Chapter 2
First-Order Equations

Certain types of first-order equations can be solved by relatively simple methods. Since, as seen in Sect. 1.2, many mathematical models are constructed with such equations, it is important to get familiarized with their solution procedures.

2.1 Separable Equations

These are equations of the form

$$\frac{dy}{dx} = f(x)g(y), \tag{2.1}$$

where f and g are given functions.

We notice that if there is any value y_0 such that $g(y_0) = 0$, then $y = y_0$ is a solution of (2.1). Since this is a constant function (that is, independent of x), we call it an *equilibrium solution.*

To find all the other (non-constant) solutions of the equation, we now assume that $g(y) \neq 0$. Applying the definition of the differential of y and using (2.1), we have

$$dy = y'(x)\,dx = \frac{dy}{dx}\,dx = f(x)g(y)\,dx,$$

which, after division by $g(y)$, becomes

$$\frac{1}{g(y)}\,dy = f(x)\,dx.$$

Next, we integrate each side with respect to its variable and arrive at the equality

$$G(y) = F(x) + C, \tag{2.2}$$

where F and G are any antiderivatives of f and $1/g$, respectively, and C is an arbitrary constant. For each value of C, (2.2) provides a connection between y and x, which defines a function $y = y(x)$ implicitly.

We have shown that every solution of (2.1) also satisfies (2.2). To confirm that these two equations are fully equivalent, we must also verify that, conversely, any function $y = y(x)$ satisfying (2.2) also satisfies (2.1). This is easily done by differentiating both sides of (2.2) with respect to x. The derivative of the right-hand side is $f(x)$; on the left-hand side, by the chain rule and bearing in mind that $G(y) = G(y(x))$, we have

C. Constanda, *Differential Equations*, Springer Undergraduate Texts in Mathematics and Technology, DOI 10.1007/978-3-319-50224-3_2

$$\frac{d}{dx}\,G(y(x)) = \frac{d}{dy}\,G(y)\frac{dy}{dx} = \frac{1}{g(y)}\,\frac{dy}{dx},$$

which, when equated to $f(x)$, yields equation (2.1).

In some cases, the solution $y = y(x)$ can be determined explicitly from (2.2).

2.1 Remark. The above handling suggests that dy/dx could be treated formally as a ratio, but this would not be technically correct. ■

2.2 Example. Bringing the DE

$$y' + 8xy = 0$$

to the form

$$\frac{dy}{dx} = -8xy,$$

we see that it has the equilibrium solution $y = 0$. Then for $y \neq 0$,

$$\int \frac{dy}{y} = \int -8x\,dx,$$

from which

$$\ln|y| = -4x^2 + C,$$

where C is the amalgamation of the arbitrary constants of integration from both sides. Exponentiating, we get

$$|y| = e^{-4x^2+C} = e^C\,e^{-4x^2},$$

so

$$y(x) = \pm e^C e^{-4x^2} = C_1 e^{-4x^2}.$$

Here, as expected, C_1 is an arbitrary nonzero constant (it replaces $\pm e^C \neq 0$), which generates all the nonzero solutions y. However, if we allow C_1 to take the value 0 as well, then the above formula also captures the equilibrium solution $y = 0$ and, thus, becomes the GS of the given equation.

Verification with Mathematica®. The input

```
y=C1*E^(-4*x^2);
D[y,x]+8*x*y
```

evaluates the difference between the left-hand and right-hand sides of our DE for the function y computed above. This procedure will be followed in all similar situations. As expected, the output here is 0, which confirms that this function is indeed the GS of the given equation.

The alternative coding

```
y[x_]=C1*E^(-4*x^2);
y'[x]+8*x*y[x]==0
```

gives the output `True`. Choosing one type of coding over the other is a matter of personal preference. Throughout the rest of the book we will use the former. ■

2.3 Example. In view of the properties of the exponential function, the DE in the IVP

$$y' + 4xe^{y-2x} = 0, \quad y(0) = 0$$

can be rewritten as

$$\frac{dy}{dx} = -4xe^{-2x}e^{y},$$

and we see that, since $e^y \neq 0$ for any real value of y, the equation has no equilibrium solutions. After separating the variables, we arrive at

$$\int e^{-y}\,dy = -\int 4xe^{-2x}\,dx,$$

from which, using integration by parts (see Sect. B.2 in Appendix B) on the right-hand side, we find that

$$-e^{-y} = 2xe^{-2x} - \int 2e^{-2x}\,dx = (2x+1)e^{-2x} + C, \quad C = \text{const}.$$

We now change the signs of both sides, take logarithms, and produce the GS

$$y(x) = -\ln[-(2x+1)e^{-2x} - C].$$

The constant C is more easily computed if we apply the IC not to this explicit expression of y but to the equality immediately above it. The value is $C = -2$, so the solution of the IVP is

$$y(x) = -\ln[2 - (2x+1)e^{-2x}].$$

Verification with Mathematica®. The input

```
y=-Log[2 - (2*x + 1)*E^(-2*x);
{D[y,x] + 4*x*E^(y - 2*x),y/. x→ 0}//Simplify
```

evaluates both the difference between the left-hand and right-hand sides (as in the preceding example) and the value of the computed function y at $x = 0$. Again, this type of verification will be performed for all IVPs and BVPs in the rest of the book with no further comment. Here, the output is, of course, $\{0, 0\}$. ∎

2.4 Example. Form (2.1) for the DE of the IVP

$$xy' = y + 2, \quad y(1) = -1$$

is

$$\frac{dy}{dx} = \frac{y+2}{x}.$$

Clearly, $y = -2$ is an equilibrium solution. For $y \neq -2$ and $x \neq 0$, we separate the variables and arrive at

$$\int \frac{dy}{y+2} = \int \frac{dx}{x};$$

hence,

$$\ln|y+2| = \ln|x| + C, \quad C = \text{const},$$

from which, by exponentiation,

$$|y+2| = e^{\ln|x|+C} = e^{C}e^{\ln|x|} = e^{C}|x|.$$

This means that

$$y + 2 = \pm e^{C}x = C_1x, \quad C_1 = \text{const} \neq 0,$$

so

$$y(x) = C_1x - 2.$$

To make this the GS, we need to allow C_1 to be zero as well, which includes the equilibrium solution $y = -2$ in the above equality. Applying the IC, we now find that $C_1 = 1$; therefore, the solution of the IVP is

$$y(x) = x - 2.$$

VERIFICATION WITH MATHEMATICA®. The input

```
y=x - 2;
{x*D[y,x] - y - 2,y/.x→ 1}//Simplify
```

generates the output $\{0, -1\}$. ■

2.5 Example. We easily see that the DE in the IVP

$$2(x+1)yy' - y^2 = 2, \quad y(5) = 2$$

has no equilibrium solutions; hence, for $x \neq -1$, we have

$$\int \frac{2y\,dy}{y^2+2} = \int \frac{dx}{x+1},$$

so

$$\ln(y^2+2) = \ln|x+1| + C, \quad C = \text{const},$$

which, after simple algebraic manipulation, leads to

$$y^2 = C_1(x+1) - 2, \quad C_1 = \text{const} \neq 0.$$

Applying the IC, we obtain $y^2 = x - 1$, or $y = \pm(x-1)^{1/2}$. However, the function with the '$-$' sign must be rejected because it does not satisfy the IC. In conclusion, the solution to our IVP is

$$y(x) = (x-1)^{1/2}.$$

If the IC were $y(5) = -2$, then the solution would be

$$y(x) = -(x-1)^{1/2}.$$

VERIFICATION WITH MATHEMATICA®. The input

```
y=(x - 1)^(1/2);
{2*(x + 1)*y*D[y,x] - y^2 - 2,y/.x→ 5}//Simplify
```

generates the output $\{0, 2\}$. ■

2.6 Example. Treating the DE in the IVP

$$(5y^4 + 3y^2 + e^y)y' = \cos x, \quad y(0) = 0$$

in the same way, we arrive at

$$\int (5y^4 + 3y^2 + e^y)\,dy = \int \cos x\,dx;$$

consequently,

$$y^5 + y^3 + e^y = \sin x + C, \quad C = \text{const}.$$

This equality describes the family of all the solution curves for the DE, representing its GS in implicit form. It cannot be solved explicitly for y.

The IC now yields $C = 1$, so the solution curve passing through the point $(0, 0)$ has equation

$$y^5 + y^3 + e^y = \sin x + 1.$$

Figure 2.1 shows the solution curves for $C = -2, -1, 0, 1, 2$. The heavier line (for $C = 1$) represents the solution of our IVP.

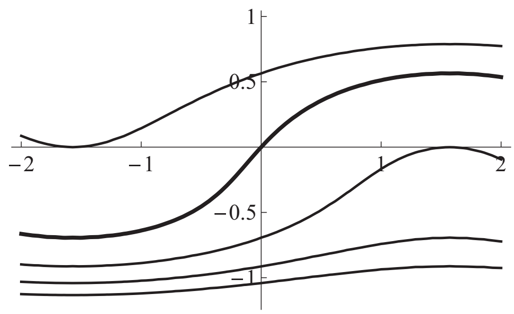

Fig. 2.1

VERIFICATION WITH MATHEMATICA®. The input

```
u=y[x]^5 + y[x]^3 + E^y[x] - Sin[x] - 1;
{(5*y[x]^4 + 3*y[x]^2 + E^y[x])*(Solve[D[u,x]==0, y'[x]])
        [[1,1,2]] - Cos[x], u/.{x→ 0, y→ 0}}//Simplify
```

generates the output $\{0, 0\}$, which shows that the function y defined implicitly above satisfies the DE and IC. ∎

Exercises

Solve the given IVP.

1 $y' = -4xy^2$, $y(0) = 1$. **2** $y' = 8x^3/y$, $y(0) = -1$.

3 $y' = -6ye^{3x}$, $y(0) = e^{-2}$. **4** $y' = y\sin(2x)$, $y(\pi/4) = 1$.

5 $y' = (3 - 2x)y$, $y(2) = e^6$. **6** $y' = y^2 e^{1-x}$, $y(1) = 1/3$.

7 $y' = (1 + e^{-x})/(2y + 2)$, $y(0) = -1$. **8** $y' = 2ye^{2x+1}$, $y(-1/2) = e^2$.

9 $y' = 2x\sec y$, $y(0) = \pi/6$. **10** $y' = 2x\sqrt{y}$, $y(1) = 0$.

11 $(1 + 2x)y' = 3 + y$, $y(0) = -2$. **12** $3(x^2 + 2)y^2 y' = 4x$, $y(1) = (\ln 9)^{1/3}$.

13 $y' = (6x^2 + 2x)/(2y + 4)$, $y(1) = \sqrt{6} - 2$. **14** $(4 - x^2)y' = 4y$, $y(0) = 1$.

15 $y' = (x - 3)(y^2 + 1)$, $y(0) = 1$. **16** $2y(x^2 + 2x + 6)^{1/2} y' = x + 1$, $y(1) = -2$.

17 $y' = 2\sin(2x)/(4y^3+3y^2)$, $y(\pi/4) = 1$. **18** $y' = (2x+1)/(2y+\sin y)$, $y(0) = 0$.

19 $y' = (x^2 + 1)/(e^{-2y} + 4y)$, $y(0) = 0$. **20** $y' = xe^{2x}/(y^4 + 2y)$, $y(0) = -1$.

2.2 Linear Equations

The standard form of this type of DE is

$$y' + p(t)y = q(t), \tag{2.3}$$

where p and q are prescribed functions. To solve the equation, we first multiply it by an unknown nonzero function $\mu(t)$, called an *integrating factor.* Omitting, for simplicity, the explicit mention of the variable t, we have

$$\mu y' + \mu p y = \mu q. \tag{2.4}$$

We now choose μ so that the left-hand side in (2.4) is the derivative of the product μy; that is,

$$\mu y' + \mu p y = (\mu y)' = \mu y' + \mu' y.$$

Clearly, this occurs if

$$\mu' = \mu p.$$

The above separable equation yields, in the usual way,

$$\int \frac{d\mu}{\mu} = \int p\, dt.$$

Integrating, we arrive at

$$\ln|\mu| = \int p\, dt,$$

so, as in Example 2.4,

$$\mu = C \exp\left\{ \int p\, dt \right\}, \quad C = \text{const} \neq 0.$$

Since we need just one such function, we may take $C = 1$ and thus consider the integrating factor

$$\mu(t) = \exp\left\{ \int p(t)\, dt \right\}. \tag{2.5}$$

With this choice of μ, equation (2.4) becomes

$$(\mu y)' = \mu q; \tag{2.6}$$

hence,

$$\mu y = \int \mu q\, dt + C,$$

or

$$y(t) = \frac{1}{\mu(t)} \left\{ \int \mu(t) q(t)\, dt + C \right\}, \quad C = \text{const}. \tag{2.7}$$

2.7 Remarks. (i) Technically speaking, C does not need to be inserted explicitly in (2.7) since the indefinite integral on the right-hand side produces an arbitrary constant, but it is good practice to have it in the formula for emphasis, and to prevent its accidental omission when the integration is performed.

(ii) It should be obvious that the factor $1/\mu$ cannot be moved inside the integral to be canceled with the factor μ already there.

(iii) Points (i) and (ii) become moot if the equality $(\mu y)' = \mu q$ (see (2.6)) is integrated from some admissible value t_0 to a generic value t. Then

$$\mu(t)y(t) - \mu(t_0)y(t_0) = \int_{t_0}^{t} \mu(\tau)q(\tau)\,d\tau,$$

from which we easily deduce that

$$y(t) = \frac{1}{\mu(t)} \left\{ \int_{t_0}^{t} \mu(\tau)q(\tau)\,d\tau + \mu(t_0)y(t_0) \right\}. \tag{2.8}$$

In the case of an IVP, it is convenient to choose t_0 as the point where the IC is prescribed.

(iv) In (2.8) we used the 'dummy' variable τ in the integrand to avoid a clash with the upper limit t of the definite integral. ■

2.8 Example. Consider the IVP

$$y' - 3y = 6, \quad y(0) = -1,$$

where, by comparison to (2.3), we have $p(t) = -3$ and $q(t) = 6$. The GS of the DE is computed from (2.5) and (2.7). Thus,

$$\mu(t) = \exp\left\{ \int -3\,dt \right\} = e^{-3t},$$

so

$$y(t) = e^{3t}\left\{ \int 6e^{-3t}\,dt + C \right\} = e^{3t}(-2e^{-3t} + C) = Ce^{3t} - 2,$$

where C is an arbitrary constant. Applying the IC, we find that $C = 1$, which yields the IVP solution

$$y(t) = e^{3t} - 2.$$

Alternatively, we could use formula (2.8) with μ as determined above and $t_0 = 0$, to obtain directly

$$\begin{aligned} y(t) &= e^{3t}\left\{ \int_0^t 6e^{-3\tau}\,d\tau + \mu(0)y(0) \right\} \\ &= e^{3t}\big[-2e^{-3\tau}\big|_0^t - 1 \big] = e^{3t} - 2. \end{aligned}$$

Verification with Mathematica®. The input

```
y=E^(3*t) - 2;
{D[y,t] - 3*y - 6, y/.t→0}//Simplify
```

generates the output $\{0, -1\}$. ■

2.9 Example. The DE in the IVP

$$ty' + 4y = 6t^2, \quad y(1) = 4$$

is not in the standard form (2.3). Assuming that $t \neq 0$, we divide the equation by t and rewrite it as

$$y' + \frac{4}{t}\,y = 6t.$$

This shows that $p(t) = 4/t$ and $q(t) = 6t$, so, by (2.5),

$$\mu(t) = \exp\left\{\int \frac{4}{t}\,dt\right\} = e^{4\ln|t|} = e^{\ln(t^4)} = t^4.$$

Using (2.8) with $t_0 = 1$, we now find the solution of the IVP to be

$$y(t) = t^{-4}\left\{\int_1^t 6\tau^5\,d\tau + \mu(1)y(1)\right\} = t^{-4}\big(\tau^6\big|_1^t + 4\big) = t^{-4}(t^6 + 3) = t^2 + 3t^{-4}.$$

VERIFICATION WITH MATHEMATICA®. The input

```
y=t^2 + 3*t^(-4);
{t*D[y,t] + 4*y - 6*t^2, y/.t→1}//Simplify
```

generates the output $\{0,4\}$. ■

2.10 Example. To bring the DE in the IVP

$$y' = (2+y)\sin t, \quad y(\pi/2) = -3$$

to the standard form, we move the y-term to the left-hand side and write

$$y' - y\sin t = 2\sin t.$$

This shows that $p(t) = -\sin t$ and $q(t) = 2\sin t$; hence, by (2.5),

$$\mu(t) = \exp\left\{-\int \sin t\,dt\right\} = e^{\cos t},$$

and, by (2.8) with $t_0 = \pi/2$,

$$\begin{aligned} y(t) &= e^{-\cos t}\left\{2\int_{\pi/2}^t e^{\cos\tau}\sin\tau\,d\tau + \mu(\pi/2)y(\pi/2)\right\} \\ &= e^{-\cos t}\left\{-2\int_{\pi/2}^t e^{\cos\tau}\,d(\cos\tau) - 3\right\} = e^{-\cos t}\big(-2e^{\cos\tau}\big|_{\pi/2}^t - 3\big) \\ &= e^{-\cos t}(-2e^{\cos t} - 1) = -e^{-\cos t} - 2. \end{aligned}$$

VERIFICATION WITH MATHEMATICA®. The input

```
y=-E^(-Cos[t]) - 2;
{D[y,t] - (2 + y)*Sin[t], y/.t→ Pi/2}//Simplify
```

generates the output $\{0, -3\}$. ■

2.11 Example. Consider the IVP

$$(t^2+1)y' - ty = 2t(t^2+1)^2, \quad y(0) = \tfrac{2}{3}.$$

Proceeding as in Example 2.9, we start by rewriting the DE in the standard form

$$y' - \frac{t}{t^2+1}\,y = 2t(t^2+1).$$

Then, with $p(t) = -t/(t^2+1)$ and $q(t) = 2t(t^2+1)$, we have, first,

$$\begin{aligned}\mu(t) &= \exp\left\{-\int \frac{t}{t^2+1}\,dt\right\} = \exp\left\{-\frac{1}{2}\int \frac{d(t^2+1)}{t^2+1}\right\}\\ &= e^{-(1/2)\ln(t^2+1)} = e^{\ln[(t^2+1)^{-1/2}]} = (t^2+1)^{-1/2},\end{aligned}$$

followed by

$$\begin{aligned}y(t) &= (t^2+1)^{1/2}\left\{\int_0^t (\tau^2+1)^{-1/2}2\tau(\tau^2+1)\,d\tau + \mu(0)y(0)\right\}\\ &= (t^2+1)^{1/2}\left\{\int_0^t (\tau^2+1)^{1/2}\,d(\tau^2+1) + \tfrac{2}{3}\right\}\\ &= (t^2+1)^{1/2}\left\{\tfrac{2}{3}\,(\tau^2+1)^{3/2}\Big|_0^t + \tfrac{2}{3}\right\} = \tfrac{2}{3}\,(t^2+1)^2.\end{aligned}$$

VERIFICATION WITH MATHEMATICA®. The input

```
y = (2/3) * (t^2 + 1)^2;
{(t^2 + 1) *D[y,t] - t*y - 2*t*(t^2 + 1)^2, y/.t→0} // Simplify
```

generates the output $\{0,\ 2/3\}$. ■

2.12 Example. For $t \neq -1$, the standard form of the DE in the IVP

$$(t-1)y' + y = (t-1)e^t, \quad y(2) = 3$$

is

$$y' + \frac{1}{t-1}\,y = e^t,$$

so $p(t) = 1/(t-1)$ and $q(t) = e^t$; consequently,

$$\mu(t) = \exp\left\{\int \frac{1}{t-1}\,dt\right\} = e^{\ln|t-1|} = |t-1| = \begin{cases} t-1, & t>1,\\ -(t-1), & t<1.\end{cases}$$

Since formula (2.8) uses the value of μ at $t_0 = 2 > 1$, we take $\mu(t) = t-1$ and, after integration by parts and simplification, obtain the solution

$$y(t) = \frac{1}{t-1}\left\{\int_2^t (\tau-1)e^\tau\,d\tau + \mu(2)y(2)\right\} = \frac{(t-2)e^t+3}{t-1}.$$

VERIFICATION WITH MATHEMATICA®. The input

```
y = ((t - 2)*E^t + 3)/(t - 1);
{(t-1)*D[y,t] + y - (t - 1)*E^t, y/.t→2} // Simplify
```

generates the output $\{0,\ 3\}$. ■

2.13 Remarks. (i) If we do not have an IC and want to find only the GS of the equation in Example 2.12, then it does not matter whether we take μ to be $t-1$ or $-(t-1)$ since μ has to be replaced in (2.6) and, in the latter case, the "$-$" sign would cancel out on both sides.

(ii) It is easily seen that the DEs in Examples 2.8 and 2.10 have an equilibrium solution (in both cases, it is $y=-2$), but its presence does not interfere with the solution process for those IVPs. ■

Exercises

Solve the given IVP, or find the GS of the DE if no IC is given.

1 $y'+4y+16=0,\quad y(0)=-2.$ **2** $y'+3y=9,\quad y(0)=1.$

3 $3y'+y=t^2/6,\quad y(0)=5.$ **4** $y'-2y=4t-e^t,\quad y(0)=2.$

5 $y'+y=4te^{-3t},\quad y(0)=3.$ **6** $2y'+y=8te^{-t/2}+6,\quad y(0)=-7.$

7 $y'-ty=(4t-3)e^{t^2/2},\quad y(1)=-\sqrt{e}.$

8 $y'-y\sin t=4t\sin(2t)e^{-\cos t},\quad y(0)=1/e.$

9 $ty'-4y=6t^7-2t^6,\quad y(-1)=-2.$ **10** $2ty'-y=2/\sqrt{t},\quad y(1)=1.$

11 $2ty'+y+12t\sqrt{t}=0,\quad y(1)=-1.$ **12** $y'+y\cot t=2\cos t,\quad y(\pi/2)=1/2.$

13 $(t-2)y'+y=8(t-2)\cos(2t),\quad y(\pi)=2/(\pi-2).$

14 $(3t+1)y'+y=(3t+1)^{2/3}\cos t,\quad y(0)=2.$

15 $t^2y'+3ty=4e^{2t},\quad y(1)=e^2.$ **16** $t^2y'+ty=\pi t\sin(\pi t)-1.$

17 $(t^2+2)y'+2ty=3t^2-4t,\quad y(0)=3/2.$

18 $ty'+(2t-1)y=9t^3e^t,\quad y(1)=2e^{-2}+2e.$

19 $(t^2-1)y'+4y=3(t+1)^2(t^2-1),\quad y(0)=0.$ **20** $(t^2+2t)y'+y=\sqrt{t},\quad y(2)=0.$

2.3 Homogeneous Polar Equations

These are DEs of the form

$$y'(x)=f\left(\frac{y}{x}\right),\quad x\neq 0, \tag{2.9}$$

where f is a given one-variable function. Making the substitution

$$y(x)=xv(x) \tag{2.10}$$

and using the fact that, by the product rule, $y'=v+xv'$, from (2.9) we see that the new unknown function v satisfies the DE

$$xv'+v=f(v), \tag{2.11}$$

or

$$\frac{dv}{dx}=\frac{f(v)-v}{x}.$$

This is a separable equation, so, for $f(v) - v \neq 0$,

$$\int \frac{dv}{f(v) - v} = \int \frac{dx}{x}, \tag{2.12}$$

which, with v replaced by y/x in the result, produces y as an explicit or implicit function of x.

2.14 Remark. If $f(v) - v = 0$, then (2.11) implies that $v' = 0$, or $v = c = \text{const}$; therefore, the equation has *singular solutions* of the form $y = cx$. The GS of the DE consists of the aggregate of the solutions obtained by the procedure described above and any singular solutions of this type that may exist. ■

2.15 Example. The DE in the IVP

$$xy' = x + 2y, \quad y(1) = 3$$

can be written as

$$y' = 1 + 2\,\frac{y}{x},$$

so $f(v) = 1 + 2v$. Then $f(v) - v = v + 1$ and, for $v \neq -1$, (2.12) becomes

$$\int \frac{dv}{v + 1} = \int \frac{dx}{x},$$

which yields

$$\ln|v + 1| = \ln|x| + C, \quad C = \text{const}.$$

Exponentiating and simplifying, we find that

$$v + 1 = C_1 x,$$

where, as in other similar situations (see Example 2.2), C_1 is an arbitrary nonzero constant; hence, by (2.10),

$$y(x) = C_1 x^2 - x. \tag{2.13}$$

The case $v = -1$ set aside earlier is equivalent to $y = -x$ (see Remark 2.14), and is also covered by (2.13) with $C_1 = 0$. Consequently, (2.13) is the GS of the DE if C_1 is any real number. The value of C_1 is found from the IC; specifically, $C_1 = 4$, so the solution of the IVP is

$$y(x) = 4x^2 - x.$$

Verification with Mathematica®. The input

```
y = 4 * x^2 - x;
{x * D[y,x] - x - 2 * y, y /. x → 1} // Simplify
```

generates the output $\{0,\, 3\}$. ■

2.16 Example. Consider the IVP

$$(x^2 + 2xy)y' = 2(xy + y^2), \quad y(1) = 2.$$

We notice that the DE has the equilibrium solution $y = 0$, but this is of no interest to us since it does not satisfy the IC, so we may assume that $y \neq 0$. Solving for y' and then dividing both numerator and denominator by x^2 brings the DE to the form

$$y' = \frac{2(xy + y^2)}{x^2 + 2xy} = \frac{2\,\dfrac{y}{x} + 2\left(\dfrac{y}{x}\right)^2}{1 + 2\,\dfrac{y}{x}} = \frac{2(v + v^2)}{1 + 2v} = f(v).$$

Clearly, f is well defined since $1 + 2v = 0$ means that $y = -x/2$, which does not satisfy the DE.

Next,

$$f(v) - v = \frac{2(v + v^2)}{1 + 2v} - v = \frac{v}{1 + 2v}.$$

By (2.12) and the fact that $y \neq 0$ implies that $v \neq 0$, we have

$$\int \frac{1 + 2v}{v}\,dv = \int \left(2 + \frac{1}{v}\right) dv = \int \frac{dx}{x};$$

therefore,

$$2v + \ln|v| = \ln|x| + C,$$

or, according to (2.10),

$$2\,\frac{y}{x} + \ln\left|\frac{y}{x^2}\right| = C.$$

Applying the IC, we find that $C = 4 + \ln 2$; hence, the solution of the IVP is defined implicitly by the equality

$$2\,\frac{y}{x} + \ln\left|\frac{y}{2x^2}\right| = 4.$$

Verification with Mathematica®. The input

```
u = 2 * y[x]/x + Log[y[x]/(2 * x^2)] - 4;
{(x^2 + 2 * x * y[x]) * (Solve[D[u,x]==0, y'[x]])[[1,1,2]]
        - 2 * (x * y[x] + y[x]^2), u/. {x → 1, y[x] → 2}} // Simplify
```

generates the output $\{0, 0\}$. ■

2.17 Remark. The DE (2.9) is sometimes written in the form

$$f_1(x, y)y' = f_2(x, y). \tag{2.14}$$

If there is a function $\tilde{y}(x)$ with the property that

$$f_1(x, \tilde{y}(x)) = f_2(x, \tilde{y}(x)) = 0,$$

then, technically speaking, $\tilde{y}$ would qualify as a solution of (2.14). But this type of 'solution' would be caused by the algebraic structure of the equation and not by its differential nature. In general, correctly formulated mathematical models are not expected to exhibit such anomalies. ■

2.18 Example. The equation in the IVP

$$(xy - 3x^2)y' = 2y^2 - 5xy - 3x^2, \quad y(1) = 3 \tag{2.15}$$

is of the form (2.14), with

$$\begin{aligned} f_1(x, y) &= xy - 3x^2 = x(y - 3x), \\ f_2(x, y) &= 2y^2 - 5xy - 3x^2 = (x + 2y)(y - 3x), \end{aligned}$$

so it is obvious that the function $y(x) = 3x$ satisfies it. 'Cleaning' the DE algebraically—in other words, assuming that $y \neq 3x$ and dividing by $y - 3x$ on both sides—we arrive at the IVP

$$xy' = x + 2y, \quad y(1) = 3$$

solved in Example 2.15, with solution $y = 4x^2 - x$. But the discarded function $y(x) = 3x$ also satisfies the prescribed IC, so, at first glance, it would appear that the IVP (2.15) does not have a unique solution. While raising no theoretical concerns, this situation, as mentioned in Remark 2.17, is unacceptable in mathematical modeling, where $y(x) = 3x$ is normally considered a spurious 'solution' and ignored. ■

Exercises

Solve the given IVP, or find the GS of the DE if no IC is given.

1 $xy' = 3y - x, \quad y(1) = 1.$ **2** $2xy' = x + y, \quad y(1) = 3.$

3 $3xy' = x + 3y.$ **4** $(x + y)y' = 2x - y, \quad y(1) = -1 - \sqrt{2}.$

5 $(2x + y)y' = 3x - 2y, \quad y(1) = -4.$ **6** $x^2y' = xy + y^2.$

7 $x^2y' - 2xy - y^2 = 0, \quad y(1) = 1.$ **8** $2x^2y' = 3x^2 - 2xy + y^2, \quad y(1) = -1.$

9 $x^2y' = 2x^2 - 2xy + y^2, \quad y(1) = 3/2.$

10 $(2x^2 - 2xy)y' = 4xy - 3y^2, \quad y(3) = -3.$

11 $x^2y' = 4x^2 + xy + y^2, \quad y(1) = 0.$ **12** $x^2y' = x^2 + xy + y^2, \quad y(1) = 0.$

13 $x^2y' = x^2 - xy + y^2.$ **14** $(2xy - x^2)y' = 3y^2 - 2xy - 2x^2, \quad y(1) = 3/2.$

15 $(x^2 + 2xy)y' = 2xy - x^2 + 3y^2, \quad y(2) = -4.$

16 $(2xy - 3x^2)y' = 2x^2 - 6xy + 3y^2, \quad y(3) = 3/2.$

17 $(2x^2 - xy)y' = xy - y^2.$ **18** $(2x^2 - 3xy)y' = x^2 + 2xy - 3y^2, \quad y(1) = 0.$

19 $3xy^2y' = x^3 + 3y^3.$ **20** $2xy^2y' = x^3 + y^3, \quad y(2) = 2^{4/3}.$

2.4 Bernoulli Equations

The general form of a Bernoulli equation is

$$y' + p(t)y = q(t)y^n, \quad n \neq 1. \tag{2.16}$$

Making the substitution

$$y(t) = (w(t))^{1/(1-n)} \tag{2.17}$$

and using the chain rule, we have

$$y' = \frac{1}{1-n}\, w^{1/(1-n)-1}w' = \frac{1}{1-n}\, w^{n/(1-n)}w',$$

so (2.16) becomes

$$\frac{1}{1-n}\, w^{n/(1-n)}w' + pw^{1/(1-n)} = qw^{n/(1-n)}.$$

Since $1/(1-n) - n/(1-n) = 1$, after division by $w^{n/(1-n)}$ and multiplication by $1-n$ this simplifies further to

$$w' + (1-n)pw = (1-n)q. \tag{2.18}$$

Equation (2.18) is linear and can be solved by the method described in Sect. 2.2. Once its solution w has been found, the GS y of (2.16) is given by (2.17).

2.19 Example. Comparing the DE in the IVP

$$y' + 3y + 6y^2 = 0, \quad y(0) = -1$$

to (2.16), we see that this is a Bernoulli equation with $p(t) = 3$, $q(t) = -6$, and $n = 2$. Substitution (2.17) in this case is $y = w^{-1}$; hence,

$$y' = -w^{-2}w', \quad w(0) = (y(0))^{-1} = -1,$$

so the IVP becomes

$$w' - 3w = 6, \quad w(0) = -1.$$

This problem was solved in Example 2.8, and its solution is

$$w(t) = e^{3t} - 2;$$

hence, the solution of the IVP for y is

$$y(t) = (w(t))^{-1} = \frac{1}{e^{3t} - 2}.$$

Verification with Mathematica®. The input

```
y = (E^(3*t) - 2)^(-1);
{D[y,t] + 3*y + 6*y^2, y /. t → 0} // Simplify
```

generates the output $\{0, -1\}$. ■

2.20 Example. For the DE in the IVP

$$ty' + 8y = 12t^2\sqrt{y}, \quad y(1) = 16$$

we have

$$p(t) = 8/t, \quad q(t) = 12t, \quad n = 1/2;$$

therefore, by (2.17), we substitute $y = w^2$ and, since $y' = 2ww'$, arrive at the new IVP

$$tw' + 4w = 6t^2, \quad w(1) = 4.$$

From Example 2.9 we see that

$$w(t) = t^2 + 3t^{-4},$$

so

$$y(t) = (w(t))^2 = (t^2 + 3t^{-4})^2.$$

Verification with Mathematica®. The input

```
y = (t^2 + 3*t^(-4))^2;
Simplify[{t*D[y,t] + 8*y - 12*t^2*Sqrt[y], y /. t → 1}, t > 0]
```

generates the output $\{0, 16\}$. ■

Exercises

Solve the given IVP, or find the GS of the DE if no IC is given.

1 $y' + y = -y^3$, $y(0) = 1$. **2** $y' - 3y = -e^{-4t}y^2$, $y(0) = -1/2$.

3 $y' - y + 6e^t y^{3/2} = 0$, $y(0) = 1/4$.

4 $9y' + 2y = 30e^{-2t}y^{-1/2}$, $y(0) = 4^{1/3}$.

5 $8y' - y = 4te^{t/2}y^{-3}$, $y(0) = 2^{1/4}$. **6** $3ty' - y = 4t^2y^{-2}$, $y(1) = 6^{1/3}$.

7 $2ty' + 3y = 9y^{-1/3}$, $y(1) = 1$. **8** $t^2y' + 4ty = 2e^t\sqrt{y}$, $y(1) = (1+e)^2$.

9 $5t^2y' + 2ty = 2y^{-3/2}$. **10** $(t+1)y' + 3y = 9y^{2/3}$, $y(0) = -8$.

2.5 Riccati Equations

The general form of these DEs is

$$y' = q_0(t) + q_1(t)y + q_2(t)y^2, \tag{2.19}$$

where q_0, q_1, and q_2 are given functions, with $q_0, q_2 \neq 0$. After some analytic manipulation, we can rewrite (2.19) as

$$v'' + p_1(t)v' + p_2(t)v = 0. \tag{2.20}$$

This is a second-order DE whose coefficients p_1 and p_2 are combinations of q_0, q_1, q_2, and their derivatives. In general, the solution of (2.20) cannot be obtained be means of integrals. However, when we know a PS y_1 of (2.19), we are able to compute the GS of that equation by reducing it to a linear first-order DE by means of the substitution

$$y = y_1 + \frac{1}{w}. \tag{2.21}$$

In view of (2.19) and (2.21), we then have

$$y' = y_1' - \frac{w'}{w^2} = q_0 + q_1\left(y_1 + \frac{1}{w}\right) + q_2\left(y_1 + \frac{1}{w}\right)^2.$$

Since y_1 is a solution of (2.19), it follows that

$$q_0 + q_1y_1 + q_2y_1^2 - \frac{w'}{w^2} = q_0 + q_1y_1 + \frac{q_1}{w} + q_2y_1^2 + 2\frac{q_2y_1}{w} + \frac{q_2}{w^2},$$

which, after a rearrangement of the terms, becomes

$$w' + (q_1 + 2q_2y_1)w = -q_2. \tag{2.22}$$

Equation (2.22) is now solved by the method described in Sect. 2.2.

The matrix version of the Riccati equation occurs in optimal control. Its practical importance and the fact that it cannot be solved by means of integrals have led to the development of the so-called *qualitative theory* of differential equations.

2.21 Example. The DE in the IVP

$$y' = -1 - t^2 + 2(t^{-1} + t)y - y^2, \quad y(1) = \tfrac{10}{7}$$

is of the form (2.19) with $q_0(t) = -1 - t^2$, $q_1(t) = 2(t^{-1} + t)$, and $q_2(t) = -1$, and it is easy to check that $y_1(t) = t$ satisfies it; hence, according to (2.22),

$$w' + 2t^{-1}w = 1,$$

whose solution, constructed by means of (2.5) and (2.7), is

$$w(t) = \tfrac{1}{3}\,t + Ct^{-2}, \quad C = \text{const}.$$

Next, by (2.21),

$$y(t) = t + \frac{3t^2}{t^3 + 3C}.$$

The constant C is determined from the IC as $C = 2$, so the solution of the IVP is

$$y(t) = \frac{t^4 + 3t^2 + 6t}{t^3 + 6}.$$

VERIFICATION WITH MATHEMATICA®. The input

```
y = (t^4 + 3*t^2 + 6*t)/(t^3 + 6);
Simplify[{D[y,t] + 1 + t^2 - 2*(t^(-1) + t)*y + y^2, y/.t→ 1}]
```

generates the output $\{0,\ 10/7\}$. ■

2.22 Example. The IVP

$$y' = -\cos t + (2 - \tan t)y - (\sec t)y^2, \quad y(0) = 0$$

admits the PS $y_1(t) = \cos t$. Then substitution (2.21) is $y = \cos t + 1/w$, and the linear first-order equation (2.22) takes the form

$$w' - (\tan t)w = \sec t,$$

with GS

$$w(t) = \frac{t + C}{\cos t},$$

from which

$$y(t) = \left(1 + \frac{1}{t + C}\right)\cos t.$$

The IC now yields $C = -1$, so the solution of the IVP is

$$y(t) = \frac{t\cos t}{t - 1}.$$

VERIFICATION WITH MATHEMATICA®. The input

```
y = (t*Cos[t])/(t - 1);
Simplify[{D[y,t] + Cos[t] - (2 - Tan[t])*y + Sec[t]*y^2,
          y/.t→ 0}]
```

generates the output $\{0,\ 0\}$. ■

Exercises

In the given IVP, verify that y_1 is a solution of the DE, use the substitution $y = y_1 + 1/w$ to reduce the DE to a linear first-order equation, then find the solution of the IVP.

1 $y' = t^{-2} + 3t^{-1} - (4t^{-1} + 3)y + 2y^2,\quad y(1) = 5/2;\quad y_1(t) = 1/t.$

2 $y' = 2 - 4t - 4t^2e^{-t} + (2 + 4te^{-t}y - e^{-t}y^2,\quad y(1) = 2 + e;\quad y_1(t) = 2t.$

3 $y' = 3t^{-2} - t^{-4} + 2(t^{-1} - t^{-3})y - t^{-2}y^2,\quad y(1) = -1/2;\quad y_1(t) = -1/t.$

4 $y' = 4t - 4e^{-t^2} + (2t - 4e^{-t^2})y - e^{-t^2}y^2,\quad y(0) = -1;\quad y_1(t) = -2.$

5 $y' = 1 - 2t + (4t - 1)y - 2ty^2,\quad y(0) = 2;\quad y_1(t) = 1.$

6 $y' = -(t^2 + 6t + 4) + 2(t + 3)y - y^2,\quad y(0) = 9/5;\quad y_1(t) = t + 1.$

7 $y' = 2t^2 - 1 - t\tan t + (4t - \tan t)y + 2y^2,\quad y(\pi/4) = -1/2 - \pi/4;\quad y_1(t) = -t.$

8 $y' = 2-4\cos t+4\sin t+(4\cos t-4\sin t-1)y+(\sin t-\cos t)y^2,\quad y(0) = 3;\quad y_1(t) = 2.$

9 $y' = 3t^2 - 2t^3 - t - 1 + (3t - 4t^2)y - 2ty^2,\quad y(0) = 0;\quad y_1(t) = 1 - t.$

10 $y' = -3e^t - e^{2t} + (2e^t + 4)y - y^2,\quad y(0) = -1/3;\quad y_1(t) = e^t.$

2.6 Exact Equations

Consider an equation of the form

$$P(x, y) + Q(x, y)y' = 0, \tag{2.23}$$

where P and Q are given two-variable functions. Recalling that the differential of a function $y = y(x)$ is $dy = y'(x)\,dx$, we multiply (2.23) by dx and rewrite it as

$$P\,dx + Q\,dy = 0. \tag{2.24}$$

The DE (2.23) is called an *exact equation* when the left-hand side above is the differential of a function $f(x, y)$. If f is found, then (2.24) becomes

$$df(x, y) = 0,$$

with GS

$$f(x, y) = C, \quad C = \text{const}. \tag{2.25}$$

2.23 Remark. Suppose that such a function f exists; then (see item (iv) in Sect. 1.1)

$$df = f_x\,dx + f_y\,dy,$$

so, by comparison to (2.24), this happens if

$$f_x = P, \quad f_y = Q. \tag{2.26}$$

In view of the comment made in item (iii) in Sect. 1.1, we have $f_{xy} = f_{yx}$, which, by (2.26), translates as

$$P_y = Q_x. \tag{2.27}$$

Therefore, if a function of the desired type exists, then equality (2.27) must hold.

The other way around, it turns out that for coefficients P and Q continuously differentiable in an open disc in the (x, y)-plane, condition (2.27), if satisfied, guarantees the existence of a function f with the required property. Since in all our examples P and Q meet this degree of smoothness, we simply confine ourselves to checking that (2.27) holds and, when it does, determine f from (2.26). ■

2.24 Example. For the DE in the IVP

$$y^2 - 4xy^3 + 2 + (2xy - 6x^2y^2)y' = 0, \quad y(1) = 1$$

we have

$$P(x, y) = y^2 - 4xy^3 + 2, \quad Q(x, y) = 2xy - 6x^2y^2,$$

so

$$P_y = 2y - 12xy^2 = Q_x,$$

which means that the equation is exact. Then, according to Remark 2.23, there is a function $f = f(x, y)$ such that

$$\begin{aligned} f_x(x, y) &= P(x, y) = y^2 - 4xy^3 + 2, \\ f_y(x, y) &= Q(x, y) = 2xy - 6x^2y^2. \end{aligned} \tag{2.28}$$

Integrating, say, the first equation (2.28) with respect to x, we find that

$$\begin{aligned} f(x, y) &= \int f_x(x, y)\, dx = \int P(x, y)\, dx \\ &= \int (y^2 - 4xy^3 + 2)\, dx = xy^2 - 2x^2y^3 + 2x + g(y), \end{aligned}$$

where, as mentioned in item (vi) in Sect. 1.1, g is an arbitrary function of y. To find g, we use this expression of f in the second equation (2.28):

$$f_y(x, y) = 2xy - 6x^2y^2 + g'(y) = 2xy - 6x^2y^2;$$

consequently, $g'(y) = 0$, from which

$$g(y) = c = \text{const}.$$

Since, by (2.25), we equate f to an arbitrary constant, it follows that, without loss of generality, we may take $c = 0$. Therefore, the GS of the DE is defined implicitly by the equality

$$xy^2 - 2x^2y^3 + 2x = C.$$

Using the IC, we immediately see that $C = 1$; hence,

$$xy^2 - 2x^2y^3 + 2x = 1$$

is the equation of the solution curve for the given IVP.

Instead of integrating f_x, we could equally start by integrating f_y from the second equation (2.28); that is,

$$\begin{aligned} f(x, y) &= \int f_y(x, y)\, dy = \int Q(x, y)\, dy \\ &= \int (2xy - 6x^2y^2)\, dy = xy^2 - 2x^2y^3 + h(x), \end{aligned}$$

where h is a function of x to be found by means of the first equation (2.28). Using this expression of f in that equation, we have

$$f_x(x,y) = y^2 - 4xy^3 + h'(x) = y^2 - 4xy^3 + 2,$$

so $h'(x) = 2$, giving $h(x) = 2x$. (Just as before, and for the same reason, we suppress the integration constant.) This expression of h gives rise to the same function f as above.

VERIFICATION WITH MATHEMATICA®. The input

```
u=x*y[x]^2 - 2*x^2*y[x]^3 + 2*x - 1;
{y[x]^2 - 4*x*y[x]^3 + 2 + (2*x*y[x] - 6*x^2*y[x]^2)
        *(Solve[D[u,x]==0, y'[x]])[[1,1,2]],
        u/.{x→ 1, y[x] → 1}}//Simplify
```

generates the output $\{0, 0\}$, which confirms that the function y defined implicitly by the equation of the solution curve satisfies both the DE and the IC. ■

2.25 Example. The DE in the IVP

$$6xy^{-1} + 8x^{-3}y^3 + (4y - 3x^2y^{-2} - 12x^{-2}y^2)y' = 0, \quad y(1) = \tfrac{1}{2}$$

has

$$P(x,y) = 6xy^{-1} + 8x^{-3}y^3, \quad Q(x,y) = 4y - 3x^2y^{-2} - 12x^{-2}y^2.$$

Obviously, here we must have $x,\ y \neq 0$.

Since

$$P_y(x,y) = -6xy^{-2} + 24x^{-3}y^2 = Q_x(x,y),$$

it follows that this is an exact equation. The function f we are seeking, obtained as in Example 2.24, is

$$\begin{aligned} f(x,y) &= \int f_x(x,y)\,dx = \int P(x,y)\,dx \\ &= \int (6xy^{-1} + 8x^{-3}y^3)\,dx = 3x^2y^{-1} - 4x^{-2}y^3 + g(y), \end{aligned}$$

with g determined from

$$f_y(x,y) = -3x^2y^{-2} - 12x^{-2}y^2 + g'(y) = 4y - 3x^2y^{-2} - 12x^{-2}y^2;$$

hence, $g'(y) = 4y$, so $g(y) = 2y^2$, which produces the GS of the DE in the implicit form

$$3x^2y^{-1} - 4x^{-2}y^3 + 2y^2 = C.$$

The IC now yields $C = 6$.

VERIFICATION WITH MATHEMATICA®. The input

```
u=3*x^2*y[x]^(-1) - 4*x^(-2)*y[x]^3 + 2*y[x]^2-6;
{6*x*y[x]^(-1) + 8*x^(-3)*y[x]^3 + (4*y[x]
        - 3*x^2*y[x]^(-2) - 12*x^(-2)*y[x]^2)
        *(Solve[D[u,x]==0, y'[x]])[[1,1,2]],
        u/.{x→ 1, y[x] → 1/2}}//Simplify
```

generates the output $\{0, 0\}$. ■

2.26 Example. Consider the IVP

$$x\sin(2y) - 3x^2 + \big(y + x^2\cos(2y)\big)y' = 0, \quad y(1) = \pi.$$

Since, as seen from the left-hand side of the DE, we have $P(x,y) = x\sin(2y) - 3x^2$ and $Q(x,y) = y + x^2\cos(2y)$, we readily verify that

$$P_y(x,y) = 2x\cos(2y) = Q_x(x,y),$$

so this is an exact equation. Then, integrating, say, the y-derivative of the desired function f, we find that

$$\begin{aligned} f(x,y) &= \int f_y(x,y)\,dy = \int Q(x,y)\,dy \\ &= \int [y + x^2\cos(2y)]\,dy = \tfrac{1}{2}\,y^2 + \tfrac{1}{2}\,x^2\sin(2y) + g(x). \end{aligned}$$

The function g is determined by substituting this expression in the x-derivative of f; that is,

$$f_x(x,y) = x\sin(2y) + g'(x) = x\sin(2y) - 3x^2,$$

which yields $g'(x) = -3x^2$; therefore, $g(x) = -x^3$, and we obtain the function

$$f(x,y) = \tfrac{1}{2}\,y^2 + \tfrac{1}{2}\,x^2\sin(2y) - x^3.$$

Writing the GS of the DE as $f(x,y) = C$ and using the IC, we find that $C = \pi^2/2 - 1$, so the solution of the IVP is given in implicit form by

$$y^2 + x^2\sin(2y) - 2x^3 = \pi^2 - 2.$$

VERIFICATION WITH MATHEMATICA®. The input

```
u=y[x]^2 + x^2*Sin[2*y[x]] - 2*x^3 - Pi^2 + 2;
{x*Sin[2*y[x]] - 3*x^2 + (y[x] + x^2*Cos[2*y[x]])
        *(Solve[D[u,x]==0,y'[x]])[[1,1,2]],
        u/.{x→ 1,y[x] →Pi}}//Simplify
```

generates the output $\{0,\,0\}$. ■

2.27 Example. The general procedure does not work for the IVP

$$2y^{-2} - 4xy + 1 + (2xy^{-1} - 6x^2)y' = 0, \quad y(1) = 1$$

because here, with $P(x,y) = 2y^{-2} - 4xy + 1$ and $Q(x,y) = 2xy^{-1} - 6x^2$, we have

$$P_y(x,y) = -4y^{-3} - 4x \neq Q_x(x,y) = 2y^{-1} - 12x,$$

so the DE is not exact. However, it may be possible to transform the equation into an exact one by using an *integrating factor* $\mu(x,y)$. Writing the DE as $P + Qy' = 0$ and multiplying it by μ, we arrive at $P_1 + Q_1y' = 0$, where $P_1 = P\mu$ and $Q_1 = Q\mu$. We now try to find a function μ such that $(P_1)_y = (Q_1)_x$; that is, $(P\mu)_y = (Q\mu)_x$, which leads to the partial differential equation

$$P\mu_y + P_y\mu = Q\mu_x + Q_x\mu. \tag{2.29}$$

Since, in general, this equation may be difficult to solve, we attempt to see if μ can be found in a simpler form, for example, as a one-variable function. In our case, let us try $\mu = \mu(y)$. Then (2.29) simplifies to

$$P\mu' = (Q_x - P_y)\mu,$$

and with our specific P and Q we get

$$\mu' = \frac{2y^{-1} - 12x + 4y^{-3} + 4x}{2y^{-2} - 4xy + 1}\,\mu = \frac{2(y^{-1} - 4x + 2y^{-3})}{y(2y^{-3} - 4x + y^{-1})}\,\mu = \frac{2}{y}\,\mu,$$

which is a separable equation with solution $\mu(y) = y^2$. Multiplying the DE in the given IVP by y^2, we arrive at the new problem

$$2 - 4xy^3 + y^2 + (2xy - 6x^2y^2)y' = 0, \quad y(1) = 1.$$

This IVP was solved in Example 2.24. ■

2.28 Remark. It should be pointed out that, in general, looking for an integrating factor of a certain form is a matter of trial and error, and that, unless some special feature of the equation gives us a clear hint, this type of search might prove unsuccessful. ■

Exercises

In **1–12**, solve the given IVP, or find the GS of the DE if no IC is given.

1 $4xy + (2x^2 - 6y)y' = 0, \quad y(1) = 1.$

2 $6 - 6x + y + (x + 4y)y' = 0, \quad y(1) = 1.$

3 $2x - 5y + (2 - 5x - 6y)y' = 0, \quad y(2) = -1.$

4 $2 + 3y + e^{2y} + (3x - 1 + 2xe^{2y})y' = 0, \quad y(-2) = 0.$

5 $2 + 3y + ye^{x/2} + (3x - 2y + 2e^{x/2})y' = 0, \quad y(0) = -1.$

6 $3x^{-1/2}y - 2x + (6x^{1/2} + 4y)y' = 0, \quad y(4) = 2.$

7 $2x\sin y - y\cos x + (x^2\cos y - \sin x)y' = 0, \quad y(\pi/2) = \pi/2.$

8 $12x^{-4} + 2xy^{-3} + 6x^2y + (2x^3 - 3x^2y^{-4})y' = 0, \quad y(1) = 2.$

9 $3x^2y^{-2} + x^{-2}y^2 - 2x^{-3} + (6y^{-4} - 2x^3y^{-3} - 2x^{-1}y)y' = 0, \quad y(1) = -1.$

10 $(x + y)^{-1} - x^{-2} - 4\cos(2x - y) + [(x + y)^{-1} + 2\cos(2x - y)]y' = 0.$

11 $2e^{2x} - 2\sin(2x)\sin y + [2y^{-3} + \cos(2x)\cos y]y' = 0, \quad y(0) = \pi/2.$

12 $(x + 1)e^{x-2y} + 4x - 4y + (2y - 4x - 2xe^{x-2y})y' = 0, \quad y(0) = 0.$

In **13–20**, find an integrating factor (of the indicated form) that makes the DE exact, then solve the given IVP, or find the GS of the DE if no IC is given.

13 $y + 3y^{-1} + 2xy' = 0, \quad y(1) = 1, \quad \mu = \mu(y).$

14 $2xy + y^2 + xyy' = 0, \quad \mu = \mu(y).$

15 $1 + 2x^2 - (x + 4xy)y' = 0, \quad \mu = \mu(x).$

16 $-2x^3 - y + (x + 2x^2)y' = 0, \quad y(1) = 1, \quad \mu = \mu(x).$

17 $3 - 4x^{-2}yy' = 0, \quad y(2) = 1, \quad \mu = \mu(x).$

18 $x^{-3} - 2x^{-4}y + x^{-1}y^2 + (x^{-3} + 2y)y' = 0, \quad y(1) = 1, \quad \mu = \mu(x).$

19 $2xy^2 - y^3 + 2y^4 + (1 - xy^2 + 4xy^3)y' = 0, \quad y(1) = -1, \quad \mu = \mu(y).$

20 $1 - 4xye^{-y} + (x - 2x^2e^{-y})y' = 0, \quad y(-2) = 0, \quad \mu = \mu(y).$

2.7 Existence and Uniqueness Theorems

Before attempting to solve an IVP or BVP for a mathematical model, it is essential to convince ourselves that the problem is uniquely solvable. This requirement is based on the reasonable expectation that, as mentioned at the beginning of Sect. 1.3, a physical system should have one and only one response to a given set of admissible constraints.

Assertions that provide conditions under which a given problem has a unique solution are called *existence and uniqueness theorems*. We discuss the linear and nonlinear cases separately.

2.29 Theorem. *Let J be an open interval of the form $a < t < b$, let t_0 be a point in J, and consider the IVP*

$$y' + p(t)y = q(t), \quad y(t_0) = y_0, \tag{2.30}$$

where y_0 is a given initial value. If p and q are continuous on J, then the IVP (2.30) *has a unique solution on J for any y_0.* ■

2.30 Remark. In fact, we already know that the unique solution mentioned in Theorem 2.29 can be constructed by means of formulas (2.5) and (2.8). ■

2.31 Definition. The largest open interval J on which an IVP has a unique solution is called the *maximal interval of existence* for that solution. ■

2.32 Remarks. (i) Theorem 2.29 gives no indication as to what the maximal interval of existence for the solution might be. This needs to be determined by other means—for example, by computing the solution explicitly when such computation is possible.

(ii) If no specific mention is made of an interval associated with an IVP, we assume that this is the maximal interval of existence as defined above.

(iii) Many IVPs of the form (2.30) model physical processes in which the DE is meaningful only for $t > 0$. This would seem to create a problem when we try to apply Theorem 2.29 because an open interval of the form $0 < t < b$ does not contain the point $t_0 = 0$ where the IC is prescribed. A brief investigation, however, will easily convince us that, in fact, there is no inconsistency here. If the IVP in question is correctly formulated, we will find that the maximal interval of existence for the solution is larger than $0 < t < b$, extending to the left of the point $t_0 = 0$. The DE is formally restricted to the interval $0 < t < b$ simply because that is where it makes physical sense. ■

2.33 Example. In the equation of the IVP

$$y' - (t^2 + 1)y = \sin t, \quad y(1) = 2$$

we have

$$p(t) = -(t^2 + 1), \quad q(t) = \sin t.$$

Since both p and q are continuous on the interval $-\infty < t < \infty$, from Theorem 2.29 it follows that this IVP has a unique solution on the entire real line. ■

2.34 Example. Bringing the equation in the IVP

$$(t + 1)y' + y = e^{2t}, \quad y(2) = 3$$

to the standard form, we see that

$$p(t) = \frac{1}{t + 1}, \quad q(t) = \frac{e^{2t}}{t + 1}.$$

The functions p and q are continuous on each of the open intervals $-\infty < t < -1$ and $-1 < t < \infty$; they are not defined at $t = -1$. Since the IC is given at $t_0 = 2 > -1$, from Theorem 2.29 we conclude that the IVP has a unique solution in the interval $-1 < t < \infty$. ■

2.35 Remark. If the IC in Example 2.34 were replaced by, say, $y(-5) = 2$, then, according to Theorem 2.29, the IVP would have a unique solution in the open interval $-\infty < t < -1$, which contains the point $t_0 = -5$. ■

2.36 Example. To get a better understanding of the meaning and limitations of Theorem 2.29, consider the IVP

$$ty' - y = 0, \quad y(t_0) = y_0,$$

where $p(t) = -1/t$ and $q(t) = 0$. The function q is continuous everywhere, but p is continuous only for $t > 0$ or $t < 0$ since it is not defined at $t = 0$. Treating the DE as either a separable or linear equation, we find that its GS is

$$y(t) = Ct, \quad C = \text{const}. \tag{2.31}$$

The IC now yields $y_0 = Ct_0$, which gives rise to three possibilities.

(i) If $t_0 \neq 0$, then, by Theorem 2.29, the IVP is guaranteed to have a unique solution

$$y(t) = \frac{y_0}{t_0}\, t \tag{2.32}$$

on any open interval containing t_0 but not containing 0; more specifically, on any interval of the form $0 < a < t < b$ (if $t_0 > 0$) or $a < t < b < 0$ (if $t_0 < 0$). However, direct verification shows that the function (2.32) satisfies the DE at every real value of t, so its maximal interval of existence is the entire real line.

(ii) If $t_0 = 0$ but $y_0 \neq 0$, then the IVP has no solution since the equality $y_0 = Ct_0 = 0$ is impossible for any value of C.

(iii) If $t_0 = y_0 = 0$, then the IVP has infinitely many solutions, given by (2.31) with any constant C, each of them existing on the entire real line.

The 'anomalous' cases (ii) and (iii) are explained by the fact that they prescribe the IC at the point where p is undefined, so Theorem 2.29 does not apply. ■

We now turn our attention to the nonlinear case.

2.37 Theorem. *Consider the IVP*

$$y' = f(t, y), \quad y(t_0) = y_0, \tag{2.33}$$

where f is a given function such that f and f_y are continuous in an open rectangle

$$R = \{(t, y) :\ a < t < b,\ c < y < d\}. \tag{2.34}$$

If the point (t_0, y_0) lies in R, then the IVP (2.33) has a unique solution in some open interval J of the form $t_0 - h < t < t_0 + h$ contained in the interval $a < t < b$. ■

2.38 Example. For the IVP

$$2(y-1)y'(t) = 2t + 1, \quad y(2) = -1$$

we have

$$f(t, y) = \frac{2t+1}{2(y-1)}, \quad f_y(t, y) = -\frac{2t+1}{2(y-1)^2},$$

both continuous everywhere in the (t, y)-plane except on the line $y = 1$. By Theorem 2.37 applied in any rectangle R of the form (2.34) that contains the point $(2, -1)$ and does not intersect the line $y = 1$, the given IVP has a unique solution on some open interval J centered at $t_0 = 2$. Figure 2.2 shows such a rectangle and the arc of the actual solution curve lying in it. The open interval $2 - h < t < 2 + h$ indicated by the heavy line is the largest of the form mentioned in Theorem 2.37 for the chosen rectangle. But the solution exists on a larger interval than this, which can be determined by solving the IVP.

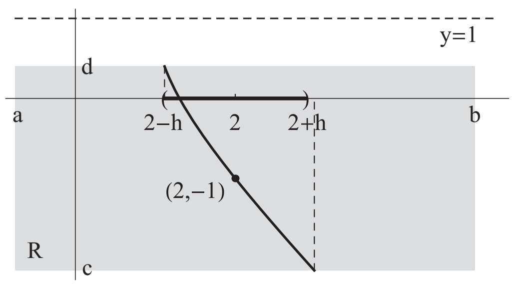

Fig. 2.2

We separate the variables in the DE and write

$$\int 2(y-1)\, dy = \int (2t+1)\, dt,$$

so

$$(y-1)^2 = t^2 + t + C, \quad C = \text{const.} \tag{2.35}$$

The constant, computed from the IC, is $C = -2$. Replacing it in (2.35), we then find that

$$y(t) = 1 \pm (t^2 + t - 2)^{1/2}.$$

Of these two functions, however, only the one with the negative root satisfies the IC; therefore, the unique solution of the IVP is

$$y(t) = 1 - (t^2 + t - 2)^{1/2}. \tag{2.36}$$

We establish the maximal interval of existence for this solution by noticing that the square root in (2.36) is well defined only if

$$t^2 + t - 2 = (t-1)(t+2) \geq 0;$$

that is, for $t \leq -2$ or $t \geq 1$. Since $t_0 = 2$ satisfies the latter, we conclude that the maximal interval of existence is $1 < t < \infty$. ∎

2.39 Example. The situation changes if the IC in Example 2.38 is replaced by $y(0) = 1$. Now the point $(t_0, y_0) = (0, 1)$ lies on the line $y = 1$ where f and f_y are undefined, so every rectangle R of the form (2.34) that contains this point will also contain a portion of the line $y = 1$ (see Figure 2.3). Consequently, Theorem 2.37 cannot be applied.

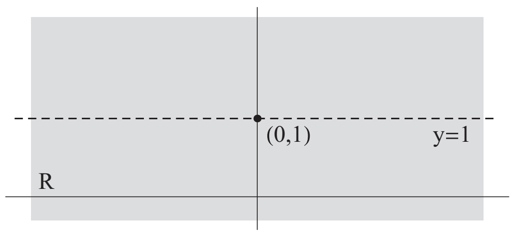

Fig. 2.3

To see what kind of 'pathology' attaches to the problem in this case, we note that the new IC leads to $C = 0$ in (2.35), which means that

$$y(t) = 1 \pm (t^2 + t)^{1/2}. \tag{2.37}$$

The square root is well defined for either $t \leq -1$ or $t \geq 0$, and the two functions given by (2.37) are continuously differentiable and satisfy the DE in each of the open intervals $-\infty < t < -1$ and $0 < t < \infty$. The point $t_0 = 0$ is well outside the former, but is a *limit point* for the latter. Hence, we conclude that our IVP has a pair of distinct solutions, whose maximal interval of existence is $0 < t < \infty$. Both these solutions are right-continuous (though not right-differentiable) at 0 and comply with the IC in the sense that $y(0+) = 1$. The nonuniqueness issue we came across here is connected with the fact that the conditions in Theorem 2.37 are violated.

Verification with Mathematica®. The input

```
{y1,y2}={1 + (t^2 + t)^(1/2),1 - (t^2 + t)^(1/2)};
Simplify[{2*({y1,y2} - 1)*D[{y1,y2},t] - 2*t - 1,
        {y1,y2}/.t→0},t>0]
```

generates the output $\{\{0, 0\}, \{1, 1\}\}$. ∎

2.40 Remarks. (i) The conditions in Theorems 2.29 and 2.37 are *sufficient* but not *necessary*. In other words, if they are satisfied, the existence of a unique solution of the kind stipulated in these assertions is guaranteed. If they are not, then, as illustrated by Examples 2.36(ii),(iii) and 2.39, a more detailed analysis is needed to settle the question of solvability of the IVP.

The restrictions imposed on f in Theorem 2.37 can be relaxed to a certain extent. It is indeed possible to prove that the theorem remains valid for functions f subjected to somewhat less stringent requirements.

(ii) Theorems 2.29 and 2.37 imply that when the existence and uniqueness conditions are satisfied, the graphs of the solutions of a DE corresponding to distinct ICs do not intersect. For if two such graphs intersected, then the intersection point, used as an IC, would give rise to an IVP with two different solutions, which would contradict the statement of the appropriate theorem. Consequently, if y_1, y_2, and y_3 are the PSs of the same DE on an open interval J, generated by initial values y_{01}, y_{02}, and y_{03}, respectively, with $y_{01} < y_{02} < y_{03}$, then $y_1(t) < y_2(t) < y_3(t)$ at all points t in J.

Figure 2.4 shows the graphs of the solutions of the equation in Example 2.8 with $y_0 = -1, 0, 1$ at $t_0 = 0$.

(iii) Existence of a unique solution is one of the conditions that an IVP or BVP representing a mathematical model needs to satisfy to be regarded as *well posed*—that is, correctly formulated. The IVP in Example 2.39 is not well posed. ■

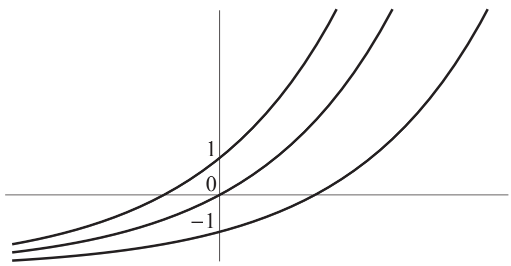

Fig. 2.4

Exercises

In **1**–**6**, find the largest open interval on which the conditions of Theorem 2.29 are satisfied, without solving the IVP itself.

1 $(2t+1)y' - 2y = \sin t, \quad y(0) = -2.$

2 $(t-1)y' - (t-2)^{1/2}y = t^2 + 4, \quad y(3) = -2.$

3 $(t^2 - 3t + 2)y' + ty = e^t, \quad y(3/2) = -1.$

4 $(t^2 - t - 2)^{1/2}y' + 3y = (t-3)^{1/2}, \quad y(4) = 1.$

5 $(2 - \ln t)y' + 3y = 5, \quad y(1) = e.$ **6** $y' + 3ty = 2\tan t, \quad y(\pi) = 1.$

In **7**–**12**, indicate the regions in the (t, y)-plane where the conditions of Theorem 2.37 are not satisfied.

7 $(2y + t)y' = 2t + y.$ **8** $y' = t(y^2 - 1)^{1/2}.$ **9** $(t^2 + y^2 - 9)y' = \ln t.$

10 $y' = (t+2)\tan(2y).$ **11** $y' = (t+y)\ln(t+y).$ **12** $y' = \sqrt{ty}\,.$

In **13**–**20**, solve the DE with each of the given ICs and find the maximum interval of existence of the solution.

13 $y' = 4ty^2$; $y(0) = 2$; $y(-1) = -2$; $y(3) = -1$.

14 $y' = 8y^3$; $y(0) = -1$; $y(0) = 1$.

15 $(y-2)y' = t$; $y(2) = 3$; $y(-2) = 1$; $y(0) = 1$; $y(1) = 2$.

16 $(y-3)y' = 2t+1$; $y(-1) = 3$; $y(-1) = 1$; $y(-1) = 5$; $y(2) = 1$.

17 $y' = 4(y-1)^{1/2}$; $y(0) = 5$; $y(-1) = 2$.

18 $y' = (y^2-1)/t$; $y(1) = 0$; $y(-1) = 2$; $y(2) = -3$.

19 $y' = (2t+2)/(3y^2)$; $y(0) = 1$; $y(1) = 1$; $y(-1) = 2$; $y(3) = -1$.

20 $y' = (y^2+y)/99(t+1)$; $y(0) = 1$; $y(-2) = 1$; $y(4) = -2$; $y(-2) = -2$.

2.8 Direction Fields

Very often, a nonlinear first-order DE cannot be solved by means of integrals; therefore, to obtain information about the behavior of its solutions we must resort to qualitative analysis methods. One such technique is the sketching of so-called *direction fields*, based on the fact that the right-hand side of the equation $y' = f(t, y)$ is the slope of the tangent to the solution curve $y = y(t)$ at a generic point (t, y). Drawing short segments of the line with slope $f(t, y)$ at each node of a suitably chosen lattice in the (t, y)-plane, and examining the pattern formed by these segments, we can build up a useful pictorial image of the family of solution curves of the given DE.

2.41 Example. In Sect. 2.5, we mentioned the difficulty that arises when we try to solve a Riccati equation for which no PS is known beforehand. The method described above, applied to the equation

$$y' = e^{-2t} - 3 + (5 - 2e^{-2t})y + (e^{-2t} - 2)y^2$$

in the rectangle defined by $-1.5 \le t \le 2$ and $0.5 \le y \le 1.6$, yields the direction field shown in Figure 2.5, where several solution curves are also graphed.

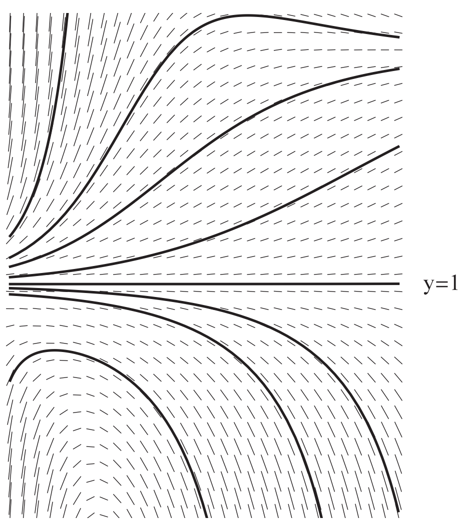

Fig. 2.5

Figure 2.5 suggests that $y(t) = 1$ might be a PS of the given DE, and direct verification confirms that this is indeed the case. Consequently, proceeding as in Sect. 2.5, we determine that the GS of our equation is

$$y(t) = 1 + \frac{1}{2 + e^{-2t} + Ce^{-t}}.$$

The solution curves in Figure 2.5 correspond, from top to bottom, to $C = -2.78$, -1.05, 1, 10, $10{,}000$, -30, -15, -6.

Of course, other equations may not benefit from this type of educated guess, however refined a point lattice is employed to generate their direction fields. ■

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Differential Equations
A Primer for Scientists and Engineers
Constanda, C.
2017, XVII, 297 p. 40 illus., Hardcover
ISBN: 978-3-319-50223-6