The Null Space Pursuit Algorithm Based on an Arbitrary Order Differential Operator

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Abstract. The Null Space Pursuit (NSP) algorithm based on the differential operator is an important method of signal denoising and signal separation. In this paper, we propose an arbitrary order differential operator and use it in a complex signal which is a sum of simple signal. By solving an optimization problem, we also estimate the parameters of the differential operator. Finally, we confirmed the practicability of the algorithm through experimental simulation.

Keywords: Null space pursuit · Differential operator · Signal separation

1 Introduction

In the recent years, the methods of signal denoising and the separation of signal have drawn greater attention of scholars. The process of signal separation involves breaking down a complex signal into a sum of simple signals. The methods used to separate signals vary because of the construct of the simple signal which is decomposed from complex signal are different. For instance in the empirical mode decomposition (EMD) method [1, 2], an oscillatory signal is resolved into a sum of intrinsic mode functions (IMFs) and in the Matching Pursuit (MP) method [3], a signal is resolved into a total of time-frequency atoms. The Null Space Pursuit (NSP) algorithm on the basis of a differential operator is of particular interest to us in all methods of signal separation.

Silong Peng and Wen-Liang Hwang (2008) conceptualized the NSP algorithm based on a differential operator [4], two years later during 2010 in the next course of attempt they further improved the NSP algorithm. This method makes use of an adaptive operator to separate a complex signal into a sum of simple signals, and these simple signals belong to the null space in the above. The important steps of the method include estimating the adaptive operator T_s from the input signal S and decomposing S into R and U, where R is the residual signal and U is extracted signal, which $T_s(U) = 0$. In [5], they developed the following second-order differential operator:

$$T_s = \frac{d^2}{dt^2} + \omega^2(t).$$

which can eliminate the FM signal $\cos(\phi(t))$, where $\phi(t)$ is a local linear function.

The NSP algorithm on the basis of a differential operator has attracted wide attention due to its adaptability. In 2011, Xiyuan Hu [6] put forwarded the null space

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pursuit algorithm and then further expanded the range of signal that could be decomposed. He empirically established the following second-order differential operator:

$$T_s = \frac{d^2}{dt^2} + P(t)\frac{d}{dt} + Q(t).$$

It can eliminate an AM-FM signal.

In this paper, we improve the algorithm [7] to use an arbitrary order differential operator. It can annihilate the signal:

$$A_1(t)\cos(\phi_1(t)) + A_2(t)\cos(\phi_2(t)) + \cdots + A_m(t)\cos(\phi_m(t)).$$

What's more,

This algorithm expands the range of signal that could be decomposed.

2 The Null Space Pursuit Algorithm Based on an Arbitrary Order Differential Operator

In this paper, we propose the following arbitrary order differential operator:

$$T_s = \frac{d^m}{dt^m} + a_{m-1}(t)\frac{d^{m-1}}{dt^{m-1}} + \cdots + a_1(t)\frac{d}{dt} + a_0(t). \tag{1}$$

This algorithm can estimate the orders of the above operator and the values of parameters $a_0, a_1, \cdots a_{m-1}$.

In a discrete case, the form of the operator is expressed as:

$$T_s = D_m + P_{a_{m-1}(t)}D_{m-1} + \cdots + P_{a_1(t)}D + P_{a_0(t)} \tag{2}$$

Where P_{a_i} is a diagonal matrix whose diagonal elements are a_i, where $i = 0, 1, \cdots m-1$, and $D_m(m = 1, 2, 3, \cdots)$ is the matrix of the n-order difference.

Then, the values of parameters $a_0, a_1, \cdots a_{m-1}$ and the signal R is estimated by the following optimization method that minimizes the problem:

$$\begin{aligned} \min_{a_0,\cdots,a_{m-1},R,\lambda_1,\gamma,\lambda_2} \Big\{ & \|T_s(S-R)\|^2 + \lambda_1\left(\|R\|^2 + \gamma\|S-R\|^2\right) \\ & + \lambda_2\left(\|D_{m-1}a_{m-1}\|^2 + \cdots + \|D_1 a_1\|^2 + \|a_0\|^2\right)\Big\} \end{aligned} \tag{3}$$

where S is the input signal, R is the residual signal, γ is the leakage parameter and λ_1, λ_2 are Lagrange parameters. Let

$$
\begin{aligned}
F(a_0,\cdots a_{m-1},R) &= \|T_s(S-R)\|^2+\lambda_1\left(\|R\|^2+\gamma\|S-R\|^2\right)\\
&+\lambda_2\left(\|D_{m-1}a_{m-1}\|^2+\cdots+\|D_1a_1\|^2+\|a_0\|^2\right)\Big\}
\end{aligned} \tag{4}
$$

For convenience, we let ϕ be the column vector as follows:

$$\phi=\left[a_{m-1}^T,a_{m-2}^T,\cdots a_1^T a_0^T\right]^T.$$

Then (4) becomes

$$F(\Phi,R)=\|(D_m+B_\Phi M_1)(S-R)\|^2+\lambda_1\left(\|R\|^2+\gamma\|S-R\|^2\right)+\lambda_2\left(\|M_2\Phi\|^2\right) \tag{5}$$

where $B_\Phi=[P_{a_{m-1}},P_{a_{m-2},}\cdots P_{a_1},P_{a_0}]$ $M_1=\left[D_{m-1}^T,D_{m-2}^T,\cdots D_2^T,D_1^T,E^T\right]^T$ and $M_2=\begin{pmatrix} D_{m-1} & & & & \\ & D_{m-2} & & & \\ & & \ddots & & \\ & & & D_1 & \\ & & & & E\end{pmatrix}$, in which E is the identity matrix.

For convenience, we rewrite the first item of (5) as the following:

$$
\begin{aligned}
&(D_m+B_\Phi M_1)(S-R)=D_m(S-R)+B_\Phi M_1(S-R)\\
&=D_m(S-R)+[P_{a_{m-1}},P_{a_{m-2}}\cdots,P_{a_1},P_{a_0}][D_{m-1}(S-R),\cdots,D_1(S-R),E(S-R)]^{\mathrm{T}}\\
&=D_m(S-R)+P_{D_{m-1}(S-R)}a_{m-1}+\cdots+P_{D_1(S-R)}a_1+P_{(S-R)}a_0\\
&=D_m(S-R)+\left[P_{D_{m-1}(S-R)},\cdots,P_{D_1(S-R)},P_{(S-R)}\right][a_{m-1},\cdots,a_1,a_0]^T\\
&=D_m(S-R)+A\phi
\end{aligned}
$$

where $A=[P_{D_{m-1}(S-R)},\cdots,P_{D_1(S-R)},P_{(S-R)}]$. Then (5) becomes

$$F(\phi,R)=\|D_m(S-R)+A\phi\|^2+\lambda_1\left(\|R\|^2+\gamma\|S-R\|^2\right)+\lambda_2\left(\|M_2\phi\|^2\right). \tag{6}$$

We assume

$$\frac{\partial F}{\partial \phi}=2A^T(D_m(S-R)+A\phi)+2\lambda_2M_2^TM_2\phi=0. \tag{7}$$

Then

$$\hat{\phi}=-\left(A^TA+\lambda_2M_2^TM_2\right)^{-1}A^TD_m(S-R). \tag{8}$$

Similarly, we let $\frac{\partial F}{\partial R}\Big|_{\phi=\hat{\phi}}=0$ to estimate $\hat{R}$ and obtain

$$\begin{aligned}\hat{R} &= \left(T_s^T T_s + \lambda_1(1+\gamma)E\right)^{-1}\left(T_s^T T_s S + \lambda_1 \gamma S\right) \\ &= M\left(\hat{\lambda}_1, \hat{\gamma}\right)\left(T_s^T T_s S + \lambda_1 \gamma S\right)\end{aligned} \tag{9}$$

Where $T_s = D_m + P_{a_{m-1}} D_{m-1} + \cdots + P_{a_1} D + P_{a_0}$, $M\left(\hat{\lambda}_1, \hat{\gamma}\right) = \left(T_s^T T_s + \lambda_1(1+\gamma)E\right)^{-1}$. For the NSP algorithm, parameters λ_1 and γ can be counted as follows:

$$\lambda_1 = \frac{1}{1+\hat{\gamma}} \frac{S^T M(\lambda_1, \hat{\gamma})^T S}{S^T M(\lambda_1, \hat{\gamma})^T M(\lambda_1, \hat{\gamma}) S} \tag{10}$$

$$\gamma = \frac{\left(S - \hat{R}\right)^T S}{\left\|S - \hat{R}\right\|^2} - 1 \tag{11}$$

However, the optimal value of λ_2 cannot be estimated by the above procedure. So, we can try more than one value of λ_2, according to the result to select the optimal solution. In practice, we find the best solution of (4) is not sensitive to the value of λ_2. Thus, the value of λ_2 can be fixed.

3 Experiment Result and Analysis

We assume that the value of λ_2 is given.

(1) Input: the signal $S(t)$, the initial values of λ_1^0 and γ^0, give the stopping threshold ε; Let $k = 1$;
(2) Let $j = 0, \hat{R}_j = 0, \lambda_1^j = \lambda_1^0, \gamma^j = \gamma^0$;
(3) Compute ϕ_j according to (8) and denoted by signal $\hat{R}_j$; Then, obtain the values of $\hat{a}_0, \hat{a}_1, \cdots, \hat{a}_{m-1}$;
(4) Compute λ_1^{j+1} according to (10) and denoted by $\phi_j, \hat{R}_j, \lambda_1^{j+1}, \lambda^j$;
(5) Compute $\hat{R}_{j+1}$ according to (9) and denoted by $\phi_j, \gamma^j, \lambda_1^{j+1}$;
(6) Compute γ^{j+1} according to (11) and denoted by $\hat{R}_{j+1}$, then set $j = j+1$;
(7) If $\left\|\hat{R}_{j+1} - \hat{R}_j\right\| > \varepsilon$, go to step (3); otherwise, go to the next step;
(8) Output the extracted signal $\hat{U} = (1+\gamma^{j+1})\left(S - \hat{R}_j\right)$ and the residual signal $\hat{R} = S - \hat{U}$;
(9) If $\left\|\hat{R}\right\|^2 > \varepsilon\|S\|^2$, set $k = k+1$ and go to step (2); otherwise, stop this program.

We illustrate several examples to demonstrate the results achieved by our algorithm. In the first example, the input signal is the signal $t\cos(t) + 3t\cos(3t) + 5t\cos(5t)$ in additive Gaussian random noise, as shown in Fig. 1. By running a Matlab program, we obtain the PSNR of input signal is 9.42217 dB, and the PSNR of extracted signal is 16.1682 dB.our algorithm have a good effect to signal denoising.

In this example, $\gamma^0 = 1, \lambda_1 = 0.0001, \lambda_2 = 1000, \varepsilon = 0.285$, and the orders of differential operator is six.

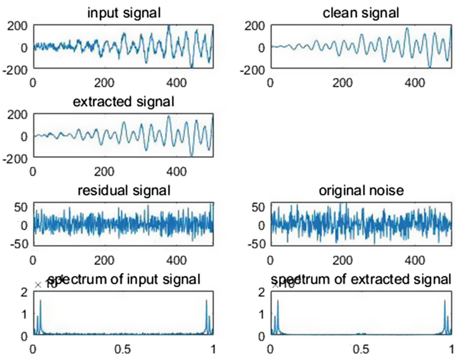

Fig. 1. Signal denoising

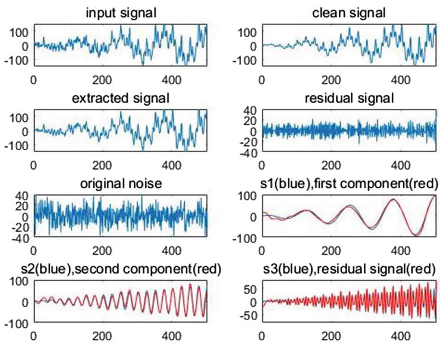

Fig. 2. Signal denoising and signal separation

In the second example, our algorithm can denoise from a noisy chirp signal and make the extracted signal into its coherent subcomponents. First, we input the signal $s1 + s2 + s3$ in additive Gaussian random noise, which $s1 = 4t\cos t$, $s2 = 3t\cos 5t$, $s3 = 2t\cos 15t + t\cos 20t$. We can see that the extracted is closed to the clean signal in Fig. 2. And it can estimate a eighth-order differential operator. In this example, we can obtain the PSNR of input signal is 11.5881 dB, and the PSNR of extracted signal is 13.6667 dB by running a Matlab program. In this process, let $\gamma^0 = 1, \lambda_1 = 0.1, \lambda_2 = 0.1$, $\varepsilon = 0.16$. Then, we separate the signal $s1 + s2 + s3$ into three subcomponents. In Fig. 2, the first component is closed to the signal $s1$, the second component is closed to the signal $s2$ and the residual signal is closed to the signal $s3$. The values of parameters for extraction of the first and second subcomponents are set at $\gamma^0 = 1, \lambda_1 = 0.0001, \lambda_2 = 0.1, \varepsilon = 0.01$ and $\gamma^0 = 1, \lambda_1 = 0.001, \lambda_2 = 0.01, \varepsilon = 0.1$.

4 Summary

In the paper, we put forward an NSP algorithm based on an arbitrary order differential operator. It improves the order of differential operator and expands the scope of signal that could be decomposed. In our future work, we will extend the method to images.

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