Preface

Geometry Over Nonclosed Fields is a reference to an active area at the interface of classical algebraic geometry and arithmetic geometry. In recent years, there has been a rapid exchange of ideas between these domains: many questions concerning integral or rational solutions of Diophantine equations become accessible via complex or even symplectic geometric techniques. On the other hand, deep problems such as the rationality problem in algebraic geometry are now understood to be related to arithmetic properties of function fields.

The papers in this proceedings volume of a Simons Symposium that took place in 2015 are concerned with different aspects of this interaction.

The joint work of Bogomolov, Kamenova, Lu, and Verbitsky is devoted to the study of the Kobayashi metric on compact hyperkähler manifolds. This metric is defined via holomorphic maps of complex one-dimensional discs with standard metric into the manifold. For any compact complex manifold, the Kobayashi metric defines a unique compact metric space with a metric-continuous surjection of the manifold to that space which induces the Kobayashi metric. Conjecturally, this metric is trivial for compact hyperkähler manifolds. This would follow immediately from a version of the SYZ conjecture for such manifolds, i.e., from the existence of a smooth complex deformation to a hyperkähler manifold with a Lagrangian fibration. However, this is not yet known for all manifolds of this type. In this paper, the authors establish a partial result. Namely, they show that a compact hyperkähler manifold with an automorphism of infinite order has everywhere degenerate Kobayashi metric, i.e., the fibers of the projection to the compact metric space have positive dimension.

The article of Debarre, Laface, and Roulleau is devoted to a classical problem of describing lines on a cubic hypersurface. It is a well-known fact that a smooth cubic surface has exactly 27 lines. This also holds for cubic surfaces over arbitrary algebraically closed fields. However, there are examples of cubic surfaces over nonclosed fields without lines, in particular, over arbitrarily large finite fields. Here, the authors consider the following question: “Over which finite fields does every smooth cubic hypersurface of dimension at least three contain at least one line?” Their answer is close to optimal: they show that in dimension three a line exists when the finite field contains at least 11 elements and that there exist smooth cubic threefolds without lines for fields with 2, 3, 4 or 5 elements. For fields with 7, 8, and 9 elements, the situation is still unclear. Cubic fourfolds contain lines over fields with 2 or at least 5 elements, while there exist counterexamples over fields with 3 and 4 elements. All smooth cubics of dimension ≥ 5 contain lines.

Harder, Katzarkov, and Liu use long-standing rationality problems as an impetus to develop a theory of perverse sheaves of categories, inspired by recent progress in mathematical physics. This offers the prospect of connecting cohomological and cycle-theoretic techniques (like decomposition of the diagonal) with geometric structures arising from homological mirror symmetry, e.g., derived categories of coherent sheaves and Lagrangian fibrations. These connections are fleshed out in key examples like Fano threefolds and cubic hypersurfaces.

The contribution of de Jong and Starr is motivated by a desire to understand Kontsevich moduli spaces of genus zero stable maps to smooth projective varieties. These can have many irreducible components of varying dimensions but nevertheless carry a virtual fundamental class. When the moduli space happens to be irreducible of the expected dimension, it is important to understand its place in the Kodaira classification. Higher-order notions like rational simple connectedness hinge on the rational connectedness of the moduli spaces, which often holds when they admit negative canonical classes. This paper develops “virtual canonical classes”, i.e., formulas in terms of tautological divisors that make sense even when the moduli space is not integral.

Lieblich and Olsson explore modern formulations of the Torelli theorem for $K3$ surfaces. The original formulation states that two complex $K3$ surfaces with isomorphic Hodge structures are in fact isomorphic. But if only the transcendental cohomologies are isomorphic, then the $K3$ surfaces have equivalent derived categories of coherent sheaves. The notion of derived equivalence makes sense over more general algebraically closed fields. Lieblich and Olsson explore these, isolating a class of derived equivalences, strongly filtered equivalences, that suffice to recover isomorphism classes of $K3$ surfaces. This builds on dramatic recent progress on the Tate conjecture for $K3$ surfaces.

Liedtke’s manuscript explores the observation that Galois-invariant globally generated line bundles are associated with morphisms to Brauer–Severi varieties. He considers this over arbitrary fields and carefully analyzes the associated homomorphisms of Picard groups. He also revisits the classification of del Pezzo surfaces from the perspective of morphisms to Brauer–Severi varieties.

Várilly–Alvarado’s article provides a survey of results and conjectures in the arithmetic of $K3$ surfaces. The first topic concerns the structure and the computation of Picard groups of $K3$ surfaces defined over arithmetic fields and their behavior under reduction modulo primes. The second topic is Brauer groups of $K3$ surfaces over arithmetic fields, their relation to the Hasse principle and Brauer–Manin obstruction to the existence of rational points, and possible effective bounds on the transcendental parts of Brauer groups. The last problem is analogous to the Mazur–Merel theorem concerning effective uniform bounds for torsion of elliptic curves. An abundance of explicit examples will make this a great reference for researchers working in this area.

Zarhin considers the odd-dimensional étale cohomology of an algebraic variety, twisted by tensoring with a power of roots of unity. He generalizes a result of Serre: if an abelian variety defined over K contains a point of order precisely m over K, then K contains roots of unity of order m. This result follows from the existence of a non-degenerate pairing on the one-dimensional cohomology group of an abelian variety, modulo m, with values in the multiplicative group of m-th roots of unity. Zarhin found a similar pairing for arbitrary twisted odd-dimensional cohomology. Thus if the variety is defined over K and the Galois action on such cohomology modulo m is trivial, then K contains roots of unity of order m.

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