are no worse off. Furthermore, every other group faces a (weakly) diminished probability of winning and all members of other groups are no better off.*

Of course, the conclusion of this proposition depends critically on the assumption that valuations do not alter when extra members join a group. In the case where, for example, the prize is a private good which is divided equally amongst members of the winning group, the increased probability of being a member of the winning group is offset by the reduced share of the prize and which effect dominates will depend on fine details of cost and production functions. Nevertheless, the decomposition theorem offers a useful tool for studying such issues, though for reasons of space we do not do so here.

8 Conclusion

In this article, we have presented a theorem on collective contests which shows that they can be decomposed into a cost minimization problem for each group and a reduced contest between the groups. This theorem clarifies the analysis of existence, uniqueness and comparative statics and simplifies the study of rent dissipation. For example, although omitted here, we can use the theorem to explore how the internal structure of groups affects the group cost function and thereby the group's probability of winning. Furthermore, the theorem is readily extended to the case where some or all players have multi-dimensional strategy spaces, and allows us to extend strategic models such as those used to study ethnic conflict by Esteban and Ray (2008).

Finally, we note that the decomposition theorem relies heavily on risk neutrality of contestants or equivalently on having divisible prizes. Indeed, pure separation breaks down if contestants are no longer risk neutral and alternative methods must then be used. However, the constrained efficiency of intra-group strategy profiles continues to hold.

Appendix

Proof of Lemma 1 Since Assumptions **A1** and **A2** imply that we are minimizing a continuous and strictly increasing function on a closed set, the minimum is achieved, though not necessarily uniquely. We will write $\widehat{\mathbf{x}}_i(y)$ for a minimizing $\mathbf{x}_i$ in (3).

To show that C_i is strictly increasing we first observe that, since f_i is increasing (by Assumption **A1**), $f_i\left(\widehat{\mathbf{x}}_i(y)\right) = y$. Now suppose that $y^0 \in (0, y)$, which entails $f_i\left(\widehat{\mathbf{x}}_i(y)\right) > y^0$. Since f_i is continuous, there must be an $\mathbf{x}_i^0$ satisfying $\mathbf{x}_i^0 \leq \widehat{\mathbf{x}}_i(y)$ and $f_i\left(\mathbf{x}_i^0\right) \geq y^0$. Since each $c_{i\ell}$ is strictly increasing, we have

$$C_i\left(y^0\right) \leq \sum_{\ell=1}^{N_i} c_{i\ell}\left(x_{i\ell}^0\right) < \sum_{\ell=1}^{N_i} c_{i\ell}\left(\widehat{x}_{i\ell}(y)\right) = C_i(y).$$

To prove convexity, observe that, if $\mu \in (0, 1)$, concavity of f_i implies that

$$f_i\left(\mu\widehat{\mathbf{x}}_i(y) + \mu^0\widehat{\mathbf{x}}_i\left(y^0\right)\right) \geq \mu y + \mu^0 y^0,$$

where $\mu^0 = 1 - \mu$. Convexity of $c_{i\ell}$ implies that:

$$\begin{aligned} C_i\left(\mu y + \mu^0 y^0\right) &\leq \sum_{\ell=1}^{N_i} c_{i\ell}\left(\mu\widehat{x}_{i\ell}(y) + \mu^0\widehat{x}_{i\ell}\left(y^0\right)\right) \\ &\leq \sum_{\ell=1}^{N_i} \mu c_{i\ell}\left(\widehat{x}_{i\ell}(y)\right) + \mu^0 c_{i\ell}\left(\widehat{x}_{i\ell}\left(y^0\right)\right) \\ &= \mu C_i(y) + \mu^0 C_i\left(y^0\right). \end{aligned}$$

The assertion that C_i satisfies $C_i(0) = 0$ is an immediate consequence of Assumptions **A1** and **A2**. ■

In the proof of Theorem 2, it will prove convenient to use the following lemma.

Lemma 9 *Suppose Assumptions **A1** and **A2** hold and $y > 0$. If $i = 1, \ldots, n$ and $\widehat{\mathbf{x}}_i(y)$ achieves the minimum in (3) and $k = 1, \ldots, N_i$, we have*

$$C_i'(y) \leq c_{ik}'\left(\widehat{x}_{ik}(y)\right) / \frac{\partial f_i}{\partial x_{ik}}\left(\widehat{\mathbf{x}}_i(y)\right),$$

with equality if $\widehat{x}_{ik}(y) > 0$. Furthermore,

$$C_i'(0) \geq \min_{k=1,\ldots,N_i}\left\{c_{ik}'(0) / \frac{\partial f_i}{\partial x_{ik}}(\mathbf{0})\right\}.$$

Proof For any $y > 0$, the assumption that f_i is unbounded above means that there is some $\mathbf{x}_i^0$ for which $f_i\left(\mathbf{x}_i^0\right) > y$. This says that the Slater constraint qualification for the optimization problem in (3) holds (cf. Rockafellar 1972), which means that there is a Lagrange multiplier $\lambda \geq 0$ such that the (necessary) first-order conditions for this optimization problem read:

$$c_{ik}'\left(\widehat{x}_{ik}(y)\right) \geq \lambda \frac{\partial f_i}{\partial x_{ik}}\left(\widehat{\mathbf{x}}_i(y)\right),$$

with equality if $\widehat{x}_{ik} > 0$. Furthermore, marginal group cost is the slope of the minimum function and is therefore equal to the Lagrange multiplier: $C_i'(y) = \lambda$. This observation completes the proof that the first displayed inequality is necessary and sufficient.

To complete the proof, note that Assumption **A2** (specifically convexity and zero cost of zero input) applied to c_{ik} implies that $c_{ik}(x_{ik}) \geq x_{ik}c'_{ik}(0)$ for all $x_{ik} \geq 0$. Similarly, from **A1** we have

$$f_i(\mathbf{x}_i) \leq \sum_{\ell=1}^{N_i} x_{i\ell} \frac{\partial f_i}{\partial x_{i\ell}}(\mathbf{0}).$$

Hence, for any $y > 0$, we have

$$\begin{aligned} C_i(y) &\geq \min_{\mathbf{x}_i} \left\{ \sum_{\ell=1}^{N_i} x_{i\ell} c'_{i\ell}(0) : \sum_{\ell=1}^{N_i} x_{i\ell} \frac{\partial f_i}{\partial x_{i\ell}}(\mathbf{0}) \geq y \right\} \\ &= y \min_k \left\{ c'_{ik}(0) \Big/ \frac{\partial f_i}{\partial x_{ik}}(\mathbf{0}) \right\}. \end{aligned}$$

(Note that Assumption **A1** implies $(\partial f_i/\partial x_{ik})(\mathbf{0}) > 0$.) Since $C_i(0) = 0$, we can divide by y and let $y \longrightarrow 0$ to obtain the second inequality in the statement of the lemma. ■

Proof of Theorem 2 Throughout the proof, we use the fact that the convexity/concavity conditions from Assumptions **A1** and **A2** and Lemma 1 mean that first order conditions are necessary and sufficient to characterize best responses.

To prove sufficiency in the Separation Theorem, suppose $\widetilde{\mathbf{x}}$ satisfies requirements 1 and 2. Since the first of these says that $(f_1(\widetilde{\mathbf{x}}_1), \ldots, f_n(\widetilde{\mathbf{x}}_n))$ is an equilibrium of $\mathcal{D}$, the first-order conditions for best responses give

$$C'_i(\widetilde{y}_i) \geq \sum_{j \neq i} \widetilde{y}_j \left[\sum_{j=1}^{n} \widetilde{y}_j \right]^{-2},$$

with equality if $\widetilde{y}_i > 0$, where $\widetilde{y}_i = f_i(\widetilde{\mathbf{x}}_i)$ for each $i = 1, \ldots, n$. Requirement 2 says that $\widetilde{\mathbf{x}}_i$ achieves the minimum in the definition of $C_i(\widetilde{y}_i)$ and Lemma 1 implies that, for any k,

$$C'_i(\widetilde{y}_i) \leq c'_{ik}(\widetilde{x}_{ik}) \Big/ \frac{\partial f_i}{\partial x_{ik}}(\widetilde{\mathbf{x}}_i)$$

with equality if $\widetilde{x}_{ik} > 0$. Combining these inequalities gives

$$\sum_{j \neq i} f_j(\widetilde{\mathbf{x}}_j) \left[\sum_{j=1}^{n} f_j(\widetilde{\mathbf{x}}_j) \right]^{-2} \frac{\partial f_i}{\partial x_{ik}}(\widetilde{\mathbf{x}}_i) \leq c'_{ik}(\widetilde{x}_{ik}), \tag{6}$$

with equality if $\widetilde{x}_{ik} > 0$. These are the first-order conditions for best responses in $\mathcal{C}$ and show that $\mathbf{x}^*$ is a Nash equilibrium.

To prove necessity, let $\widetilde{\mathbf{x}}$ be a Nash equilibrium of $\mathcal{C}$ and write $\widetilde{y}_i = f_i(\widetilde{\mathbf{x}}_i)$ for $i = 1, \ldots, n$. If $\widetilde{y}_i > 0$, the first order conditions for best responses for members of group i in $\mathcal{C}$ are (6) and

$$\lambda = \sum_{j \neq i} f_j(\widetilde{\mathbf{x}}_j) \left[\sum_{j=1}^{n} f_j(\widetilde{\mathbf{x}}_j) \right]^{-2},$$

these can be expressed as

$$c'_{ik}(\widetilde{x}_{ik}(y)) \geq \lambda \frac{\partial f_i}{\partial x_{ik}} (\widetilde{\mathbf{x}}_i(y)),$$

with equality if $\widehat{x}_{ik} > 0$. Since $\lambda \geq 0$, these are the first order conditions for the minimization problem in (3). These conditions are necessary and sufficient by Assumptions **A1** and **A2**, which means $\widetilde{\mathbf{x}}_i$ achieves the minimum in the definition of $C_i(f_i(\widetilde{\mathbf{x}}_i))$. It follows from Lemma 1 that

$$C'_i(\widetilde{y}_i) = c'_{ik}(\widetilde{x}_{i\ell}) / \frac{\partial f_i}{\partial x_{ik}} (\widetilde{\mathbf{x}}_i).$$

Combining these observations, we get

$$C'_i(\widetilde{y}_i) = \sum_{j \neq i} \widetilde{y}_j \left[\sum_{j=1}^{n} \widetilde{y}_j \right]^{-2}.$$

These are the first order conditions for $\widetilde{y}_i$ being a best response in $\mathcal{D}$.

In the case $\widetilde{y}_i = 0$, we must have $\widetilde{\mathbf{x}}_i = \mathbf{0}$ (which achieves the minimum in (3)) and the first-order conditions imply that, for all k,

$$\sum_{j \neq i} f_j(\widetilde{\mathbf{x}}_j) \left[\sum_{j=1}^{n} f_j(\widetilde{\mathbf{x}}_j) \right]^{-2} \frac{\partial f_i}{\partial x_{ik}} (\mathbf{0}) \leq c'_{ik}(0).$$

Combined with the second inequality in Lemma 1, we deduce

$$\sum_{j \neq i} \widetilde{y}_j \left[\sum_{j=1}^{n} \widetilde{y}_j \right]^{-2} \leq C'_i(0),$$

which implies that 0 is a best response by group i in $\mathcal{D}$. We have demonstrated that $\widetilde{\mathbf{y}}$ is an equilibrium of $\mathcal{D}$. ■

Proof of Lemma 5 To see that $c_{ik}(\widehat{x}_{ik}(y))$ is increasing in y when Assumption **A1*** holds, we start by noting that the group cost function can be rewritten:

$$C_i(y) = \min_{\mathbf{x}_i} \left\{ \sum_{\ell=1}^{N_i} c_{i\ell}(x_{i\ell}) : \sum_{\ell=1}^{N_i} g_{i\ell}(x_{i\ell}) \geq \phi_i^{-1}(y) \right\}, \tag{7}$$

where ϕ_i^{-1} is the inverse function of ϕ_i. Convexity of the cost functions and concavity of the $g_{i\ell}$ imply the existence of a Lagrange multiplier $\lambda_i(y) \geq 0$ such that

$$\sum_{\ell=1}^{N_i} c_{i\ell}(\widehat{x}_{i\ell}(y)) - \lambda_i(y) \sum_{\ell=1}^{N_i} g_{i\ell}(\widehat{x}_{i\ell}(y)) < \sum_{\ell=1}^{N_i} c_{i\ell}(x_{i\ell}) - \lambda_i(y) \sum_{\ell=1}^{N_i} g_{i\ell}(x_{i\ell}) \tag{8}$$

for any $\mathbf{x}_i \geq \mathbf{0}$ satisfying $\mathbf{x}_i \neq \widehat{\mathbf{x}}_i(y)$ the constraint in (7). Now suppose $y^* > y$.

It is possible that $\widehat{\mathbf{x}}_i(y) = \widehat{\mathbf{x}}_i(y^*)$. If not, combining (8) with the constraint in (7), we have

$$\sum_{\ell=1}^{N_i} c_{i\ell}(\widehat{x}_{i\ell}(y)) - \lambda_i(y)\phi_i^{-1}(y) < \sum_{\ell=1}^{N_i} c_{i\ell}\left(\widehat{x}_{i\ell}\left(y^*\right)\right) - \lambda_i(y)\phi_i^{-1}\left(y^*\right),$$

$$\sum_{\ell=1}^{N_i} c_{i\ell}\left(\widehat{x}_{i\ell}\left(y^*\right)\right) - \lambda_i\left(y^*\right)\phi_i^{-1}\left(y^*\right) < \sum_{\ell=1}^{N_i} c_{i\ell}(\widehat{x}_{i\ell}(y)) - \lambda_i\left(y^*\right)\phi_i^{-1}(y),$$

where we have used the fact that $\sum_l g_{il}(\widehat{x}_{il}(y)) = \phi_i^{-1}(y)$ (since g_i is increasing). Adding these inequalities shows that

$$\left[\phi_i^{-1}\left(y^*\right) - \phi_i^{-1}(y)\right]\left[\lambda_i\left(y^*\right) - \lambda_i(y)\right] > 0.$$

Since ϕ_i is strictly increasing, we have $\phi_i^{-1}(y^*) > \phi_i^{-1}(y)$ and may conclude that $\lambda_i(y^*) > \lambda_i(y)$.

It follows from (4) and (8) that $\widehat{x}_{ik}(y)$ minimizes $c_{ik}(x_{i\ell}) - \lambda_i(y) g_{ik}(x_{ik})$ subject to $x_{ik} \geq 0$ and therefore, for any k for which $\widehat{x}_{ik}(y) \neq \widehat{x}_{ik}(y^*)$, we have

$$c_{ik}(\widehat{x}_{ik}(y)) - \lambda_i(y) g_{ik}(\widehat{x}_{ik}(y)) < c_{ik}\left(\widehat{x}_{ik}\left(y^*\right)\right) - \lambda_i(y) g_{ik}\left(\widehat{x}_{ik}\left(y^*\right)\right),$$

$$c_{ik}\left(\widehat{x}_{ik}\left(y^*\right)\right) - \lambda_i\left(y^*\right) g_{ik}\left(\widehat{x}_{ik}\left(y^*\right)\right) < c_{ik}(\widehat{x}_{ik}(y)) - \lambda_i\left(y^*\right) g_{ik}(\widehat{x}_{ik}(y)).$$

In the case $\lambda_i(y) > 0$, dividing the first inequality by $\lambda_i(y)$, the second by $\lambda_i(y^*)$, adding the results and rearranging gives

$$\left[\frac{1}{\lambda_i(y)} - \frac{1}{\lambda_i(y^*)}\right]\left[c_{ik}\left(\widehat{x}_{ik}\left(y^*\right)\right) - c_{ik}\left(\widehat{x}_{ik}(y)\right)\right] > 0.$$

Since we have already shown that $\lambda_i(y^*) > \lambda_i(y)$, we can deduce that $c_{ik}(\widehat{x}_{ik}(y^*)) > c_{ik}(\widehat{x}_{ik}(y))$. Since c_{ik} is a strictly increasing function, this implies $\widehat{x}_{ik}(y^*) > \widehat{x}_{ik}(y)$. In the case $\lambda(y) = 0$, we have $\widehat{x}_{ik}(y) = 0$ and $c_{ik}(\widehat{x}_{ik}(y^*)) > 0 = c_{ik}(\widehat{x}_{ik}(y))$. Hence, $\widehat{x}_{ik}(y^*) > \widehat{x}_{ik}(y)$. ∎

Proof of Lemma 7 By definition,

$$C_i(y) = c_{i1}(\widehat{x}_{i1}(y)) + D_i(y - g_{i1}(\widehat{x}_{i1}(y)))$$

for any $y \geq 0$, where $\widehat{x}_{ik}(y)$ is the optimal solution of (3). For any $y' > y$, the definition of C_i implies

$$\begin{aligned} C_i\left(y'\right) &= c_{i1}\left(\widehat{x}_{i1}\left(y'\right)\right) + D_i\left(y' - g_{i1}\left(\widehat{x}_{i1}\left(y'\right)\right)\right) \\ &\leq c_{i1}(\widehat{x}_{i1}(y)) + D_i\left(y' - g_{i1}(\widehat{x}_{i1}(y))\right). \end{aligned}$$

Hence,

$$\begin{aligned} C_i\left(y'\right) - C_i(y) &\leq D_i\left(y' - g_{i1}(\widehat{x}_{i1}(y))\right) - D_i(y - g_{i1}(\widehat{x}_{i1}(y))) \\ &\leq D_i\left(y'\right) - D_i(y), \end{aligned}$$

using convexity of D_i (Lemma 1) and the fact that $g_{i1}(\widehat{x}_{i1}(y)) > 0$. Dividing by $y' - y$ and letting $y' \longrightarrow y$ gives the result. ∎

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