

Chapter 2

Towards a Relativistic Theory of Gravity

The equations of Newton's gravitational theory, which provide the theoretical foundations of Kepler's celestial mechanics and seem to describe so well the gravitational force at all macroscopical scales, are *not* compatible, unfortunately, with the principles of Einstein's special relativity.

The Newton equations, in fact, predict for the gravitational effects an infinite speed of propagation in any medium. Also, they do not take into account the possible transformation properties of the gravitational field of forces from one reference frame to another. The Newtonian theory defines indeed the forces generated by static matter sources, but gives us no hint about the forces produced by *moving* sources. Hence, the theory may describe the gravitational field of a mass M through the static potential $\phi(r) = -GM/r$ only in the non-relativistic approximation, i.e. in the regime where the modulus of the potential energy $m\phi$ of a test mass m is negligible with respect to its rest energy mc^2 , namely:

$$\frac{GM}{rc^2} \ll 1. \quad (2.1)$$

A correct description of gravity in the relativistic regime thus require an appropriate generalization of Newton's theory. Which kind of generalization? A natural answer seems to be suggested by the close formal analogy existing between the Newton force among static masses and the Coulomb electrostatic force among electric charges. In the same way as the Coulomb potential corresponds to the fourth component of the electromagnetic vector potential, the Newton potential might correspond to the component of a four-vector, and the relativistic gravitational interaction might be represented by an appropriate *vector* field, in close analogy with the electromagnetic theory.

Such an attractive speculation, however, has to be immediately discarded for a very simple reason: vector-like interactions produce *repulsive* static interactions between sources of the same sign, while—as is well known—the static gravitational interaction between masses of the same sign is *attractive*.

Another simple (and formally consistent) possibility is based on the assumption that the Newton potential may be treated as an invariant under a general change of frame, i.e. that gravity may be correctly described by a relativistic *scalar* field. However, also this hypothesis has to be discarded on the ground of phenomenological results, even if the reasons, this time, are more subtle. In view of our subsequent applications it is worthwhile to recall here one of these reasons, concerning the precession of planetary orbits.

Let us consider the motion of a relativistic test body of mass m , interacting with a central (i.e. radially oriented) field of forces described by the scalar potential $U = U(r)$. The dynamics of the problem is controlled by the relativistic Lagrangian

$$L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} - mU, \quad (2.2)$$

where $v^2 = v_i v^i$ and $v^i = dx^i/dt$. The kinetic term of this Lagrangian can be directly obtained from the free-particle action (1.118), by parametrizing the particle trajectory with the time coordinate t of a generic inertial system, $x^\mu = x^\mu(t)$.

It can be easily shown that the angular momentum of this dynamical system is conserved, and that the motion is confined on a plane (as $\mathbf{r} \times \nabla U = 0$). By introducing on that plane a system of polar coordinates $\{r, \varphi\}$,

$$x = r \cos \varphi, \quad y = r \sin \varphi, \quad (2.3)$$

and specializing U as the gravitational potential produced by a central body of mass M , i.e. $U = -GM/r$, we obtain the explicit Lagrangian

$$L = -mc^2 \left[1 - \frac{1}{c^2} (\dot{r}^2 + r^2 \dot{\varphi}^2) \right]^{1/2} + \frac{GMm}{r}, \quad (2.4)$$

where the dot denotes differentiation with respect to t .

In the above Lagrangian φ and t are cyclic coordinates, and the corresponding conjugate momenta are thus associated to constants of motion which define, respectively, the total angular momentum and the total energy (or Hamiltonian) of our system. We can set, in particular,

$$\frac{\partial L}{\partial \dot{\varphi}} = m\gamma r^2 \dot{\varphi} = mh = \text{const}, \quad (2.5)$$

$$H = v^i \frac{\partial L}{\partial v^i} - L = m\gamma c^2 + mU = m\alpha = \text{const}, \quad (2.6)$$

where γ is the Lorentz factor,

$$\gamma = \left[1 - \frac{1}{c^2} (\dot{r}^2 + r^2 \dot{\varphi}^2) \right]^{-1/2}, \quad (2.7)$$

and where h and α are constant parameters whose particular values are determined by the initial conditions.

Let us now combine the two relations (2.5) and (2.6) by eliminating $\dot{\varphi}^2$, differentiate the result with respect to φ , and define $u = 1/r$. By excluding the (possible) case of circular orbits, with $r = \text{const}$, we then arrive at the following equation of motion (see Exercise 2.1),

$$u'' + k^2 u = \frac{k^2}{p}, \quad (2.8)$$

where a prime denotes differentiation with respect to φ , and where the two constants k and p are defined by:

$$k^2 = 1 - \frac{c^2 r_0^2}{4h^2}, \quad \frac{k^2}{p} = \frac{\alpha r_0}{2h^2}, \quad r_0 = \frac{2GM}{c^2}. \quad (2.9)$$

The general solution of Eq. (2.8) depends on two integration constants (which we will call e and φ_0), and can be obtained by adding to the particular solution $u = p^{-1}$ the general solution of the corresponding homogeneous equation. Since we are interested in describing planetary orbits, we can choose initial conditions such that the motion keeps confined on a finite portion of space, and we can conveniently write the general solution as follows:

$$u = \frac{1}{p} \left\{ 1 + e \cos [k(\varphi - \varphi_0)] \right\}, \quad (2.10)$$

where $0 < e < 1$. In the non-relativistic limit ($c \rightarrow \infty$) we have $r_0 \rightarrow 0$, $\alpha \rightarrow \infty$, the product αr_0 stays finite and we find $k^2 \rightarrow 1$. We thus recover, in this limit, the well known equation describing (in polar coordinates) an ellipse of eccentricity e and perihelion position $\varphi = \varphi_0$.

If we take into account the relativistic corrections, using for k the exact value (2.9), $k \neq 1$, we find that the motion is still confined in a portion of space extending between a minimum and a maximum distance from the origin, but such a motion does not correspond anymore to a closed trajectory. The orbit of the planet is not a stationary ellipse but rather an open curve with flower petal shape, i.e. a “rosette-like” path. In that case the position of closest approach to the source—the so called perihelion—is periodically reached after the angle subtended by the motion is not $\varphi - \varphi_0 = 2\pi$, but rather $k(\varphi - \varphi_0) = 2\pi$ (see Eq. (2.10)). To each planetary revolution is thus associated an angular shift of the perihelion position given by

$$\Delta\varphi = \frac{2\pi}{k} - 2\pi = 2\pi \left(\frac{1}{k} - 1 \right) \simeq 2\pi \left(\frac{c^2 r_0^2}{8h^2} \right) = \frac{\pi G^2 M^2}{c^2 h^2} \quad (2.11)$$

(we have used the approximation $c^2 r_0^2 / h^2 \ll 1$, which is well satisfied by the parameters of a typical planetary orbit).

A theory of gravity based on a relativistic scalar potential thus predicts for the planetary orbits a (small) effect of perihelion precession, described by Eq. (2.11). An effect of this type actually exists in our solar system, and has been measured for various planets with a series of long (more than secular) and precise astronomical observations.

However, the prediction (2.11) following from our (tentative) scalar version of relativistic gravity is in sharp quantitative contrast with all observational results: for the planet Mercury, for instance, the observed precession amounts to 43 arc-seconds per century, while Eq. (2.11) only gives 7 arc-seconds per century. This difference largely exceeds all possible experimental and systematic errors.¹ A scalar model of the gravitational interaction cannot thus represent a satisfactory relativistic generalization of Newton's theory.

An alternative approach to a relativistic gravitational theory, which is (up to now) in good agreement with all available observations, and which admits, at a classical level, also an interesting geometric interpretation, is the model of *tensor* gravitational interactions adopted by the so-called theory of general relativity.

The starting point of this successful approach is a radical extension of the symmetry principles of special relativity, asserting the physical equivalence of all *inertial* frames. Such an equivalence is lifted to a more general level by the assumption that:

- *the physical laws are the same in all frames,*

without special restrictions to the class of the inertial frames. This assumption necessarily leads to the so-called “principle of general covariance”, stating that:

- *the physical laws are covariant with respect to general coordinate transformations,*

and not only covariant with respect to the Lorentz transformations. The two above assumptions, which represent a rather natural—and even trivial, apparently—generalization of the special-relativistic postulates, are at the ground of the theory of general relativity, and have a revolutionary impact on the whole physics. In fact, they necessarily imply that the rigid, pseudo-Euclidean structure of the Minkowski space–time has to be abandoned in favor of a more general geometrical structure.

For an explicit illustration of this important point let us recall that, under a general coordinate transformation $x^\mu \rightarrow x'^\mu$, the coordinate differentials are related by

$$dx^\mu = \left(\frac{\partial x^\mu}{\partial x'^\nu} \right) dx'^\nu, \quad (2.12)$$

where the term in round brackets represents the inverse Jacobian matrix of the given transformation. Let us suppose that we start with the coordinates x^μ of an inertial frame, where the infinitesimal space–time interval assumes the Minkowski form:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu. \quad (2.13)$$

¹As we shall see in Chap. 10, the perihelion precession predicted by general relativity is approximately six times larger than the special-relativistic result (2.11), and is thus in good agreement with the astronomical observations.

The same interval, given as a function of the new coordinates x'^{μ} , will acquire in general a different form. From Eq. (2.12) we obtain, in fact,

$$ds^2 = \eta_{\mu\nu} \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} dx'^{\alpha} dx'^{\beta} \equiv g_{\alpha\beta}(x') dx'^{\alpha} dx'^{\beta}, \quad (2.14)$$

where we have defined

$$g_{\alpha\beta}(x') = \eta_{\mu\nu} \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\beta}}, \quad (2.15)$$

and where $g_{\alpha\beta} \neq \eta_{\alpha\beta}$, in general. This explicitly shows that a generic coordinate transformation—unlike the special case of the Lorentz transformations—does not preserve the infinitesimal interval of the Minkowski space–time.

If we plan to extend the class of physically equivalent systems so as to include also non-inertial frames, then we must be prepared to accept a space–time interval (also called “line-element”) ds^2 which is not rigidly determined as a pseudo-Euclidean combination of the quadratic differentials dx_i^2 and dt^2 , but which corresponds to a more general (possibly position-dependent) combination of dx_i and dt .

2.1 The Postulates of the Riemannian Geometry

As pointed out in the previous section, a relativistic model based on the principle of general covariance induces on the space–time a geometrical structure which includes the Minkowski one, but which is compatible also with generalized geometrical schemes. In order to formulate a predictive physical model we need then to introduce some “working hypotheses”, and provide a more precise characterization of the space–time geometry we are willing to assume.

Let us consider, for this purpose, the following basic postulates:

- the space–time interval ds^2 can be expressed as a homogeneous quadratic form (with coefficients which are position-dependent, in general) of the coordinate differentials:

$$ds^2 = g_{\mu\nu}(x) dx^{\mu} dx^{\nu}; \quad (2.16)$$

- the space–time interval ds^2 is invariant under general coordinate transformations:

$$\begin{aligned} ds^2 &= g_{\mu\nu}(x) dx^{\mu} dx^{\nu} = g_{\mu\nu}(x) \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} dx'^{\alpha} dx'^{\beta} = \\ &\equiv g'_{\alpha\beta}(x') dx'^{\alpha} dx'^{\beta} = ds'^2. \end{aligned} \quad (2.17)$$

As we shall see below, this second assumption is exactly equivalent to the requirement that the coefficients $g_{\mu\nu}$ of the quadratic form ds^2 —i.e. the components of the so-called “metric” of the space–time manifold—transform as the components of a second-rank (covariant) tensor, namely:

$$g'_{\alpha\beta}(x') = g_{\mu\nu}(x) \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} \quad (2.18)$$

(see Chap. 3, Sect. 3.1).

Starting with the above hypotheses we are led to a geometric model of Riemannian type, namely a model which extends to manifolds with four (or more) space–time dimensions the approach suggested by Gauss for the intrinsic description of two-dimensional surfaces.

It should be recalled, in fact, that the geometric properties of a generic n -dimensional hypersurface Σ_n can be described in two ways. A first possibility is based on the *extrinsic* approach, based on the embedding of Σ_n into a higher-dimensional Euclidean or pseudo-Euclidean manifold $\mathcal{M}_D$, with $D > n$, parametrized by the coordinates X^A and with line-element:

$$ds^2 = \eta_{AB} dX^A dX^B, \quad A, B = 1, \dots, D \quad (2.19)$$

(we have assumed a pseudo-Euclidean signature, using the Minkowski metric η). Let us consider the case $D = n + 1$, for simplicity. The hypersurface Σ_n can be represented as a submanifold of $\mathcal{M}_{n+1}$ defined by one condition relating the $n + 1$ coordinates X^A , namely by a relation of the type $f(X^A) = 0$. We can take, as a simple example, a two-dimensional spherical surface S_2 of radius a , embedded in the external 3-dimensional Euclidean space $\mathcal{R}_3$ parametrized by the Cartesian coordinates X^i , $i = 1, 2, 3$. The given surface is then defined by the coordinate condition

$$f(X^i) \equiv X_1^2 + X_2^2 + X_3^2 - a^2 = 0, \quad (2.20)$$

representing indeed a sphere of radius a with center at the origin $X^i = 0$.

There is, however, a second possible approach, of *intrinsic* type, able to describe the geometry of Σ_n without referring to the coordinates X^A of the external manifold, and using instead a set of coordinates ξ^μ , $\mu = 1, \dots, n$ defined on Σ_n itself. What we need, to this purpose, is a set of parametric equations,

$$X^A = X^A(\xi^\mu), \quad A, B = 1, \dots, n + 1, \quad \mu = 1, \dots, n, \quad (2.21)$$

describing the immersion of Σ_n into $\mathcal{M}_{n+1}$. We can then write the line-element (2.19) restricted to the hypersurface Σ_n , i.e. imposing on the coordinates X^A to satisfy the parametric equations (2.21), so that

$$ds^2 = \left[\eta_{AB} \frac{\partial X^A(\xi)}{\partial \xi^\mu} \frac{\partial X^B(\xi)}{\partial \xi^\nu} \right] d\xi^\mu d\xi^\nu \equiv g_{\mu\nu}(\xi) d\xi^\mu d\xi^\nu. \quad (2.22)$$

The variable $g_{\mu\nu}(\xi)$, defined by the terms in square bracket of the above equation, is the so-called “induced metric” on the given hypersurface.

We may thus conclude that it is always possible to describe the geometry of Σ_n in terms of its intrinsic coordinates ξ^μ only, introducing, however, a line-element

which is not, in general, Euclidean (or pseudo-Euclidean) like that of the embedding space $\mathcal{M}_{n+1}$. In fact, let us consider again the example of the spherical surface S_2 embedded in $\mathcal{R}_3$. We may choose as intrinsic coordinates on S_2 the angular variables $\xi^\mu = \{\theta, \varphi\}$ of a polar coordinate system, related to the Cartesian coordinates of $\mathcal{R}_3$ by the following parametric equations:

$$X_1 = a \sin \theta \cos \varphi, \quad X_2 = a \sin \theta \sin \varphi, \quad X_3 = a \cos \theta. \quad (2.23)$$

By differentiating these equations, and inserting the result into the Euclidean line-element of $\mathcal{R}_3$, we obtain for the spherical surface S_2 the interval:

$$ds^2 = dX_1^2 + dX_2^2 + dX_3^2 = a^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (2.24)$$

Hence we find on the S_2 sphere an intrinsic non-Euclidean geometry, described by the Riemannian metric $g_{\mu\nu} = g_{\mu\nu}(\theta, \varphi)$ with components:

$$g_{11} = a^2, \quad g_{22} = a^2 \sin^2 \theta, \quad g_{12} = g_{21} = 0. \quad (2.25)$$

We can thus notice, at this point, that the two postulates introduced at the beginning of this section are compatible with a description of intrinsic type of the space–time geometry, and that the associated Riemannian metric structure simply generalizes the (intrinsic) Gaussian description of two-dimensional surfaces to the case of manifolds with an arbitrary number of space–time dimensions (and arbitrary signature). It should be clearly stressed, however, that the above postulates are neither the only possible nor the more general ones.

In fact, we could even start with less restrictive assumptions, and modify the first hypothesis (2.16) by imposing (for instance) on the interval ds to be a homogeneous form of first degree in the coordinate differentials. In that case we can write $ds = F(x, dx)$, where the function F satisfies the condition

$$F(x, \lambda dx) = \lambda F(x, dx), \quad (2.26)$$

for any value of the parameter λ . As a simple example of this type we may consider the interval:

$$ds = (dx_1^4 + dx_2^4 + \dots)^{1/4}. \quad (2.27)$$

The condition (2.26) defines the so-called *Finsler geometry*, and characterizes a space–time structure which is different from (and more general than) the Riemann one. In fact, the Riemann assumption (2.16) satisfies as a particular case the Finsler condition (2.26) (just like the Minkowski line-element being only a particular case of the more general Riemannian interval). Conversely, there are examples of space–time intervals—like that of Eq. (2.27)—which satisfy the Finsler hypothesis but not the Riemannian postulates introduced before.

In view of the various possible types of geometric structure, which include the Minkowski and the Riemann geometries within schemes of increasing level of

generality and complexity, one may wonder what is the geometric model more appropriate to the physical space–time in which we live. The principle of general covariance tell us that the Minkowski geometry has to be generalized, but does not select a unique prescription. Can we obtain useful hints from other physical principles?

An answer to this question will be presented in the following section.

2.2 The Equivalence Principle

In order to formulate a theory of gravity based on a generalized relativity principle we need a generalized space–time geometry, “adapted” to the physical properties of the gravitational interaction.

One of the most important (and most typical) property of such interaction is expressed by the so-called “equivalence principle”, stating that:

the gravitational interaction can always be locally eliminated,

where *locally* means at any given space–time point and in its infinitesimal neighborhood. Basically, this property arises from the fact that the effects of the gravitational interactions are *locally* indistinguishable from the effects of an accelerated frame, so that the gravitational effects can be eliminated by the simple application of an acceleration of appropriate sign and intensity.

It is important to stress that such a complete elimination of the interaction, for any given physical system, is achieved as a consequence of the *universality* of the gravitational coupling. In fact, as is well known since Galileo, all test bodies respond to a given external gravitational field with the same acceleration, which means that the ratio between the “gravitational charge” (i.e. the gravitational mass) and the inertial mass is the same for all bodies.

Gravity is the only fundamental interaction characterized by this type of universality. In the case of the electromagnetic interaction, for instance, there is no analogous of the equivalence principle because test bodies with different electric charges are differently affected by the same external field: by introducing an accelerated frame we can then locally eliminate at most the electromagnetic force acting on a particular type of charge, but not on *all* the charges of a given physical system. Hence, the electromagnetic interaction *is not* always locally eliminable, unlike the gravitational interaction.

If we want to formulate a relativistic gravitational theory by introducing a new geometrical space–time structure, more general than the Minkowski one, we must require—in agreement with the principle of equivalence—that the effects of the new geometry can always be locally eliminated, i.e. that the new geometry can always be reduced, locally, to the Minkowski one.

This property is not in general satisfied, for instance, in the Finsler case, but is always satisfied in the case of a Riemann space–time. In fact, if the space–time interval satisfies the properties (2.16), (2.17), then we can always introduce an appropriate

system of coordinates—the so-called “locally inertial” system—where the Riemann metric $g_{\mu\nu}$ exactly reduces to $\eta_{\mu\nu}$ at a chosen space–time point. In the neighborhood of that point the geometry locally approaches the Minkowskian form.

For a simple visualization of this geometric property we may recall the example of the spherical surface S_2 already considered in the previous section. The intrinsic geometry of S_2 is not described by an Euclidean line-element; however, at any given point of S_2 we can always introduce an Euclidean plane tangent to the sphere, and approximate the local geometry of the sphere, in the neighborhood of that point, with the Euclidean geometry of the tangent plane. In the same way, in the case of a four-dimensional Riemann space–time we can always introduce at any given point a “flat” tangent manifold characterized by a Minkowskian metric structure, and locally approximate the Riemann geometry with the tangent Minkowski geometry.

For a more explicit illustration of the local reduction of the metric to the Minkowski form we shall now consider a metric g which satisfy the Riemann postulates (2.16), (2.17), and show that we can always find a coordinate transformation $x \rightarrow x'$ such that the transformed metric $g'(x')$ exactly reduces to the Minkowski metric η at a given point x_0 , namely that $g'(x_0) = \eta$. We can choose, for simplicity, a coordinate system x' which coincides with x at the reference position x_0 .

Let us consider the inverse transformation $x = x(x')$, expanded in a Taylor series around $x' = x_0$:

$$\begin{aligned} x^\mu(x') \simeq & x_0^\mu + \left(\frac{\partial x^\mu}{\partial x'^\nu} \right)_{x'=x_0} (x'^\nu - x_0^\nu) + \\ & + \frac{1}{2} \left(\frac{\partial^2 x^\mu}{\partial x'^\alpha \partial x'^\beta} \right)_{x'=x_0} (x'^\alpha - x_0^\alpha)(x'^\beta - x_0^\beta) + \dots \end{aligned} \quad (2.28)$$

Such a transformation is locally determined, to first order, once we are given the 16 (constant) matrix coefficients

$$I_\nu^\mu = \left(\frac{\partial x^\mu}{\partial x'^\nu} \right)_{x'=x_0}. \quad (2.29)$$

The metric transformation under a general change of coordinates, on the other hand, is fixed by Eq. (2.18). Such a transformation, evaluated at $x' = x = x_0$, and applied to the condition $g'(x_0) = \eta$, leads to

$$g'_{\alpha\beta}(x_0) = I_\alpha^\mu I_\beta^\nu g_{\mu\nu}(x_0) = \eta_{\alpha\beta}. \quad (2.30)$$

Since the initial metric $g_{\mu\nu}(x_0)$ is known, the above non-homogeneous system of equations provides 10 equations for the 16 unknown components of the matrix $I^\mu{}_\nu$. Such a system always admits non-trivial solutions for the coefficients $I^\mu{}_\nu$, so that we can always determine the infinitesimal coordinate transformation reducing the given metric to the Minkowski form at the chosen space–time point.

It should be noted that the system of equations (2.30) does not completely fix all the coefficients of $I^\mu{}_\nu$, but determines a class of solution depending on $16 - 10 = 6$ parameters. Otherwise stated, the coordinate transformation reducing the metric to the Minkowski form is only defined modulo six arbitrary degrees of freedom. This freedom physically corresponds to the residual possibility of performing local changes of frames, even after having imposed $g(x_0) = \eta$, by applying an arbitrary Lorentz transformation (which depends indeed on six parameters and which, as is well known, does not modify the local Minkowski metric).

More generally, if the two coordinate systems are not assumed to coincide at the point x_0 , the sought transformation will be determined modulo the four additional constant parameters $x^\mu(x_0)$ which are associated to the zeroth-order term of the Taylor expansion (2.28), and which are to be added to the six parameters mentioned before. In fact, the most general transformations preserving the Minkowski metric are those of the Poincaré group, which include Lorentz rotations plus space–time translations and which depend indeed on $6 + 4 = 10$ parameters.

In conclusion we can say that the Riemann geometry, thanks to its local properties, provides the space–time manifold with a structure which is able to incorporate and to generalize (consistently with the equivalence principle) the geometry of special relativity, and is then suitable (at least in principle) for a possible geometric description of the gravitational interaction. A few notions about the formalism and the computational techniques to be used in the context of a Riemann manifold will be introduced in the next chapter.

Exercises Chap. 2

2.1 Relativistic Motion in a Central Gravitational Field Deduce the equation of motion (2.8) by combining equations (2.5) and (2.6) which define, respectively, the constants of motion h and α .

2.2 Four-Dimensional Pseudo-Sphere Let us consider a four-dimensional hypersurface Σ_4 with pseudo-Euclidean signature, $g_{\mu\nu} = (+, -, -, -)$, parametrized by the four intrinsic coordinates $x^\mu = (ct, x^i)$, and embedded in a five-dimensional Minkowski space–time with coordinates z^A , $A = 0, 1, 2, 3, 4$. The hypersurface is described by the following parametric equations:

$$\begin{aligned} z^0 &= \frac{c}{H} \sinh(Ht) + \frac{H}{2c} e^{Ht} x_i x^i, \\ z^i &= e^{Ht} x^i, \\ z^4 &= \frac{c}{H} \cosh(Ht) - \frac{H}{2c} e^{Ht} x_i x^i, \end{aligned} \tag{2.31}$$

where H is a constant. Show that the given hypersurface Σ_4 represents a four-dimensional pseudo-sphere (or hyperboloid), and determine its intrinsic metric, i.e. the metric induced on Σ_4 by the embedding equations (2.31).

Solutions

2.1 Solution By expressing the derivatives with respect to t as

$$\dot{r} = r' \dot{\varphi}, \quad r' = \frac{dr}{d\varphi}, \quad (2.32)$$

we can rewrite Eq. (2.5) in the form

$$\dot{\varphi}^2 = \frac{h^2}{r^4} \left(1 - \frac{r'^2}{c^2} \varphi^2 - \frac{r^2}{c^2} \dot{\varphi}^2 \right), \quad (2.33)$$

and obtain

$$\dot{\varphi}^2 = \frac{h^2}{r^4} \left[1 + \frac{h^2}{r^4 c^2} (r'^2 + r^2) \right]^{-1}. \quad (2.34)$$

We can also conveniently obtain, from Eq. (2.6),

$$\frac{1}{\gamma^2} \equiv 1 - \frac{1}{c^2} (r'^2 + r^2) \dot{\varphi}^2 = \frac{c^4}{\left(\alpha + \frac{GM}{r} \right)^2}. \quad (2.35)$$

By inserting the expression (2.34) for $\dot{\varphi}^2$, and inverting the previous equation, we are led to

$$\frac{1}{c^4} \left(\alpha + \frac{GM}{r} \right)^2 = 1 + \frac{h^2}{r^4 c^2} (r'^2 + r^2). \quad (2.36)$$

Let us now replace the radial variable r with the new variable $u = 1/r$, such that $r' = -u'/u^2$, and differentiate with respect to φ both terms of the above equation. We obtain:

$$u' (u'' + u) = u' \left(\frac{r_0 \alpha}{2h^2} + \frac{r_0^2 c^2}{4h^2} u \right), \quad (2.37)$$

where $r_0 = 2GM/c^2$. This equation always admits the trivial solution $u' = 0$, namely $r = \text{const}$, which describes a circular trajectory in the orbital plane. If we exclude the case of circular orbits, $u' \neq 0$, we can divide by u' and we arrive at the equation

$$u'' + u = \frac{r_0 \alpha}{2h^2} + \frac{r_0^2 c^2}{4h^2} u, \quad (2.38)$$

which, by using the definitions (2.9), exactly reduces to the equation of motion (2.8).

2.2 Solution By squaring the coordinates z^A of Eq. (2.31), and computing the squared modulus $z^A z_A$ in the five-dimensional Minkowski space-time, we easily find that the hypersurface Σ_4 satisfies the equation:

$$\eta_{AB} z^A z^B = (z^0)^2 - (z^1)^2 - (z^2)^2 - (z^3)^2 - (z^4)^2 = -\frac{c^2}{H^2} = \text{const.} \quad (2.39)$$

This equation describes a four-dimensional pseudo-sphere of radius $R = c/H$ (compare for instance this result with Eq. (2.20), describing a two-dimensional spherical surface). Due to the pseudo-Euclidean signature of the external manifold, the space-time sections of Σ_4 —e.g. the sections defined by $z^2 = z^3 = z^4 = 0$ —are represented by hyperbolas instead of circles. The given hypersurface can thus be interpreted as a four-dimensional hyperboloid of revolution.

The intrinsic metric $g_{\mu\nu}$ induced by the parametric equations $z^A = z^A(x^\mu)$ is defined by

$$g_{\mu\nu} = \frac{\partial z^A}{\partial x^\mu} \frac{\partial z^B}{\partial x^\nu} \eta_{AB}, \quad (2.40)$$

according to Eq. (2.22). A simple differentiation of the equations (2.31) with respect to the components of x^μ gives

$$\begin{aligned} g_{00} &= \frac{1}{c^2} \left(\frac{\partial z^0}{\partial t} \right)^2 - \frac{1}{c^2} \left| \frac{\partial z^i}{\partial t} \right|^2 - \frac{1}{c^2} \left(\frac{\partial z^4}{\partial t} \right)^2 = 1, \\ g_{ij} &= \frac{\partial z^0}{\partial x^i} \frac{\partial z^0}{\partial x^j} - \frac{\partial z^k}{\partial x^i} \frac{\partial z^l}{\partial x^j} \delta_{kl} - \frac{\partial z^4}{\partial x^i} \frac{\partial z^4}{\partial x^j} = -\delta_{ij} e^{2Ht}, \\ g_{0i} &= 0. \end{aligned} \quad (2.41)$$

The intrinsic geometry of our hyperboloid, in the given system of coordinates x^μ , is then described by the line-element:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = c^2 dt^2 - e^{2Ht} |d\mathbf{x}|^2. \quad (2.42)$$

This space-time interval represents a possible parametrization of the so-called de Sitter geometry (see for instance [19]), which has important applications in a cosmological context (see for instance [12]).

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