Chapter 2
Thermodynamics and Phase Transition

An important quantity for this model is the logarithmic moment generating function λ of $\omega(n,x)$,

$$\lambda(\beta) = \ln \mathbb{P}[\exp(\beta\omega(n,x))], \tag{2.1}$$

which is finite for all real β by assumption (1.1). For example, $\lambda(\beta) = \ln(pe^{-\beta} + (1-p)e^{\beta})$ for the Bernoulli environment and $\lambda(\beta) = \frac{1}{2}\beta^2$ for the Gaussian environment. All through these notes, we will restrict to

$$\boxed{\beta \geq 0}\,.$$

Doing this we do not lose generality, since the opposite case being obtained by considering environment $-\omega$.

2.1 Preliminaries

On the space Ω of environments, define for $i \geq 1, x \in \mathbb{Z}^d$, the **shift operator** $\theta_{i,x} : \Omega = \mathbb{R}^{\mathbb{N}^* \times \mathbb{Z}^d} \to \Omega$ given by $\omega \mapsto \theta_{i,x}\omega$,

$$(\theta_{i,x}\omega)(t,y) = \omega(i+t, x+y)\ . \tag{2.2}$$

Thus, $\theta_{i,x}\omega$ is simply the field of environment variables which is seen from the "point" (i,x). Its law is the same as the law of ω itself, i.e., the product law $\mathbb{P}$.

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2.1.1 Markov Property and the Partition Function

For $n, m \geq 1, x \in \mathbb{Z}^d$, the random variable

$$Z_m \circ \theta_{n,x}(\omega) = Z_m(\theta_{n,x}\omega; \beta) = P_x\left[\exp\left(\sum_{1\leq t\leq m} \beta\omega(t+n, S_t)\right)\right],$$

is the partition function of the polymer of length m starting at x at time n. Since ω and its shift $\theta_{n,x}\omega$ have the same law, $Z_m \circ \theta_{n,x}$ has the same law as Z_m. By the Markov property of the random walk, we can also write it in the form of a conditional expectation given $\mathscr{F}_n = \sigma\{S_t, t \leq n\}$,

$$Z_m \circ \theta_{n,x}(\omega) = P\left[e^{\beta\{H_{n+m}(S)-H_n(S)\}}|\mathscr{F}_n\right] \quad \text{on the event } \{S_n = x\}.$$

For $n, m \geq 1$, we can express the partition function of the polymer of length $n+m$ by conditioning:

$$\begin{aligned} Z_{n+m} &= P\left[e^{\beta H_{n+m}(S)}\right] \\ &= P\left[e^{\beta H_n(S)} P\left[e^{\beta\{H_{n+m}(S)-H_n(S)\}}|\mathscr{F}_n\right]\right] \\ &= P\left[e^{\beta H_n(S)} \times Z_m \circ \theta_{n,S_n}\right] \end{aligned}$$

using the previous observation. This important identity will be referred to as the Markov property. It can be reformulated as

$$Z_{n+m} = Z_n \times P_n^{\beta,\omega}\left[Z_m \circ \theta_{n,S_n}\right]. \tag{2.3}$$

2.1.2 The Polymer Measure as a Markov Chain

In this section we discuss some basic properties of the polymer measure $P_n^{\beta,\omega}$.

Let us fix the environment ω. Then, under the polymer measure $P_n^{\beta,\omega}$ the path S is a Markov chain, with transition probabilities

$$P_n^{\beta,\omega}(S_{i+1}=y|S_i=x) = \frac{e^{\beta\omega(i+1,y)} Z_{n-i-1} \circ \theta_{i+1,y}}{Z_{n-i} \circ \theta_{i,x}} P(S_1=y|S_0=x) \tag{2.4}$$

for $0 \le i < n$, and $P_n^{\beta,\omega}(S_{i+1} = y | S_i = x) = P(S_1 = y | S_0 = x)$ for $i \ge n$. Indeed, one directly sees that, for all path $(x_0 = 0, x_1, \dots, x_n)$, the following product is telescopic,

$$\prod_{i=0}^{n-1} \frac{e^{\beta\omega(i+1,x_{i+1})} Z_{n-i-1} \circ \theta_{i+1,x_{i+1}}}{Z_{n-i} \circ \theta_{i,x_i}} P(S_1 = x_{i+1} | S_0 = x_i) = P_n^{\beta,\omega}(S_{[1,n]} = x_{[1,n]}) ,$$

which proves our claim. (Recall that $Z_0 = 1$.) We can re-write (2.4) as

$$P_n^{\beta,\omega}(S_{i+1} = y | S_i = x) = P_{n-i}^{\beta,\theta_{i,x}\omega}(S_1 = y - x)$$

The transition probabilities depend on the environment at the current time and at later times. Since the transition probability at step i depends on i, the chain is *time-inhomogeneous*. Since the transition probabilities also depend on the time horizon n, the family $P_n^{\beta,\omega}$ is not consistent: Except for the trivial case $\beta = 0$, there exists no Markov chain on the set of paths with infinite length which, for all $n \ge 1$, its marginal on the set of paths of length n is $P_n^{\beta,\omega}$. Therefore we consider the full sequence $(P_n^{\beta,\omega}; n \ge 1)$ of Markov chains. Finally, note that for $0 \le m, n$,

$$P_{m+n}^{\beta,\omega}(S_{[1,n]} = \cdot \, | S_n = y) = P_n^{\beta,\omega}(S_{[1,n]} = \cdot \, | S_n = y) , \tag{2.5}$$

$$P_{m+n}^{\beta,\omega}(S_{[n,n+m]} = y + \cdot \, | S_n = y) = P_m^{\beta,\theta_{n,y}\omega}(S_{[0,m]} = \cdot \,) . \tag{2.6}$$

2.2 Free Energy

As it is well known in statistical mechanics, a great amount of information on the Gibbs measure is encoded in the following quantity,

$$p_n = p_n(\omega; \beta) = \frac{1}{n} \ln Z_n(\omega; \beta) , \tag{2.7}$$

that we will call the (finite volume, specific) free energy[1] for the polymer of length n. The first step to understand its limit as the polymer length tends to infinity, and if it depends on the realization of the environment ω.

[1] In physics, the free energy is rather defined as $-\beta^{-1} p_n$, it has the same unit as the energy $-H_n$. The name specific means that it has been normalized by the number n of monomers.

Theorem 2.1 *As $n \to \infty$,*

$$p_n(\omega;\beta) \longrightarrow p(\beta) = \sup_n \frac{1}{n}\mathbb{P}\big[\ln Z_n(\omega;\beta)\big]$$

$\mathbb{P}$-a.s. and in L^p-norm, for all $p \in [1,\infty)$.

The theorem states that the sequence $p_n(\omega;\beta)$ converges a.s. to a limit, the limit is deterministic and given as a supremum over the polymer length. The limit p is called the (infinite volume, specific) free energy.[2]

Proof The proof splits in two steps, with the first one showing that expectations converge, and the second one showing that random fluctuations are negligible.

- *Step 1:* We first consider expected values and show that:

$$\lim_{n\to\infty} \mathbb{P}[p_n] = \sup_{n\in\mathbb{N}} \mathbb{P}[p_n] \in \mathbb{R}$$

For $m, n \geq 1$, recall the identity (2.3), and also that Z_m and $Z_m \circ \theta_{n,x}$ have the same law. By Jensen's inequality, we obtain

$$\ln Z_{n+m} \geq \ln Z_n + \sum_x P_n^{\beta,\omega}\{S_n = x\} \ln Z_m(\theta_{n,x}\omega) \ .$$

Taking expectation and using independence of the $\omega(i,y)$'s, we obtain

$$\begin{aligned}\mathbb{P}[\ln Z_{n+m}] &\geq \mathbb{P}[\ln Z_n] + \sum_x \mathbb{P}[P_n^{\beta,\omega}\{S_n = x\}] \times \mathbb{P}[\ln Z_m] \\ &= \mathbb{P}[\ln Z_n] + \mathbb{P}[\ln Z_m] \sum_x \mathbb{P}[P_n^{\beta,\omega}\{S_n = x\}] \\ &= \mathbb{P}[\ln Z_n] + \mathbb{P}[\ln Z_m] \end{aligned} \tag{2.8}$$

i.e., $\mathbb{P}[\ln Z_n]$ is super-additive. From the superadditive Lemma (see Toolbox, Sect. A.1), we see that

$$\lim_{n\nearrow\infty} \frac{1}{n}\mathbb{P}[\ln Z_n] = \sup_n \frac{1}{n}\mathbb{P}[\ln Z_n].$$

Now, the finiteness of p follows from the annealed bound (2.13) below.

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[2] A much more detailed account will be given in Theorem 9.1 in the last chapter, defining the free energy in fixed directions.

- *Step 2:* We will apply to $\ln Z_n$ a concentration inequality, given in Theorem 2.2 below. Then, by Borel-Cantelli lemma, this implies that $\limsup_n \big|p_n - \mathbb{P}[p_n]\big| \leq \epsilon$, $\mathbb{P}$-a.s. Hence,

$$\limsup_{n\to\infty} \big|p_n - \mathbb{P}[p_n]\big| = 0 \qquad \mathbb{P} - \text{a.s.},$$

which, together with Step 1 above, completes the proof of almost sure convergence in Theorem 2.1. To get L^p convergence, one checks from the concentration inequality that the sequence $(p_n(\omega;\beta))_n$ is uniformly integrable. For definiteness, we will use the concentration inequality (2.9) in the second line below: since $EZ = \int_0^\infty P(Z > r)dr$ for $Z \geq 0$,

$$\begin{aligned}
\mathbb{P}\big(\big|p_n - \mathbb{P}[p_n]\big|^p\big) &= \int_0^\infty \mathbb{P}\big(\big|p_n - \mathbb{P}[p_n]\big| > r^{1/p}\big)dr \\
&\leq 2\int_0^\infty \exp\{-nC(r^{1/p} \wedge r^{2/p})\}dr \\
&\leq 2\int_0^\infty \exp\{-nCr^{2/p}\}dr + 2\int_1^\infty \exp\{-nCr^{1/p}\}dr \\
&= C'n^{-p/2} + \mathcal{O}(\exp\{-Cn\})
\end{aligned}$$

with $C' = \int_0^\infty \exp\{-Cv^{2/p}\}dv < \infty$. This implies L^p-convergence. □

We have used a result of Liu and Watbled [167].

Theorem 2.2 (Theorem 1.4 in [167]) *(General concentration inequality for the free energy) Assume that the environment has all exponential moments (1.1). Then,*

$$\mathbb{P}\big[|p_n - \mathbb{P}[p_n]| \geq r\big] \leq \begin{cases} 2\exp\{-nCr^2\} & \text{if } 0 \leq r \leq 1, \\ 2\exp\{-nCr\} & \text{if } r \geq 1, \end{cases} \tag{2.9}$$

for some constant $C > 0$.

We will prove it here, only in two particular but important cases in Sect. A.3.1—Gaussian environment and bounded ones—, where the tails are subGaussian all the way (i.e., the bound $2\exp\{-nCr^2\}$ holds for all $r > 0$) and where the result follows from general principles. For bounded environment, it is a consequence of Azuma's Lemma A.2 as explained in Exercise A.1. For Gaussian environment the result follows from the Gaussian concentration inequality (A.4), and it takes the simplest possible form.

Remark 2.1 (Gaussian Environments) For a standard gaussian environment $\omega(t,x) \sim \mathcal{N}(0,1)$, we can apply the well-known Gaussian concentration inequality (A.4), which yields

$$\mathbb{P}\big[|p_n - \mathbb{P}[p_n]| \geq \varepsilon\big] \leq 2\exp\{-n\varepsilon^2/2\beta^2\}, \qquad \forall\varepsilon > 0. \tag{2.10}$$

Remark 2.2 (Self-averaging) The property that the random sequence $p^{\omega}_{n,\beta}$ becomes nonrandom in the limit is called self-averaging. Here, $p^{\omega}_{n,\beta}$ is the sum of $\mathbb{P}p^{\omega}_{n,\beta} = \mathcal{O}(n)$ and of a term of lower order $n^{-1/2}$ in probability according to the concentration inequality.

2.3 Upper Bounds

Computing the value of the free energy is impossible in general, hence we need to estimate it. We focus on upper bounds, and refer to Remark 2.5 for lower bounds.

2.3.1 The Annealed Bound

For all path S,

$$\mathbb{P}[\exp\{\beta H_n(S)\}] = \exp n\lambda(\beta), \tag{2.11}$$

and by Fubini's theorem,

$$\mathbb{P}[Z_n] = \mathbb{P}[P\exp\{\beta H_n(S)\}] = \exp n\lambda(\beta).$$

By Jensen inequality,

$$\mathbb{P}[p_n(\omega,\beta)] = \frac{1}{n}\mathbb{P}[\ln Z_n] \leq \frac{1}{n}\ln\mathbb{P}[Z_n] = \lambda(\beta)\ , \tag{2.12}$$

hence

$$p(\beta) \leq \lambda(\beta). \tag{2.13}$$

This bound, comparing the quenched free energy p to the annealed free energy λ is central in the realm of random medium, it is known as the **annealed bound**. It appears in all standard disordered systems.

It is based on the single fact (2.11), and then averaging S under P. This bound is very crude, it does not depends either on: (1) the fine structure of P—except for being a probability measure—, (2) what are the correlations between $H_n(S)$ for different path S, (3) how rare are the deviations of $H_n(S)$ which are implicitly present in the value of $\mathbb{P}[\exp\{\beta H_n(S)\}]$.

In our case, the P-expectation performs an average on a collection of paths S which cardinality grows exponentially in n. Recall that Jensen's inequality is almost an equality when integrating a function which does not fluctuate much. Note also that correlations between the summands typically increase fluctuations of the average. Then, the annealed bound can be reasonable when the correlation between $H_n(S)$ for different S is small—e.g. if they are independent for different S's, cf. Lemmas 5.1, 5.3, 5.4 in [235]—and when the values of $H_n(S)$ contributing to that of $\mathbb{P}\exp\{\beta H_n(S)\}$ are not so rare in comparison with the entropy of P. For instance, for a stationary ergodic sequence $(\mathscr{E}_i)_{i\geq 1}$ with positive bounded values, we have as $n \to \infty$,

$$\mathbb{E}\ln\left(\frac{1}{n}\sum_{i=1}^{n}\mathscr{E}_i\right) \longrightarrow \ln\mathbb{E}(\mathscr{E})$$

by the ergodic theorem, and then the annealed bound is sharp in this asymptotics. In this example, we observe that the $(\mathscr{E}_i)_{i\geq 1}$ have a fixed correlation (which therefore should be viewed as weak as n diverges), and that the contributing values of $\mathscr{E}$ to the expectation are typical.

On the other hand, the annealed bound will be crude when the $H_n(S)$'s have strong correlation, or when the values of $H_n(S)$ contributing to that of $\mathbb{P}\exp\{\beta H_n(S)\}$ are too rare to be compensated by the entropy of P. To illustrate the first effect, consider the extreme case when the $H_n(S)$ are equal for all S to a nondegenerate random variable: Then, $\mathbb{E}\ln\left(\frac{1}{n}\sum_{i=1}^{n}\mathscr{E}_i\right) = \mathbb{E}\ln\mathscr{E}$ does not depend on n and is strictly less than $\ln\mathbb{E}(\mathscr{E})$ by Jensen inequality. In our framework, the annealed bound will not be sharp in general, due to these two effects. The first effect is more difficult to control than the second one, that we address in the next section.

Remark 2.3 (Annealed Bound Is Not Always Sharp) The inequality in annealed bound (2.13) may be strict. In this case of a Gaussian $\mathscr{N}(0,1)$ distribution for ω, we have $\lambda(\beta) = \beta^2/2$, and for all path s, $H_n(\mathbf{x}) \sim \mathscr{N}(0,n)$. Recall the standard Gaussian tail estimate: if X has density $g(x) = (2\pi)^{-1/2}\exp\{-x^2/2\}$,

$$\frac{x}{1+x^2}g(x) \leq \mathbb{P}(X \geq x) \leq (\frac{1}{x} \wedge \sqrt{2\pi})g(x)\ , \quad x > 0\ ,$$

—see (A.2)—. Then, for all $a > 0$, we have by the union bound,

$$\begin{aligned}\sum_{n\geq 1} \mathbb{P}\left(\max_s H_n(s) > na\right) &\leq \sum_{n\geq 1} \sum_{s:\text{length}\, n} \mathbb{P}\left(H_n(s) > na\right) \\ &\leq \sum_{n\geq 1} (2d)^n \frac{1}{a\sqrt{n}} \exp\{-na^2/2\} \\ &< \infty \qquad \text{if } a > \sqrt{2\ln(2d)}.\end{aligned}$$

By Borel-Cantelli's lemma, we see that a.s.,

$$n^{-1} \ln Z_n \leq n^{-1} \beta \max_s \{H_n(s)\} \leq \beta a$$

for n large enough and such a's. Hence,

$$p(\beta) \leq \beta \sqrt{2\ln(2d)}\ . \tag{2.14}$$

Since this bound is of smaller order as $\beta \to \infty$ than $\lambda(\beta) = \beta^2/2$, we conclude that $p(\beta) < \lambda(\beta)$ for β large enough.

2.3.2 *Improving the Annealed Bound*

To improve (2.13) we will use monotonicity properties which are standard in thermodynamics:

Proposition 2.1 *For all fixed ω, we have:*

(i) The function $p_n(\omega; \beta)$ is convex in β, with $p_n(\omega; 0) = 0$.
(ii) $\beta \mapsto \beta^{-1} p_n(\omega; \beta)$ is increasing.
(iii) $\beta \mapsto \beta^{-1}[p_n(\omega; \beta) + \ln(2d)]$ is decreasing.

The function $p(\beta)$ also satisfies (i)–(iii).

Proof Note that $\beta \mapsto p_n(\omega; \beta)$ is C^∞. By differentiation, one gets

$$\frac{d}{d\beta} np_n = P_n^{\beta,\omega}[H_n]\ , \quad \frac{d^2}{d\beta^2} np_n = \mathrm{Var}_{P_n^{\beta,\omega}}[H_n] > 0\ ,$$

proving (i). By convexity, $\beta^{-1} p_n(\beta, \omega) = \beta^{-1}[p_n(\beta, \omega) - p_n(0, \omega)]$ is non-decreasing in β. Turning to (iii), we have the identity

$$\begin{aligned}\frac{d}{d\beta}\left(\frac{1}{\beta}\,[p_n + \ln(2d)]\right) &= -\frac{1}{\beta^2}\,[p_n + \ln(2d)] + \frac{1}{n\beta}\, P_n^{\beta,\omega}[H_n] \\ &= \frac{1}{n\beta^2}\, h(P_n^{\beta,\omega})\ ,\end{aligned}$$

where $h(\nu)$ is the Boltzmann entropy of a probability measure ν on the n steps path space,

$$h(\nu) := \sum_{\mathbf{x}} \nu(\mathbf{x}) \ln \nu(\mathbf{x}) \; . \tag{2.15}$$

Clearly, $h(\nu) \leq 0$ for all ν, which ends the proof. □

We derive an upper bound, which is improving on the annealed bound (2.13). It yields a sufficient condition for the strict inequality to hold in the annealed bound. It improves on the argument in the Remark 2.3, being however less transparent than the argument used in Remark 2.3. It is essentially of the same nature, in the sense that it relies on entropy considerations on a finite state space, and does not take into account the correlation structure of the random vector $(H_n(s))_s$.

Proposition 2.2 *We have*

$$p(\beta) \leq \beta \inf_{b \in (0,\beta]} \frac{\lambda(b) + \ln(2d)}{b} - \ln(2d) \; . \tag{2.16}$$

Hence, under Condition (T),

$$(\mathbf{T}) : \qquad \beta\lambda'(\beta) - \lambda(\beta) > \ln(2d) \; , \tag{2.17}$$

we have

$$p(\beta) < \lambda(\beta).$$

More precisely, if there exists a positive root β_1 *to the equation* $\beta\lambda'(\beta) = \lambda(\beta) + \ln(2d)$*, then for all* $\beta > \beta_1$ *it holds*

$$p(\beta) \leq \frac{\beta}{\beta_1}\big[\lambda(\beta_1) + \ln(2d)\big] - \ln(2d) < \lambda(\beta) \; . \tag{2.18}$$

We summarize the above bounds in Figs. 2.1 and 2.2.

Proof To simplify the notation, we introduce

$$g(\beta) = \beta\lambda'(\beta) - \lambda(\beta) \; , \qquad f(\beta) = \frac{\lambda(\beta) + \ln(2d)}{\beta}. \tag{2.19}$$

Since λ is a smooth and convex, $g'(\beta) = \beta\lambda''(\beta)$ has the sign of β, and $g(\beta)$ is increasing on $\mathbb{R}_+$. In fact, g can be expressed from the convex conjugate λ^* of λ, with $\lambda^*(u) = \sup\{\beta u - \lambda(\beta)\}$ for $u \in \mathbb{R}$. We have

$$g(\beta) = \lambda^*(u) \; , \qquad u = \lambda'(\beta).$$

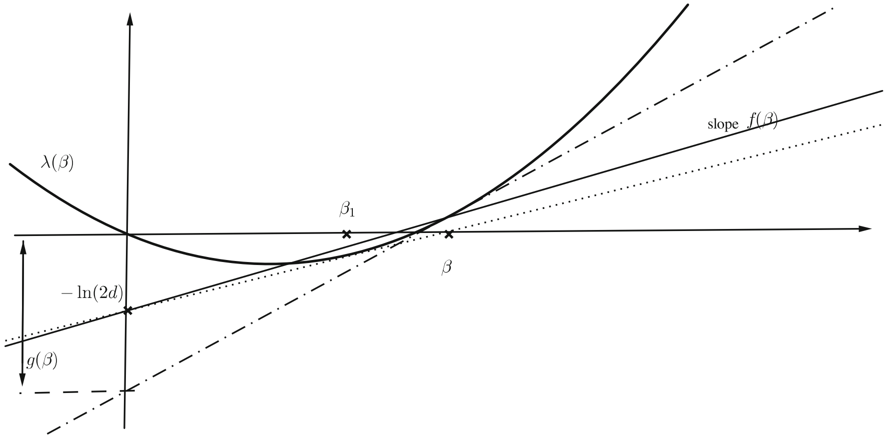

Fig. 2.1 Constructing the upper bound of Proposition 2.2; $f(\beta)$ is the slope of the *solid line* through the points $(0, -\ln(2d))$ and $(\beta, \lambda(\beta))$; $g(\beta)$ is the y-intercept of the tangent to λ at β. For β as in the figure, the infimum of $f(\beta') = \frac{\lambda(\beta')+\ln(2d)}{\beta'}$ over the interval $[0, \beta]$ is achieved at $\beta' = \beta_1$ where the tangent to λ intersects the vertical axis at $-\ln(2d)$

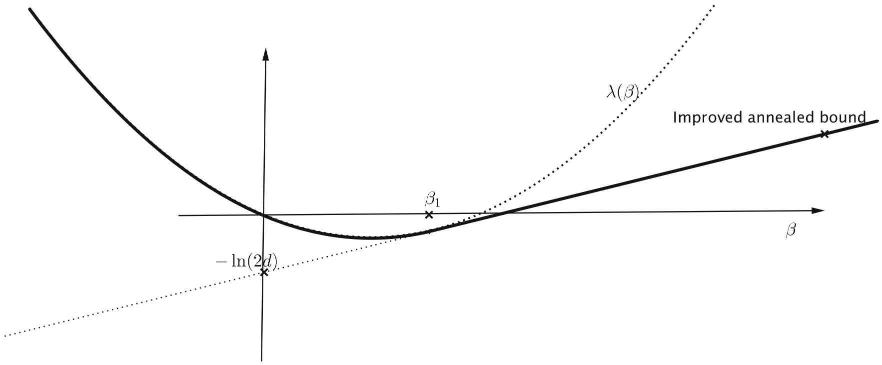

Fig. 2.2 Improved annealed bound. If λ has a tangent at some $\beta_1 > 0$ which intersects the vertical axis at height $-\ln(2d)$, the bound for p on $\mathbb{R}^+$ is improved on the annealed bound $\lambda(\beta)$ by that tangent for $\beta > \beta_1$

Moreover, $f'(\beta) = (g(\beta) - \ln(2d))/\beta^2$. We can write

$$\begin{aligned}
\mathbb{P}[p_n(\omega,\beta) + \ln(2d)] &= \beta \times \frac{1}{\beta}\mathbb{P}[p_n(\omega,\beta) + \ln(2d)] \\
&\overset{\text{Proposition 2.1 (iii)}}{=} \beta \times \inf_{b\in(0,\beta]} \frac{1}{b}\,\mathbb{P}[p_n(\omega,b) + \ln(2d)] \\
&\overset{(2.12)}{\leq} \beta \inf_{b\in(0,\beta]} f(b)\ .
\end{aligned}$$

Letting $n \to \infty$, we obtain

$$p(\beta) + \ln(2d) \le \beta \inf_{b \in (0,\beta]} \frac{\lambda(b) + \ln(2d)}{b}, \tag{2.20}$$

which is the bound (2.16). Now, let us define $\beta_1 \in (0, \infty]$ by

$$\beta_1 = \inf\{\beta \ge 0 : g(\beta) \ge \ln(2d)\},$$

allowing an infinite value. Then, f has its minimum at β_1. Then, as shown in Fig. 2.1,

$$\inf_{\beta' \in]0,\beta]} f(\beta') = \begin{cases} f(\beta) & \text{if } \beta \le \beta_1, \\ f(\beta_1) & \text{if } \beta \ge \beta_1. \end{cases}$$

With (2.20), this yields the bound in (2.18). Note that $\beta > \beta_1$ is equivalent to Condition **(T)**. For such a β, the last inequality is strict by strict convexity of λ. □

Remark 2.4

(i) We could have used the notation $\beta_1 = \beta_c^{\mathrm{tr}}$, since it is equal to the critical value for the polymer model on the tree, see Theorem 4.2. In fact the bound given in (2.16) is the free energy for the model on the tree.

(ii) The improved upper bound follows from Proposition 2.1 (iii) and an explicit computation for the simple random walk. Similar improvement can be implemented for general a priori laws P of the path, as long-range walks. The only requirement is an upper bound on the entropy rate.

Example 2.1 We consider again the Gaussian case, $\omega(i,x) \sim \mathcal{N}(0,1)$. We easily compute $\beta_1 = \sqrt{2\ln(2d)}$, we check that the bound in (2.18) is equal to $\beta\sqrt{2\ln(2d)} - \ln(2d)$, and therefore is strictly smaller than the one from (2.14).

We now look for conditions ensuring $p(\beta) < \lambda(\beta)$ for large β, in terms of the marginal distribution q of ω. A sufficient condition is that the environment is unbounded from above, or bounded from above with a large enough mass on its essential supremum.

Corollary 2.1 *Set* $q(dh) = \mathbb{P}(\omega(n,x) \in dh)$ *and* $s = \sup \operatorname{supp}[q]$.
If $s = +\infty$ *or* $q(\{s\}) < \frac{1}{2d}$*, then, there exists* $\beta_1 \in (0,\infty)$ *such that* $p(\beta) < \lambda(\beta)$ *for* $\beta > \beta_1$.

Proof We check condition (2.17). Let λ^* the Legendre transform of λ,

$$\lambda^*(u) = \sup\{u\beta - \lambda(\beta)\}$$

Then,

$$\begin{aligned}\beta\lambda'(\beta) - \lambda(\beta) &= \lambda^*(\lambda'(\beta))\\ &= \tilde{q}^\beta\big[\ln\frac{d\tilde{q}^\beta}{dq}\big]\end{aligned}$$

where $\tilde{q}^\beta$ is the tilted probability measure on $\mathbb{R}$ given by

$$\tilde{q}^\beta(d\omega) = e^{\beta\omega-\lambda(\beta)}q(d\omega)$$

As illustrated in Fig. 2.1, $\lambda'(\beta) \to \sup\operatorname{supp} q$ when $\beta \to +\infty$.

1. If $s = +\infty$, then $\lambda^* \to \infty$ at infinity, and (2.17) holds for large β.
2. If $s < +\infty$,

$$\lim_{\beta\to+\infty} \lambda^*(\lambda'(\beta)) = -\ln q(\{s\})$$

and (2.17) holds for large β if we assume $q(\{s\}) < 1/2d$. □

Remark 2.5 Lower bounds are less useful for our analysis. We can use the formula as a supremum from Theorem 2.1, and the simplest application leads to

$$p(\beta) \geq \mathbb{P}[p_1(\beta)] = \mathbb{P}\ln P[\exp\{\beta\omega(1, S_1)\}]$$

which is already better than using Jensen inequality

$$p(\beta) \geq \ln P[\exp\{\beta\mathbb{P}[\omega(1, S_1)]\}] = \beta\mathbb{P}[\omega(t, x)].$$

But all these bounds correspond to local optimization in contrast with the polymer measure which performs optimization in highly non local manner.

2.4 Monotonicity

We have seen in Proposition 2.1 that $\beta \mapsto p_n(\omega;\beta)$ is convex and differentiable. It is plain to compute its derivative,

$$\frac{\partial}{\partial\beta}p_n(\omega;\beta) = \frac{1}{n}\frac{\partial}{\partial\beta}\ln Z_n(\omega;\beta) = P_n^{\beta,\omega}\big[H_n(S)/n\big],$$

which is the specific (internal) energy. Let $\mathscr{D} = \mathscr{D}(p)$ be the set of β's such that the limit p is differentiable at β. Since p is convex, it is a general fact from convex functions theory that its complement $\mathscr{D}^c$ is at most countable. In fact, p is $\mathscr{C}^1$ on $\mathscr{D}$. Since p is the limit of $p_n(\omega;\beta)$ a.s. and in the L^1-norm, we deal with a third convex function, e.g., $\mathbb{P}[p_n(\omega;\beta)]$. The following is a direct consequence of general results on convergence of convex functions and their derivatives.

Proposition 2.3 *For all $\beta \in \mathscr{D}$ and almost every environment ω,*

$$\lim_{n\to\infty} P_n^{\beta,\omega}\big[H_n(S)/n\big] = \lim_{n\to\infty} \mathbb{P}\big[P_n^{\beta,\omega}\big(H_n(S)/n\big)\big] = p'(\beta).$$

Moreover, for all $\beta \in \mathbb{R}$,

$$p'(\beta^-) \le \liminf_{n\to\infty} P_n^{\beta,\omega}\big[H_n(S)/n\big] \le \limsup_{n\to\infty} P_n^{\beta,\omega}\big[H_n(S)/n\big] \le p'(\beta^+).$$

The notation $p'(\beta^+)$ denotes the limit of $p'(b)$ as b decreases to β in $\mathscr{D}$. We could as well write bounds involving the left and right derivatives. In this section, we show that

$$\mathbb{P}[\omega(t,x)] \le p'(\beta) \le \lambda'(\beta)\ , \qquad \beta \ge 0, \beta \in \mathscr{D},$$

where the last inequality follows from (2.22) below. But the main point is that the difference $\lambda - p$ has a nice monotonicity property.

Theorem 2.3 *The functions $\beta \mapsto \lambda(\beta) - \mathbb{P}[p_n(\omega;\beta)]$ and $\beta \mapsto \lambda(\beta) - p(\beta)$ are non-decreasing on $\mathbb{R}^+$, and non-increasing on $\mathbb{R}^-$.*

Proof Recall the notation q for the law of $\omega(t,x)$ under $\mathbb{P}$. With

$$\zeta_n(S) = \exp \beta H_n(S)\ ,$$

it is straightforward to check

$$\begin{aligned}\frac{\partial}{\partial\beta}\mathbb{P}[\ln Z_n] &= \mathbb{P}\Big[\frac{\partial}{\partial\beta}\ln Z_n\Big]\\ &= \mathbb{P}[(Z_n)^{-1}\frac{\partial}{\partial\beta}Z_n]\\ &= P\left[\mathbb{P}[(Z_n)^{-1}H_n\zeta_n]\right]\end{aligned}$$

At this point, we will use the fact that independent variables are positively associated, and satisfy the Harris-FKG inequality given in the toolbox.

Fix a path $\mathbf{x}$. Then, the probability measure $\tilde{\mathbb{P}}^{\mathbf{x}}$,

$$d\tilde{\mathbb{P}}^{\mathbf{x}} = \zeta_n(\mathbf{x})e^{-n\lambda(\beta)}d\mathbb{P}$$

is product, and therefore the family ω satisfies the FKG inequality. Indeed, the variables $\omega(t,x), t \geq 0, x \in \mathbb{Z}^d$, are independent under $\tilde{\mathbb{P}}^{\mathbf{x}}$—though not identically distributed—with $\omega(t,x)$ distributed as q or $d\tilde{q}^{\beta}(w) = e^{\beta w - \lambda}dq(w)$ according to $x_t \neq x$ or $x_t = x$. Note that the function H_n is increasing in ω, while $(Z_n)^{-1}$ is a decreasing for $\beta \geq 0$. We apply Proposition A.1 for fixed $\mathbf{x}$, and we find

$$\begin{aligned}\mathbb{P}[(Z_n)^{-1}H_n\zeta_n] &\leq e^{-n\lambda(\beta)}\mathbb{P}[(Z_n)^{-1}\zeta_n] \times \mathbb{P}[H_n\zeta_n]\\ &= \mathbb{P}[(Z_n)^{-1}\zeta_n] \times n\lambda'(\beta)\end{aligned}$$

using independence and also that

$$\mathbb{P}[\omega(t,x)e^{\beta\omega(t,x)}] = \lambda'(\beta)e^{\lambda(\beta)} \ . \tag{2.21}$$

Integrating with respect to P, we get

$$\begin{aligned}\frac{\partial}{\partial\beta}\mathbb{P}\ln Z_n &\leq n\lambda'(\beta)P\left[\mathbb{P}[(Z_n)^{-1}\zeta_n]\right]\\ &= n\lambda'(\beta)\mathbb{P}\left[(Z_n)^{-1}P[\zeta_n]\right]\\ &= n\lambda'(\beta)\end{aligned} \tag{2.22}$$

which yields the desired result, since $p_n(\omega;\beta)$ and λ are both equal to zero when $\beta = 0$. □

For more information on FKG inequality, see [165, pp. 77–83].

Remark 2.6 It is proved in [188] that the r.v.'s $Z_n(\beta,\omega)e^{-n\lambda(\beta)}$ are increasing in β in the convex order. For two integrable r.v.'s X, X', we say that X is smaller than X' in convex order if

$$\mathbb{P}[\phi(X)] \leq \mathbb{P}[\phi(X')],$$

for all convex $\phi : \mathbb{R} \to \mathbb{R}$ such that the expectations exist. We write $X \leq_{cx} X'$. First, this implies that

$$\mathbb{P}[X] = \mathbb{P}[X'],$$

since both functions $\phi(x) = x$ and $\phi(x) = -x$ are convex, and furthermore,

$$\mathrm{Var}(X) \leq \mathrm{Var}(X'), \quad \mathbb{P}[|X-a|^q] \leq \mathbb{P}[|X'-a|^q] \quad (a \in \mathbb{R}, q \geq 1).$$

Since they have same expectation, this is a natural quantitative way to express that X is less dispersed than X'.
In our case, the process $\beta \mapsto Z_n(\beta,\omega)e^{-n\lambda(\beta)}$ is increasing for the convex order. It is a mathematical formulation of the intuitive property that

the fluctuations of $Z_n(\beta,\omega)e^{-n\lambda(\beta)}$ increase as β grows.

An increasing process for the convex order is called a peacock. It is well known that it has the same marginals as some martingale: There exists a martingale $(M(\beta), \beta \in \mathbb{R}_+)$ such that for all β,

$$Z_n(\beta,\omega)e^{-n\lambda(\beta)} \overset{\text{law}}{=} M(\beta).$$

(Of course, M depends also on n.) Such a martingale M is explicitly known in a few examples (Gaussian or Bernoulli environment), but not in general. See [188] for more details.

2.5 Phase Transition

In this section, we elaborate a phase diagram based on whether the annealed bound (2.13) is achieved or not.

Theorem 2.4 (Critical Temperature) *There exists* $\beta_c = \beta_c(\mathbb{P}, d) \in [0,\infty]$ *such that*

$$\begin{cases} p(\beta) = \lambda(\beta) & \text{if } 0 \le \beta \le \beta_c, \\ p(\beta) < \lambda(\beta) & \text{if } \beta > \beta_c \end{cases} \tag{2.23}$$

Proof Define

$$\beta_c = \inf\{\beta \ge 0 : p(\beta) < \lambda(\beta)\}. \tag{2.24}$$

The claim is a direct consequence of Proposition 2.3. □

The terminology in the next definition becomes transparent.

Definition 2.1 We call *high temperature region* (or small β region) the set of β's such that $p = \lambda$, and the *low temperature region* (or large β region) the set of β's such that $p < \lambda$.

We can already make a few observations:

1. We have trivially $p(0) = 0 = \lambda(0)$, showing that $\beta = 0$ is in the high temperature region.
2. We have seen sufficient conditions for $\beta_c < \infty$, e.g., condition (T) in (2.17).

3. Theorem 2.4 implies the absence of reentrant phase transition in the phase diagram of the model. Of course, in complete generality, we may have $\beta_c = 0$ or ∞, i.e., absence of one of the two regimes in the interval $(0, \infty)$.

One expects that the polymer measure has completely different behavior in these two regions. In the high temperature region, the Gibbs measure is a small perturbation of the simple random walk. At low temperature, the polymer strongly feels the environment, and should have quite different scaling limits.

Breaking the Analyticity Recall that a function is analytic at a point β_0 in the interior of its domain if it is equal in a neighborhood of β_0 to the sum of a power series in $\beta - \beta_0$ with a positive radius of convergence. For instance, λ is analytic in $\mathbb{R}$. Recall that $\beta \mapsto p_n(\omega; \beta)$ is infinitely differentiable; since it is the logarithm of a finite sum of smooth terms, it is easy to see that it is even analytic for all n. However, the limit p may not be analytic.

If β_c is strictly positive and finite, both high and low temperature regions have a nonempty interior, the function p is nonanalytic at $\beta = \beta_c$. (Indeed, it is given by $p(\beta) = \lambda(\beta)$ for $\beta \in [0, \beta_c]$, which analytic continuation on $\mathbb{R}$ is $\lambda(\beta)$, under the assumption (1.1).) The value β_c is called **critical**.

Physical Picture In physics, a phase transition is the passage of a thermodynamic system from one phase to another. The distinguishing characteristic of a phase transition is an abrupt change in one or more physical properties, in particular the heat capacity, with a small change in a thermodynamic variable such as the temperature. To keep our discussion simple, we take some freedom with statistical mechanics, considering the energy H_n as only observable, and β as the only parameter.

- The function p is analytic at a point β_0 if it is the sum of a power series in $\beta - \beta_0$ with a positive radius of convergence.
- If p is non-analytic at β_0, β_0 is called a critical point.
- Ehrenfest proposed a classification scheme, grouping phase transitions based on the degree of non-analyticity: phase transitions are labelled by the lowest derivative of the free energy that is discontinuous at the transition. First-order phase transitions exhibit a discontinuity in the first derivative of the free energy (with respect to some thermodynamic variable).

Important questions are:

- What phenomenon is responsible for the phase transition? In what respect the polymer measures below and above the critical value are different? If we observe a sample of paths from the Gibbs measure, can we decide if $\beta > \beta_c$ or not?
- What happens at the critical value? Critical phenomena are believed to depend on very few details of the model, which are then grouped into universality classes. Loosely speaking, an universality class is the collection of all models sharing the same scalings and/or the same limits at criticality.

We anticipate the next chapters and mention at this point that high temperature corresponds to the delocalized phase, and low temperature to the localized phase. We conclude this section with a simple phase diagram, depicting phases according to the parameter value β, that we restrict to $[0, +\infty)$ for the case of negative β is covered by changing the environment ω into $-\omega$:

β	0	β_c $+\infty$
Temperature	High temperature	Low temperature
Free energy	$p = \lambda$	$p < \lambda$
Phase	Delocalized	Localized

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