Chapter 2
One-Sided Reflected Brownian Motions and Related Models

For reflected Brownian motions with index set $\mathbb{Z}_+$ we properly define the dynamics using an iterated Skorokhod construction. The full index set $\mathbb{Z}$ requires a proof of well-posedness. In addition, we establish that the uniform Poisson process is stationary under the dynamics and discuss previous results for the model.

2.1 Skorokhod Construction

Our definition of reflected Brownian motions is through the so-called Skorokhod representation (Skorokhod 1961; Anderson and Orey 1976), which is a deterministic function of the driving Brownian motions. This representation is the following: the process $x(t)$, driven by the Brownian motion $B(t)$, starting from $x(0) \in \mathbb{R}$ and being reflected (in the positive direction) at some continuous function $f(t)$ with $f(0) \le x(0)$ is defined as:

$$\begin{aligned} x(t) &= x(0) + B(t) - \min\big\{0, \inf_{0\le s\le t}(x(0) + B(s) - f(s))\big\} \\ &= \max\big\{x(0) + B(t), \sup_{0\le s\le t}(f(s) + B(t) - B(s))\big\}. \end{aligned} \tag{2.1.1}$$

Let $B_n, n \in \mathbb{Z}$, be independent standard Brownian motions starting at 0. Throughout this work, B_n always denotes these Brownian motions, allowing for coupling arguments. The non-positive indices will be used from Sect. 2.3 on.

Definition 2.1 The half-infinite system of one-sided reflected Brownian motions $\{x_n(t), n \ge 1\}$ with initial condition $\vec{x}(0) = \vec{\zeta}$, $\zeta_n \le \zeta_{n+1}$ is defined recursively by $x_1(t) = \zeta_1 + B_1(t)$ and, for $n \ge 2$,

$$x_n(t) = \max\big\{\zeta_n + B_n(t), \sup_{0\le s\le t}(x_{n-1}(s) + B_n(t) - B_n(s))\big\}. \tag{2.1.2}$$

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Introducing the random variables

$$Y_{k,n}(t) = \sup_{0\le s_k\le\ldots\le s_{n-1}\le t} \sum_{i=k}^{n} (B_i(s_i) - B_i(s_{i-1})) \tag{2.1.3}$$

for $k \le n$, with the convention $s_{k-1} = 0$ and $s_n = t$, allows for an equivalent explicit expression:

$$x_n(t) = \max_{k\in[1,n]} \{Y_{k,n}(t) + \zeta_k\}. \tag{2.1.4}$$

Although we are constructing an infinite system of particles, well-definedness is clear in this case, as each process $x_n(t)$ is a deterministic function of only finitely many Brownian motions B_k, $1 \le k \le n$.

Adopting a stochastic analysis point of view, the system $\{x_n(t), n \ge 1\}$ satisfies

$$x_n(t) = \zeta_n + B_n(t) + L^n(t), \quad \text{for } n \ge 0, \tag{2.1.5}$$

Here, $L^1(t) = 0$, while L^n, $n \ge 2$, are continuous non-decreasing processes increasing only when $x_n(t) = x_{n-1}(t)$. In fact, L^n is twice the semimartingale local time at zero of $x_n - x_{n-1}$.

2.2 Packed Initial Conditions

A canonical and in fact the most studied initial condition for the system $\{x_n(t), n \ge 1\}$ is the one where all particles start at zero. We call this *packed initial condition*:

$$\vec{x}(0) = \vec{\zeta}^{\text{packed}} = 0. \tag{2.2.1}$$

Using the monotonicity,

$$\begin{aligned} Y_{k-1,n}(t) &= \sup_{0\le s_{k-1}\le s_k\le\ldots\le s_{n-1}\le t} \sum_{i=k-1}^{n} (B_i(s_i) - B_i(s_{i-1})) \\ &\ge \sup_{0=s_{k-1}\le s_k\le\ldots\le s_{n-1}\le t} \sum_{i=k}^{n} (B_i(s_i) - B_i(s_{i-1})) = Y_{k,n}(t), \end{aligned} \tag{2.2.2}$$

and inserting this initial condition into (2.1.4), leads to:

$$x_n(t) = Y_{1,n}(t) = \sup_{0\le s_1\le\ldots\le s_{n-1}\le t} \sum_{i=1}^{n} (B_i(s_i) - B_i(s_{i-1})), \tag{2.2.3}$$

again with the convention $s_0 = 0$ and $s_n = t$.

2.2.1 Queues, Last Passage Percolation and Directed Polymers

There are other interpretations for the quantity $x_n(t)$ than the system of reflected Brownian motions focused on in this work, and each interpretation has inspired different results over the last decades. One of these is seeing it as a sequence of *Brownian queues in series*. A famous theorem of queueing theory, *Burke's theorem*, which states that the output of a stable, stationary M/M/1 queue is Poisson, can be adapted to the Brownian setting (O'Connell and Yor 2001). This will be employed in Sect. 2.3.4.

Furthermore, $x_n(t)$ can be viewed as a model of directed last-passage percolation through a random medium, or equivalently a zero-temperature directed polymer in a random environment. This model is constructed as follows:

Consider the space $\mathbb{R}_+ \times \mathbb{Z}$ and assign white noise $\mathrm{d}B_n$ as random background weight on each line $\mathbb{R}_+ \times \{n\}$ for $n \geq 1$. An up-right path is characterized by its jumping points s_i and consists of line segments $[s_{n-1}, s_n] \times \{n\}$, see Fig. 2.1. The set of up-right paths going from (t_1, n_1) to (t_2, n_2) can then be parameterized by

$$\Pi(t_1, n_1; t_2, n_2) = \left\{ \vec{s} \in \mathbb{R}^{n_2 - n_1 + 2} | t_1 = s_{n_1 - 1} \leq s_{n_1} \leq \cdots \leq s_{n_2} = t_2 \right\}. \quad (2.2.4)$$

The *percolation time* or *weight* of a path $\vec{\pi} \in \Pi$ is the integral over the background weights along the path. Explicitly, we have:

$$w(\vec{\pi}) = \sum_{i=n_1}^{n_2} \left(B_i(s_i) - B_i(s_{i-1}) \right). \quad (2.2.5)$$

The *last passage percolation time* is given by the supremum over all such paths:

$$L_{(t_1,n_1)\to(t_2,n_2)} := \sup_{\vec{\pi} \in \Pi(t_1,n_1;t_2,n_2)} w(\vec{\pi}). \quad (2.2.6)$$

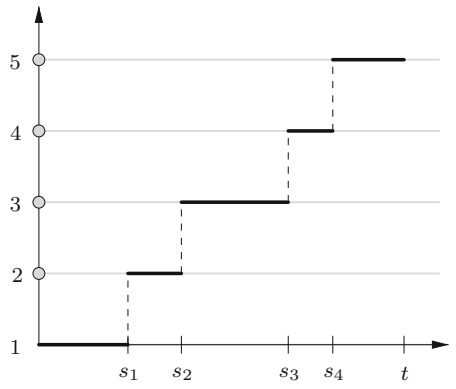

Fig. 2.1 A path $\pi \in \Pi(0, 1; t, 5)$ (*thick black*) and the random background noise (*gray*)

The supremum is almost surely attained by a unique path $\vec{\pi}^*$, called the maximizer. It exists because the supremum can be rewritten as a composition of a finite maximum and a supremum of a continuous function over a compact set. Uniqueness follows from elementary properties of the Brownian measure. Most importantly, from the definition, we have

$$x_n(t) = L_{(0,1)\to(t,n)}. \tag{2.2.7}$$

This representation will be used repeatedly throughout this work as it nicely visualizes coupling arguments, however, it also gives some connections to different works. Our model can be seen as the semi-continuous limit of a more widely studied discrete last passage percolation model (see for example Johansson (2000, 2003)).

This last passage percolation model is also the zero temperature limit of a directed polymer model, which has been studied thoroughly in the recent past (Seppäläinen and Valkó 2010; Borodin et al. 2015). In the directed polymer setting we have a parameter β representing the inverse temperature, consider $w(\vec{\pi})$ as an *energy* and assign a Gibbs measure on the set of paths according to the density $e^{\beta w(\vec{\pi})}$, i.e. paths with higher energy have a higher probability. The partition function of the polymer is given by

$$Z_{(t_1,n_1)\to(t_2,n_2)}(\beta) = \int_{\Pi(t_1,n_1;t_2,n_2)} \mathrm{d}\vec{\pi}\, e^{\beta w(\vec{\pi})}, \tag{2.2.8}$$

and satisfies the limit

$$\lim_{\beta\to\infty} \frac{1}{\beta} \log Z_{(t_1,n_1)\to(t_2,n_2)}(\beta) = L_{(t_1,n_1)\to(t_2,n_2)}. \tag{2.2.9}$$

Although apparently more difficult to handle, attention has been turning to these positive temperature models recently, because, among other things, they allow for a scaling limit to the KPZ equation by tuning the parameter β in the right way.

2.2.2 *Previous Results*

The behaviour of $x_n(t)$ under the packed initial condition is quite well understood by now. Notice that by Brownian scaling we have the distributional identity $\{x_n(t), n \geq 1\} \stackrel{d}{=} \{\sqrt{t}x_n(1), n \geq 1\}$. The first result has been a law of large numbers, i.e. $x_n(1)/\sqrt{n}$ converges to a constant (Glynn and Whitt 1991), therein it was already conjectured it equals 2, which has been proven subsequently (Seppäläinen 1997). Much of the results that followed exploited connections to a random matrix model, so let us introduce it in full generality right away:

Let $b_{i,i}(t)$ for $1 \leq i \leq N$ and $b_{i,j}(t)$, $b'_{i,j}(t)$ for $1 \leq i < j \leq N$, be independent Brownian motions. Define a stochastic process $H(t)$, $t \geq 0$, on the space of $N \times N$ Hermitian matrices by

$$\begin{aligned} H_{i,i}(t) &= b_{i,i}(t) && \text{for } 1 \le i \le N \\ H_{i,j}(t) &= \sqrt{2}\left(b_{i,j}(t) + \mathrm{i}b'_{i,j}(t)\right) && \text{for } 1 \le i < j \le N \\ H_{i,j}(t) &= \sqrt{2}\left(b_{i,j}(t) - \mathrm{i}b'_{i,j}(t)\right) && \text{for } 1 \le j < i \le N \end{aligned} \tag{2.2.10}$$

Denote by $\lambda_1^n(t) \le \lambda_2^n(t) \le \cdots \le \lambda_n^n(t)$ the ordered eigenvalues of the $n \times n$ principal minor of the matrix $H(t)$. The process $\{\lambda_k^n(t), 1 \le k \le n \le N\}$ is called the *Dyson Brownian minor process*. It is a classical result, that the eigenvalues of consecutive minors are interlaced, $\lambda_k^{n+1}(t) \le \lambda_k^n(t) \le \lambda_{k+1}^{n+1}(t)$, so this process lives in the Gelfand–Tsetlin cone:

$$\mathrm{GT}_N = \left\{x_k^n \in \mathbb{R}, 1 \le k \le n \le N, x_k^{n+1} \le x_k^n \le x_{k+1}^{n+1}\right\}. \tag{2.2.11}$$

Restricted to one layer n, the process $\{\lambda_k^n(t), 1 \le k \le n\}$ is called *Dyson's Brownian motion* (Dyson 1962). It is a Markov process and satisfies the coupled stochastic differential equation

$$\mathrm{d}\lambda_k^n = \mathrm{d}B_k + \sum_{i \ne k} \frac{\mathrm{d}t}{\lambda_k^n - \lambda_i^n}. \tag{2.2.12}$$

Its fixed time distribution is the eigenvalue distribution of the Gaussian unitary ensemble (GUE):

$$\mathbb{P}\left(\lambda_k^n(t) \in \mathrm{d}y_k, 1 \le k \le n\right) = \frac{1}{Z_n(t)} \prod_{k=1}^{N} e^{-\frac{y_k^2}{2t}} \prod_{k<i} (y_i - y_k)^2 \mathrm{d}\vec{y}. \tag{2.2.13}$$

Recognizing the second product as the square of the Vandermonde determinant, one is able to describe this eigenvalue distribution as a determinantal point process governed by the Hermite kernel. Asymptotic analysis of the distribution of the largest eigenvalue in the appropriate *edge scaling* gives the GUE Tracy–Widom distribution (Tracy and Widom 1994).

The first connections between our system of reflecting Brownian motions and the matrix diffusion were found by Gravner et al. (2001), proving that for every $n \ge 1$, $x_n(1) \stackrel{d}{=} \lambda_n^n(1)$, and Baryshnikov (2001) generalizing this to $\{x_n(1), n \ge 1\} \stackrel{d}{=} \{\lambda_n^n(1), n \ge 1\}$ by a combinatorial procedure originating from group representation theory, called the Robinson–Schensted–Knuth (RSK) correspondence. Baryshnikov (2001) also showed the remarkable fact that conditioned on the top layer $\{\lambda_k^N(1), 1 \le k \le N\}$ the distribution of $\{\lambda_k^n(1), 1 \le k \le n < N\}$ is uniform on the compact set given by the Gelfand–Tsetlin interlacing inequality. Restricting the Dyson Brownian minor process to a fixed time gives the *GUE minor process* $\{\lambda_k^n(1), 1 \le k \le n \le N\}$, whose full distribution has been found in Johansson and Nordenstam (2006), again in the form of a determinantal point process.

There is a natural extension of the system of reflecting Brownian motions $\{x_n(t), n \ge 1\}$ to a process in the Gelfand–Tsetlin cone that is constructed in the following way: Let $B_1^1(t)$ be a Brownian motion. Let $B_1^2(t)$ and $B_2^2(t)$ be Brownian

motions, which are reflected downwards resp. upwards from $B_1^1(t)$. Iteratively construct $B_k^n(t)$ as a Brownian motion reflected downwards from $B_k^{n-1}(t)$ and upwards from $B_{k-1}^{n-1}(t)$, with the peripheral processes $B_k^n(t)$ for $k = 1$ or $k = n$ being reflected from one process only. By construction, we have $x_n(t) = B_n^n(t)$. The process $\{B_k^n(t), 1 \le k \le n \le N\}$ is called *Warren's process*, and has been introduced and studied in Warren (2007). Restricted to one layer n, it is distributed as a Dyson's Brownian motion. Warren's process shares the fixed time distribution with the GUE minor process,

$$\{\lambda_k^n(1), 1 \le k \le n \le N\} \stackrel{d}{=} \{B_k^n(1), 1 \le k \le n \le N\}. \tag{2.2.14}$$

There are also formulas for the transition density of the system along the edge, $\{B_n^n(t), 1 \le n \le N\}$ as well as for the system of two consecutive layers $\{B_n^k(t), 1 \le k \le n, N-1 \le n \le N\}$.

The connection between Warren's process and the Dyson Brownian minor process does not, however, hold in full generality. The common dynamics of any amount of consecutive layers of Warrens process, i.e. the process $\{B_n^k(t), 1 \le k \le n, N_1 \le n \le N_2\}$, is simply given by Dyson's SDE (2.2.12) for the layer $n = N_1$ and the reflection SDE's for the higher order layers. In the Dyson Brownian minor process, on the other hand, the common dynamics of two consecutive layers is given by a more complicated SDE (see (2.30) in Adler et al. (2014)) and the evolution of three or more consecutive layers is not even a Markov process anymore. Interestingly, both processes still show the same distribution along so-called space-like paths, i.e. sequences of points (n_i, t_i) satisfying $t_i \le t_{i+1}$ and $n_i \ge n_{i+1}$, in which case also determinantal formulas exist (Adler et al. 2014; Ferrari and Frings 2010).

The determinantal formulas coming from the random matrix model are suitable for asymptotic analysis to show multi-point scaling limits, where the Airy_2 process arises. It appears both for correlations of $x_n(t)$ along the n direction and the t direction, as well as along general space-like paths. Reference (Johansson 2003) gives a sketch of the proof for both directions, (Adler and Moerbeke 2005) prove the scaling limit with correlations along t rigorously. A complete proof for correlations along n is given in Sect. 5.1.

2.3 Infinite Particle Systems

The main focus of this work is showing determinantal formulas and scaling limits for other initial conditions. Unfortunately the connection to random matrices breaks down in this case. In fact, neither the Airy_1 process nor the $\text{Airy}_{\text{stat}}$ process have ever been found in a scaling limit of a random matrix model. This has been rather surprising, as the one-point distribution of the Airy_1 process, the GOE Tracy–Widom distribution, *does* arise in such a model, namely as the limiting distribution of the largest eigenvalue of a Gaussian real symmetric matrix, the GOE ensemble (Bornemann et al. 2008).

2.3.1 Definition

At first sight it might seem trivial to extend the definition of the half-infinite system of one-sided reflected Brownian motions to an infinite number of particles $\{x_n(t), n \in \mathbb{Z}\}$, by letting the index k run over $(-\infty, n]$ in (2.1.4). However, it has to be shown that this maximum is finite, which is only the case for initial conditions which are not too closely spaced together. Roughly said, the growth rate of $-\zeta_{-k}$ has to be faster than $\sqrt{k}$ for large k. We call those initial conditions *admissible*.

Knowing the law of large numbers under packed initial conditions, it is reasonable to expect such a behaviour. It implies that $Y_{k,n}(t)$ grows roughly as $2\sqrt{(n-k)t}$, so in order for the maximum in (2.3.2) being attained by a finite k for we need $-\zeta_{-k}$ to grow faster than $\sqrt{k}$.

Definition 2.2 A random vector $\vec{\zeta} \in \mathbb{R}^{\mathbb{Z}}$ with $\zeta_n \leq \zeta_{n+1}$ for all $n \in \mathbb{Z}$ is called an *admissible initial condition*, if there exists a $\chi > \frac{1}{2}$ such that for any $n \in \mathbb{Z}$ the sum

$$\sum_{M \geq 0} \mathbb{P}\left(\zeta_n - \zeta_{-M} \leq M^{\chi}\right) \tag{2.3.1}$$

is finite.

Definition 2.3 Let $\vec{\zeta} \in \mathbb{R}^{\mathbb{Z}}$ be an admissible initial condition. The infinite system of one-sided reflected Brownian motions $\{x_n(t), n \in \mathbb{Z}\}$ with initial condition $\vec{x}(0) = \vec{\zeta}$ is defined by

$$x_n(t) = \max_{k \leq n} \{Y_{k,n}(t) + \zeta_k\}. \tag{2.3.2}$$

By Proposition 2.4 below, which is proven in Sect. 2.3.2, this maximum exists and is finite. More specifically, we will show that for $\vec{\zeta}$ being any admissible initial condition, the infinite system $\{x_n(t), n \in \mathbb{Z}\}$ is the limit of certain half-infinite systems $\{x_n^{(M)}(t), n \geq -M\}$ as $M \to \infty$, where

$$x_n^{(M)}(t) = \max_{k \in [-M,n]} \{Y_{k,n}(t) + \zeta_k\}, \quad n \geq -M. \tag{2.3.3}$$

Notice that these processes indeed satisfy the Skorokhod equation,

$$x_n^{(M)}(t) = \max\left\{\zeta_n + B_n(t), \sup_{0 \leq s \leq t} (x_{n-1}^{(M)}(s) + B_n(t) - B_n(s))\right\}, \tag{2.3.4}$$

for $n > -M$, while the leftmost process is simply

$$x_{-M}^{(M)}(t) = \zeta_{-M} + B_{-M}(t). \tag{2.3.5}$$

Thus as desired $x_n^{(M)}(t)$ is a Brownian motion starting from ζ_n and reflected off by $x_{n-1}^{(M)}$ for $n > -M$.

Proposition 2.4 *For any $t > 0$, $n \in \mathbb{Z}$ there exists almost surely a $k \leq n$ maximizing $Y_{k,n}(t) + \zeta_k$, i.e. the maximum in (2.3.2) exists. Furthermore, for any $T > 0$,*

$$\sup_{t\in[0,T]} |x_n(t)| < \infty, \quad \text{a.s.}, \tag{2.3.6}$$

as well as

$$\lim_{M\to\infty} \sup_{t\in[0,T]} |x_n^{(M)}(t) - x_n(t)| = 0, \quad \text{a.s.}. \tag{2.3.7}$$

The convergence result allows for taking the limit in (2.3.4), implying that the system $\{x_n(t), n \in \mathbb{Z}\}$ satisfies the Skorokhod equation, too.

The initial conditions we are actually interested in are corresponding to the two remaining fundamental geometries in the KPZ universality class. The first one is the flat surface, which translates into periodic initial conditions $\vec{x}(0) = \vec{\zeta}^{\text{flat}}$, defined by

$$\zeta_n^{\text{flat}} = n, \text{ for } n \in \mathbb{Z}, \tag{2.3.8}$$

which is obviously admissible.

Finally we will study the case where the model starts in its random stationary distribution, which is in our case a Poisson point process on the real line. However, as already familiar from other models in the KPZ universality class (Borodin et al. 2015; Baik et al. 2010; Ferrari and Spohn 2006; Imamura and Sasamoto 2004), Theorem 6.1 will be proven via a sequence of approximating initial conditions.

Let therefore be $\{\text{Exp}_n, n \in \mathbb{Z}\}$ be i.i.d. random variables with exponential distribution with parameter 1. For parameters $\lambda > 0$ and $\rho > 0$ define the initial condition $\vec{x}(0) = \vec{\zeta}^{\text{stat}}(\lambda, \rho)$ by

$$\begin{aligned} \zeta_0^{\text{stat}} &= 0, \\ \zeta_n^{\text{stat}} - \zeta_{n-1}^{\text{stat}} &= \begin{cases} \lambda^{-1}\,\text{Exp}_n, & \text{for } n > 0, \\ \rho^{-1}\,\text{Exp}_n, & \text{for } n \leq 0. \end{cases} \end{aligned} \tag{2.3.9}$$

Admissibility of this initial condition is also not hard to prove.

To recover the uniform Poisson process on the whole real line, we will set $\lambda = 1$ and carefully take the limit $\rho \to 1$ in the determinantal formulas that hold in the case $\rho < \lambda$. Finally, setting $\zeta_0 = 0$ will induce a difference of order one as compared to the true Poisson process case. By Proposition 7.11 this difference will stay bounded at all times, and consequently be irrelevant in the scaling limit. Thus it is enough to prove Theorem 6.1 for the initial conditions $\vec{x}(0) = \vec{\zeta}^{\text{stat}}(1, 1)$.

2.3.2 Well-Definedness

For the proof of Proposition 2.4 we first need the following concentration inequality:

Proposition 2.5 *For each $T > 0$ there exists a constant $C > 0$ such that for all $k < m$, $\delta > 0$,*

$$\mathbb{P}\left(\frac{Y_{k,m}(T)}{\sqrt{(m-k+1)T}} \geq 2+\delta\right) \leq \text{const} \cdot e^{-(m-k+1)^{2/3}\delta}. \tag{2.3.10}$$

This proposition is proven in Sect. 5.1.2. Another necessary lemma, that will be proven an intuitive way in Sect. 2.3.3, is:

Lemma 2.6 *Consider $0 \leq t_1 \leq t_2$ and m, M_{t_1}, M_{t_2} such that*

$$x_m(t_i) = x_m^{(M_{t_i})}(t_i) = \widetilde{x}_m^{(M_{t_i})}(t_i), \quad \text{for } i = 1, 2. \tag{2.3.11}$$

Then

$$x_m(t_1) = x_m^{(M_{t_2})}(t_1) = \widetilde{x}_m^{(M_{t_2})}(t_1). \tag{2.3.12}$$

Proof (*Proof of Proposition* 2.4) Let us define an auxiliary system of processes, which we will use later in proving Proposition 2.7, by

$$\widetilde{x}_{-M}^{(M)}(t) = \zeta_{-M} + B_{-M}(t) + \rho t, \tag{2.3.13}$$

and

$$\widetilde{x}_n^{(M)}(t) = \max\left\{\zeta_n + B_n(t), \sup_{0\leq s\leq t}(\widetilde{x}_{n-1}^{(M)}(s) + B_n(t) - B_n(s))\right\} \tag{2.3.14}$$

for $n > -M$. This system differs from $x_n^{(M)}(t)$ just in the drift of the leftmost particle, which of course influences all other particles as well (the choice of the extra drift is because the system with infinite many particles in $\mathbb{R}_-$ generates a drift ρ). This system of particles satisfies

$$\widetilde{x}_n^{(M)}(t) = \max\left\{\widetilde{Y}_{-M,n}(t) + \zeta_{-M}, \max_{k\in[-M+1,n]}\{Y_{k,n}(t) + \zeta_k\}\right\}, \tag{2.3.15}$$

with

$$\widetilde{Y}_{k,n}(t) = \sup_{0\leq s_{k+1}\leq\ldots\leq s_m\leq t}\left(\rho s_{k+1} + \sum_{i=k}^{n}(B_i(s_{i+1}) - B_i(s_i))\right). \tag{2.3.16}$$

Also, we have the inequalities

$$Y_{k,n}(t) \leq \widetilde{Y}_{k,n}(t) \leq Y_{k,n}(t) + \rho t. \tag{2.3.17}$$

Consider the event

$$\begin{aligned} A_M := &\{Y_{-M,n}(T) \geq 3\sqrt{(M+n+1)T}\} \cup \{\zeta_n - \zeta_{-M} \leq M^\chi\} \\ &\cup \{Y_{n,n}(T) \leq \rho T + 3\sqrt{(M+n+1)T} - M^\chi\}. \end{aligned} \tag{2.3.18}$$

It is now straightforward to show $\sum_{M=0}^{\infty} \mathbb{P}(A_M) < \infty$. In fact, summability of the probabilities of the first set in (2.3.18) is a consequence of Proposition 2.5, applied with $\delta = 1$, while the second set is covered by the definition of an admissible initial condition. For the third set, notice that the left hand side is a Gaussian distribution independent of M, while the right hand side is dominated by the M^{χ} term for large M. Finiteness of the sum allows applying Borel–Cantelli, i.e. A_M occurs only finitely many times almost surely. This means, that a.s. there exists a M_T, such that for all $M \geq M_T$ the following three inequalities hold:

$$\begin{aligned} Y_{-M,n}(T) &< 3\sqrt{(M+n+1)T} \\ M^{\chi} &< \zeta_n - \zeta_{-M} \\ -Y_{n,n}(T) &< -\rho T - 3\sqrt{(M+n+1)T} + M^{\chi} \end{aligned} \tag{2.3.19}$$

Adding up these, $Y_{-M,n}(T) + \zeta_{-M} + \rho T < Y_{n,n}(T) + \zeta_n$ for all $M \geq M_T$ and dropping the term ρT shows us that the maximizing element in (2.3.2) cannot be a $k \leq -M_T$, or

$$x_n(T) = x_n^{(M_T)}(T). \tag{2.3.20}$$

Moreover, applying (2.3.17), gives

$$\widetilde{Y}_{-M_T,n}(t) + \zeta_{-M_T} \leq Y_{-M_T,n}(t) + \zeta_{-M_T} + \rho T < Y_{n,n}(T) + \zeta_n, \tag{2.3.21}$$

resulting in

$$\widetilde{x}_n^{(M_T)}(T) = x_n^{(M_T)}(T). \tag{2.3.22}$$

Repeating the same argument, we see that for every $t \in [0, T]$ there exists M_t such that $x_n(t) = x_n^{(M_t)}(t) = \widetilde{x}_n^{(M_t)}(t)$. Applying Lemma 2.6 then gives $x_n(t) = x_n^{(M_T)}(t) = \widetilde{x}_n^{(M_T)}(t)$ for every $t \in [0, T]$. This settles the convergence and the existence of a finite maximizing k in (2.3.2).

To see (2.3.6), which is equivalent to $\sup_{t\in[0,T]} |x_m^{(M_T)}(t)| < \infty$, we apply the bound

$$|Y_{k,n}(t)| \leq \sum_{i=k}^{n} \Big(\sup_{0\leq s\leq t} B_i(s) - \inf_{0\leq s\leq t} B_i(s) \Big) < \infty. \tag{2.3.23}$$

2.3.3 *Last Passage Percolation*

It is also possible to extend the last passage percolation interpretation to nontrivial initial conditions. In order to do this, add non-negative Dirac background weights $\zeta_k - \zeta_{k-1}$ on $(0, k)$, $k \in \mathbb{Z}$. The weight of a path is explicitly given by

$$w(\vec{\pi}) = \sum_{i=n_1}^{n_2} \left(B_i(s_i) - B_i(s_{i-1}) + (\zeta_i - \zeta_{i-1}) \mathbb{1}_{s_{i-1}=0} \right), \tag{2.3.24}$$

and the percolation time $L_{(0,n_1)\to(t,n_2)}$ again as the supremum over the weight of all paths. As $t \to 0$ it is clear that any contribution from the Brownian background weight will converge to 0, so the path tries to accumulate as much of the Dirac weights as possible, i.e. we have the initial condition

$$\lim_{t\to 0} L_{(0,n_1)\to(t,n_2)} = \sum_{i=n_1}^{n_2} (\zeta_i - \zeta_{i-1}) = \zeta_{n_2} - \zeta_{n_1-1}. \tag{2.3.25}$$

By defining a normalized percolation time,

$$\widehat{L}_{(0,n_1)\to(t,n_2)} = L_{(0,n_1)\to(t,n_2)} + \zeta_{n_1-1}, \tag{2.3.26}$$

we recover the system

$$x_n^{(M)}(t) = \widehat{L}_{(0,-M)\to(t,n)}. \tag{2.3.27}$$

For any $M \le n$, $M = -\infty$ included, we can define an *exit point* of a path $\pi = (\ldots, s_{n_2-1}, s_{n_2}) \in \Pi(0, -M; t, n)$ by

$$\inf\{k \in [n_1, n_2], s_k > 0\}, \tag{2.3.28}$$

which is of course the maximizing index k in (2.3.3).

We also can reproduce the system $\widetilde{x}_n^{(M)}(t)$ by adding a Lebesgue measure of density ρ on the line $\{-M\} \times \mathbb{R}_+$.

Proof (*Proof of Lemma* 2.6) For $t_1 = t_2$ there is nothing to prove, so let $t_1 < t_2$. For each i, the equation $x_m(t_i) = x_m^{(M_{t_i})}(t_i)$ implies that the maximizing paths of the LHS and the RHS are equal on the restriction to $(s_i, i \ge M_i)$, i.e. they have the same exit point e_i that satisfies $e_i \ge M_{t_i}$. Now if $e_1 \ge e_2$, then also $e_1 \ge M_{t_2}$, which means that the path maximizing $x_m(t_1)$ is contained in the set $\Pi(0, -M_{t_2}; t_1, m)$, resulting in $x_m(t_1) = x_m^{(M_{t_2})}(t_1)$.

If, however, $e_1 < e_2$, then the maximizing path segments $(0, e_1) \to (t_1, m)$ and $(0, e_2) \to (t_2, m)$ would need to have an intersection point (t^*, m^*). We can then construct a new maximizing path for $x_m(t_1)$ by stringing together the segments $(0, e_1) \to (0, e_2) \to (t^*, m^*) \to (t_1, m)$, where the middle segment is part of the $x_m(t_2)$-maximizing path and the last segment is part of the original $x_m(t_1)$-maximizing path. This contradicts the uniqueness of the maximizing path.

The equality $x_m(t_1) = \widetilde{x}_m^{(M_{t_2})}(t_1)$ is shown in the same way.

2.3.4 Stationarity

We establish a useful property which will allow us to study our system of interacting Brownian motions through a system with a left-most Brownian particle.

Proposition 2.7 *Under the initial condition* $\vec{x}(0) = \vec{\zeta} = \vec{\zeta}^{\text{stat}}(\lambda, \rho)$ *defined in* (2.3.9)*, for each* $n \leq 0$ *the process*

$$x_n(t) - \zeta_n - \rho t \tag{2.3.29}$$

is a standard Brownian motion.

Remark 1 Proposition 2.7 allows us to restrict our attention to the half-infinite system. In fact, conditioned on the path of x_0, the systems of particles $\{x_n(t), n < 0\}$ and $\{x_n(t), n > 0\}$ are independent, as it is clear by the definition of the system. Then (2.3.29) implies that the law of $\{x_n(t), n > 0\}$ is the same as the one obtained replacing the infinitely many particles $\{x_m(t), m \leq 0\}$ with a single Brownian motion $x_0(t)$ which has a drift ρ. This property will be used to derive our starting result, Proposition 6.2.

Remark 2 From a stochastic analysis point of view, we find that the system $\{x_n(t), n \geq 0\}$ satisfies

$$\begin{aligned} x_n(t) &= \zeta_n + B_n(t) + L^n(t), \quad \text{for } n \geq 1, \\ x_0(t) &= \widetilde{B}_0(t) + \rho t. \end{aligned} \tag{2.3.30}$$

Here L^n, $n \geq 2$, are continuous non-decreasing processes increasing only when $x_n(t) = x_{n-1}(t)$. In fact, L^n is twice the semimartingale local time at zero of $x_n - x_{n-1}$. Notice that $\widetilde{B}_0(t)$ is a standard Brownian motion independent of $\{\zeta_n, B_n(t), n \geq 1\}$, but not equal to $B_0(t)$.

Proof (*Proof of Proposition* 2.7) First notice that for any M,

$$\widetilde{x}^{(M)}_{-M}(t) - \zeta_{-M} - \rho t, \tag{2.3.31}$$

is a Brownian motion. Now assume $\widetilde{x}^{(M)}_{n-1}(t) - \zeta_{n-1} - \rho t$ is a Brownian motion. By definition,

$$\begin{aligned} &\widetilde{x}^{(M)}_n(t) - \zeta_{n-1} \\ &\quad = \max\left\{\zeta_n - \zeta_{n-1} + B_n(t), \sup_{0 \leq s \leq t} (\widetilde{x}^{(M)}_{n-1}(s) - \zeta_{n-1} + B_n(t) - B_n(s))\right\}, \end{aligned} \tag{2.3.32}$$

which allows us to apply Proposition 2.8, i.e., we have that

$$\widetilde{x}^{(M)}_n(t) - \zeta_{n-1} - (\zeta_n - \zeta_{n-1}) - \rho t = \widetilde{x}^{(M)}_n(t) - \zeta_n - \rho t \tag{2.3.33}$$

is a Brownian motion. Since $\widetilde{x}_n^{(M_T)}(t) = x_n(t)$ the proof is completed.

It is clear, that in the case $\lambda = \rho$ the process (2.3.29) is a Brownian motion for $n > 0$, too, i.e., the system is stationary in n. We also have stationarity in t, in the sense that for each $t \geq 0$ the random variables $\{x_n(t) - x_{n-1}(t), n \in \mathbb{Z}\}$ are independent and distributed exponentially with parameter ρ. The following result is a small modification of Theorem 2 in O'Connell and Yor (2001).

Proposition 2.8 (Burke's theorem for Brownian motions) *Fix $\rho > 0$ and let $B(t)$, $C(t)$ be standard Brownian motions, as well as $\zeta \sim \exp(\rho)$, independent. Define the process*

$$D(t) = \max\left\{\zeta + C(t), \sup_{0\leq s\leq t}(B(s) + \rho s + C(t) - C(s))\right\}. \tag{2.3.34}$$

Then

$$D(t) - \zeta - \rho t \tag{2.3.35}$$

is distributed as a standard Brownian motion.

Proof Extend the processes $B(t)$, $C(t)$ to two-sided Brownian motions indexed by $\mathbb{R}$. Defining

$$q(t) = \sup_{-\infty<s\leq t}\{B(t) - B(s) + C(t) - C(s) - \rho(t-s)\} \tag{2.3.36}$$

and

$$d(t) = B(t) + q(0) - q(t), \tag{2.3.37}$$

we can apply Theorem 2 (O'Connell and Yor 2001), i.e., $d(t)$ is a Brownian motion. Now,

$$q(0) = \sup_{s\leq 0}\{-B(s) - C(s) + \rho s\} \stackrel{d}{=} \sup_{s\geq 0}\{\sqrt{2}B(s) - \rho s\} \stackrel{d}{=} \sup_{s\geq 0}\left\{B(s) - \frac{\rho}{2}s\right\}, \tag{2.3.38}$$

so by Lemma 2.9 it has exponential distribution with parameter ρ. As it is independent of the processes $\{B(t), C(t), t \geq 0\}$ we can write $q(0) = \zeta$. Dividing the supremum into $s < 0$ and $s \geq 0$ we arrive at:

$$\begin{aligned} -d(t) &= q(t) - B(t) - q(0) \\ &= \max\left\{C(t) - \rho t, \sup_{0\leq s\leq t}\{-B(s) + C(t) - C(s) - \rho(t-s)\} - \zeta\right\}, \end{aligned} \tag{2.3.39}$$

which is (2.3.35) up to a sign flip of $B(s)$.

Lemma 2.9 *Fix $\rho > 0$ and let $B(t)$ be a standard Brownian motion. Then*

$$\sup_{s\geq 0}(B(s) - \rho s) \sim \exp(2\rho). \tag{2.3.40}$$

Proof This formula can be found in Borodin and Salminen (2012), Part II, Sect. 2, Eq. (1.1.4). We provide here two different proofs of the claim.
(a) The random variable

$$\sup_{0\le s\le t}(B(s)-\rho s) \tag{2.3.41}$$

is distributed as $\tilde{B}(t)$, where $\tilde{B}$ is a Brownian motion starting at 0, reflected (upwards) at zero and drift $-\rho$. This follows from the Skorokhod construction. Indeed,

$$\begin{aligned}\tilde{B}(t) &\stackrel{d}{=} \sup_{0\le s\le t}(B(t)-B(s)-\rho(t-s))\\ &= \sup_{0\le s\le t}(B(t)-B(t-s)-\rho s) \stackrel{d}{=} \sup_{0\le s\le t}(B(s)-\rho s),\end{aligned} \tag{2.3.42}$$

where we changed s into $t-s$ and used the fact that $B(t)-B(t-s)$ has the same distribution of $B(s)$. As $t\to\infty$, this converges to the stationary distribution of this process, which is the exponential distribution with parameter 2ρ.
(b) For $x\ge 0$, define the stopping time $\tau_x=\inf\{s\ge 0\,|\,B(s)-s\rho/2\ge x\}$. Then,

$$\mathbb{P}\left(\sup_{s\ge 0}(B(s)-\rho s)\ge x\right)=\mathbb{P}(\tau_x<\infty). \tag{2.3.43}$$

The process $s\mapsto M_s=e^{\rho B(s)-\frac{1}{2}\rho^2 s}$ is a martingale (the geometric Brownian motion), with $M_0=1$. Thus by applying the optional sampling theorem (apply it first to $\tau_x\wedge T$ and then take $T\to\infty$) one obtains

$$1=\mathbb{E}(M_{\tau_x})=e^{\rho x}\mathbb{P}(\tau_x<\infty)+0\mathbb{P}(\tau_x=\infty), \tag{2.3.44}$$

from which $\mathbb{P}(\tau_x<\infty)=e^{-\rho x}$ as claimed (just replace ρ by 2ρ).

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