Chapter 2
Small-Gain and Passivity for Input–Output Maps

In this chapter we give the basic versions of the classical *small-gain* (Sect. 2.1) and *passivity* theorems (Sect. 2.2) in the study of closed-loop stability. Section 2.3 briefly touches upon the "loop transformations" which can be used to expand the domain of applicability of the small-gain and passivity theorems. Finally, Sect. 2.4 deals with the close relation between passivity and L_2-gain via the scattering representation.

2.1 The Small-Gain Theorem

A straightforward, but very important, theorem is as follows.

Theorem 2.1.1 (Small-gain theorem) *Consider the closed-loop system $G_1 \|_f G_2$ given in Fig. 1.1, and let $q \in \{1, 2, \dots, \infty\}$. Suppose that G_1 and G_2 have L_q-gains $\gamma_q(G_1)$, respectively $\gamma_q(G_2)$. Then the closed-loop system $G_1 \|_f G_2$ has finite L_q-gain (see Definition 1.2.11) if*

$$\gamma_q(G_1) \cdot \gamma_q(G_2) \; < \; 1 \tag{2.1}$$

Remark 2.1.2 Inequality (2.1) is known as the *small-gain condition*. Two stable systems G_1 and G_2 which are interconnected as in Fig. 1.1 result in a stable closed-loop system provided the "*loop gain*" is "small" (i.e., less than 1). Note that the small-gain theorem implies an inherent *robustness* property: the closed-loop system remains stable for all *perturbed* input–output maps, as long as the small-gain condition remains satisfied.

Proof By the definition of $\gamma_q(G_1)$, $\gamma_q(G_2)$ and (2.1) there exist constants $\gamma_{1q}, \gamma_{2q}, b_{1q}, b_{2q}$ with $\gamma_{1q} \cdot \gamma_{2q} < 1$, such that for all $T \geq 0$

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$$
\begin{aligned}
||(G_1(u_1))_T||_q &\leq \gamma_{1q}||u_{1T}||_q + b_{1q} \ , \qquad \forall u_1 \in L_{qe}(U_1) \\
||(G_2(u_2))_T||_q &\leq \gamma_{2q}||u_{2T}||_q + b_{2q} \ , \qquad \forall u_2 \in L_{qe}(U_2)
\end{aligned}
\tag{2.2}
$$

For simplicity of notation we will drop the subscripts "q." Since $u_{1T} = e_{1T} - (G_2(u_2))_T$

$$
\begin{aligned}
||u_{1T}|| \leq ||e_{1T}|| + ||(G_2(u_2))_T|| &\leq \ ||e_{1T}|| + \gamma_2||u_{2T}|| + b_2 \\
||u_{2T}|| \leq ||e_{2T}|| + ||(G_1(u_1))_T|| &\leq \ ||e_{2T}|| + \gamma_1||u_{1T}|| + b_1.
\end{aligned}
$$

Combining these two inequalities, using the fact that $\gamma_2 \geq 0$, yields

$$
||u_{1T}|| \ \leq \ \gamma_1\gamma_2||u_{1T}|| + (||e_{1T}|| + \gamma_2||e_{2T}|| + b_2 + \gamma_2 b_1).
$$

Since $\gamma_1\gamma_2 < 1$ this implies

$$
||u_{1T}|| \ \leq \ (1 - \gamma_1\gamma_2)^{-1}(||e_{1T}|| + \gamma_2||e_{2T}|| + b_2 + \gamma_2 b_1). \tag{2.3}
$$

Similarly we derive

$$
||u_{2T}|| \ \leq \ (1 - \gamma_1\gamma_2)^{-1}(||e_{2T}|| + \gamma_1||e_{1T}|| + b_1 + \gamma_1 b_2). \tag{2.4}
$$

This proves finite L_q-gain of the relation R_{eu}, and thus by Lemma 1.2.12 finite L_q-gain of $G_1\|_f G_2$. □

Remark 2.1.3 Note that in (2.3) and (2.4) we have actually derived a bound on the L_q-gain of the relation R_{eu}. Substituting $y_1 = G_1(u_1),\ y_2 = G_2(u_2)$, and combining (2.2) with (2.3) and (2.4), we also obtain the following bound on the L_q-gain of the relation R_{ey}:

$$
\begin{aligned}
||y_{1T}|| &\leq (1 - \gamma_1\gamma_2)^{-1}\gamma_1(||e_{1T}|| + \gamma_2||e_{2T}|| + b_2 + \gamma_2 b_1) + b_1 \\
||y_{2T}|| &\leq (1 - \gamma_1\gamma_2)^{-1}\gamma_2(||e_{2T}|| + \gamma_1||e_{1T}|| + b_1 + \gamma_1 b_2) + b_2.
\end{aligned}
\tag{2.5}
$$

Remark 2.1.4 Theorem 2.1.1 remains valid for *relations* $R_{u_1y_1}$ and $R_{u_2y_2}$, instead of maps G_1 and G_2.

Note that in many situations, e_1 and e_2 are *given* and $u_1,\ u_2$ (as well as $y_1,\ y_2$) are *derived*. The above formulation of the small-gain theorem (as well as the definition of L_q-stability of the closed-loop system $G_1\|_f G_2$, cf. Definition 1.2.11) *avoids* the question of *existence* of solutions $u_1 \in L_{qe}(U_1), u_2 \in L_{qe}(U_2)$ to $e_1 = u_1 + G_2(u_2),\ e_2 = u_2 - G_1(u_1)$ for given $e_1 \in L_{qe}(E_1), e_2 \in L_{qe}(E_2)$. As we will see, a *stronger* version of the small-gain theorem does also answer this question, as well as some other issues. First, we extend the definition of L_q-gain to its *incremental* version.

Definition 2.1.5 *(Incremental L_q-gain)* The input–output map $G : L_{qe}(U) \to L_{qe}(Y)$ is said to have *finite incremental L_q-gain* if there exists a constant $\Gamma_q \geq 0$ such that

$$||(G(u))_T - (G(v))_T||_q \le \Gamma_q ||u_T - v_T||_q \,, \;\; \forall T \ge 0, \; u, v \in L_{qe}(U) \tag{2.6}$$

Furthermore, its *incremental L_q-gain* $\Gamma_q(G)$ is defined as the infimum over all such Γ_q.

The property of finite incremental L_q-gain is seen to imply *causality*.

Proposition 2.1.6 *Let $G : L_{qe}(U) \to L_{qe}(Y)$ have finite incremental L_q-gain. Then it is causal.*

Proof Let $u, v \in L_{qe}(U)$ be such that $u_T = v_T$. Then by (2.6)

$$||(G(u))_T - G(v))_T||_q \;\le\; \Gamma_q ||u_T - v_T||_q = 0,$$

and thus $(G(u))_T = (G(v))_T$, implying by Lemma 1.1.4 causality of G. □

Remark 2.1.7 Hence, finite incremental L_q-gain for causal maps is the same as requiring that for all $T \ge 0$

$$||(G(u_T))_T - (G(v_T))_T||_q \le \Gamma_q ||u_T - v_T||_q \,, \;\; \forall u, v \in L_{qe}(U) \tag{2.7}$$

Theorem 2.1.8 (Incremental form of small-gain theorem) *Let $G_1 : L_{qe}(U_1) \to L_{qe}(Y_1)$, $G_2 : L_{qe}(U_2) \to L_{qe}(Y_2)$ be input–output maps with incremental L_q-gains $\Gamma_q(G_1)$, respectively $\Gamma_q(G_2)$. Consider the closed-loop system $G_1 \|_f G_2$. Then, if $\Gamma_q(G_1) \cdot \Gamma_q(G_2) < 1$,*

(i) For all $(e_1, e_2) \in L_{qe}(E_1 \times E_2)$ there exists a unique solution $(u_1, u_2, y_1, y_2) \in L_{qe}(U_1 \times U_2 \times Y_1 \times Y_2)$.
(ii) The map $(e_1, e_2) \mapsto (u_1, u_2)$ is uniformly continuous on the space $L_{qe}(E_1 \times E_2)$.
(iii) If the solution (u_1, u_2) to $e_1 = e_2 = 0$ is in $L_q(U_1 \times U_2)$, then $(e_1, e_2) \in L_q(E_1 \times E_2)$ implies that $(u_1, u_2) \in L_q(U_1 \times U_2)$.

Proof First we note that since $\Gamma_q(G_1) \cdot \Gamma_q(G_2) < 1$, there exist constants Γ_{1q}, Γ_{2q} with $\Gamma_{1q} \cdot \Gamma_{2q} < 1$ such that for all $T \ge 0$ and for all $u_1, v_1 \in L_{qe}(U_1), u_2, v_2 \in L_{qe}(U_2)$

$$\begin{aligned} ||(G_1(u_1))_T - (G_1(v_1))_T||_q &\le \Gamma_{1q} ||u_{1T} - v_{1T}||_q \\ ||(G_2(u_2))_T - (G_2(v_2))_T||_q &\le \Gamma_{2q} ||u_{2T} - v_{2T}||_q \end{aligned} \tag{2.8}$$

Furthermore, by Proposition 2.1.6 G_1, G_2 are causal. The statements (i), (ii) and (iii) are now proved as follows.
(i) Since $u_2 = e_2 + G_1(e_1 - G_2(u_2))$ it follows that

$$u_{2T} = e_{2T} + [G_1(e_1 - G_2(u_2))]_T$$

Using causality of G_1 and G_2 this yields

$$u_{2T} = e_{2T} + \{G_1[e_{1T} - (G_2(u_{2T}))_T]\}_T \tag{2.9}$$

For every e_1, e_2 this is an equation of the form $u_{2T} = C(u_{2T})$. We claim that C is a *contraction* on $L_{q,[0,T]}(U_2)$ (the space of L_q-functions on $[0, T]$). Indeed for all $u_{2T}, v_{2T} \in L_{q,[0,T]}(U_2)$

$$\begin{aligned} &||G_1[e_{1T} - (G_2(u_{2T}))_T] - G_1[e_{1T} - (G_2(v_{2T}))_T]||_{q,[0,T]} \\ &\le \Gamma_{1q}||(G_2(v_{2T}))_T - (G_2(u_{2T}))_T||_q \le \Gamma_{1q} \cdot \Gamma_{2q}||u_{2T} - v_{2T}||_q \end{aligned}$$

by (2.8). By assumption $\Gamma_{1q} \cdot \Gamma_{2q} < 1$, and thus C is a contraction. Therefore, for all $T \geq 0$, and all $(e_1, e_2) \in L_{qe}(E_1 \times E_2)$, there is a uniquely defined element of $u_{2T} \in L_{q,[0,T]}(U_2)$ solving $u_{2T} = C(u_{2T})$. The same holds trivially for u_{1T} since

$$u_{1T} = e_{1T} - (G_2(u_{2T}))_T$$

Thus for all $(e_1, e_2) \in L_{qe}(E_1 \times E_2)$ there exists a unique solution $(u_1, u_2) \in L_{qe}(U_1 \times U_2)$ to (1.30).

(ii) Since $u_{1T} = e_{1T} - (G_2(u_{2T}))_T,\ u'_{1T} = e'_{1T} - (G_2(u'_{2T}))_T$ we obtain by subtraction and the triangle inequality

$$||u_{1T} - u'_{1T}|| \le ||e_{1T} - e'_{1T}|| + \Gamma_{2q}||u_{2T} - u'_{2T}||$$

for all $(e_1, e_2),\ (e'_1, e'_2)$ and corresponding solutions $(u_1, u_2),\ (u'_1, u'_2)$. Similarly

$$||u_{2T} - u'_{2T}|| \le ||e_{2T} - e'_{2T}|| + \Gamma_{1q}||u_{1T} - u'_{1T}||$$

and thus

$$||u_{1T} - u'_{1T}|| \le (1 - \Gamma_{1q}\Gamma_{2q})^{-1}(||e_{1T} - e'_{1T}|| + \Gamma_{2q}||e_{2T} - e'_{2T}||), \tag{2.10}$$

and similarly for $||u_{2T} - u'_{2T}||$. This yields (ii).

(iii) Insert $e'_1 = e'_2 = 0$ in (2.10) and in the same inequality for the expression $||u_{2T} - u'_{2T}||$. □

Remark 2.1.9 For a *linear* map G, property (2.6) is equivalent to

$$||(G(u))_T||_q \le \Gamma_q||u_T||_q$$

and thus to the property that G has L_q-gain $\le \Gamma_q$ (with zero bias). Note also that in this case the solution to $e_1 = e_2 = 0$ is $u_1 = u_2 = 0$, and thus (iii) is always satisfied.

2.2 Passivity and the Passivity Theorems

While the small-gain theorem is naturally concerned with *normed* (finite-dimensional) linear spaces $\mathcal{V}$ and the corresponding Banach spaces $L_q(\mathcal{V})$ for every $q = 1, 2, \ldots, \infty$, passivity is, *at least in first instance*, independent of any norm, but, at the same time, requires a duality between the input and output space.

Indeed, let us consider any finite-dimensional linear input space U (of dimension m), and let the output space Y be the *dual space* U^* (the set of linear functions on U). Denote the duality product between U and $U^* = Y$ by $< y \mid u >$ for $y \in U^*, u \in U$. (That is, $< y \mid u >$ is the linear function $y : U \to \mathbb{R}$ evaluated at $u \in U$.) Furthermore, take any linear space of functions $u : \mathbb{R}^+ \to U$, denoted by $L(U)$, and any linear space of functions $y : \mathbb{R}^+ \to Y = U^*$, denoted by $L(U^*)$. Define the extended spaces $L_e(U)$, respectively $L_e(U^*)$, similar to Definition 1.1.2, that is, $u \in L_e(U)$ if $u_T \in L(U)$ for all $T \geq 0$ and $y \in L_e(U^*)$ if $y_T \in L(U^*)$ for all $T \geq 0$. Define a duality pairing between $L_e(U)$ and $L_e(U^*)$ by defining for $u \in L_e(U)$, $y \in L_e(U^*)$

$$< y \mid u >_T := \int_0^T < y(t) \mid u(t) > dt\,, \tag{2.11}$$

assuming that integral on the right-hand side exists. In examples, the duality product $< y(t) \mid u(t) >$ usually is the (instantaneous) *power* (electrical power if the components of u, y are voltages and currents, or mechanical power if the components of u, y are forces and velocities). In these cases, $< y \mid u >_T$ will denote the externally *supplied energy* during the time interval $[0, T]$.

Definition 2.2.1 *(Passive input–output maps)* Let $G : L_e(U) \to L_e(U^*)$. Then G is *passive* if there exists some constant β such that

$$< G(u) \mid u >_T \geq -\beta, \quad \forall u \in L_e(U), \quad \forall T \geq 0, \tag{2.12}$$

where additionally it is assumed that the left-hand side of (2.12) is well defined.

Note that (2.12) can be rewritten as

$$- < G(u) \mid u >_T \leq \beta, \quad \forall u \in L_e(U), \quad \forall T \geq 0, \tag{2.13}$$

with the interpretation that the *maximally extractable energy* is bounded by a finite constant β. Hence, G is passive iff only a *finite amount* of energy can be extracted from the system defined by G. This interpretation, together with its ramifications, will become more clear in Chaps. 3 and 4.

Definition 2.2.1 directly extends to relations.

Definition 2.2.2 *(Passive relation)* A relation $R \subset L_e(U) \times L_e(U^*)$ is said to be passive if $< y \mid u >_T \geq -\beta$, for all $(u, y) \in R$ and $T \geq 0$, assuming that $< y \mid u >_T$ is well defined for all $(u, y) \in R$ and all $T \geq 0$.

Remark 2.2.3 In many applications $L_e(U)$ will be defined as $L_{2e}(U)$ for some norm $||\ \ ||_U$ on U. Then $L_e(U^*)$ can be taken to be $L_{2e}(U^*)$, with $||\ \ ||_{U^*}$ the norm on U^* canonically induced by $||\ \ ||_U$, that is,

$$||y||_{U^*} := \max_{u \neq 0} \frac{< y \mid u >}{||u||_U}.$$

This implies $| < y \mid u > | \leq ||y||_{U^*} \cdot ||u||_U$, yielding

$$\begin{array}{c} | < G(u) \mid u >_T | = | \int_0^T < G(u)(t) \mid u(t) > dt | \leq \\ \left(\int_0^T ||G(u)(t)||^2_{U^*} dt \right)^{\frac{1}{2}} \cdot \left(\int_0^T ||u(t)||^2_U dt \right)^{\frac{1}{2}}. \end{array} \tag{2.14}$$

Hence, in this case the left-hand side of (2.12) is automatically well defined. The same holds for a passive relation $R \subset L_{2e}(U) \times L_{2e}(U^*)$

Remark 2.2.4 For a *linear* single-input single-output map the property of passivity is equivalent to the *phase shift* of an input sinusoid being always less than or equal to 90° (see e.g., [343]). This should be contrasted with the L_q-gain of a linear input–output map, which deals with the amplification of the input signal.

Similarly to Proposition 1.2.3 we have the following alternative formulation of passivity for *causal* maps G.

Proposition 2.2.5 *Let $G : L_e(U) \to L_e(U^*)$ satisfy (2.12). Then also*

$$< G(u) \mid u > \ \geq \ -\beta \ , \quad \forall\, u \in L(U), \tag{2.15}$$

if the left-hand side of (2.15) is well defined. Conversely, if G is causal, *then (2.15) implies (2.12).*

Proof Suppose (2.12) holds. By letting $T \to \infty$ we obtain (2.15) for $u \in L(U)$. Conversely, suppose (2.15) holds and G is causal. Then for $u \in L_e(U)$

$$\begin{aligned} < G(u) \mid u >_T &= < (G(u))_T \mid u_T > = < (G(u_T))_T \mid u_T > \\ &= < G(u_T) \mid u_T > \ \geq -\beta. \end{aligned}$$

□

We are ready to state the first version of the Passivity theorem.

Theorem 2.2.6 (Passivity theorem; first version) *Consider the closed-loop system $G_1 \|_f G_2$ in Fig. 1.1, with $G_1 : L_e(U_1) \to L_e(U_1^*)$ and $G_2 : L_e(U_2) \to L_e(U_2^*)$ passive, and $E_1 = U_2^* = U_1$, $E_2 = U_1^* = U_2$.*

(a) Assume that for any $e_1 \in L_e(U_1)$, $e_2 \in L_e(U_2)$ there are solutions $u_1 \in L_e(U_1)$ and $u_2 \in L_e(U_2)$. Then $G_1 \|_f G_2$ with inputs (e_1, e_2) and outputs (y_1, y_2) is passive.

(b) *Assume that for any* $e_1 \in L_e(U_1)$ *and* $e_2 = 0$ *there are solutions* $u_1 \in L_e(U_1),\ u_2 \in L_e(U_2)$. *Then* $G_1\|_f G_2$ *with* $e_2 = 0$ *and input* e_1 *and output* y_1 *is passive.*

Proof The definition of standard negative feedback, cf. (1.30), implies the key property

$$\begin{aligned} &< y_1 \mid u_1 >_T + < y_2 \mid u_2 >_T \\ &= < y_1 \mid e_1 - y_2 >_T + < y_2 \mid e_2 + y_1 >_T \\ &= < y_1 \mid e_1 >_T + < y_2 \mid e_2 >_T\ , \end{aligned} \tag{2.16}$$

and thus for any $e_1 \in L_e(U_1),\ e_2 \in L_e(U_2)$ and any $T \geq 0$

$$\begin{aligned} &< y_1 \mid u_1 >_T + < y_2 \mid u_2 >_T \\ &= < y_1 \mid e_1 >_T + < y_2 \mid e_2 >_T \end{aligned} \tag{2.17}$$

with $y_1 = G_1(u_1)$, $y_2 = G_2(u_2)$. By passivity of G_1 and G_2, $< y_1 \mid u_1 >_T \geq -\beta_1$, $< y_2 \mid u_2 >_T \geq -\beta_2$, and thus by (2.17)

$$< y_1 \mid e_1 >_T + < y_2 \mid e_2 >_T \geq -\beta_1 - \beta_2 \tag{2.18}$$

implying part (a). For part (b) take $e_2 = 0$ in (2.17). □

Remark 2.2.7 Theorem 2.2.6 expresses an inherent *robustness* property of passive systems: the closed-loop system $G_1\|_f G_2$ remains passive for all perturbations of the input–output maps G_1, G_2, as long as they *remain passive* (compare with Remark 2.1.2).

In order to state a stronger version of the Passivity theorem we need stronger notions of passivity. First of all, we will assume that the input space U is equipped with an *inner product* $<,>$. Using the linear bijection

$$u \in U \longmapsto < u, \cdot > \in U^*, \tag{2.19}$$

we may then identify $Y = U^*$ with U. That is, $Y = U^* = U$, and $< y \mid u > = < y, u >$. Furthermore, for any input function $u \in L_{2e}(U)$ and corresponding output function $y = G(u) \in L_{2e}(U)$ we will have $< y \mid u >_T = \int_0^T < y(t), u(t) > dt$, which will be throughout denoted by $< y, u >_T$.

Definition 2.2.8 *(Output and input strict passivity)* Let $U = Y$ be a linear space with inner product $<,>$ and corresponding norm $||\cdot||$. Let $G : L_{2e}(U) \to L_{2e}(Y)$ be an input–output map. Then G is *input strictly passive* if there exists β and $\delta > 0$ such that

$$< G(u), u >_T \geq \delta ||u_T||_2^2 - \beta, \quad \forall u \in L_{2e}(U),\ \forall T \geq 0, \tag{2.20}$$

and *output strictly passive* if there exists β and $\varepsilon > 0$ such that

$$< G(u), u >_T \geq \varepsilon ||(G(u))_T||_2^2 - \beta, \quad \forall u \in L_{2e}(U),\ \forall T \geq 0. \tag{2.21}$$

Furthermore, $G : L_{2e}(U) \to L_{2e}(Y)$ is merely *passive* if there exists β such that (2.21) holds for $\varepsilon = 0$ (or equivalent (2.20) for $\delta = 0$). Whenever we want to emphasize the role of the constants δ, ε we will say that G is δ*-input strictly passive* or ε*-output strictly passive*. In the same way we define (δ-)input and (ε-)output strict passivity for *relations* $R \subset L_{2e}(U) \times L_{2e}(Y)$.

Remark 2.2.9 Note that by Remark 2.2.3 the left-hand sides of (2.20) and (2.21) are well defined.

Remark 2.2.10 Proposition 2.2.5 immediately generalizes to input, respectively, output strict passivity.

We obtain the following extension of Theorem 2.2.6.

Theorem 2.2.11 (Passivity theorem; second version) *Consider the closed-loop system $G_1\|_f G_2$ in Fig. 1.1, with $G_1 : L_{2e}(U_1) \to L_{2e}(U_1)$, $G_2 : L_{2e}(U_2) \to L_{2e}(U_2)$, and $E_1 = U_1 = U_2 = E_2 =: U$ an inner product space.*

(a) Assume that for any $e_1, e_2 \in L_{2e}(U)$ there are solutions $u_1, u_2 \in L_{2e}(U)$. If G_1 and G_2 are respectively ε_1- and ε_2-output strictly passive, then $G_1\|_f G_2$ with inputs (e_1, e_2) and outputs (y_1, y_2) is ε-output strictly passive, with $\varepsilon =$ $\min(\varepsilon_1, \varepsilon_2)$.

(b) Assume that for any $e_1 \in L_{2e}(U)$ and $e_2 = 0$ there are solutions $u_1, u_2 \in L_{2e}(U)$. If G_1 is passive and G_2 is δ_2-input strictly passive, or if G_1 is ε_1-output strictly passive and G_2 is passive, then $G_1\|_f G_2$ for $e_2 = 0$, with input e_1 and output y_1, is δ_2-input, respectively ε_1-output strictly passive.

Proof Equation (2.17) becomes

$$< y_1, u_1 >_T + < y_2, u_2 >_T = < y_1, e_1 >_T + < y_2, e_2 >_T \tag{2.22}$$

(a) Since G_1 and G_2 are output strictly passive (2.22) implies

$$< y_1, e_1 >_T + < y_2, e_2 >_T = < y_1, u_1 >_T + < y_2, u_2 >_T$$

$$\geq \varepsilon_1 ||y_{1T}||_2^2 + \varepsilon_2 ||y_{2T}||_2^2 - \beta_1 - \beta_2$$

$$\geq \varepsilon(||y_{1T}||_2^2 + ||y_{2T}||_2^2) - \beta_1 - \beta_2$$

for $\varepsilon = \min(\varepsilon_1, \varepsilon_2) > 0$.

(b) Let G_1 be passive and G_2 be δ_2-input strictly passive. By (2.22) with $e_2 = 0$

$$< y_1, e_1 >_T = < y_1, u_1 >_T + < y_2, u_2 >_T$$

$$\geq -\beta_1 + \delta_2 ||u_{2T}||_2^2 - \beta_2 = \delta_2 ||y_{1T}||_2^2 - \beta_1 - \beta_2$$

If G_1 is ε_1-output strictly passive and G_2 is passive, then the same inequality holds with δ_2 replaced by ε_1. □

Remark 2.2.12 A similar theorem can be stated for *relations* R_1 and R_2.

For statements regarding the L_2-stability of the feedback interconnection of passive systems a key observation will be the fact that *output strict passivity* implies *finite L_2-gain*.

Theorem 2.2.13 *Let $G : L_{2e}(U) \to L_{2e}(U)$ be ε-output strictly passive. Then G has L_2-gain $\leq \frac{1}{\varepsilon}$.*

Proof Since G is ε-output strictly passive there exists β such that $y = G(u)$ satisfies

$$\begin{aligned} \varepsilon||y_T||_2^2 &\leq < y, u >_T + \beta \\ &\leq < y, u >_T + \beta + \tfrac{1}{2}||\tfrac{1}{\sqrt{\varepsilon}}u_T - \sqrt{\varepsilon}y_T||_2^2 \\ &= \beta + \tfrac{1}{2\varepsilon}||u_T||_2^2 + \tfrac{\varepsilon}{2}||y_T||_2^2, \end{aligned} \tag{2.23}$$

whence $\frac{\varepsilon}{2}||y_T||_2^2 \leq \frac{1}{2\varepsilon}||u_T||_2^2 + \beta$, proving that $\gamma_2(G) \leq \frac{1}{\varepsilon}$. □

Remark 2.2.14 As a partial converse statement, note that if G is δ-*input* strictly passive *and* has L_2-gain $\leq \gamma$, then

$$< G(u), u > \ \geq \delta\|u\|_2^2 - \beta \ \geq \ \frac{\delta}{\gamma}\|G(u)\|_2^2 - \beta,$$

implying that G is $\frac{\delta}{\gamma}$-output strictly passive.

Combining Theorems 2.2.11 and 2.2.13 one directly obtains the following.

Theorem 2.2.15 (Passivity theorem; third version) *Consider the closed-loop system $G_1\|_f G_2$ in Fig. 1.1, with $G_1 : L_{2e}(U_1) \to L_{2e}(U_1)$, $G_2 : L_{2e}(U_2) \to L_{2e}(U_2)$, and $E_1 = E_2 = U_1 = U_2 =: U$ an inner product space.*

(a) Assume that for any $e_1, e_2 \in L_{2e}(U)$ there exist solutions $u_1, u_2 \in L_{2e}(U)$. If G_i is ε_i-output strictly passive, $i = 1, 2$, then $G_1\|_f G_2$ with inputs (e_1, e_2) and outputs (y_1, y_2) has L_2-gain $\leq \frac{1}{\varepsilon}$ with $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. For $e_1, e_2 \in L_2(U)$ it follows that $u_1, u_2, y_1, y_2 \in L_2(U)$.

(b) Assume that for any $e_1 \in L_{2e}(U)$ and $e_2 = 0$ there are solutions $u_1, u_2 \in L_{2e}(U)$. If G_1 is passive and G_2 is δ_2-input strictly passive, or if G_1 is ε_1-output strictly passive and G_2 is passive, then $G_1\|_f G_2$ for $e_2 = 0$ with input e_1 and output y_1 has L_2-gain $\leq \frac{1}{\delta_2}$, respectively $\leq \frac{1}{\varepsilon_1}$. Furthermore, if $e_1 \in L_2(U)$ then also $y_1 = u_2 \in L_2(U)$.

Remark 2.2.16 Suppose G_1 and G_2 are causal. Then by Propositions 2.2.5 and 1.2.14 we can relax the assumption in (a) to assuming that for any $e_1, e_2 \in L_2(U)$ there exist solutions $u_1, u_2 \in L_{2e}(U)$. Similarly, we can relax the assumption in (b) to assuming that for any $e_1 \in L_2(U)$ and $e_2 = 0$ there exist solutions $u_1, u_2 \in L_{2e}(U)$. If G_1 and/or G_2 are not causal, then this relaxation of assumptions will guarantee at least L_2-stability.

Example 2.2.17 Note that in Theorem 2.2.15 (b) it is *not* claimed that u_1 and $y_2 = G_2(u_2)$ are in $L_2(U)$. In fact, a *physical counterexample* to such a claim can be given as follows. Consider a mass moving in one-dimensional space. Let the mass be subject to a friction force which is the sum of an ideal Coulomb friction and a linear damping. Furthermore, let the mass be actuated by a force $u_1 = e_1 - y_2$, where e_1 is an external force and y_2 is the force delivered by a linear spring. Defining y_1 as the velocity of the mass, the input–output map G_1 from u_1 to y_1 for zero initial condition (velocity zero) is output strictly passive, as follows from the definition of the friction force. Furthermore, let G_2 be the passive input–output map defined by the linear spring for zero initial extension, with the spring attached at one end to a wall and with the velocity of the other end being its input u_2 and with output y_2 being the spring force (acting on the mass). Now let $e_1(\cdot)$ be an external force time function with the shape of a pulse, of magnitude h and width w. Then by taking h large enough the force e_1 will overcome the total friction force (in particular the Coulomb friction force), resulting in a motion of the mass and thus of the free end of the spring. On the other hand by taking the width w of the pulse small enough the extension of the spring will be such that the spring force does not overcome the Coulomb friction force. As a result, the velocity of the mass y_1 will converge to zero, while the spring force y_2 will converge to a nonzero constant value (smaller than the Coulomb friction constant). Hence, y_2 and u_1 will *not* be in $L_2(\mathbb{R})$.

A useful generalization of the Passivity Theorems 2.2.11 (a) and 2.2.15 (a), where we do *not* necessarily require passivity of G_1 and G_2 separately, can be stated as follows.

Theorem 2.2.18 *Suppose there exist constants $\varepsilon_i, \delta_i, \beta_i, i = 1, 2$, satisfying*

$$\varepsilon_1 + \delta_2 > 0, \ \varepsilon_2 + \delta_1 > 0 \tag{2.24}$$

such that

$$< G_i(u_i), u_i >_T \ \geq \ \varepsilon_i ||(G_i(u_i))_T||_2^2 + \delta_i ||u_{iT}||_2^2 - \beta_i \ , \tag{2.25}$$

for all $u_i \in L_{2e}(U_i)$ and all $T \geq 0, i = 1, 2$. Then $G_1 \|_f G_2$ has finite L_2-gain from (e_1, e_2) to (y_1, y_2).

Proof Addition of (2.25) with $y_i = G_i(u_i)$ for $i = 1, 2$ yields

$$\begin{aligned} &< y_1, u_1 >_T + < y_2, u_2 >_T \\ &\quad \geq \varepsilon_1 \|y_{1T}\|_2^2 + \delta_1 \|u_{1T}\|_2^2 + \varepsilon_2 \|y_{2T}\|^2 + \delta_2 \|u_{2T}\|^2 - \beta_1 - \beta_2. \end{aligned} \tag{2.26}$$

Substitution of the negative feedback $u_1 = e_1 - y_2,\ u_2 = e_2 + y_1$ results in

$$\begin{aligned} &< y_1, e_1 >_T + < y_2, e_2 >_T + \beta_1 + \beta_2 \\ &\quad \geq \varepsilon_1 \|y_{1T}\|_2^2 + \delta_1 \|e_1 - y_2\|_2^2 + \varepsilon_2 \|y_2\|_2^2 + \delta_2 \|e_2 + y_1\|_2^2. \end{aligned} \tag{2.27}$$

Writing out and rearranging terms leads to

$$\begin{aligned} &-\delta_1 \|e_{1T}\|_2^2 - \delta_2 \|e_{2T}\|_2^2 + \beta_1 + \beta_2 \\ &\quad \geq (\varepsilon_1 + \delta_2) \|y_{1T}\|_2^2 + (\varepsilon_2 + \delta_1) \|y_{2T}\|_2^2 \\ &\quad -2\delta_1 < y_2, e_1 >_T -2\delta_2 < y_1, e_2 >_T - < y_1, e_1 >_T - < y_2, e_2 >_T . \end{aligned}$$

By the positivity assumption on $\alpha_1^2 := \varepsilon_1 + \delta_2,\ \alpha_2^2 := \varepsilon_2 + \delta_1$ we can perform "completion of the squares" on the right-hand side of this inequality, to obtain an expression of the form

$$\| \begin{bmatrix} \alpha_1 y_{1T} \\ \alpha_2 y_{2T} \end{bmatrix} - A \begin{bmatrix} e_{1T} \\ e_{2T} \end{bmatrix} \|_2^2 \leq c^2 \| \begin{bmatrix} e_{1T} \\ e_{2T} \end{bmatrix} \|_2^2 + \beta_1 + \beta_2, \tag{2.28}$$

for a certain 2×2 matrix A and constant c. In combination with the triangle inequality

$$\| \begin{bmatrix} \alpha_1 y_{1T} \\ \alpha_2 y_{2T} \end{bmatrix} \|_2 \leq \| \begin{bmatrix} \alpha_1 y_{1T} \\ \alpha_2 y_{2T} \end{bmatrix} - A \begin{bmatrix} e_{1T} \\ e_{2T} \end{bmatrix} \|_2 + \| A \begin{bmatrix} e_{1T} \\ e_{2T} \end{bmatrix} \|_2, \tag{2.29}$$

this yields finite L_2-gain from (e_1, e_2) to (y_1, y_2). □

Remark 2.2.19 Clearly, Theorem 2.2.18 includes Part (a) of Theorems 2.2.11 and 2.2.15 by taking $\delta_1 = \delta_2 = 0$. Importantly, it shows that $\varepsilon_1, \varepsilon_2, \delta_1, \delta_2$ need not all be nonnegative. Negativity of ε_1 ("lack of passivity" of G_1) can be "compensated" by a sufficiently large positive δ_2 ("surplus of passivity" of G_2).

Notice that the last version of the Passivity Theorem 2.2.15 still assumes the *existence of solutions* $u_1, u_2 \in L_{2e}(U)$. In the small-gain case this was remedied, cf. Theorem 2.1.8, by replacing finite L_q-gain and the small-gain condition by their *incremental versions*. Similarly this can be done by invoking a notion of *incremental* passivity defined as follows.

Definition 2.2.20 *(Incremental passivity)* An input–output map $G : L_{2e}(U) \to L_{2e}(Y)$ is $\mathfrak{E}$-*output strictly incrementally passive* for some $\mathfrak{E} > 0$ if there exists β such that

$$\mathfrak{E} ||y_T - z_T\|_2^2 \ \leq \ < y - z, u - v >_T \ + \ \beta \tag{2.30}$$

for all $u, v \in L_{2e}(U)$ and corresponding outputs $y = G(u), z = G(v)$. If $\mathfrak{E} = 0$ then G is *incrementally passive*.

Furthermore, G is called Δ-*input strictly incrementally passive* for some $\Delta > 0$ if there exists β such that

$$\Delta ||u_T - v_T\|_2^2 \ \leq \ < y - z, u - v >_T \ + \ \beta \tag{2.31}$$

for all $u, v \in L_{2e}(U)$ and corresponding outputs $y = G(u), z = G(v)$.

We immediately obtain the following incremental version of Theorem 2.2.15.

Proposition 2.2.21 *Consider the closed-loop system $G_1 \|_f G_2$ in Fig. 1.1, with $G_1 : L_{2e}(U_1) \to L_{2e}(U_1)$, $G_2 : L_{2e}(U_2) \to L_{2e}(U_2)$, and $E_1 = U_1 = U_2 = E_2 =: U$ an inner product space.*

(a) Assume that for any $e_1, e_2 \in L_{2e}(U)$ there are solutions $u_1, u_2 \in L_{2e}(U)$. If G_1 and G_2 are respectively $\mathfrak{E}_1$- and $\mathfrak{E}_2$-output strictly incrementally passive, then $G_1 \|_f G_2$ with inputs (e_1, e_2) and outputs (y_1, y_2) is $\mathfrak{E}$-output strictly incrementally passive, with $\mathfrak{E} = \min(\mathfrak{E}_1, \mathfrak{E}_2)$.
(b) Assume that for any $e_1 \in L_{2e}(U)$ and $e_2 = 0$ there are solutions $u_1, u_2 \in L_{2e}(U)$. If G_1 is incrementally passive and G_2 is Δ_2-input strictly incrementally passive, or if G_1 is $\mathfrak{E}_1$-output strictly incrementally passive and G_2 is incrementally passive, then $G_1 \|_f G_2$ with $e_2 = 0$ and input e_1 and output y_1 is $\mathfrak{E}$-output strictly incrementally passive, with $\mathfrak{E}$ equal to Δ_2 respectively $\mathfrak{E}_1$.

The following crucial step is the observation that output strict incremental passivity implies finite incremental L_2-gain in the same way as output strict passivity implies finite L_2-gain, cf. Theorem 2.2.13.

Proposition 2.2.22 *Let $G : L_{2e}(U) \to L_{2e}(U)$ be $\mathfrak{E}$-output strictly incrementally passive. Then G has incremental L_2-gain $\leq \frac{1}{\mathfrak{E}}$.*

Proof Repeat the same argument as in the proof of Theorem 2.2.13, but now in the incremental setting, to conclude that

$$\mathfrak{E}\|y_T - z_T\|_2^2 \leq \beta + \frac{1}{2\mathfrak{E}}\|u_T - v_T\|_2^2 + \frac{\mathfrak{E}}{2}\|y_T - z_T\|_2^2,$$

where $y = G(u), z = G(v)$. This proves that the incremental L_2-gain of G is $\leq \frac{1}{\mathfrak{E}}$. □

By combining Propositions 2.2.21 and 2.2.22 with Theorem 2.1.8 we immediately obtain the following corollary.

Corollary 2.2.23 *Consider the closed-loop system $G_1 \|_f G_2$ in Fig. 1.1, with $G_1 : L_{2e}(U_1) \to L_{2e}(U_1)$, $G_2 : L_{2e}(U_2) \to L_{2e}(U_2)$, and $E_1 = E_2 = U_1 = U_2 =: U$ an inner product space.*

Assume that G_1 and G_2 are $\mathfrak{E}_1$-, respectively $\mathfrak{E}_2$-, output strictly incrementally passive, and that

$$\mathfrak{E}_1 \cdot \mathfrak{E}_2 \ > \ 1. \tag{2.32}$$

Then

(i) For all $(e_1, e_2) \in L_{2e}(E_1 \times E_2)$ there exists a unique solution $(u_1, u_2, y_1, y_2) \in L_{2e}(U_1 \times U_2 \times Y_1 \times Y_2)$.

(ii) *The map* $(e_1, e_2) \mapsto (u_1, u_2)$ *is uniformly continuous on the domain* $L_{2e}(E_1 \times E_2)$.
(iii) *If the solution* (u_1, u_2) *to* $e_1 = e_2 = 0$ *is in* $L_2(U_1 \times U_2)$, *then* $(e_1, e_2) \in L_2(E_1 \times E_2)$ *implies that* $(u_1, u_2) \in L_2(U_1 \times U_2)$.

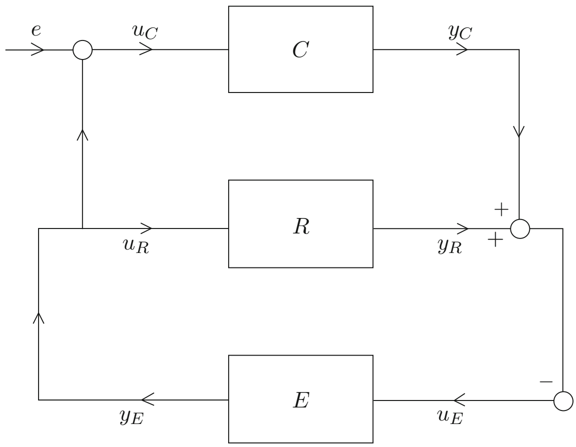
Remark 2.2.24 (General power-conserving interconnections) All the derived passivity theorems can be generalized to interconnections which are more general than the standard feedback interconnection of Fig. 1.1. This relies on the observation that the essential requirement in the proof of Theorem 2.2.6 is the identity (2.16), expressing the fact that the feedback interconnection $u_1 = -y_2 + e_1, u_2 = y_1 + e_2$ is *power-conserving*. Many other interconnections share this property, and as a result the interconnected systems share the same passivity properties as the closed-loop systems arising from standard feedback interconnection. As an example, consider the following system (taken from [355]) given in Fig. 2.1. Here R represents a robotic system and C is a controller, while E represents the environment interacting with the controlled robotic mechanism. The external signal e denotes a velocity command. We assume R and E to be passive, and C to be a output strictly passive controller. By the interconnection constraints $u_C = y_E + e, \; u_R = y_E$ and $u_E = -y_R - y_C$ we obtain

$$< y_C \mid u_C > + < y_R \mid u_R > + < y_E \mid u_E > \; = \; < y_C \mid e >$$

and hence, as in Theorem 2.2.15 part (b), the interconnected system with input e and output y_C is output strictly passive, and therefore has finite L_2-gain.

This idea will be further developed in the subsequent chapters, especially in Chaps. 4, 6 and 7 in the passive and port-Hamiltonian systems context.

Fig. 2.1 An alternative power-conserving interconnection



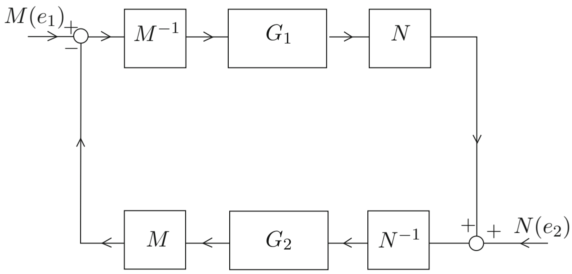


Fig. 2.2 Feedback system with multipliers

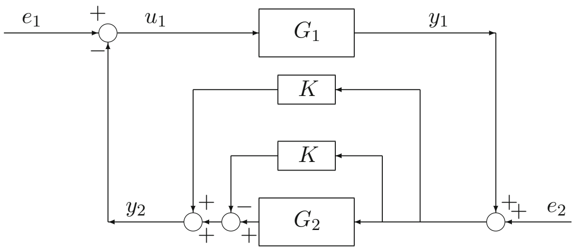


Fig. 2.3 Transformed closed-loop configuration

2.3 Loop Transformations

The range of applicability of the small-gain and passivity theorems can be considerably enlarged using *loop transformations*. We will only indicate two basic ideas.

The first possibility is to insert *multipliers* in Fig. 1.1 by pre- and post-multiplying G_1 and G_2 by L_q-stable input–output mappings M and N and their inverses M^{-1} and N^{-1}, which are also assumed to be L_q-stable input–output mappings, see Fig. 2.2.

By L_q-stability of M, M^{-1}, N and N^{-1} it follows that $e_1 \in L_q(E_1)$, $e_2 \in L_q(E_2)$ if and only if $M(e_1) \in L_q(E_1),\ M(e_2) \in L_q(E_2)$. Thus stability of $G_1 \|_f G_2$ is equivalent to stability of $G_1 \|_f G_2$, with $G_1' = NG_1M^{-1},\ G_2' = MG_2N^{-1}$.

A second idea is to introduce an additional L_q-stable and *linear* operator K in the closed-loop system $G_1 \|_f G_2$ by first subtracting and then adding to G_2 (see Fig. 2.3).

Using the linearity of K, this can be redrawn as in Fig. 2.4. Clearly, by stability of K, $e_1 - K(e_2)$ and e_2 are in L_q if and only if e_1, e_2 are in L_q. Thus stability of $G_1 \|_f G_2$ is equivalent to stability of $G_1' \|_f G_2'$.

2.4 Scattering and the Relation Between Passivity and L_2-Gain

Let us return to the basic setting of passivity, as exposed in Sect. 2.2, starting with a finite-dimensional linear input space U (without any additional structure such as inner product or norm) and its dual space $Y := U^*$ defining the space of outputs.

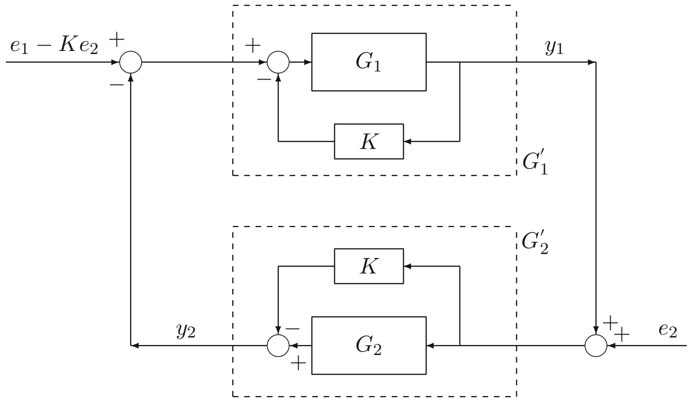


Fig. 2.4 Redrawn transformed closed-loop configuration

On the product space $U \times Y$ of inputs and outputs there exists a canonically defined symmetric bilinear form $\ll, \gg$, given as

$$\ll (u_1, y_1), (u_2, y_2) \gg := < y_1 \mid u_2 > + < y_2 \mid u_1 > \tag{2.33}$$

with $u_i \in U,\ y_i \in Y,\ i = 1, 2$, and $< \mid >$ denoting the duality pairing between $Y = U^*$ and U. With respect to a basis $e_1, \ldots, e_m$ of U (where $m = \dim U$), and the corresponding dual basis $e_1^*, \ldots, e_m^*$ of $Y = U^*$, the bilinear form $\ll, \gg$ has the matrix representation

$$\begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} \tag{2.34}$$

It immediately follows that $\ll, \gg$ has singular values $+1$ (with multiplicity m) and -1 (also with multiplicity m), and thus defines an *indefinite* inner product on the space $U \times Y$ of inputs and outputs. *Scattering* is based on decomposing the combined vector $(u, y) \in U \times Y$ with respect to the positive and negative singular values of this indefinite inner product. More precisely, we obtain the following definition.

Definition 2.4.1 Any pair (V, Z) of subspaces $V, Z \subset U \times Y$ is called a pair of *scattering subspaces* if

(i) $V \oplus Z = U \times Y$
(ii) $\ll v_1, v_2 \gg > 0$, for all $v_1, v_2 \in V$ unequal to 0,
$\ll z_1, z_2 \gg < 0$, for all $z_1, z_2 \in Z$ unequal to 0
(iii) $\ll v, z \gg = 0$, for all $v \in V, z \in Z$.

It follows from (2.34) that any pair of scattering subspaces (V, Z) satisfies

$$\dim V = \dim Z = m$$

Given a pair of scattering subspaces (V, Z) it follows that any combined vector $(u, y) \in U \times Y$ also can be represented, in a unique manner, as a pair $v \oplus z \in V \oplus Z$, where v is the projection along Z of the combined vector $(u, y) \in U \times Y$ on V, and z is the projection of (u, y) along V on Z. The representation $(u, y) = v \oplus z$ is called a *scattering* representation of (u, y), and v, z are called the *wave vectors* of the combined vector (u, y).

Using orthogonality of V with respect to Z it immediately follows that for all $(u_i, y_i) = v_i \oplus z_i, i = 1, 2,$

$$\ll (u_1, y_1), (u_2, y_2) \gg = < v_1, v_2 >_V - < z_1, z_2 >_Z \tag{2.35}$$

where $< , >_V$ denotes the inner product on V defined as the restriction of $\ll , \gg$ to V, and $< , >_Z$ denotes the inner product on Z defined as *minus* the restriction of $\ll , \gg$ to Z.

In particular, taking $(u_1, y_1) = (u_2, y_2) = (u, y)$, we obtain for any $(u, y) = v \oplus z$ the following fundamental relation between (u, y) and its wave vectors v, z

$$< y \mid u > = \frac{1}{2} \ll (u, y), (u, y) \gg = \frac{1}{2}||v||_V^2 - \frac{1}{2}||z||_Z^2 , \tag{2.36}$$

where $|| \; ||_V, \; || \; ||_Z$ are the norms on V, Z, defined by $< , >_V$, respectively $< , >_Z$.

Identifying as before $< y \mid u >$ with *power*, the vector v thus can be regarded as the *incoming* wave vector, with half times its norm being the *incoming power*, and the vector z is the *outgoing* wave vector, with half times its norm being the *outgoing power*.

Now let $G : L_e(U) \to L_e(Y)$, with $Y = U^*$, be an input–output map as before. Expressing $(u, y) \in U \times Y$ in a scattering representation as $v \oplus z \in V \oplus Z$, it follows that G transforms into the *relation*

$$\begin{aligned} R_{vz} = \{ & v \oplus z \in L_e(V) \oplus L_e(Z) \mid \\ & v(t) \oplus z(t) = (u(t), y(t)), \; t \in \mathbb{R}^+, \; y = G(u)\}, \end{aligned} \tag{2.37}$$

with the function spaces $L_e(V)$ and $L_e(Z)$ yet to be defined. As a direct consequence of (2.36) we obtain the following relation between G and R_{vz}:

$$< G(u) \mid u >_T = \frac{1}{2}||v_T||_V^2 - \frac{1}{2}||z_T||_Z^2, \quad T \geq 0. \tag{2.38}$$

In particular, if u and $y = G(u)$ are such that $v \in L_{2e}(V)$ and $z \in L_{2e}(Z)$ then, since the right-hand side of (2.38) is well defined, also the expression $< G(u) \mid u >_T$ is well defined for all $T \geq 0$.

We obtain from (2.38) the following key relation between passivity of G and the L_2-gain of R_{vz}.

Proposition 2.4.2 *Consider the relation $R_{vz} \subset L_{2e}(V) \oplus L_{2e}(Z)$ as defined in (2.37), with L_e replaced by L_{2e}. Then G is passive if and only if R_{vz} has L_2-gain ≤ 1.*

Proof By (2.38), $||z_T||_Z^2 \leq ||v_T||_V^2 + c$ if and only if $< G(u) \mid u >_T \geq -\frac{c}{2}$. □

If the relation R_{vz} can be written as the graph of an input–output map

$$S : L_{2e}(V) \to L_{2e}(Z), \tag{2.39}$$

(with respect to the intrinsically defined norms $|| \, ||_V$ and $|| \, ||_Z$) then we call S the *scattering operator* of the input–output map G. We obtain the following fundamental relation between passivity and L_2-gain.

Corollary 2.4.3 *The scattering operator S has L_2-gain ≤ 1 if and only if G is passive.*

As noted before, the choice of scattering subspaces V, Z, and therefore of the scattering representation, is not unique. Particular choices of scattering subspaces are given as follows. Take any basis $e_1, \ldots, e_m$ for U, with dual basis $e_1^*, \ldots, e_m^*$ for $U^* = Y$. Then it can be directly checked that the pair (V, Z) given as

$$\begin{aligned} V &= \text{span}\left\{\left(\tfrac{e_i}{\sqrt{2}}, \tfrac{e_i^*}{\sqrt{2}}\right), i = 1, \ldots, m\right\} \\ Z &= \text{span}\left\{\left(\tfrac{-e_i}{\sqrt{2}}, \tfrac{e_i^*}{\sqrt{2}}\right), i = 1, \ldots, m\right\} \end{aligned} \tag{2.40}$$

defines a pair of scattering subspaces. (In the above the factors $\frac{1}{\sqrt{2}}$ were inserted in order that the vectors spanning V, respectively Z, are *orthonormal* with respect to the intrinsically defined inner products $< , >_V$ and $< , >_Z$.) In these bases for U, Y and V, Z the relation between (u, y) and its scattering representation (v, z) is given as

$$\begin{aligned} v &= \tfrac{1}{\sqrt{2}}(u + u^*) \\ z &= \tfrac{1}{\sqrt{2}}(-u + u^*). \end{aligned} \tag{2.41}$$

Hence, with $y = G(u)$, the relation R_{vz} has the coordinate expression

$$\begin{aligned} R_{vz} = \{(v, z) : \mathbb{R}^+ \to V \times Z \mid \\ v(t) = \tfrac{1}{\sqrt{2}}(G + I)(u)(t),\ z(t) = \tfrac{1}{\sqrt{2}}(G - I)(u)(t)\}, \end{aligned} \tag{2.42}$$

where I denotes the identity operator. In particular, R_{vz} can be expressed as the graph of a scattering operator S if and only if the operator $G + I : L(U) \to L(V)$ is *invertible*, in which case S takes the standard form

$$S = (G - I)(G + I)^{-1}. \tag{2.43}$$

In case U is equipped with an inner product $< , >_U$, and U^* can be identified with U (see Sect. 2.2), we obtain the following relation between passivity of G and L_2-gain of R_{vz}.

Proposition 2.4.4 *Let U be endowed with an inner product $< , >_U$. Consider an input–output mapping $G : L_{2e}(U) \to L_{2e}(U)$ and the corresponding relation $R_{vz} \subset L_{2e}(V) \times L_{2e}(Z)$. Then G is input and output strictly passive if and only if the L_2-gain of R_{vz} (or, if $G + I$ is invertible, the L_2-gain of the scattering operator S) is strictly less than* 1.

Proof Let the L_2-gain of R_{vz} be $\leq 1 - \delta$, with $1 \geq \delta > 0$. Then $||z_T||_2^2 \leq (1 - \delta)||v_T||_2^2 + c$, and thus by (2.38)

$$2 < G(u) \mid u > \geq \ \delta||v_T||_2^2 - c$$

Since $||v_T||_2^2 = ||u_T + (G(u))_T||_2^2 = ||u_T||_2^2 + ||G(u)_T||_2^2 + 2 < G(u) \mid u >$, this implies for some $\epsilon > 0$ and β

$$< G(u) \mid u > \geq \ \varepsilon||G(u)||_2^2 + \varepsilon||u||_2^2 - \beta$$

The converse statement follows similarly. □

Remark 2.4.5 Since "input strict passivity" plus "finite L_2-gain" implies output strict passivity, cf. Remark 2.2.14, and conversely output strict passivity implies finite L_2-gain, the condition of input and output strict passivity in the above proposition can be replaced by input strict passivity and finite L_2-gain.

2.5 Notes for Chapter 2

1. The treatment of Sects. 2.1 and 2.2 is largely based on Vidyasagar [343], with extensions from Desoer & Vidyasagar [83]. We have emphasized a "coordinate-free" treatment of the theory, which in particular has some impact on the formulation of passivity. See also Sastry [267] and Khalil [160] for expositions. The developments regarding incremental passivity, in particular Corollary 2.2.23, seem to be relatively new.
2. The small-gain theorem is usually attributed to Zames [362, 363], and in its turn is closely related to the Nyquist stability criterion. See also Willems [348]. A classical treatise on passivity and its implications for stability is Popov [255].
3. Theorem 2.2.18 is treated in Sastry [267], Vidyasagar [343].
4. An interesting generalization of the small-gain theorem (Theorem 2.1.1) is obtained by considering input–output maps G_1 and G_2 that have a finite "*nonlinear gain*" in the following sense. Suppose there exist functions $\gamma_i : \mathbb{R}^+ \to \mathbb{R}^+$

of class[1] $\mathcal{K}$ and constants b_i, $i = 1, 2$, such that

$$\|G_i(u_T)\| \leq \gamma_i(\|u_T\|) + b_i, \quad T \geq 0, \tag{2.44}$$

for $i = 1, 2$, where $\|\ \|$ denotes some L_q-norm. Note that by taking *linear* functions $\gamma_i(z) = \gamma_i z$, with constant $\gamma_i > 0$, we recover the usual definition of finite gain. Then, similar to the proof of Theorem 2.1.1, we derive the following inequalities for the closed-loop system $G_1\|_f G_2$:

$$\begin{aligned} \|u_{1T}\| &\leq \|y_{2T}\| + \|e_{1T}\| \\ \|u_{2T}\| &\leq \|y_{1T}\| + \|e_{2T}\| \end{aligned} \tag{2.45}$$

and thus by (2.44)

$$\begin{aligned} \|y_{1T}\| &\leq \gamma_1(\|y_{2T}\| + \|e_{1T}\|) + b_1 \\ \|y_{2T}\| &\leq \gamma_2(\|y_{1T}\| + \|e_{2T}\|) + b_2 \end{aligned} \tag{2.46}$$

which by cross-substitution yields

$$\begin{aligned} \|y_{1T}\| &\leq \gamma_1(\gamma_2(\|y_{1T}\| + \|e_{2T}\|) + \|e_{1T}\| + \ b_2) + b_1 \\ \|y_{2T}\| &\leq \gamma_2(\gamma_1(\|y_{2T}\| + \|e_{1T}\|) + \|e_{2T}\| + \ b_1) + b_2. \end{aligned} \tag{2.47}$$

One may wonder under what conditions on γ_1 and γ_2 the inequalities (2.47) imply that

$$\begin{aligned} \|y_{1T}\| &\leq \delta_1(\|e_{1T}\|, \|e_{2T}\|) + d_1 \\ \|y_{2T}\| &\leq \delta_2(\|e_{1T}\|, \|e_{2T}\|) + d_2 \end{aligned} \tag{2.48}$$

for certain constants d_1, d_2 and functions $\delta_i : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $i = 1, 2$, which are of class $\mathcal{K}$ in *both* their arguments. Indeed, this would imply that the closed-loop system $G_1\|_f G_2$ has finite nonlinear gain from e_1, e_2 to y_1, y_2. As shown in Mareels & Hill [194] this is the case if there exist functions $g, h \in \mathcal{K}$ and a constant $c \geq 0$, such that

$$\gamma_1 \circ (\mathrm{id} + g) \circ \gamma_2(z) \leq z - h(z) + c, \ \text{ for all } z, \tag{2.49}$$

with id denoting the identity mapping. Condition (2.49) can be interpreted as a direct generalization of the small-gain condition $\gamma_1 \cdot \gamma_2 < 1$. See also [149] for another formulation.

5. There is an extensive literature related to the theory presented in Sects. 2.1 and 2.2. Among the many contributions we mention the work of Safonov [262] & Teel [337] on conic relations, the work on nonlinear small-gain theorems in Mareels & Hill [194], Jiang, Teel & Praly [149], Teel [336] briefly discussed in the pre-

[1] A function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is of class $\mathcal{K}$ (denoted $\gamma \in \mathcal{K}$) if it is zero at zero, strictly increasing and continuous.

vious Note 4, and work on robust stability, see e.g., Georgiou [111], Georgiou & Smith [112], as well as the important contributions on stability theory within the "Petersburg school", see e.g., the classical paper Yakubovich [359], and developments inspired by this, see e.g., Megretski & Rantzer [215]. The developments stemming from dissipative systems theory will be treated in Chaps. 3, 4, and 8.

6. For further ramifications and implications of the loop transformations sketched in Sect. 2.3 we refer to Vidyasagar [343], Scherer, Gahinet & Chilali [306], Scherer [307], and the references quoted therein.

7. The scattering relation between L_2-gain and passivity is classical, and can be found in Desoer & Vidyasagar [83], see also Anderson [6]. The geometric, coordinate-free, treatment given in Sect. 2.4 is developed in Maschke & van der Schaft [208], Stramigioli, van der Schaft, Maschke & Melchiorri [190, 331], Cervera, van der Schaft & Banos [63].

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