

Chapter 2

Applications of the Mechanics of a String

“Books on dynamics usually include a number of problems on the motion of chains. Though hardly important in themselves, such problems furnish excellent illustrations of dynamical principles.”

H. Lamb [193, Page 142].

2.1 Introduction

In this chapter we present tractable applications of the theory developed in Chapter 1. Our primary aim is to demonstrate a systematic procedure to establish and analyze the motion of a string using the balance laws. We start with the simplest dynamic problem of a steady axial motion of a string. After exploring recent results for this problem, we then turn to examining a classic set of chain problems. An example of the latter class of problems is shown in Figure 2.1. All but one of the examples we explore are adopted from classic texts in dynamics and mechanics. The chain is modeled as a heavy inextensible string and we shall find that the string is often in states of rectilinear motion which are separated

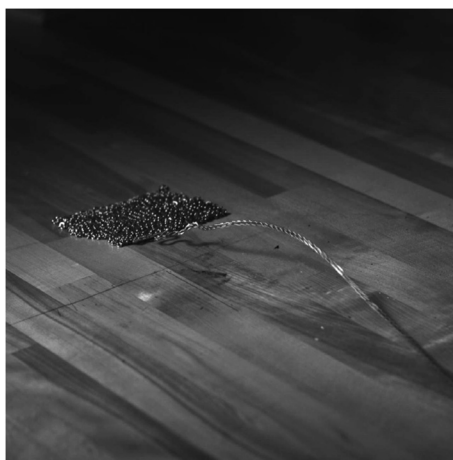


Fig. 2.1 A link chain which is subject to a steady pull from the right (out of frame) emerges from a stationary pile and forms an arch-like shape. Image courtesy of James Hanna and Wes Royston.

by shocks where the velocity vector \mathbf{v} is discontinuous (cf. Figure 2.2). The methodology we use to arrive at the equations of motion contrasts to the semi-inverse method used in textbooks. With the help of the balance laws for material momentum and linear momentum along with some kinematical considerations, we demonstrate a systematic framework with which to arrive at the governing equations of motion for these problems. The resulting equations of motion are then analyzed and several interesting features of the individual problems are explored. The set of problems we consider is far from exhaustive and some additional examples are explored in the exercises at the end of this chapter.

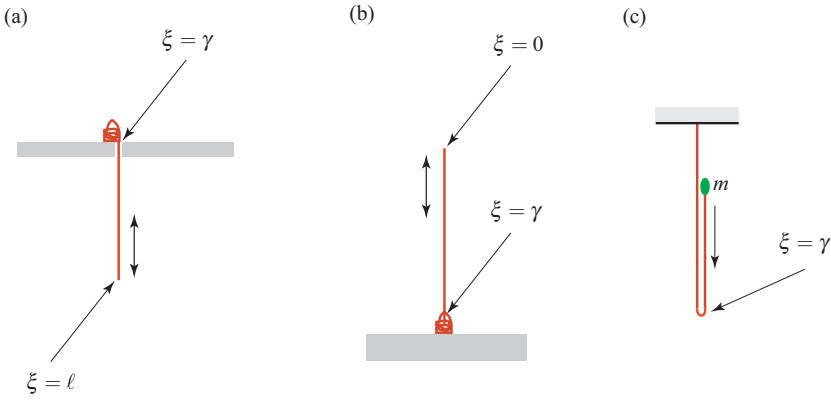


Fig. 2.2 Three singularities commonly encountered in classic problems in the dynamics of inextensible strings. (a) A string issuing from a heap, (b) a string descending into or being drawn from a pile, and (c) a string with a fold.

Typically, the class of problems we consider involve impacting segments of the chain - especially when a stationary heap of chain is present in the problem. In our model for the chain, which is an inextensible, perfectly flexible string, the dissipation associated with the impact is modeled using the material momentum B_γ and the associated source of power Φ_{E_γ} at a point of discontinuity $\xi = \gamma$. We find that we will need to prescribe B_γ and, based on experimental results in the literature, some of the constitutive quantities associated with the prescription for B_γ will need to be found from experiments. In this chapter, the prescription we use is

$$B_\gamma = -2e\rho_0\dot{\gamma}^2 \frac{\dot{\gamma}}{|\dot{\gamma}|},$$

where e is a constant which must be determined by experiments. We also take this opportunity to acknowledge that our exposition in this chapter was greatly facilitated by the recent experimental works of Biggins [23], Grewal et al. [142], and Hamm and Géminard [151]. These works provided invaluable insights and helpful

experimental results. We also note that many of the paradoxes and much of the controversy associated with this class of problems stems from the inappropriate use of energy conservation.¹

2.2 Steady Axial Motions of Elastic Strings and Inextensible Strings

In a class of motions of strings, which occupies a central role in many applications, the string only moves along its length (cf. Figures 1.2 and 2.3). Such motions are known as axial motions. For motions of this type which are classified as steady axial motions, the following three conditions hold²:

- (i) The material curve \mathcal{L} of the string moves along a fixed space curve.
- (ii) Suppose the position vector \mathbf{c} of any point on the space curve is uniquely identified by the arc-length parameter χ (i.e., $\mathbf{c} = \mathbf{c}(\chi)$), then the velocity vector \mathbf{v} of a material point of \mathcal{L} is a function of χ only.
- (iii) The vector \mathbf{r}' is a function of χ only.

It has been proven for both inextensible and elastic strings that these three conditions are equivalent to the statement that the position vector \mathbf{r} of a material point of the string has the representation³

$$\mathbf{r} = \tilde{\mathbf{r}}(\chi) = \mathbf{r}(\xi, t) \quad (2.1)$$

where $\chi = \xi + ct$ and c is a constant (which has units of speed).

For steady axial motions, straightforward application of the chain rule can be used to show that

$$\frac{\partial \tilde{\mathbf{r}}}{\partial \chi} = \frac{\partial \mathbf{r}}{\partial \xi}, \quad \mathbf{v} = \dot{\mathbf{r}} = c \frac{\partial \tilde{\mathbf{r}}}{\partial \chi}, \quad \mathbf{a} = \ddot{\mathbf{r}} = c^2 \frac{\partial^2 \tilde{\mathbf{r}}}{\partial \chi^2}. \quad (2.2)$$

Thus, the speed of a material point $v = \|\mathbf{v}\| = \mu |c|$. From the compatibility condition $\left[[\mathbf{v} + \dot{\gamma} \mathbf{r}'] \right]_{\gamma} = \mathbf{0}$ for a steady motion, we can conclude that

$$\dot{\gamma} = -c. \quad (2.3)$$

We also find, from the definition of \mathbf{v}_{γ} (see Eqn. (1.54)), that during a steady motion $\mathbf{v}_{\gamma} = \mathbf{0}$. In other words, the point of discontinuity (if one exists) is stationary - as expected. Because c is a constant for a steady axial motion, only one of the examples

¹ Our perspective on this matter is influenced by the works of Troger and his coworkers [315, 331] and statements on the lack of energy conservation in some chain problems in the textbooks authored by Lamb [193] and Love [212].

² These conditions were first enunciated in a work by Green and Laws [129] on ideal jets of fluid.

³ The proof is presented in [250].

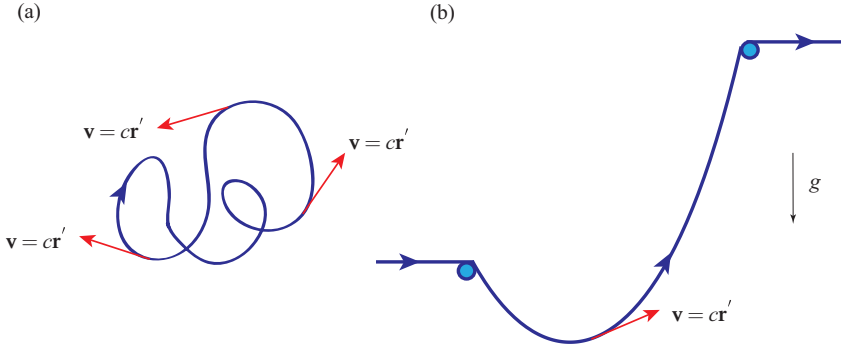


Fig. 2.3 Two representative examples of steady axial motions of strings. (a) A closed loop of string propagating along its length studied by Healey and Papadopoulos [160] and (b) the axially moving catenary studied by Perkins and Mote [287, 288].

that we consider in the sequel qualifies as an example of a steady axial motion. This example, which is discussed in Section 2.8, is known as the chain fountain and involves an inverted catenary.

The representation (2.1) has been used for inextensible strings dating to the mid-19th century (cf. [305, Sections 594 & 595]) and for extensible strings it was first used by Healey and Papadopoulos [160]. It has since presented itself in a large number of studies on the dynamics of strings and rods (cf., e.g., [63, 65, 157, 279] and references therein). One of the reasons for the popularity of these motions lies in a result noted by Routh [305, Sections 594 & 595] for inextensible strings which has since been applied to extensible rods and strings.

To see Routh's result and its extensions, we consider the balance of linear momentum (1.115) for a homogeneous string undergoing a steady axial motion:

$$\mathbf{n}' + \rho_0 \mathbf{f} = \rho_0 \dot{\mathbf{v}}. \quad (2.4)$$

Appealing to Eqn. (2.1) and noting that

$$\frac{\partial f(\xi, t)}{\partial \xi} = \frac{\partial \tilde{f}(\chi, t)}{\partial \chi}, \quad \rho_0 = \mu \rho, \quad \mathbf{n} = n \mathbf{e}_t = \frac{n}{\mu} \frac{\partial \mathbf{r}}{\partial \xi} = \frac{n}{\mu} \frac{\partial \tilde{\mathbf{r}}}{\partial \chi}, \quad (2.5)$$

we can write the balance law as follows:

$$\frac{\partial}{\partial \chi} \left((n - \rho_0 \mu c^2) \frac{1}{\mu} \frac{\partial \tilde{\mathbf{r}}}{\partial \chi} \right) + \rho_0 \mathbf{f} = \mathbf{0}. \quad (2.6)$$

Thus, it is possible for the shape taken by the string performing a steady axial motion to be identical to that taken by a stationary string provided the tension n is replaced by $n - \rho_0 \mu c^2$. For an inextensible string, this result was proven by Routh, while for an elastic string it is the extension to Routh's theorem published by Healey and Papadopoulos [160] nearly a century later.

Referring to the summary of the governing equations in Section 1.7, we note that several of the jump conditions (1.116) remain to be discussed for the aforementioned steady axial motion. For the motions of interest, no sources of linear or material momentum are assumed to be present: $\mathbf{F}_\gamma = \mathbf{0}$ and $\mathbf{B}_\gamma = 0$. Thus, the jump conditions (1.116) imply that

$$[(n - \rho_0 \mu c^2) \mathbf{e}_t]_\gamma = \mathbf{0}, \quad [[\rho_0 \psi - \mu (n - \rho_0 \mu c^2)]]_\gamma = 0. \quad (2.7)$$

We conclude from this pair of conditions that if $n \neq \rho_0 \mu c^2$, then the tangent vector \mathbf{e}_t varies continuously along the material curve and that, depending on ψ , the stretch μ might be continuous. For an inextensible string, one can deduce that n is continuous and that, provided $n \neq \rho_0 c^2$, \mathbf{e}_t must be continuous.

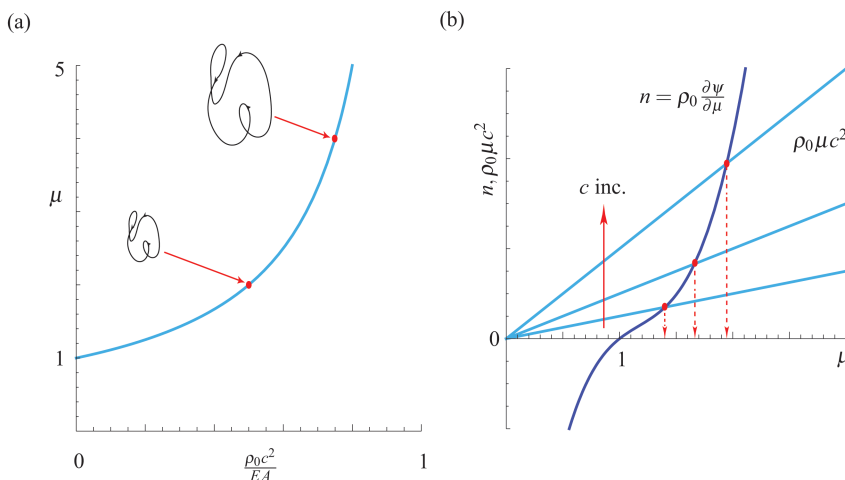


Fig. 2.4 (a) The stretch μ as a function of $\frac{\rho_0 c^2}{EA}$ for a steady axial motion of a string whose strain energy function is $\rho_0 \psi = \frac{EA}{2} (\mu - 1)^2$ (cf. Eqn. (2.9)). The inset images are representative steady axial motions of loops of string. These solutions are stable according to the stability criterion discussed on Page 56. (b) Schematic of a graphical solution procedure for the equation $n = \rho_0 \mu c^2$ (cf. Eqn. (2.8)) to determine the stretch μ in an steady axial motion of a closed loop of elastic string as the parameter c varies. The strain energy function shown in the figure is identical to one shown in Figure 1.14.

2.2.1 Closed Loops of String

As a specific example, consider the case where $\mathbf{f} = \mathbf{0}$. If the string is inextensible, then the string can take the form of *any* closed loop provided $n = \rho_0 c^2$, a result which dates to Routh [305]. Similarly, as shown more recently by Healey and

Papadopoulos [160], an elastic string can take the form of *any* closed loop provided the stretch μ satisfies

$$\rho_0 \frac{\partial \psi}{\partial \mu} = \rho_0 \mu c^2. \quad (2.8)$$

That is, $n - \rho_0 \mu c^2 = 0$. For the elastic string with the simple strain energy function $\rho_0 \psi = \frac{EA}{2} (\mu - 1)^2$ mentioned earlier (cf. Eqn. (1.100)), Eqn. (2.8) can be solved analytically:

$$\mu = \frac{EA}{EA - \rho_0 c^2}. \quad (2.9)$$

This solution is shown in Figure 2.4(a). We observe that as $c^2 \rightarrow \frac{EA}{\rho_0}$ the stretch will become infinite. However, the speed in this case approaches the speed of longitudinal waves in the string and the constitutive relation for $\rho_0 \psi$ is questionable for such high stretches.⁴ For more complex strain energy functions, it is easy to conceive a graphical representation of Eqn. (2.8) for a given ψ (cf. Figure 2.4(b)). In the example shown, there is a unique μ for every c . Furthermore as c increases, the stretch in the axial motion increases from its value of 1 when $c = 0$.

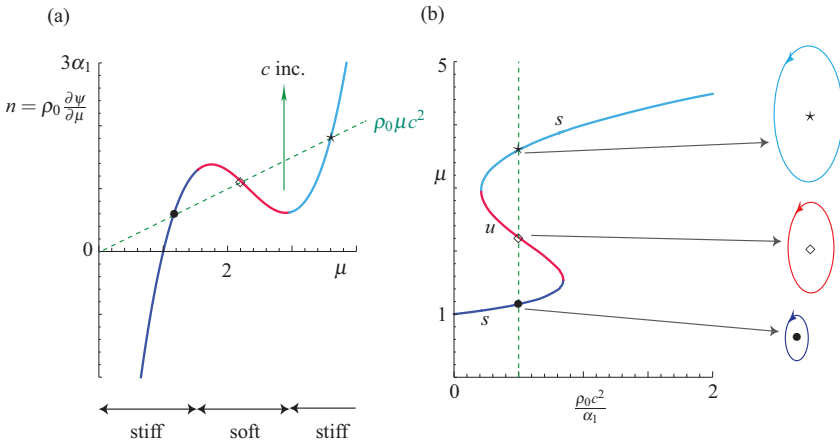


Fig. 2.5 (a) The tension n as a function of the stretch μ for a specific strain energy function given by Eqn. (2.10). (b) Solutions $\mu(\rho_0 c^2)$ of Eqn. (2.14) which determine the stretch μ in a steady axial motion of an elastic string as the parameter c varies. Observe the pair of saddle-node bifurcations as $\rho_0 c^2$ is varied. At these bifurcations, the number of possible solutions changes from 1 to 3 and 3 to 1, respectively. The labels s and u on this figure denote stable and unstable steady motions and the inset images are representative images of the simplest possible axial motions.

⁴ For a string modeling a three-dimensional linearly elastic body with a cross-sectional area A and mass density per unit volume ρ_0^* , we find that $\rho_0 = \rho_0^* A$ and so $\frac{EA}{\rho_0} = \frac{E}{\rho_0^*}$. We also recall that $\sqrt{\frac{E}{\rho_0^*}}$ is the propagation speed for longitudinal waves in an infinitely long bar [346, Section 168].

As a further example, consider the strain energy function that produces the function $n(\mu)$ shown in Figure 2.5(a).⁵ Here,

$$\rho_0 \psi = \alpha_1 \left(\frac{1}{4} (\mu - \alpha_2)^4 + \mu \left(1 + \alpha_2 - \frac{\mu}{2} \right) \right), \quad (2.10)$$

where α_1 and α_2 are constants. The strain energy function is a simplified version of the function which arises in Exercise 1.9. A differentiation of (2.10) with respect to μ shows that

$$\begin{aligned} n &= \rho_0 \frac{\partial \psi}{\partial \mu} \\ &= \alpha_1 \left((\mu - \alpha_2)^3 + 1 + \alpha_2 - \mu \right). \end{aligned} \quad (2.11)$$

Because we assume that the force n is zero when the string is unstretched, α_2 is the real root of the equation $(1 - \alpha_2)^3 + \alpha_2 = 0$:

$$\alpha_2 \approx 2.32472 \quad (2.12)$$

We also note that the stiffness of the string when $\mu = 1$ can be found from the second partial derivative of μ ⁶:

$$\begin{aligned} \rho_0 \frac{\partial^2 \psi}{\partial \mu \partial \mu} &= \alpha_1 \left(3(\alpha_2 - 1)^2 - 1 \right) \\ &= 4.26463 \alpha_1. \end{aligned} \quad (2.13)$$

For a given c , the stretch for a steady motion is obtained by solving Eqn. (2.8):

$$\alpha_1 \left((\mu - \alpha_2)^3 + 1 + \alpha_2 - \mu \right) = \rho_0 \mu c^2. \quad (2.14)$$

Depending on the value of $\rho_0 c^2$, it is possible for 1, 2, or 3 distinct values of μ to exist. Consequently, as shown in Figure 2.5(b), for a given closed loop of string, multiple axial motions are possible and the three motions are scaled versions of each other. The values of $\rho_0 c^2$ where the number of solutions changes are known as critical values. For the example shown, these critical values are $\rho_0 c^2 \approx 0.2099 \alpha_1$ and $\rho_0 c^2 \approx 0.845 \alpha_1$ and the bifurcation that occurs at these points is often referred to as a saddle-node bifurcation.

⁵ The functional form of $n(\mu)$ with its local minimum and local maximum is motivated by constitutive relations for one-dimensional continua that are used in studies on phase transformations by Abeyaratne and Knowles [2, 5], Ericksen [98], and Purohit and Bhattacharya [296] (among many others).

⁶ The corresponding stiffness for the strain energy function $\rho_0 \psi = \frac{EA}{2} (\mu - 1)^2$ is EA .

It is interesting to examine the balance of material momentum (1.72) and balance of energy (1.74) applied to the closed loop of string⁷:

$$\frac{d}{dt} \oint \rho_0 \mathbf{v} \cdot \frac{\partial \mathbf{r}}{\partial \xi} d\xi = 0, \quad (2.15)$$

and

$$\frac{d}{dt} \oint \left(\frac{1}{2} \mathbf{v} \cdot \mathbf{v} + \psi \right) \rho_0 d\xi = 0. \quad (2.16)$$

Thus, we observe that two kinematic quantities are conserved: the flow or integral of the circulation $\rho_0 \mathbf{v} \cdot \frac{\partial \mathbf{r}}{\partial \xi}$ and the integral of the energy density. These conserved quantities were used by Healey [157] to establish stability criteria for steady axial closed loops of string where the perturbations conserved both the total energy and the integral of the material momentum. Specifically, he showed that a nonlinear stability criterion could be established when the string was stiff. A string is termed stiff at $\mu = \mu_0$ if the constitutive relations satisfy the following inequality and is otherwise termed soft:

$$\rho_0 \frac{\partial^2 \psi}{\partial \mu^2} (\mu_0) - \frac{\rho_0}{\mu_0} \frac{\partial \psi}{\partial \mu} (\mu_0) > 0. \quad (2.17)$$

Thus, for the constitutive relation shown in Figure 2.5(a), the string is soft when $\mu \in [1.54, 2.96]$ and is otherwise stiff. With the help of these observations, the stability results presented in Figure 2.5(b) can be easily established. We have taken the liberty of defining the solutions where the criterion (2.17) is violated as unstable even though a formal proof of instability is not known to us. By way of contrast, the string discussed in Figure 2.4(b) is stiff and the steady motions shown in Figure 2.4(a) are stable.

2.3 Inextensible Strings with Shocks

Consider the motions of strings shown in Figure 2.2. All three examples involve inextensible strings performing piecewise rectilinear motions. In the remainder of this chapter, our goal is to establish the governing equations of motion and to solve for the motion of the string. Here, we summarize the equations needed to establish the equations of motion for the series of examples that follow.

⁷ For the homogeneous strings of interest here, $\oint \rho_0 \mathbf{v} \cdot \frac{\partial \mathbf{r}}{\partial \xi} d\xi = \rho_0 \oint \mathbf{v} \cdot \mathbf{e}_t ds$, and thus, the material momentum \mathbf{P} and its integral can be related to the circulation $\mathbf{v} \cdot \mathbf{e}_t$ and “flow” that were first defined by Kelvin in 1869 and play a prominent role in fluid mechanics. For further details on this topic, the reader is referred to [52, 194, 349].

2.3.1 General Considerations

For an inextensible string, the stretch $\mu = 1$ and consequently, $\frac{\partial \mathbf{r}}{\partial \xi} = \mathbf{e}_t$. For many of the problems of interest in this chapter, $\mathbf{v} = v\mathbf{E}$ where \mathbf{E} is a (piecewise) constant unit vector that is equal to \mathbf{e}_t . As $\mathbf{v}' \cdot \mathbf{e}_t = 0$ for inextensible strings (see Eqn. (1.40)), it follows that $v = v(t)$. Further, from the definition (1.54) of \mathbf{v}_γ , we can conclude that

$$\begin{aligned} \mathbf{v}_\gamma &= ((v + \dot{\gamma})\mathbf{E})^+ = ((v + \dot{\gamma})\mathbf{E})^-, \\ [[v\mathbf{E} + \dot{\gamma}\mathbf{E}]]_\gamma &= \mathbf{0}. \end{aligned} \quad (2.18)$$

These results are very useful for solving boundary-value problems.

We will find it convenient to use the balance of material momentum (1.83) to establish conservation laws. For an inextensible string, $\mathbf{n} = n\mathbf{e}_t$ and the strain energy function $\psi = 0$. Consequently, the material contact force \mathbf{C} is

$$\begin{aligned} \mathbf{C} &= -\frac{\rho_0}{2} \mathbf{v} \cdot \mathbf{v} - n \\ &= -\frac{\rho_0}{2} v^2 - n. \end{aligned} \quad (2.19)$$

In this chapter, we shall assume that ρ_0 is a continuous function of ξ and this implies that the prescription (1.112) for b_p simplifies considerably. As a result, Eqn. (1.83) now reads:

$$-\rho_0 \mathbf{f} \cdot \mathbf{r}' = \dot{\mathbf{p}} - \frac{\partial \mathbf{C}}{\partial \xi}. \quad (2.20)$$

The local balance law which is neither trivially nor identically satisfied comes from the balance of linear momentum:

$$\mathbf{n}' + \rho_0 \mathbf{f} = \rho_0 \ddot{\mathbf{r}}. \quad (2.21)$$

We emphasize that the material momentum balance (2.20) is identically satisfied by the solutions to Eqn. (2.21) and it finds application in establishing conservation laws (such as (2.99) for the chain fountain problem).

The jump conditions which need to be satisfied are obtained from Eqn. (1.86):

$$\begin{aligned} [[\mathbf{n} + \rho_0 v \dot{\gamma}]]_\gamma + \mathbf{F}_\gamma &= \mathbf{0}, \\ \left[\left[\mathbf{C} - \dot{\gamma} \rho_0 \mathbf{v} \cdot \mathbf{r}' \right] \right]_\gamma + \mathbf{B}_\gamma &= \mathbf{0}. \end{aligned} \quad (2.22)$$

As emphasized earlier in Section 1.5.5, the jump conditions for angular momentum and energy provide identities for \mathbf{M}_{O_γ} and Φ_{E_γ} :

$$\begin{aligned}\mathbf{M}_{O_\gamma} - \mathbf{r}(\gamma, t) \times \mathbf{F}_\gamma &= \mathbf{0}, \\ \mathbf{F}_\gamma \cdot \mathbf{v}_\gamma + B_\gamma \dot{\gamma} &= \Phi_{E_\gamma}.\end{aligned}\quad (2.23)$$

In the sequel, we will often use the identity $(2.23)_2$ to establish Φ_{E_γ} given \mathbf{F}_γ and B_γ . The force \mathbf{F}_γ will represent an impulse on the string at a point where there is an abrupt change in its motion and B_γ will represent an energy loss associated with impacts of the links of the chain that the string is modeling. We shall find that the following prescription for B_γ is useful for problems where the string's motion has an abrupt change in speed:

$$B_\gamma = -2e\rho_0\dot{\gamma}^2 \frac{\dot{\gamma}}{|\dot{\gamma}|}. \quad (2.24)$$

While the constant e in this expression has a value of $e = 0.25$ for many classic formulations, recent experimental results by Biggins [23] and Hamm and Géminard [151] have lead us to believe that e can have values lower than this ideal number depending on the type of chain that the string is modeling and the nature of the shock that the string is experiencing. The chains in question are primarily of two types: a link chain or a bead chain (cf. Figure 2.6). The latter type of chain is sometimes known as a ball chain. Among others, we anticipate that the value of e will depend on the nature of the connections between the segments of the chain. In all of the examples considered in the sequel, there will be several points where singular supplies are present and so the subscript β in \mathbf{F}_β , B_β , and Φ_{E_β} will be used to distinguish the point $\xi = \beta$ where the supply acts.

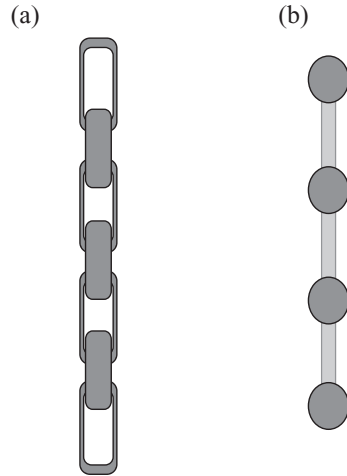


Fig. 2.6 Two types of chain: (a) a link chain and (b) a bead chain.

2.3.2 Boundary Conditions, Sources, Sinks, and Reservoirs

A primary use of the jump conditions (which is hard to find a published discussion of) is to establish boundary conditions for terminally loaded strings. For instance, suppose a force $-P_0\mathbf{E}_1$ is applied at the end $\xi = 0$ of a string and a force $P_1\mathbf{E}_1$ is applied at the end $\xi = \ell$. Questions arise as to what precisely $\mathbf{n}(0^+, t)$ and $\mathbf{n}(\ell^-, t)$ are for these loadings. To answer such questions, we can appeal to the jump condition $(2.22)_1$, $[[\mathbf{n} + \rho_0\mathbf{v}\dot{\gamma}]]_\gamma + \mathbf{F}_\gamma = \mathbf{0}$, and apply this condition twice, once at each end.

To elaborate, let us consider the end $\xi = 0$. As the string terminates at this point, $\mathbf{n}(0^-, t)$ and $\mathbf{v}(0^-, t)$ are nonexistent. Further, because $\gamma = 0$ which is fixed, $\dot{\gamma} = 0$. The last ingredient is to model the applied force by \mathbf{F}_0 : $\mathbf{F}_0 = -P_0\mathbf{E}_1$. Now invoking the jump condition (2.22)₁, we find

$$\mathbf{n}(0^+, t) = -\mathbf{F}_0 = P_0\mathbf{E}_1. \quad (2.25)$$

We typically drop the $+$ ornamenting the 0 and simply write: $\mathbf{n}(0, t) = P_0\mathbf{E}_1$. For the end $\xi = \ell$, we follow the same procedure but now $\mathbf{n}(\ell^+, t)$ and $\mathbf{v}(\ell^+, t)$ are nonexistent. Using Eqn. (2.22)₁ with $\gamma = \ell$ and $\mathbf{F}_\ell = P_1\mathbf{E}_1$, we find

$$\mathbf{n}(\ell^-, t) = \mathbf{F}_\ell = P_1\mathbf{E}_1. \quad (2.26)$$

If we now consider a string of length ℓ which is loaded by equal and opposite forces, $P_0 = P_1$, then this procedure leads to the satisfying conclusion that the tension in the string is constant: $\mathbf{n} = P_0\mathbf{E}_1$.

Several problems in the dynamics of heavy inextensible strings (also sometimes known as chains) involve a string being drawn from a heap of quiescent string (cf. the examples shown in Figure 2.2(a) & (b)). We often refer to the heap as a source, sink, or reservoir; others refer to the heap shown in Figure 2.2(a) as a top pile and the heap shown in Figure 2.2(b) as a bottom pile. To examine the boundary conditions at the exit point of the string from the heap, two cases need to be considered. The first, an example of which is shown in Figure 2.2(a), arises when the parameterization of ξ is such that γ decreases as an increasing amount of string is pulled from the heap. In this case, we denote the end point of the string by $\xi = \gamma^+$, and assume that the heap exerts a force \mathbf{F}_γ and a material force \mathbf{B}_γ on the string. The jump conditions at the point where the string exits the heap then read

$$\begin{aligned} \mathbf{n}(\gamma^+, t) + \rho_0\mathbf{v}(\gamma^+, t)\dot{\gamma} &= -\mathbf{F}_\gamma, \\ C(\gamma^+, t) - \dot{\gamma}\rho_0\mathbf{v}(\gamma^+, t) \cdot \mathbf{r}'(\gamma^+, t) &= -\mathbf{B}_\gamma, \end{aligned} \quad (2.27)$$

where

$$C(\gamma^+, t) = -\frac{\rho_0}{2}\mathbf{v}(\gamma^+, t) \cdot \mathbf{v}(\gamma^+, t) - n(\gamma^+, t). \quad (2.28)$$

The identity $\Phi_{E_\gamma} = \mathbf{B}_\gamma\dot{\gamma} + \mathbf{F}_\gamma \cdot \mathbf{v}_\gamma$ is used to identify if the source is dissipative, energetic, or energy conserving. Note that our implementation of the jump conditions (2.27) is tantamount to assuming that the string is stationary and slack in the heap: $\mathbf{n}(\gamma^-, t) = \mathbf{0}$ and $\mathbf{v}(\gamma^-, t) = \mathbf{0}$. We will shortly explore an example showing how the conditions (2.27) are used in Section 2.4.

The complementary case to the heap discussed above occurs when the parameterization of ξ is such that γ increases as an increasing amount of string is pulled from the heap (i.e., the bottom pile). An example of this situation is shown in Figure 2.2(b). We proceed as before to examine the boundary conditions on the string at the point $\xi = \gamma^-$ and assume that the heap exerts a force \mathbf{F}_γ and a material force \mathbf{B}_γ on the string. Applying the jump conditions in a similar manner to our earlier work, we find the following pair of conditions:

$$\begin{aligned} \mathbf{n}(\gamma^-, t) + \rho_0 \mathbf{v}(\gamma^-, t) \dot{\gamma} &= \mathbf{F}_\gamma, \\ C(\gamma^-, t) - \dot{\gamma} \rho_0 \mathbf{v}(\gamma^-, t) \cdot \mathbf{r}'(\gamma^-, t) &= B_\gamma, \end{aligned} \quad (2.29)$$

where

$$C(\gamma^-, t) = -\frac{\rho_0}{2} \mathbf{v}(\gamma^-, t) \cdot \mathbf{v}(\gamma^-, t) - n(\gamma^-, t). \quad (2.30)$$

As with the case discussed earlier, the identity $\Phi_{E_\gamma} = B_\gamma \dot{\gamma} + \mathbf{F}_\gamma \cdot \mathbf{v}_\gamma$ is also very helpful in determining the energetic properties of this heap. We close by remarking that the most celebrated problem associated with a heap of this type involves lowering a string into a pile atop a horizontal table. As noted by Ruina et al. [142] this problem involves chains which are “sucked” into the pile which we shall explore in Section 2.6.

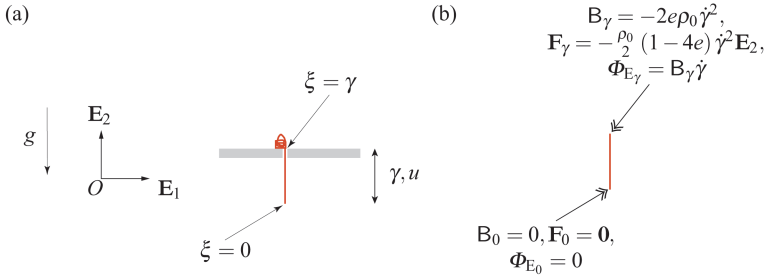


Fig. 2.7 (a) Schematic of a chain falling from a hole on a horizontal surface or table and (b) summary of the singular supplies acting on the chain. The chain is supplied from a heap which lies at rest on the surface. In classic formulations of this problem, the parameter $e = 0.25$.

2.4 Cayley’s Problem

In an influential work [53] by the celebrated English mathematician Arthur Cayley (1821–1895), he presented the equations of motion for a chain that issues from a quiescent heap and falls freely under gravity over the edge of a table. If u denotes the length of chain that is below the table surface, then Cayley established the following differential equation governing $u(t)$:

$$u(\ddot{u} - g) = -\dot{u}^2. \quad (2.31)$$

A representation of Cayley’s problem is shown in Figure 2.7. This problem has attracted much attention in the past 150 years and we refer the interested reader to [142, 171, 366, 367] for additional references and perspectives, some of which we will discuss below.

Referring to Figure 2.7(a), a segment of string of length γ is falling under gravity. A discontinuity is present where the string moves over the edge of the table. At this point, which we label as $\xi = \gamma$, in addition to possible supplies of power Φ_{E_γ} and material momentum B_γ , a reaction force \mathbf{F}_γ acts on the string.

As the shock is stationary, $\mathbf{v}_\gamma = \mathbf{0}$, and we also note that $\mathbf{r}'(\gamma^-, t) = \mathbf{E}_2$. Using the definition of \mathbf{v}_γ , we find that

$$\mathbf{v}(\gamma^-, t) = -\dot{\gamma}\mathbf{E}_2. \quad (2.32)$$

In the sequel, it is convenient to define a variable u :

$$u = \gamma, \quad \dot{u} = \dot{\gamma}. \quad (2.33)$$

Thus, for the section of chain that is freely falling,

$$\mathbf{v} = -\dot{u}\mathbf{E}_2. \quad (2.34)$$

A body force $-\rho_0 g \mathbf{E}_2$ acts on the string and so the balance of linear momentum (2.22) with $\mathbf{n} = n\mathbf{E}_2$ reads

$$(n\mathbf{E}_2)' - \rho_0 g \mathbf{E}_2 = -\rho_0 \ddot{u} \mathbf{E}_2. \quad (2.35)$$

We solve this equation for n using the boundary condition that $\mathbf{n}(0) = \mathbf{0}$:

$$n(\gamma^-, t) = -\rho_0 (\ddot{u} - g) u. \quad (2.36)$$

The tension $n(\gamma^-, t)$ is unknown and we now invoke the jump conditions (2.29) at the heap:

$$\begin{aligned} n(\gamma^-, t) - \rho_0 \dot{u}^2 &= \mathbf{F}_\gamma \cdot \mathbf{E}_2, \\ \frac{\rho_0}{2} \dot{u}^2 - n(\gamma^-, t) &= B_\gamma, \end{aligned} \quad (2.37)$$

where we used the substitution $C(\gamma^-, t) = -\frac{\rho_0}{2} \dot{u}^2 - n(\gamma^-, t)$.

To proceed with the classic formulation, we assume that the heap does not exert a force on the string:

$$\mathbf{F}_\gamma = \mathbf{0}. \quad (2.38)$$

Whence, the tension $n(\gamma^-, t)$ along with the supplies B_γ and Φ_{E_γ} are

$$\begin{aligned} n(\gamma^-, t) &= \rho_0 \dot{u}^2, \\ B_\gamma &= \frac{\rho_0}{2} \dot{u}^2 - n(\gamma^-, t) = -\frac{\rho_0}{2} \dot{u}^2, \\ \Phi_{E_\gamma} &= \mathbf{F}_\gamma \cdot \mathbf{v}_\gamma + B_\gamma \dot{\gamma} = -\frac{\rho_0}{2} \dot{u}^3. \end{aligned} \quad (2.39)$$

We used the identity (2.23)₂ to establish the result for Φ_{E_γ} . A graphical summary of the singular supplies acting on the chain is presented in Figure 2.7(b).

As an alternative to the classic formulation, we assume that the material momentum supply acting at $\xi = \gamma$ is

$$B_\gamma = -2e\rho_0\dot{\gamma}^2 \frac{\dot{\gamma}}{|\dot{\gamma}|}, \quad (2.40)$$

where e is a constant which is determined by experiment. With this prescription for B_γ , along with the assumption that $\dot{\gamma} = \dot{u} > 0$, we find from Eqns. (2.37) and (2.23)₂ that

$$\begin{aligned} n(\gamma^-, t) &= \frac{\rho_0}{2} (1 + 4e) \dot{u}^2, \\ \mathbf{F}_\gamma &= -\frac{\rho_0}{2} (1 - 4e) \dot{u}^2 \mathbf{E}_2, \\ \Phi_{E_\gamma} &= -2e\rho_0 \dot{u}^3. \end{aligned} \quad (2.41)$$

If we set $e = 0.25$, then we will recover the classic formulation.

Now that $n(\gamma^-, t)$ has been determined in terms of \dot{u} , we are in a position to combine Eqns. (2.36) and (2.41)₁ to arrive at a differential equation for $\gamma(t) = u(t)$:

$$\frac{\rho_0}{2} (1 + 4e) \dot{u}^2 + \rho_0 u (\ddot{u} - g) = 0. \quad (2.42)$$

When $e = 0.25$, this equation is identical to the differential equation (2.31) established by Cayley. We can non-dimensionalize the differential equation (2.42) by defining a dimensionless time τ and a dimensionless variable $x = x(\tau)$:

$$x = \frac{u}{\ell_0}, \quad \tau = \sqrt{\frac{g}{\ell_0}} t. \quad (2.43)$$

The dimensionless equation is

$$x \frac{d^2 x}{d\tau^2} + \frac{1}{2} (1 + 4e) \frac{dx}{d\tau} \frac{dx}{d\tau} = x. \quad (2.44)$$

Observe that this equation has an equilibrium point when $(x, \frac{dx}{d\tau}) = (0, 0)$. However, this equilibrium point is unstable: an instability which manifests in the chain falling and the heap unraveling.

Restricting attention to the classic case $e = 0.25$, we use Eqn. (2.44) to construct the phase portrait for the ordinary differential equation (2.42) which is shown in Figure 2.8. What is interesting to observe from this portrait is the asymptotic behavior of $(u(t), \dot{u}(t))$ to the following exact solution of Eqn. (2.42)⁸:

$$u(t) = \frac{g}{6} t^2, \quad \dot{u}(t) = \frac{g}{3} t. \quad (2.45)$$

⁸ This solution was first determined by Cayley [53, Page 511] and is labeled c in Figure 2.8.

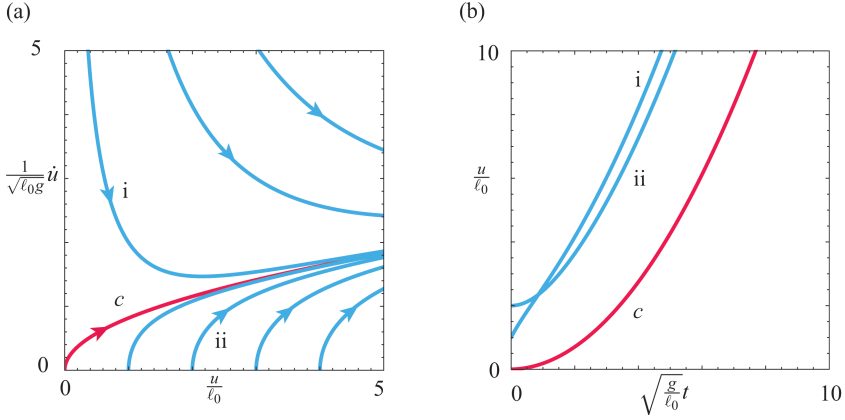


Fig. 2.8 (a) Phase portrait for the differential equation (2.42) governing $u(t) = \gamma(t)$ and (b) three representative solutions of (2.42). The solution labeled c corresponds to Cayley's exact solution (2.45). The arrows in (a) indicate the direction of increasing time t . For the solutions shown here, the parameter $e = 0.25$ in order to agree with Cayley's formulation.

This solution corresponds to a body falling freely under a gravitational force $g/3$ and asymptotes backwards in time to the equilibrium at the origin. Thus the chain falls slower than it would were it not attached to the heap. It is also interesting to return to Eqn. (2.39) and examine the asymptotic behavior of \mathbf{n} using Eqn. (2.45):

$$\begin{aligned}
 \mathbf{n}(\gamma^-, t) &= n(\gamma^-, t) \mathbf{E}_2 = \rho_0 \dot{u}^2 \mathbf{E}_2 \\
 &= \frac{\rho_0}{9} g^2 t^2 \mathbf{E}_2 \\
 &= \frac{2\rho_0 g}{3} u(t) \mathbf{E}_2.
 \end{aligned} \tag{2.46}$$

This tension is 33.3% less than its static counterpart for a string of the same length (i.e., $\rho_0 g u$).

We now revisit Eqn. (2.41) and examine the energy dissipated at the heap: $\Phi_{E_\gamma} = -2e\rho_0 \dot{u}^3$. Thus, the shock at the heap dissipates energy. To elaborate, let us examine an expression for the total energy of the string:

$$E = \frac{1}{2} \int_0^\gamma \rho_0 \mathbf{v} \cdot \mathbf{v} dy + \int_0^\gamma \rho_0 g \mathbf{E}_2 \cdot \mathbf{r} dy. \tag{2.47}$$

Substituting for $\mathbf{r} = y\mathbf{E}_2$ where $y \in (-\gamma, 0)$, $u = \gamma$, and $\mathbf{v} = -\dot{\gamma}\mathbf{E}_2$, we find that

$$E = \frac{\rho_0}{2} u \dot{u}^2 - \frac{\rho_0 g}{2} u^2. \tag{2.48}$$

Whence, with the help of the equation of motion (2.42) and our earlier result Eqn. (2.41)₃ for Φ_{E_γ} ,

$$\begin{aligned}
\dot{E} &= \rho_0 \dot{u} \left(\frac{\dot{u}^2}{2} + u\ddot{u} - gu \right) \\
&= -2e\rho_0 \dot{u}^3 \\
&= \Phi_{E_\gamma}.
\end{aligned} \tag{2.49}$$

Thus $\dot{E} = \Phi_{E_\gamma}$, and, so, assuming $e > 0$, $\Phi_{E_\gamma} < 0$ and E decreases in time. It is initially puzzling that the shock at the heap should be dissipative. However, it should be emphasized that this dissipation is responsible for the chain's retarded rate of fall that we mentioned earlier.

The dissipation present in Cayley's model has not been universally adopted. Some of the controversy can be attributed to the (far from transparent) manner in which Cayley's equation of motion (2.31) was established by him in [53]. The remainder of the controversy stems from the belief by some authors⁹ that this problem should conserve energy. As shall be shown in Exercise 2.1, for the string's motion to conserve energy, the tension in the string reduces by 50% compared to the dissipative case we have just considered and a nonzero force $\mathbf{F}_\gamma = -\frac{\rho_0}{2}\dot{u}^2\mathbf{E}_2$ is needed to push down on the chain at the point where it leaves the heap. In contrast, for the dissipative case we have just considered when $e = 0.25$, we found that $\mathbf{F}_\gamma = \mathbf{0}$ (cf. Eqn. (2.38)).

2.5 A Chain of Finite Length Falling off the Edge of a Table

The previous example's dynamics are complicated by the heap of chain that serves as a reservoir. As a result, it is interesting to consider a closely related classic problem discussed in a marvelous dynamics text from 1929 by Horace Lamb (1849–1934) (see [193, Section 49]). As shown in Figure 2.9, we consider a homogeneous inextensible string of length ℓ . At the end labeled $\xi = 0$, a particle of mass m is attached. Initially, the particle is placed over the edge of the table, and proceeds to fall under gravity, pulling the string along with it. Eventually, the end $\xi = \ell$ falls off the edge of the table. Assuming that the surface of the table is smooth, we now discuss how to establish and analyze a differential equation governing the motion of the string.

To proceed we note that a discontinuity is present where the string moves over the edge of the table. At this point, which we label by $\xi = \gamma$, a reaction force \mathbf{F}_γ is exerted on the string. Further, $\mathbf{v}_\gamma = \mathbf{0}$, $\mathbf{r}'(\gamma^-, t) = \mathbf{E}_2$, and $\mathbf{r}'(\gamma^+, t) = -\mathbf{E}_1$. Using the definition of \mathbf{v}_γ , we find that

$$\mathbf{v}(\gamma^-, t) = -\dot{\gamma}\mathbf{E}_2, \quad \mathbf{v}(\gamma^+, t) = \dot{\gamma}\mathbf{E}_1. \tag{2.50}$$

As a consequence, the velocity vector of the particle of mass m is $\mathbf{v}_0 = -\dot{\gamma}\mathbf{E}_2$.

⁹ See [142, 366, 367] for the relevant citations.

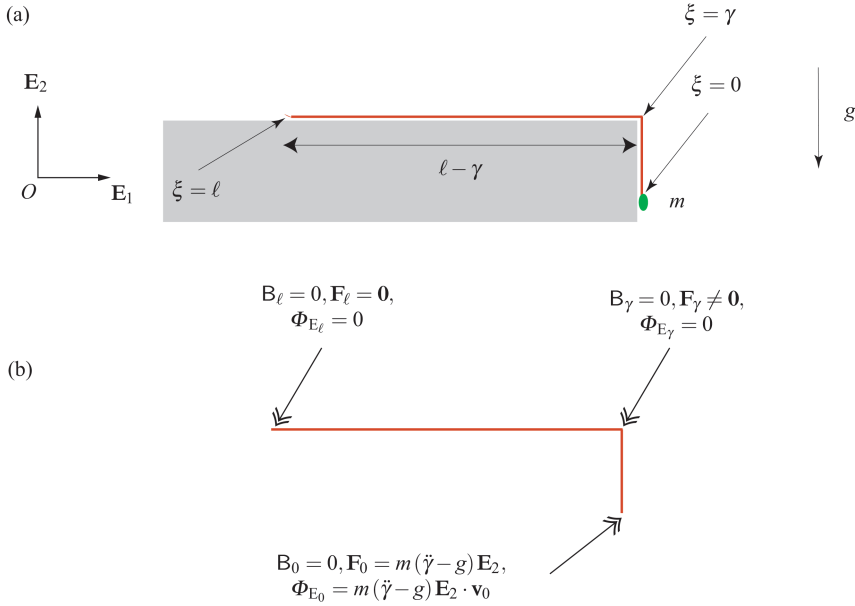


Fig. 2.9 Schematic of an inextensible string of length ℓ which is falling off the edge of a smooth horizontal table. (a) Geometric parameters of the system and (b) schematic of the singular supplies at the points $\xi = 0$, $\xi = \gamma$, and $\xi = \ell$.

To establish the boundary condition at the particle of mass m , we denote the force acting on the particle by $-\mathbf{F}_0$ and the corresponding force acting on the string at $\xi = 0$ by \mathbf{F}_0 . Thus, for the particle of mass m ,

$$-m\ddot{\gamma}\mathbf{E}_2 = -mg\mathbf{E}_2 - \mathbf{F}_0. \quad (2.51)$$

We can use this balance to determine \mathbf{F}_0 and then invoke Eqn. (2.27)₁ to find that

$$\mathbf{n}(0^+, t) = -m(\ddot{\gamma} - g)\mathbf{E}_2. \quad (2.52)$$

Now, integrating the local form of the balance of linear momentum with $\rho_0\mathbf{f} = -\rho_0g\mathbf{E}_2$ for the hanging section of the chain, we find that

$$\begin{aligned} \mathbf{n}(\gamma^-, t) &= \mathbf{n}(0^+, t) - \rho_0\gamma(\ddot{\gamma} - g)\mathbf{E}_2 \\ &= -(m + \rho_0\gamma)(\ddot{\gamma} - g)\mathbf{E}_2. \end{aligned} \quad (2.53)$$

Notice that we used Eqn. (2.52) to establish the final expression above.

For the portion of the string in contact with the table $\rho_0\mathbf{f} = (N - \rho_0g)\mathbf{E}_2$, where N is the normal force (per unit length of ξ) exerted by the table on the lateral surface of the string. Here, the assigned force $\rho_0\mathbf{f}$ is perpendicular to the tension \mathbf{n} in the

string. Integrating the local form of the balance of linear momentum, and noting that $\mathbf{n}(\ell, t) = \mathbf{0}$, we find that

$$\mathbf{n}(\gamma^+, t) = -\rho_0(\ell - \gamma) \ddot{\gamma} \mathbf{E}_1. \quad (2.54)$$

We have now completed assembling the pieces needed to solve the problem.

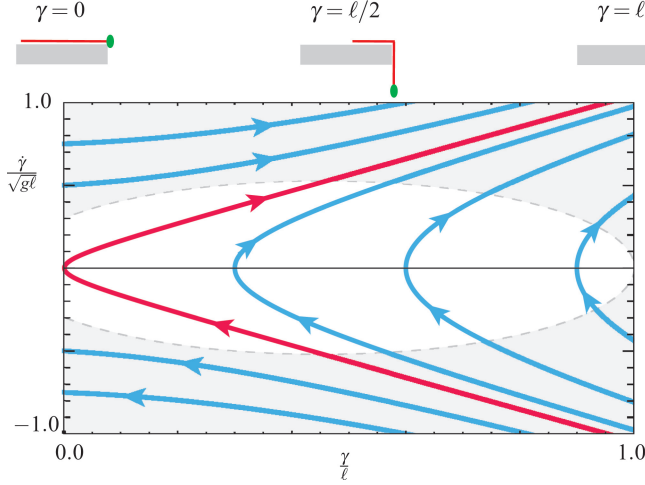


Fig. 2.10 Phase portrait of the solutions to the differential equation (2.58). When $\dot{\gamma} < 0$, the particle of mass m moves upwards, and, for some of the solutions shown in this portrait, can eventually reach the same level as the horizontal surface. The trajectories where $\dot{\gamma} > 0$ show that the string will eventually fall off the table. The shaded region in the figure corresponds to the physically unrealistic situation where $F_\gamma < 0$ (cf. Eqn. (2.61)). For the solutions shown in this figure, $\frac{m}{\rho_0 \ell} = 0.1$.

The unknowns in this problem are the reaction force \mathbf{F}_γ at $\xi = \gamma$, the material momentum supply B_γ at $\xi = \gamma$ and $\gamma(t)$. Using the identity

$$\Phi_{E_\gamma} = B_\gamma \dot{\gamma} + \mathbf{F}_\gamma \cdot \mathbf{v}_\gamma, \quad (2.55)$$

and assuming that the discontinuity at $\xi = \gamma$ does not dissipate energy (i.e., $\Phi_{E_\gamma} = 0$), we find that $B_\gamma = 0$. We then use the jump condition from the balance of linear momentum to compute \mathbf{F}_γ :

$$\begin{aligned} \mathbf{F}_\gamma &= \mathbf{n}(\gamma^-, t) + \rho_0 \dot{\gamma} \mathbf{v}(\gamma^-, t) - \mathbf{n}(\gamma^+, t) - \rho_0 \dot{\gamma} \mathbf{v}(\gamma^+, t) \\ &= (\rho_0(\ell - \gamma) \ddot{\gamma} - \rho_0 \dot{\gamma}^2) \mathbf{E}_1 - ((m + \rho_0 \gamma)(\ddot{\gamma} - g) + \rho_0 \dot{\gamma}^2) \mathbf{E}_2. \end{aligned} \quad (2.56)$$

The final equation of interest is obtained from the jump condition for the balance of material momentum at $\xi = \gamma$: $[[P\dot{\gamma} + C]]_\gamma + B_\gamma = 0$. For this problem, this equation reduces to

$$[[\mathbf{n} \cdot \mathbf{r}']]_\gamma = 0. \quad (2.57)$$

That is, the tension $n = \mathbf{n} \cdot \mathbf{e}_t$ in the string is unaltered by the discontinuity. Substituting into Eqn. (2.57) from Eqns. (2.53) and (2.54), we find a differential equation for $\gamma(t)$:

$$\ddot{\gamma} = \left(\frac{m + \rho_0 \gamma}{m + \rho_0 \ell} \right) g. \quad (2.58)$$

Observe that $m + \rho_0 \gamma$ is the mass of the string over the edge of the table, while $m + \rho_0 \ell$ is the total mass of the string. Thus as $\gamma \rightarrow \ell$, the acceleration of m approaches g .

A phase portrait of the differential equation (2.58) is shown in Figure 2.10. Some of the solutions shown in this portrait correspond to m initially moving upwards and then moving down, and others correspond to m eventually reaching the surface of the table (when $\gamma = 0$). The solutions of particular interest arise when $\gamma > 0$ and the particle is released from rest. It is easy to see from Figure 2.10 that these solutions result in $\gamma \rightarrow \ell$ in a finite time. This behavior is also evident from the analytical expression for the solution to the differential equation (2.58). This solution can be found using standard methods and provides $\gamma(t)$:

$$\gamma(t) = (\cosh(\alpha t) - 1)\varepsilon + \gamma(0)\cosh(\alpha t) + \left(\frac{\sinh(\alpha t)}{\alpha} \right) \dot{\gamma}(0), \quad (2.59)$$

where

$$\alpha = \sqrt{\frac{g}{1 + \varepsilon}}, \quad \varepsilon = \frac{m}{\rho_0 \ell}. \quad (2.60)$$

The analytical expression for $\gamma(t)$ can be used to determine $\mathbf{n}(\xi, t)$ and \mathbf{F}_γ . In particular,

$$\mathbf{F}_\gamma = F_\gamma (\mathbf{E}_1 + \mathbf{E}_2), \quad (2.61)$$

where

$$F_\gamma = \left(\frac{m + \rho_0 \gamma}{m + \rho_0 \ell} \right) \rho_0 (\ell - \gamma) g - \rho_0 \dot{\gamma}^2. \quad (2.62)$$

Observe from this expression that as the string falls off the table, the inertial term $\rho_0 \dot{\gamma}^2$ will dominate and the possibility of a physically meaningless \mathbf{F}_γ will arise.¹⁰ That is, $F_\gamma < 0$. This is the shaded region in the phase portrait shown in Figure 2.10. When $F_\gamma < 0$, we suspect that the contact condition at the end of the table is invalid. Indeed, an experimental demonstration of the loss of contact can be seen in the recent work by Cambou et al. [45]. As discussed in the recent work by Brun et al. [38], a similar lift off phenomenon occurs in a chain moving over a pulley.

Setting $m = 0$ in Eqn. (2.58), we find the differential equation established by Lamb for this problem (see [193, Section 49]). He did not use a balance of material momentum, rather he argued that the tension in the string was unaltered by the discontinuity. As we have seen, this is equivalent to using the jump condition $[[P\dot{\gamma} + C]]_\gamma + B_\gamma = 0$. In consonance with our assumption that $\Phi_{E_\gamma} = 0$, the total energy of the string is conserved for this problem.

¹⁰ I am most grateful to James Hanna for pointing this out.

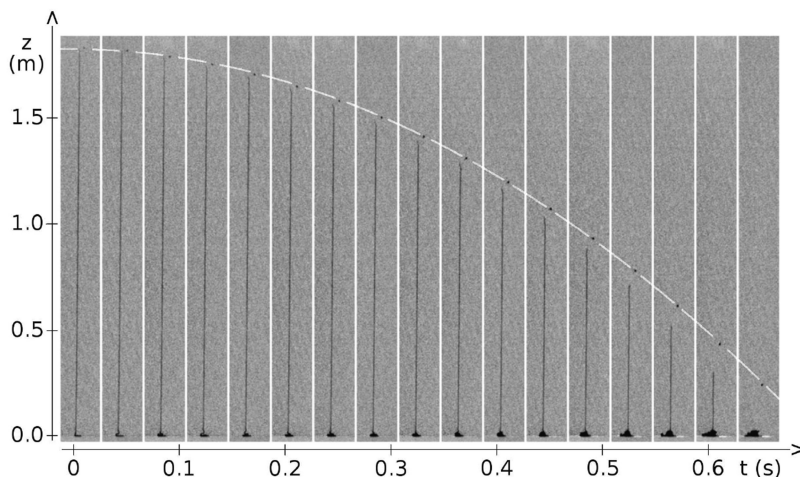


Fig. 2.11 Time lapse images of a chain falling into a heap demonstrating that the top segment of the chain is falling faster than a freely falling particle. Image courtesy of Luis Hamm and Jean-Christophe G  minard.

2.6 “Chains that Suck”

For our third example, we consider the problem of a section of string that is falling into a heap on a horizontal surface (cf. Figures 2.2(b), 2.11 and 2.12). This classic problem is discussed in some textbooks, such as [174, Problem 17, Page 252], [193, Exercise 4, Page 149], and [236, Ex. 536], and is interesting for several reasons. First, if one assumes that the chain is in free fall (i.e., accelerating at a rate of g), then the weight of the chain recorded by a weighing scale located under the heap will asymptote to three times the weight of the chain.¹¹ However, recent experiments by Hamm and G  minard [151] show that the chain actually falls faster than a free falling particle and the (dynamic) weight measured can be much higher than a factor of three times the weight of the chain. Their results on free fall are shown in Figure 2.11.

We call the problems of interest here “chains that suck” following Andy Ruina who noted that these chains appear to be sucked into the heap (cf. [142]). Referring to Figure 2.13(a), the explanation for this phenomenon in [142] is as follows: The links of the chains can be considered as pin-jointed rigid bodies. When one end of the rigid body collides with the ground, the reaction force at the contact point

¹¹ Problem 17 on Page 252 of Jeans’ textbook [174] is reproduced in Exercise 2.5 at the end of this chapter.

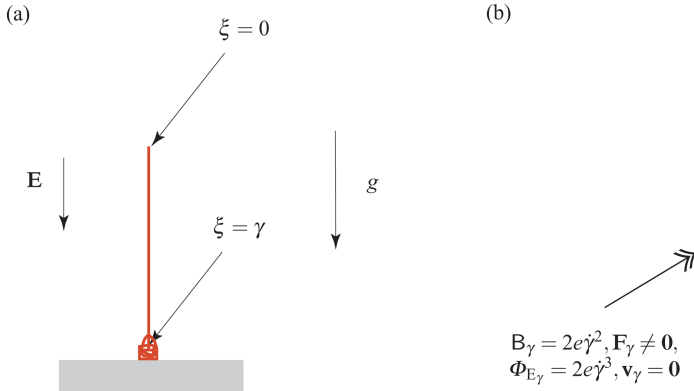


Fig. 2.12 A string falling under gravity into a stationary heap. (a) Schematic of the string and (b) schematic of the supplies at the shock $\xi = \gamma$. Referring to Eqn. (2.67), it will be shown that $\mathbf{F}_\gamma = -\frac{\rho_0}{2} \left(1 - 4e \frac{\dot{\gamma}}{|\dot{\gamma}|}\right) \gamma^2 \mathbf{E}$.

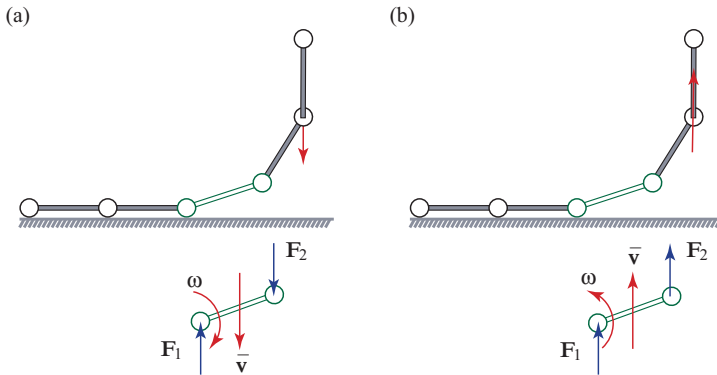


Fig. 2.13 Multi-rigid body dynamics models of chains that are used by Biggins [23] and Grewal et al. [142] to explain the anomalous behavior of chains. (a) A chain falling onto a horizontal surface. The presence of the reaction force \mathbf{F}_1 manifests in the chain segment developing a clockwise angular acceleration $\dot{\omega}$ which increases the acceleration \dot{v} of its center of mass. (b) A chain being pulled from a horizontal surface. The reaction force \mathbf{F}_1 is synonymous with a heap of stationary chain segments apparently pushing the chain segment upward.

produces a moment on the body which gives the body an angular acceleration $\dot{\omega}$ and results in the body pulling its pin-jointed neighbor downwards.¹² As noted by

¹² We use the symbol ω to denote the angular velocity of a rigid body in this discussion. It should not be confused with the same symbol for the velocity \mathbf{v} that appears in later chapters and is associated with a set of directors.

Biggins [23], the opposite effect happens when the chain is being raised out of a heap and the heap appears to push the chain upwards (cf. Figure 2.13(b)).¹³

We follow the methods used in the two previous systems to establish the equations governing the motion of the falling string and start by presenting some kinematical preliminaries. The string, which has a length ℓ , is suspended above a horizontal surface and released from rest. The string falls vertically onto the surface and collects in a heap. We assume that the heap is quiescent and label the point of the string where the transition from free falling motion to rest occurs by $\xi = \gamma$.

It is straightforward to show that

$$\mathbf{v}^- = -\dot{\gamma}\mathbf{E}, \quad (2.63)$$

where $\mathbf{r}' = \mathbf{E}$ is a unit vector which also is parallel to the gravitational force on the string. The balance of linear momentum for the string (2.21)₁ where $\rho_0\mathbf{f} = \rho_0g\mathbf{E}$ and $\rho_0\dot{\mathbf{v}} = -\rho_0\dot{\gamma}\mathbf{E}$ can be integrated to find the tension in the string:

$$\mathbf{n}(\gamma^-, t) = n(\gamma^-, t)\mathbf{E} = \mathbf{n}(0, t) - \rho_0\gamma(\dot{\gamma} + g)\mathbf{E}. \quad (2.64)$$

In the sequel, we shall assume that there is no external applied force at $\xi = 0$, so $\mathbf{n}(0, t) = \mathbf{0}$.

We appeal to the jump conditions (2.29) established earlier to find that

$$\begin{aligned} (n(\gamma^-, t) - \rho_0\dot{\gamma}^2)\mathbf{E} &= \mathbf{F}_\gamma, \\ \frac{\rho_0}{2}\dot{\gamma}^2 - n(\gamma^-, t) &= B_\gamma. \end{aligned} \quad (2.65)$$

To determine $\gamma(t)$, we now need to prescribe either \mathbf{F}_γ or B_γ . We shall assume that the process by which the string's motion ceases when $\xi = \gamma$ is one where energy is dissipated and prescribe¹⁴

$$B_\gamma = -2e\rho_0\dot{\gamma}^2 \frac{\dot{\gamma}}{|\dot{\gamma}|}. \quad (2.66)$$

Here, e is a constant and the presence of the term $\frac{\dot{\gamma}}{|\dot{\gamma}|}$ is needed to ensure that this prescription for B_γ is independent of our parameterization of the string. That is, this prescription is valid whether we label the end point of the string $\xi = 0$ (as we have done here) or $\xi = \ell$. As a result of the prescription for B_γ , Eqn. (2.65) now yields an equation for $n(\gamma^-, t)$ and \mathbf{F}_γ :

$$\begin{aligned} n(\gamma^-, t) &= \frac{\rho_0}{2} \left(1 + 4e \frac{\dot{\gamma}}{|\dot{\gamma}|} \right) \dot{\gamma}^2, \\ \mathbf{F}_\gamma &= -\frac{\rho_0}{2} \left(1 - 4e \frac{\dot{\gamma}}{|\dot{\gamma}|} \right) \dot{\gamma}^2 \mathbf{E}. \end{aligned} \quad (2.67)$$

¹³ For additional commentary, including slow-motion videos of chain fountains and falling chains, on these works see [114] and Ruina's website <http://ruina.mae.cornell.edu/>.

¹⁴ This prescription is identical to the one we used earlier with Cayley's problem (cf. Eqn. (2.40)).

As summarized in Figure 2.12(b), the dissipation produced at $\xi = \gamma$ can be computed from the identity $\Phi_{E_\gamma} = \mathbf{B}_\gamma \dot{\gamma} + \mathbf{F}_\gamma \cdot \mathbf{v}_\gamma$ where $\mathbf{v}_\gamma = \mathbf{0}$:

$$\Phi_{E_\gamma} = -2e\rho_0 |\dot{\gamma}^3|. \quad (2.68)$$

In this problem, $\dot{\gamma} < 0$, so setting $e > 0$ means that the shock at $\xi = \gamma$ will dissipate energy while setting $e = 0$ implies that energy will be conserved there. The parameter e must be determined from experiments.

Having determined that $n(\gamma^-, t) = \frac{\rho_0}{2} \left(1 + 4e \frac{\dot{\gamma}}{|\dot{\gamma}|}\right) \dot{\gamma}^2$, the governing equation for $\gamma(t)$ with $\dot{\gamma} < 0$ now follows from Eqn. (2.64):

$$\rho_0 \gamma (\ddot{\gamma} + g) + \frac{\rho_0}{2} (1 - 4e) \dot{\gamma}^2 = 0. \quad (2.69)$$

This equation has been established previously in a variety of manners (cf. [142, 151, 356]).¹⁵ It is interesting to note that if $e = \frac{1}{4}$, then $\ddot{\gamma} = -g$ and so the string would be in free fall. On the other hand, if $e = 0$, then energy is conserved during the impact process of the string with the heap.

Returning to the weighting scale, we note that the weight W recorded by a scale placed under the heap corresponds to the sum of the force $-\mathbf{F}_\gamma \cdot \mathbf{E}$ and the weight of the quiescent chain. An expression can be found for this dynamic weight:

$$W(t) = \frac{\rho_0}{2} (1 + 4e) \dot{\gamma}^2 + (\ell - \gamma) \rho_0 g. \quad (2.70)$$

In principle, and if the model is adequate, then by measuring $W(t)$ and $\gamma(t)$, the parameter e can be determined. Indeed, Hamm and Géminard [151] performed experiments on a falling chain and established a value of $4e = 0.83$ for a particular chain falling onto a variety of horizontal surfaces.

The free-falling case, which as noted by Grewal et al. [142], is discussed in several textbooks, corresponds to the assumption that $e = 0.25$. In this case, Eqns. (2.64) and (2.67) can be used to show that the string is slack and find an interesting expression for \mathbf{F}_γ :

$$\mathbf{n}(\xi, t) = \mathbf{0}, \quad \mathbf{F}_\gamma = -\rho_0 \dot{\gamma}^2 \mathbf{E}. \quad (2.71)$$

That is, the heap pushes up on the string at the point $\xi = \gamma$ where the string impacts the heap. Assuming that the string is released from rest, we can determine $\gamma(t)$, the time T when $\gamma(T^-) = 0$, and $W(t)$:

$$\gamma(t) = \ell - \frac{g}{2} t^2, \quad T = \sqrt{\frac{2\ell}{g}}, \quad \frac{W(t)}{\rho_0 g \ell} = \frac{3g}{2\ell} t^2. \quad (2.72)$$

¹⁵ For instance, Eqn. (2.65) is equivalent to the governing equation for the same problem established by Virga in [356, Eqn. (5.10)] with $(\frac{1}{2} - 2e, \gamma)$ identified with his $(1 - f, y)$. The same differential equation is also equivalent to Grewal et al.’s work with $(\dot{\gamma}, \gamma)$ identified with their $(-v = -\dot{x}, L - x)$ and $e = \frac{1}{6}$ for [142, Eqn. (18)] and $e = 0$ for [142, Eqn. (19)]. Finally, Eqn. (2.65) is equivalent to [151, Eqn. (5)] with $(\frac{1}{2} - 2e, \gamma)$ identified with their $(1 - \gamma, y)$.

Thus, $W(T^-)$ is three times the weight $\rho_0 g \ell$ of the string.

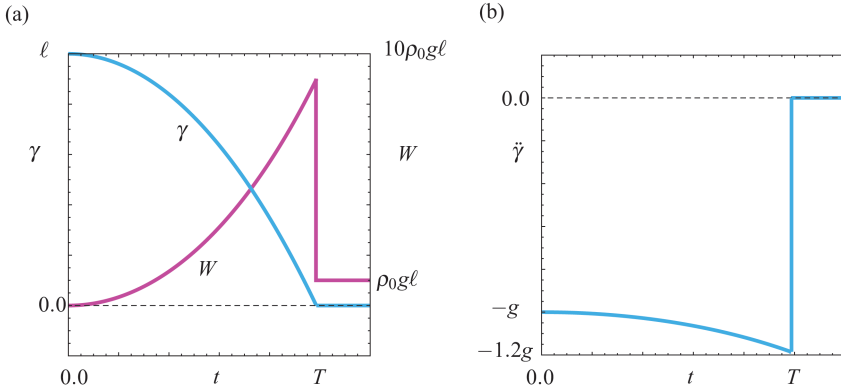


Fig. 2.14 Simulation results for the chain falling into a heap governed by the differential equation (2.69). (a) Results for $\gamma(t)$ and $W(t)$ and (b) the normalized acceleration $\ddot{\gamma}(t)$. For the results shown, $\gamma(0) = \ell$, $\dot{\gamma}(0) = 0$, $4e = 0.83$, $T = \sqrt{\frac{2\ell}{g}}$, and it takes $0.98589T$ seconds for the top of the chain (labeled $\xi = 0$) to reach the heap.

However, as shown in the experiments by Hamm and Géminard [151], e is not necessarily 0.25 and the string can fall faster than a free-falling particle. To see these results, let us now use their experimental value of $4e = 0.83$. We dimension the time t using the free-fall time for a particle $T = \sqrt{\frac{2\ell}{g}}$, $t = T\tau$. Additionally, the variable γ is non-dimensionalized using the length ℓ : $\ell u = \gamma$. After integrating the differential equation (2.69) and evaluating $W(t)$ and $\ddot{\gamma}$ using the resulting expressions for $\gamma(t)$, we find that the string falls slightly faster than a free-falling particle. Referring the reader to Figure 2.14, the time taken between the release of the tip $\xi = 0$ of the chain and the time it lies at rest on the heap is $0.98589T$. We also observe from this figure that the tip of the chain is accelerating (slightly) faster than gravity. While the difference in time of flight is arguable imperceptible to the naked eye, the same cannot be said of the weight $W(t)$. With the help of Eqn. (2.70), we find that the largest weight recorded by the scale is $W(T^-) = 8.9789$ times the weight of the chain!

2.7 The Falling Folded Chain

For our next example, we consider the string of length ℓ shown here in Figure 2.15 (and earlier in Figure 1.9(a)). One end of the string is fixed at a point which we conveniently take to be the origin, $\mathbf{r}(\xi = 0, t) = \mathbf{0}$. The other end of the string is attached to a particle of mass m which is acted upon by a force $-P_0 \mathbf{E}$. An applied

force $\rho_0 \mathbf{f} = \rho_0 g \mathbf{E}$ acts on the string. The interesting feature of this problem is the fold at $\xi = \gamma$. The question we ask is as follows: If the particle is released from rest, then what is the velocity and position of the particle as it falls?

Our presentation of this classic problem follows [277]. We modify their treatment to incorporate the notion of material momentum in this problem that was noted in [264]. Other relevant works on this problem include the treatment by Augustus E. H. Love (1863–1940) in [212] and the (more recent) papers by Reeken [297] and Hans Troger (1943–2010) and his coworkers [315, 331].

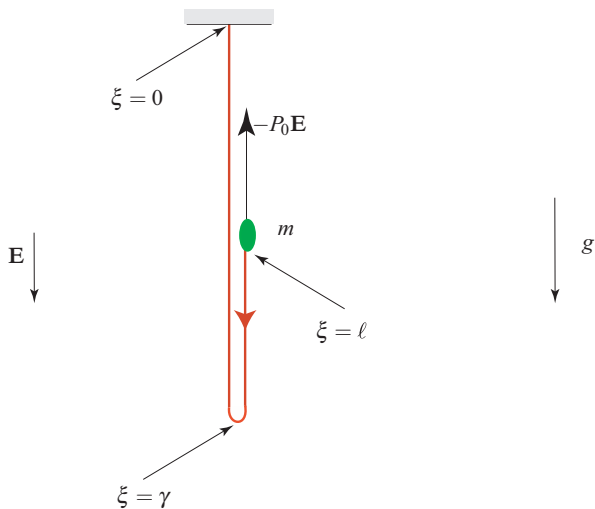


Fig. 2.15 Graphical representation of a heavy inextensible string (also known as a cable) of length ℓ with a fold located at a point $\xi = \gamma$ below the point of support. At the other end of the cable, a mass particle of mass m is attached and an applied force $-P_0 \mathbf{E}$ acts on this particle where \mathbf{E} is a unit vector in the direction of gravity.

As the chain is at rest for $\xi \in [0, \gamma^-)$ and $\mathbf{e}_t = \pm \mathbf{E}$, the following results hold:

$$\mathbf{v}(\gamma^-, t) = \mathbf{0}, \quad \mathbf{v}(\gamma^+, t) = 2\dot{\gamma} \mathbf{E}, \quad \mathbf{v}_\gamma = \dot{\gamma} \mathbf{E}. \quad (2.73)$$

It is important to note the dramatic change in velocity that occurs at $\xi = \gamma$.

Referring to Figure 2.16, at the end of the chain, we can use the jump condition $[[\mathbf{n} + \rho_0 \ell \mathbf{v}]]_\ell + \mathbf{F}_\ell = \mathbf{0}$ where $\ell = 0$ to compute the boundary condition. First, we consider a balance of linear momentum for the particle of mass m attached to the end $\xi = \ell$ of the chain:

$$-P_0 \mathbf{E} + mg \mathbf{E} - \mathbf{F}_\ell = m \ddot{\mathbf{r}}(\ell, t). \quad (2.74)$$

As $\mathbf{v}(\ell^-, t) = 2\dot{\gamma} \mathbf{E}$, we find that

$$\mathbf{F}_\ell = -(2m\ddot{\gamma} + (P_0 - mg)) \mathbf{E}. \quad (2.75)$$

Now at the end $\xi = \ell$ of the string, we have the jump condition

$$-\mathbf{n}(\ell^-, t) + \mathbf{F}_\ell = \mathbf{0}. \quad (2.76)$$

Consequently, dropping the $-$ ornamenting the ℓ and rearranging,

$$\mathbf{n}(\ell, t) = -(2m\dot{\gamma} - mg + P_0)\mathbf{E}. \quad (2.77)$$

In interpreting this result, it is important to note that for the moving section of chain $\mathbf{r}'(\xi, t) = -\mathbf{E}$.

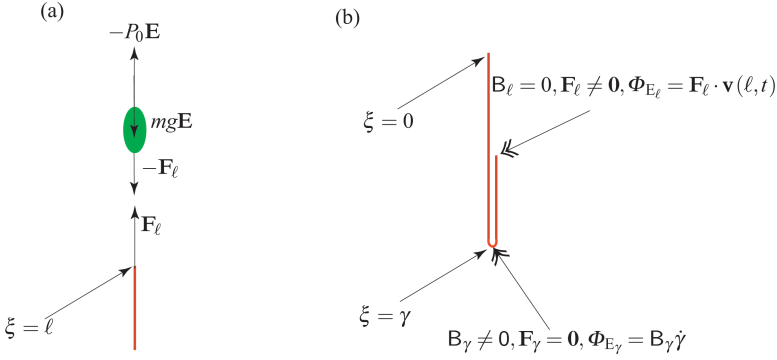


Fig. 2.16 Graphical representation of the forces acting at certain locations for the free-falling folded string problem. In (a), the particle of mass m is shown and in (b) the singular supplies acting at the points $\xi = \gamma$ and $\xi = \ell$ are shown.

Noting that an applied body force $\rho_0 g \mathbf{E}$ acts on the chain, integrating the balance of linear momentum (2.21) on the intervals $\xi \in (0, \gamma^-)$ and $\xi \in (\gamma^+, \ell)$, and then using (2.77), we find that

$$\begin{aligned} \mathbf{n}(0, t) &= \mathbf{n}(\gamma^-, t) + \rho_0 g \gamma \mathbf{E}, \\ \mathbf{n}(\gamma^+, t) &= -(P_0 + (m + \rho_0(\ell - \gamma))(2\dot{\gamma} - g))\mathbf{E}. \end{aligned} \quad (2.78)$$

The jump in \mathbf{n} across the fold can be computed using the jump condition $[[\mathbf{n} + \rho_0 \dot{\gamma} \mathbf{v}]]_\gamma + \mathbf{F}_\gamma = \mathbf{0}$. In this case $\mathbf{F}_\gamma = \mathbf{0}$, and so

$$\mathbf{n}(\gamma^-, t) = \mathbf{n}(\gamma^+, t) + 2\rho_0 \dot{\gamma}^2 \mathbf{E}. \quad (2.79)$$

The equations (2.78) and (2.79) enable us to compute $\mathbf{n}(\xi, t)$ once $\gamma(t)$ is known.

The most interesting part of this problem is to provide prescriptions for Φ_{E_γ} and B_γ at the fold. As pointed out by Love [212] in 1897, “energy is dissipated in the impulsive action at the place where the discontinuous change of motion occurs.”

He then proceeded with a semi-inverse solution where he assumed that $\mathbf{n}(\gamma^-, t) = \mathbf{0}$ to show that, in our notation,

$$\Phi_{E_\gamma} = -2\rho_0\dot{\gamma}^3. \quad (2.80)$$

To see how he arrived at this result, we start with the jump condition (1.86)₅, the results (2.73), and assume that $\mathbf{n}(\gamma^-, t) = \mathbf{0}$ in (2.79). In this case, the following set of identities hold:

$$\mathbf{n}(\gamma^+, t) = -2\rho_0\dot{\gamma}^2\mathbf{E}, \quad (2.81)$$

and

$$\begin{aligned} \Phi_{E_\gamma} &= -[\![\mathbf{n} \cdot \mathbf{v}]\!]_\gamma - \left[\left[\rho_0\psi + \frac{1}{2}\rho_0\mathbf{v} \cdot \mathbf{v} \right] \right]_\gamma \dot{\gamma} \\ &= -\frac{1}{2}\rho_0(\mathbf{v}^+ \cdot \mathbf{v}^+) \dot{\gamma} \\ &= -2\rho_0\dot{\gamma}^3. \end{aligned} \quad (2.82)$$

As was recently pointed out in [331], some formulations of this problem erroneously set $\Phi_{E_\gamma} = 0$ thereby ignoring the dissipation of energy that occurs at the fold.¹⁶ With the help of the identity $\Phi_{E_\gamma} = B_\gamma\dot{\gamma} + \mathbf{F}_\gamma \cdot \mathbf{v}_\gamma$, we see that the corresponding prescription for B_γ is

$$B_\gamma = -2\rho_0\dot{\gamma}^2. \quad (2.83)$$

We shall shortly choose more general prescriptions for Φ_{E_γ} and B_γ than those implied by Love's analysis. Our motivation for doing so is two-fold. First, it enables us to incorporate experimental results which suggest in related problems that (2.80) is too restrictive. Second, the generalization allows us to readily compare Love's formulation to the energy-conserving formulations.

It remains to compute $\gamma(t)$ and the jump condition from the balance of material momentum (i.e., Eqn. (2.22)) provides the differential equation for this quantity once B_γ has been prescribed. For the present purposes, we generalize the prescription provided by Love's analysis to¹⁷

$$B_\gamma = -2e\rho_0\dot{\gamma}^2 \left(\frac{\dot{\gamma}}{|\dot{\gamma}|} \right), \quad (2.84)$$

where e is a constant. The constant $e = 1.0$ for Love's analysis in [212] of this problem that we previously discussed. Because \mathbf{F}_γ is zero at the fold, the prescription (2.84) for B_γ implies that

¹⁶ See, for example, [150, 189, 304].

¹⁷ This prescription is identical to the prescriptions (2.40) and (2.66) for B_γ that were used in Sections 2.4 and 2.6, respectively. As with these prescriptions, observe that the prescription (2.84) accommodates cases where $\dot{\gamma} < 0$.

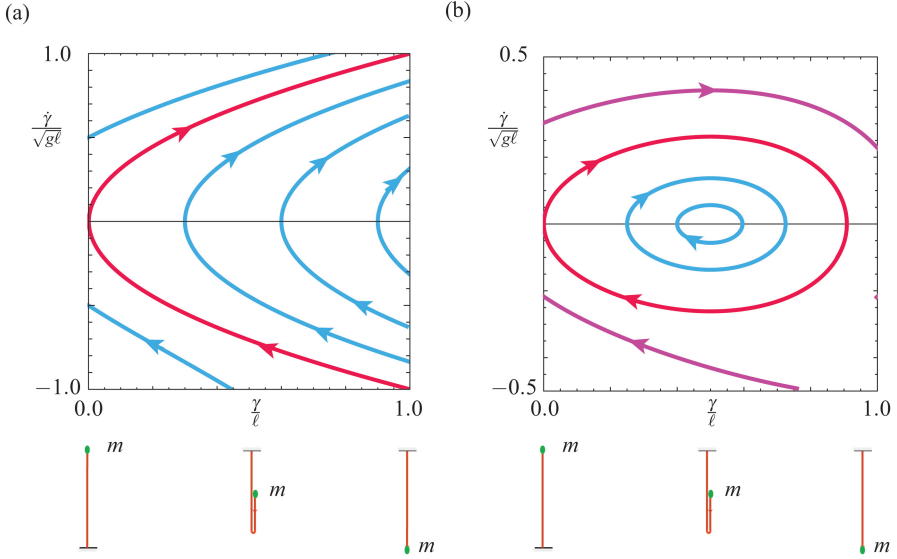


Fig. 2.17 Phase portraits of the solutions to (2.89). When $\dot{\gamma} < 0$, the particle of mass m moves upwards. For some of the trajectories shown in these portraits $\gamma \rightarrow 0$, which implies that the string will become completely vertical. For the solutions shown in this figure, $\frac{m}{\rho_0 \ell} = 1.0$ and $e = 1.0$. For (a) $\frac{P_0}{\rho_0 g \ell} = 0$ and for (b) $\frac{P_0}{\rho_0 g \ell} = 1.5$. The equilibrium in (b) corresponds to the static configuration where the applied force P_0 balances the combined weight of the particle and a length of the string.

$$\begin{aligned}\Phi_{E\gamma} &= B_\gamma \dot{\gamma} \\ &= -2e\rho_0 |\dot{\gamma}|^3.\end{aligned}\quad (2.85)$$

Thus, if $e > 0$, the fold will dissipate energy. We now start with Eqn. (2.22):

$$C(\gamma^+, t) + \dot{\gamma}P(\gamma^+, t) - C(\gamma^-, t) - \dot{\gamma}P(\gamma^-, t) + B_\gamma = 0. \quad (2.86)$$

Substituting for C and P yields an intermediate result:

$$-n(\gamma^+, t) + n(\gamma^-, t) + B_\gamma = 0. \quad (2.87)$$

We next use (2.79) to eliminate $n(\gamma^-, t)$:

$$2\rho_0 \dot{\gamma}^2 - 2n(\gamma^+, t) + B_\gamma = 0. \quad (2.88)$$

Finally, we use Eqn. (2.78)₂ and the prescription (2.84) for B_γ to find the desired differential equation for $\gamma(t)$:

$$(m + \rho_0(\ell - \gamma))(2\ddot{\gamma} - g) = -P_0 + \rho_0 \dot{\gamma}^2 \left(1 - e \frac{\dot{\gamma}}{|\dot{\gamma}|}\right). \quad (2.89)$$

Given a pair of initial conditions $\dot{\gamma}(0)$ and $\gamma(0)$, this equation can be integrated to determine $\gamma(t)$.

It is interesting to note from Eqn. (2.89) that if P_0 is sufficiently large then the string has a static equilibrium $(\gamma, \dot{\gamma}) = (\gamma_0, 0)$ where $\gamma_0 = \ell - \left(\frac{P_0 - mg}{\rho_0 g}\right)$. A representative sample of the solutions of (2.89) when $P_0 \neq 0$ is shown in Figure 2.17(b). Observe that in addition to the solutions where $\gamma(t) \rightarrow 1$, we also find solutions $\gamma(t)$ which are periodic and oscillate back and forth about the equilibrium point $(\gamma_0, 0)$. This is in stark contrast to the case where $P_0 = 0$ and no equilibrium is present (cf. Figure 2.17(a)).

Further discussion of the solutions to the ordinary differential equation (2.89) for various values of P_0 , m , and e can be found in the literature. Of particular interest in some of these works, such as [277, 315, 356] is the fact that in the energy conserving case when $m = 0$ and $P_0 = 0$ (presented in the influential textbooks by Hamel [150] and Rosenberg [304]), the tip of the chain (at $\xi = \ell$) develops an infinite speed as $\gamma \rightarrow \ell$. This singular behavior can be easily explained: in the absence of dissipation all the potential energy of the chain is converted into the kinetic energy of an increasingly smaller portion of the chain and eventually the speed of the tip becomes unbounded.

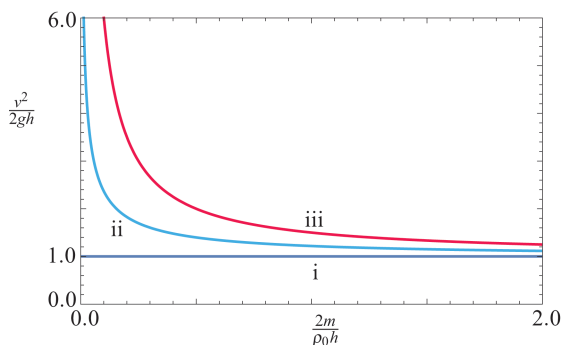


Fig. 2.18 Solutions $\frac{v^2}{2gh}$ to Eqn. (2.90) for the terminal velocity of the mass particle m at the tip of the folded chain as a function of the mass ratio and three values of the energy dissipation parameter e : i corresponds to $e = 1.0$, ii corresponds to $e = 0.5$, and iii corresponds to the energy conserving case $e = 0.0$. The choice $e = 1$ corresponds to the formulation of this problem in Love [212] and Steiner and Troger [331].

To elaborate further on this matter, we recall a result from [277]. These authors examined the solution to the differential equation (2.89) when the particle of mass m was released from a height h (i.e., $\gamma(t = 0) = \ell - h/2$) and the terminal load $-P_0 \mathbf{E}$ was absent. They showed that the velocity $v = 2\dot{\gamma}(t = T)$ that the particle has when $\gamma(t = T) = \ell$ and the mass m had fallen a height h is provided by the equation

$$\frac{v^2}{2gh} = \frac{1}{2-e} \left(\frac{2m}{\rho_0 h} \right) \left(\left(1 + \frac{\rho_0 h}{m} \right)^{2-e} - 1 \right). \quad (2.90)$$

For a free-falling mass, the corresponding velocity is $\sqrt{2gh}$. We can examine the expression for $\frac{v^2}{2gh}$ as a function of e and m provided by Eqn. (2.90). It is easy to observe from Figure 2.18 that for all cases, except $e = 1$, the mass m will fall faster than its free-falling counterpart.

2.8 The Chain Fountain

Despite the number of investigations of chains during the past two centuries, it was surprising to see Hanna and Santangelo's 2012 demonstration of an arch-like shape formed by a link chain [153, 154] (cf. Figure 2.1) and Steve Mould's 2013 video demonstration of a chain fountain formed by a chain of beads (also known as self-siphoning beads).¹⁸ In addition to a relatively large number of popular press articles, Mould's demonstration inspired a series of papers [23, 24, 356, 357] devoted to explaining the chain fountain. For our analysis, we shall assume that the motion of the chain is a steady axial motion.¹⁹

We now examine the steady motion of the chain fountain using the framework that includes the material momentum balance. Referring to Figure 2.19, we divide the string that models the chain into three parts. First, the material point $\xi = \gamma_0(t)$ where the string exits a quiescent heap; second, the material point $\xi = \gamma_1(t)$ where the string ceases its motion and forms a quiescent heap; and third, the segment of string $\xi \in (\gamma_0, \gamma_1)$ between the heaps. For the latter portion of the string, we assume that the motion is steady and the space curve formed by the string is planar:

$$\mathbf{r} = x\mathbf{E}_1 + y\mathbf{E}_2, \quad (2.91)$$

where \mathbf{E}_2 is in the direction of gravity.

2.8.1 An Inverted Catenary

To describe the dynamics of the segment between the heaps, we assume that the motion of the string is a steady axial motion (cf. Section 2.2):

$$\mathbf{r}(\xi, t) = \bar{\mathbf{r}}(\chi), \quad \chi = \xi + ct, \quad (2.92)$$

where c is a constant. From our earlier results (cf. Eqn. (2.3)) we know that $c = -\dot{\gamma}$ and so we conclude that

$$c = -\dot{\gamma}_0 = -\dot{\gamma}_1. \quad (2.93)$$

¹⁸ The video can be accessed at <http://stevemould.com/siphoning-beads/>.

¹⁹ It might be helpful for some readers to review the three conditions for a steady axial motion discussed in Section 2.2 and to see how they pertain to the chain fountain shown in Figure 2.19.

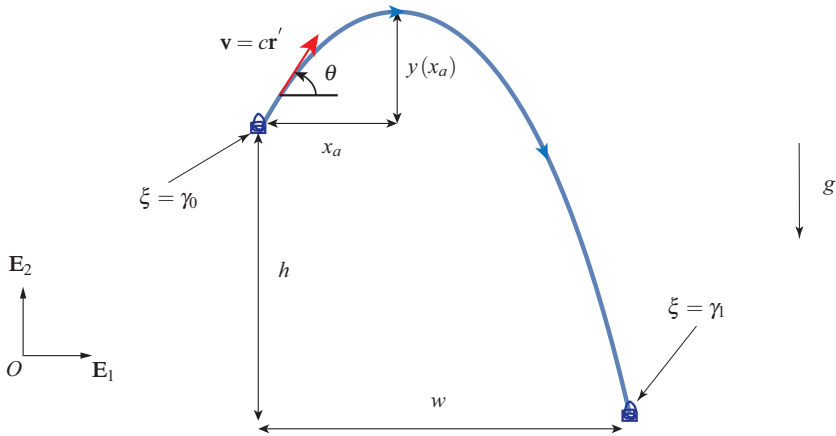


Fig. 2.19 Schematic of the chain fountain.

If ξ is taken as the arc-length parameter in a reference configuration, then χ can be identified as the arc-length parameter for the space curve formed by the string:

$$s = \chi = \xi + ct. \quad (2.94)$$

We also have the following equivalencies among partial derivatives for any function $f(\xi, t) = \tilde{f}(s = \xi + ct)$:

$$\frac{\partial f}{\partial \xi} = \frac{\partial \tilde{f}}{\partial s}, \quad \frac{\partial f}{\partial t} = c \frac{\partial \tilde{f}}{\partial s}, \quad \frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 \tilde{f}}{\partial s^2}. \quad (2.95)$$

These identities are invoked without comment in the sequel.

We can parameterize the string using ξ in such a manner that $\dot{\gamma}_0 < 0$. As $\mathbf{v} = c\mathbf{r}'_s$, we have that

$$v = c = \|\mathbf{v}\|, \quad \mathbf{v} = v\mathbf{r}', \quad \mathbf{a} = \kappa v^2 \mathbf{r}'', \quad (2.96)$$

where κ is the curvature of the space curve formed by the string. It is convenient to parameterize the unit tangent vector to the space curve by an angle θ :

$$\mathbf{r}' = \frac{\partial \mathbf{r}}{\partial \chi} = \cos(\theta)\mathbf{E}_1 + \sin(\theta)\mathbf{E}_2, \quad (2.97)$$

where the gravitational force on the string acts in the \mathbf{E}_2 direction: $\rho_0 \mathbf{f} = -\rho_0 g \mathbf{E}_2$.

The local form of the balance of material momentum $\dot{\mathbf{P}} = \mathbf{b} + \mathbf{C}'$ (cf. Eqns. (1.83) and (2.20)) can be used to establish a useful conservation. For the problem at hand,²⁰

²⁰ It may be helpful to examine Eqn. (1.112) to find the prescription for \mathbf{b} that we are using here. Note that because the string is assumed to be homogeneous, the expression for this force simplifies dramatically to the derivative of a potential energy density.

$$\begin{aligned}
P &= -\rho_0 \mathbf{v} \cdot \mathbf{r}' = -\rho_0 c, \\
C &= -\mathbf{n} \cdot \frac{\partial \mathbf{r}}{\partial \xi} - \frac{\rho_0}{2} \mathbf{v} \cdot \mathbf{v} = -n - \frac{\rho_0 c^2}{2}, \\
b &= b_p = -\rho_0 \mathbf{f} \cdot \mathbf{r}' = (\rho_0 g y)'.
\end{aligned} \tag{2.98}$$

Thus the balance of material momentum (Eqn. (2.20)) yields the conservation

$$n - \rho_0 c^2 - \rho_0 g y = U, \tag{2.99}$$

where U is a constant.

We are now in a position to involve the balance of linear momentum $\mathbf{n}' + \rho_0 \mathbf{f} = \rho_0 \dot{\mathbf{v}}$. With some rearranging, this law can be expressed in a more convenient form:

$$\left((n - \rho_0 c^2) \mathbf{r}' \right)' - \rho_0 g \mathbf{E}_2 = \mathbf{0}. \tag{2.100}$$

From this equation, we infer a conservation in the horizontal direction and a differential equation in the vertical direction:

$$(n - \rho_0 c^2) \cos(\theta) = H, \quad \left((n - \rho_0 c^2) \sin(\theta) \right)' = \rho_0 g, \tag{2.101}$$

where H is a constant and we used the earlier result that $v = c$. Because we anticipate that the solution will be an inverted catenary, $H < 0$ and $n < \rho_0 c^2$.²¹

From geometric considerations, along the space curve formed by the string, we have

$$\frac{ds}{dx} = \sqrt{1 + \frac{dy}{dx} \frac{dy}{dx}}, \quad \tan(\theta) = \frac{dy}{dx}. \tag{2.102}$$

By combining Eqns. (2.101)_{1,2} to eliminate $n - \rho_0 v^2$ and invoking Eqn. (2.95)₁ we find that

$$\begin{aligned}
\frac{d}{ds} (\tan(\theta)) &= \frac{\rho_0 g}{H}, \\
\frac{d}{dx} \left(\tan(\theta) = \frac{dy}{dx} \right) &= \frac{\rho_0 g}{H} \frac{ds}{dx} = \frac{\rho_0 g}{H} \sqrt{1 + \frac{dy}{dx} \frac{dy}{dx}}.
\end{aligned} \tag{2.103}$$

Thus, we arrive at the familiar equation for a catenary²²:

$$\frac{d^2 y}{dx^2} = \frac{\rho_0 g}{H} \sqrt{1 + \frac{dy}{dx} \frac{dy}{dx}}. \tag{2.104}$$

The solution $y(x)$ of this differential equation and the companion solution $s(x)$ are

²¹ We are closely following the discussion of the inverted axially moving catenary in Perkins and Mote [288] and its extension to the nonlinearly elastic case that is discussed in [259].

²² For extensions of the classic catenary problem to the case where the string is extensible, the reader is referred to Antman [11, 12] and Dickey [87, 88].

$$\begin{aligned}
 y(x) &= \frac{H}{\rho_0 g} \left(\cosh \left(\frac{\rho_0 g}{H} (x - x_a) \right) - 1 \right) + y(x_a), \\
 s(x) &= \frac{H}{\rho_0 g} \sinh \left(\frac{\rho_0 g}{H} (x - x_a) \right) + s(x_a).
 \end{aligned} \tag{2.105}$$

Here, $y(x_a)$, x_a , and $s(x_a)$ are constants where $(x_a, y(x_a))$ are the Cartesian coordinates of the apex of the catenary and $s(x_a)$ is the arc-length coordinate at this point (cf. Figure 2.19).

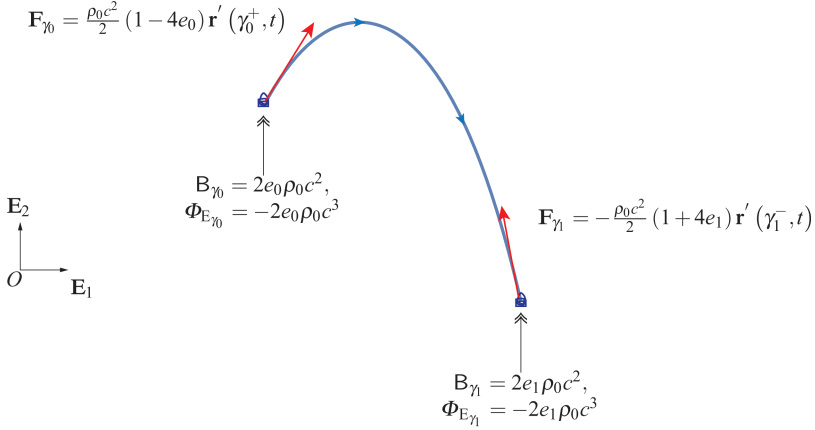


Fig. 2.20 Summary of the singular supplies at $\xi = \gamma_0$ and $\xi = \gamma_1$.

2.8.2 The String Leaving the Heap

We now turn our attention to the point $\xi = \gamma_0(t)$ where the string leaves the quiescent heap. The tangent vector to the string is

$$\mathbf{r}'(\gamma_0^+, t) = \cos(\theta_0) \mathbf{E}_1 + \sin(\theta_0) \mathbf{E}_2. \tag{2.106}$$

As the shock at $\xi = \gamma_0(t)$ is stationary (i.e., $\mathbf{v}_{\gamma_0} = \mathbf{0}$) and $\dot{\gamma}_0 = -c$ (cf. Eqn. (2.93)), we conclude that

$$\mathbf{v}(\gamma_0^+, t) = c \mathbf{r}'(\gamma_0^+, t). \tag{2.107}$$

The appropriate jump conditions were established earlier (cf. Eqn. (2.27)). Substituting for \mathbf{C} , \mathbf{v} , and \mathbf{r}' , we find that

$$\begin{aligned}
 n(\gamma_0^+, t) - \rho_0 c^2 &= -\mathbf{F}_{\gamma_0} \cdot \mathbf{r}'(\gamma_0^+, t), \\
 \frac{\rho_0}{2} c^2 - n(\gamma_0^+, t) &= -B_{\gamma_0}.
 \end{aligned} \tag{2.108}$$

Next, we proceed to prescribe B at γ_0 as²³

$$B_{\gamma_0} = -2e_0\rho_0\dot{\gamma}_0^2 \left(\frac{\dot{\gamma}_0}{|\dot{\gamma}_0|} \right), \quad (2.109)$$

where e_0 is a constant which needs to be determined experimentally. Because we expect the process that occurs at γ_0 to be dissipative, e_0 is expected to be positive: $e_0 > 0$.

Substituting into the prescription (2.109) and noting that $\dot{\gamma}_0 < 0$, we can solve for $n(\gamma_0^+, t)$ and \mathbf{F}_{γ_0} :

$$\begin{aligned} n(\gamma_0^+, t) &= \frac{\rho_0}{2} (1 + 4e_0) c^2, \\ \mathbf{F}_{\gamma_0} \cdot \mathbf{r}'(\gamma_0^+, t) &= -n(\gamma_0^+, t) + \rho_0 c^2 \\ &= \frac{\rho_0}{2} (1 - 4e_0) c^2. \end{aligned} \quad (2.110)$$

Referring to Figure 2.20, observe that the heap exerts a singular force \mathbf{F}_{γ_0} on the string which has a positive component in the direction of \mathbf{v}^- and $\rho_0 c^2 - n(\gamma_0^+, t) > 0$ provided $e_0 < 0.25$. In addition, \mathbf{F}_{γ_0} only has a component in the tangential direction $\mathbf{r}'(\gamma_0^+, t)$.

2.8.3 The String at the End of the Catenary

We now turn our attention to the point $\xi = \gamma_1(t)$ where the string comes to a halt in a quiescent heap. The tangent vector to the string at this point is

$$\mathbf{r}'(\gamma_1^-, t) = \cos(\theta_1) \mathbf{E}_1 + \sin(\theta_1) \mathbf{E}_2. \quad (2.111)$$

As the shock at $\xi = \gamma_1(t)$ is stationary (i.e., $\mathbf{v}_{\gamma_1} = \mathbf{0}$), and $\dot{\gamma}_1 = -c$ (cf. Eqn. (2.93)), we conclude that

$$\mathbf{v}(\gamma_1^-, t) = c \mathbf{r}'(\gamma_1^-, t). \quad (2.112)$$

The appropriate jump conditions were established earlier (cf. Eqn. (2.29)). Substituting for C , \mathbf{v} , and \mathbf{r}' , we find that

$$\begin{aligned} n(\gamma_1^-, t) - \rho_0 c^2 &= \mathbf{F}_{\gamma_1} \cdot \mathbf{r}'(\gamma_1^-, t), \\ \frac{\rho_0}{2} c^2 - n(\gamma_1^-, t) &= B_{\gamma_1}. \end{aligned} \quad (2.113)$$

We now proceed to prescribe B at γ_1 in a manner identical to B_{γ_0} :

²³ This prescription is identical to the prescriptions (2.40), (2.66), and (2.84) for B_γ that were used in Sections 2.4, 2.6, and 2.7, respectively.

$$B_{\gamma_1} = -2e_1\rho_0\dot{\gamma}_1^2 \left(\frac{\dot{\gamma}_1}{|\dot{\gamma}_1|} \right) = 2e_1\rho_0c^2, \quad (2.114)$$

where e_1 is a constant which needs to be determined experimentally and we substituted for $\dot{\gamma}_1 = -c$. Again, we anticipate that e_1 is positive, but there is no reason for us to expect that $e_1 = e_0$. It is straightforward at this point to solve for $n(\gamma_1^-, t)$ and $\mathbf{F}_{\gamma_1} \cdot \mathbf{r}'(\gamma_1^-, t)$:

$$\begin{aligned} n(\gamma_1^-, t) &= \frac{\rho_0}{2} (1 - 4e_1) c^2, \\ \mathbf{F}_{\gamma_1} \cdot \mathbf{r}'(\gamma_1^-, t) &= n(\gamma_1^-, t) - \rho_0 c^2 \\ &= -\frac{\rho_0}{2} (1 + 4e_1) c^2. \end{aligned} \quad (2.115)$$

As with the other shock condition, it is easy to conclude that $\rho_0 c^2 - n(\gamma_1^-, t) > 0$ provided $e_1 < 0.25$. In addition, if $e_1 > -0.25$, then \mathbf{F}_{γ_1} has a negative (retarding) component in the direction of $\mathbf{r}'(\gamma_1^-, t)$.

2.8.4 Characteristics of the Chain Fountain

To gain a complete picture of the chain fountain we need to combine the analyses of the inverted catenary and the shocks at $\xi = \gamma_0$ and $\xi = \gamma_1$.²⁴ The catenary is typically determined by prescribing $y(x)$ at two known values of x . Here, we assume that the location for $\xi = \gamma_0$ is the origin and that $\xi = \gamma_1$ is located at a known distance h below the origin (cf. Figure 2.19). Rather than assuming that we know the x coordinate of the point $\xi = \gamma_1$, we assume that $\theta_0 = \theta(\gamma_0^+, t)$ is known. In summary, the following quantities are prescribed:

$$\mathbf{r}(\xi = \gamma_0, t) = \mathbf{0}, \quad \mathbf{r}(\xi = \gamma_1, t) \cdot \mathbf{E}_2 = -h, \quad \theta(\gamma_0^+, t) = \theta_0. \quad (2.116)$$

In addition, we assume e_0 , e_1 , g , and ρ_0 are prescribed.

Applying the conservation (2.99) to points on the catenary, we find the value of the constant U and some additional physical interpretations of its value:

$$\begin{aligned} U &= -\mathbf{F}_{\gamma_0} \cdot \mathbf{r}'(\gamma_0^+, t) = -\frac{\rho_0}{2} (1 - 4e_0) c^2 \\ &= -\frac{\rho_0}{2} (1 + 4e_1) c^2 + \rho_0 g h. \end{aligned} \quad (2.117)$$

Thus, a relationship between the distance h and the speed c can be determined and U can be expressed in a convenient manner:

²⁴ Our presentation and scope here is strongly influenced by the earlier analytical and experimental work of Biggins [23].

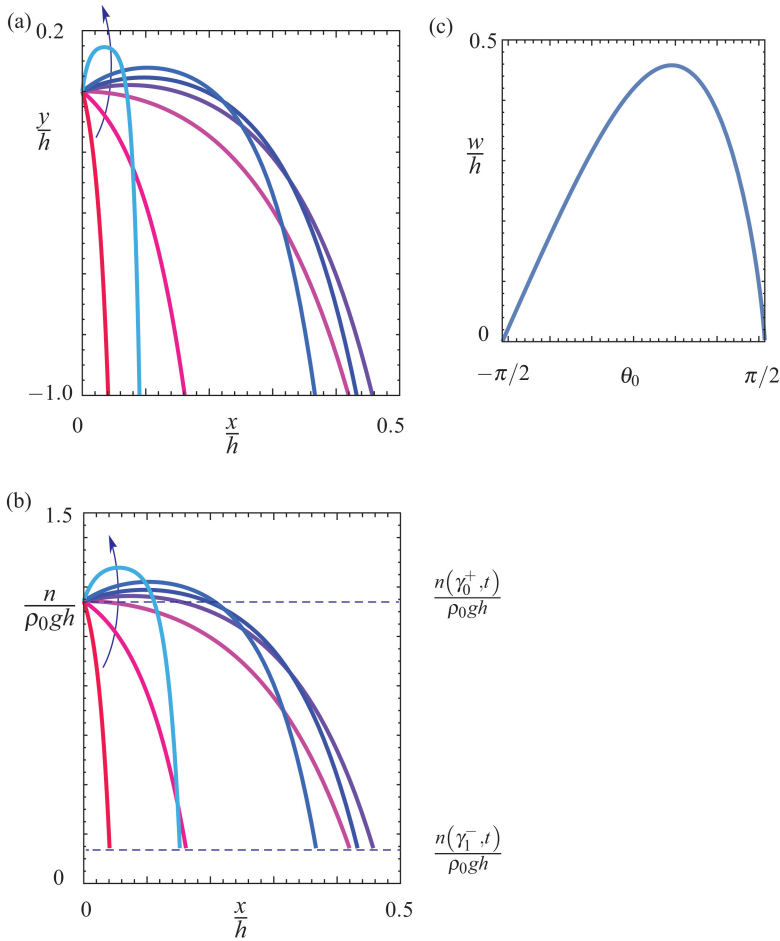


Fig. 2.21 Plots of (a) $y(x)/h$ using (2.105)₁, (b) $n(x)/\rho_0 g h$ using (2.123), and (c) w/h for various values of θ_0 . The arrow in (a) and (b) indicates the direction of increasing θ_0 . For the results shown in this figure, $e_0 = 0.190$, $e_1 = 0.195$, and θ_0 takes the discrete values $-11\pi/24$, $-\pi/3$, 0 , $\pi/6$, $\pi/4$, $\pi/3$, and $11\pi/24$. The constants y_a , x_a , and H in (2.105)₁ and (2.123) are determined using (2.120) and (2.121).

$$c^2 = \frac{gh}{2(e_0 + e_1)},$$

$$U = -\left(\frac{\rho_0 g h}{4}\right) \left(\frac{1 - 4e_0}{e_0 + e_1}\right). \quad (2.118)$$

The identity (2.118)₁ places further restrictions on e_0 and e_1 . A material point of the string gains a kinetic energy $\frac{1}{2}\rho_0 c^2$ in exiting the heap at $\xi = \gamma_0$. This energy must be less than the net loss in potential energy that the material point experiences as

it moves from $\mathbf{r} = \mathbf{0}$ to $-h\mathbf{E}_2 + w\mathbf{E}_1$. Thus, in addition to the energetic restrictions $e_0 > 0$ and $e_1 > 0$ that we discussed earlier,

$$e_0 + e_1 > \frac{1}{4}. \quad (2.119)$$

This energetic restriction was first shown by Biggins [23, Page 4].²⁵ It is interesting to note that Biggins' energetic restriction (2.119) complements the condition $e_1 < 0.25$ needed to ensure that the string does not become slack at $\xi = \gamma_1$.

Conservation of $H = (n - \rho_0 c^2) \cos(\theta)$ (cf. Eqn. (2.101)) enables us to determine H and the slope at γ_1 given the slope at γ_0 :

$$\begin{aligned} \cos(\theta_1) &= \left(\frac{1 - 4e_0}{1 + 4e_1} \right) \cos(\theta_0), \\ H &= U \cos(\theta_0), \end{aligned} \quad (2.120)$$

where $\theta_1 = \theta(\gamma_1^-, t)$. Turning to the equation (2.105) for $y(x)$, we now solve for $(x_a, y(x_a))$ by examining the slope $\frac{dy}{dx}$ at $x = 0$ and using the boundary condition $y(x = 0) = 0$ and the conservations (2.99) and (2.101)₁:

$$\begin{aligned} x_a &= -\frac{H}{\rho_0 g} \sinh^{-1}(\tan(\theta_0)) \\ &= \left(\frac{h}{4} \right) \left(\frac{1 - 4e_0}{e_0 + e_1} \right) \cos(\theta_0) \sinh^{-1}(\tan(\theta_0)), \\ y(x_a) &= -\frac{U}{\rho_0 g} (1 - \cos(\theta_0)) \\ &= \frac{h}{4} \left(\frac{1 - 4e_0}{e_0 + e_1} \right) (1 - \cos(\theta_0)). \end{aligned} \quad (2.121)$$

Note that $y(x_a)$ is the maximum height that the string attains. In addition, w as a function of the boundary conditions (2.116) and parameters e_0 , e_1 , and ρ_0 can be determined:

$$w = w(\rho_0, e_1, e_2, h, \theta_0) = -\frac{H}{\rho_0 g} \cosh^{-1} \left(-\frac{\rho_0 g}{H} (h + y(x_a)) + 1 \right) + x_a. \quad (2.122)$$

With this last result, $y(x)$ can be determined. The reader is referred to Figure 2.21(a) for examples. We found it convenient to non-dimensionalize distance using h and energies by $\rho_0 g h$ as this reduces the number of independent parameters.

²⁵ In [23], the parameters α and β are used to prescribe $n(\gamma_0^+, t) = T(0) = (1 - \alpha)\rho_0 c^2$ and $n(\gamma_1^-, t) = T(w) = \beta\rho_0 c^2$. Thus, examining Eqns. (2.110) and (2.115), we can conclude that $\alpha = \frac{1}{2} - 2e_0$ and $\beta = \frac{1}{2} - 2e_1$. The values of $\alpha = 0.12$ and $\beta = 0.11$ that are used in [23] are equivalent to $e_0 = 0.190$ and $e_1 = 0.195$ that are used to construct the results shown in Figure 2.21.

Having determined w , $y(x)$, and c , it remains to use Eqn. (2.99) to determine the variation in $(n - \rho_0 c^2)$ as we move from $\xi = \gamma_0$ to $\xi = \gamma_1$ (i.e., as x varies from 0 to w)²⁶:

$$n - \rho_0 c^2 = U + \rho_0 g y = H \cosh \left(\frac{\rho_0 g}{H} (x - x_a) \right). \quad (2.123)$$

As anticipated, and shown in Figure 2.21(b), the maximum tension in the string as it ranges from $n(\gamma_0^+, t) = \frac{\rho_0}{2} (1 + 4e_0) c^2$ to $n(\gamma_1^-, t) = \frac{\rho_0}{2} (1 - 4e_1) c^2$ that is predicted by this expression occurs at the apex $x = x_a$.

Solving the chain fountain problem involves an interesting synthesis of background on steady axial motions, the classic solution for a catenary, and models for heaps of strings that was present in some of the earlier problems, such as the one studied by Cayley, that were discussed in the present chapter. Hopefully, it also demonstrates the usefulness of the prescription for B_γ that has been used for a variety of problems throughout this chapter. In contrast to the earlier problems, the solution is assumed to be steady, $\dot{\gamma}(t) = -c$, and the differential equation that is normally provided by the material momentum jump condition is not present in the analysis.

Our discussion and analysis of the chain fountain is far from complete. In the actual chain fountain, it is easy to observe instances where the shape formed by the chain is far more complex than an inverted catenary. In addition, we have not discussed the transient phases of the fountain when it is first set into motion and when the end of the chain finally exits the reservoir at $\xi = \gamma_0$. There is also the issue of drag forces on the chain and impact phenomena that occur when the segments of chain rattle against each other during the motion. While many of these effects can be incorporated into the string model we have used, other models, such as a multibody dynamics model involving a large number of rigid bodies modeling the individual segments of the chain, are required to provide additional insight and to capture effects that are beyond the capabilities of a string model.

2.9 Closing Comments

We cannot do justice to the enormous volume of literature on the application of the string model in the limited number of pages available to us. However, we hope this chapter has given the reader a flavor of some of the problems and a thorough introduction on how to use local balance laws, constitutive relations, and jump conditions to formulate a determinate system of equations from which the motion of the string can be computed. When we use the theory of a string to model any flexible one-dimensional continuum, such as a telephone cord, it quickly becomes apparent that the resistance to an applied bending moment that is absent in the string is present in

²⁶ It is left as an exercise for the interested reader to substitute for H , x_a , and U in terms e_0 , e_1 , θ_0 , ρ_0 , and h in the expressions for w and $n - \rho_0 c^2$ to convince themselves that knowledge of these parameters suffices to determine w and $n - \rho_0 c^2$.

the phone cord. While more elaborate theories, known as rod theories, are needed to accommodate bending, and we shall devote considerable attention to such theories, the structure of the rod theories will be similar to those we have just encountered for a string.

2.10 Exercises

Exercise 2.1: In direct contrast to Cayley's formulation of the problem of a chain falling from a heap that was discussed in Section 2.4, let us turn to formulating this problem *assuming* that energy is conserved.²⁷ For this problem, one can parallel the developments leading to Eqn. (2.37). However, we now make the assumption that

$$\Phi_{E_\gamma} = 0. \quad (2.124)$$

That is, the shock at the heap is not dissipative.

- (a) Show that the assumption that the shock is not dissipative implies that

$$\mathbf{F}_\gamma = -\frac{\rho_0}{2}\dot{\gamma}^2\mathbf{E}_2, \quad \mathbf{B}_\gamma = 0. \quad (2.125)$$

- (b) With the help of the previous results, show that the equation governing $\gamma = w$ is

$$w(\ddot{w} - g) = -\frac{1}{2}\dot{w}^2. \quad (2.126)$$

Construct a phase portrait for the ordinary differential equation (2.126) and compare the solutions to those for Eqn. (2.42) that are shown in Figure 2.8.

- (c) By paralleling the developments leading to Eqn. (2.48), show that the total energy E of the chain is

$$E = \frac{\rho_0}{2}w\dot{w}^2 - \frac{\rho_0 g}{2}w^2. \quad (2.127)$$

Verify that $\dot{E} = 0$.

- (d) Show that the differential equation (2.126) admits the exact solution

$$w(t) = \frac{g}{4}t^2. \quad (2.128)$$

Identify this solution on your phase portrait and then show the well-known result that this solution implies that the terminal acceleration of the chain is $\frac{g}{2}$. How does this compare to the case considered in Section 2.4 with $e = 0.25$?

²⁷ For further references on, and discussion of, the energy-conserving formulation, we refer interested readers to [142, 366, 367].

Exercise 2.2: An inextensible string of mass density ρ_0 per unit length hangs over the edge of a smooth horizontal table (see Figure 2.22). A particle of mass m is attached at the end $\xi = 0$ of the string. At time $t = 0$, the string and the particle are

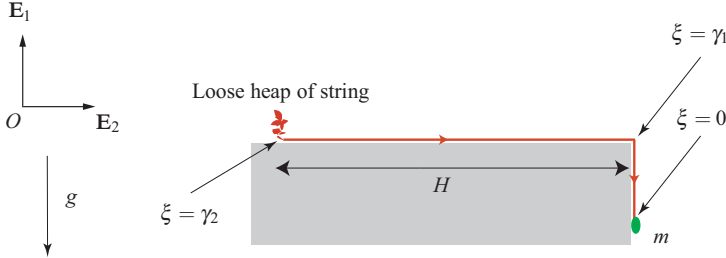


Fig. 2.22 A string leaves a loose heap of stationary string and moves a distance H along the smooth table before falling off the edge.

released from rest. The particle subsequently falls, and as it does it draws additional string from the loose heap of string. This is in contrast to the problem considered in Section 2.5, and the energy loss at the exit point from the heap changes the nature of solutions from those considered previously.

- For an inextensible string, show that if $\mathbf{r} = r(\xi, t)\mathbf{c}$ where \mathbf{c} is a constant, then $\mathbf{v} = v(t)\mathbf{c}$. HINT: Use the inextensibility constraint.
- The velocity vector of the material points of the string experiences two points of discontinuity, one at $\xi = \gamma_1$ and the other where the string exits the heap at $\xi = \gamma_2$. Give representations for $\mathbf{v}_{\gamma_{1,2}}$, $\mathbf{v}(\gamma_1^+, t)$, and $\mathbf{v}(\gamma_2^-, t)$, and show that $\dot{\gamma}_1 = \dot{\gamma}_2$.
- Give prescriptions for the applied force $\rho_0 \mathbf{f}$ acting on the string. Using the local form of the balance of linear momentum, establish expressions for

$$\mathbf{n}(\gamma_2^-, t) - \mathbf{n}(\gamma_1^+, t), \quad \mathbf{n}(\gamma_1^-, t) - \mathbf{n}(0, t). \quad (2.129)$$

- At $\xi = \gamma_1$, assume that no energy is lost (i.e., $\Phi_{E_{\gamma_1}} = 0$). Determine the supply of material momentum B_{γ_2} at this point.
- At $\xi = \gamma_2$, prescribe $\Phi_{E_{\gamma_2}} = -2e\rho_0\dot{\gamma}_2^3$. Determine an expression for the material force B_{γ_2} at this point.
- Using the results of (a)–(d), the appropriate jump conditions, and assuming that $e = 0.25$, show that

$$\dot{\gamma} = \frac{1}{m + \rho_0 H + \rho_0 \gamma} ((m + \rho_0 \gamma)g - \rho_0 \dot{\gamma}^2), \quad (2.130)$$

where $\gamma = \gamma_1$. HINT: The jump condition from the material momentum balance law plays a key role in establishing the desired differential equation.

- (g) After nondimensionalizing the differential equation in (f), numerically integrate the resulting equation and determine the motion of the mass particle m .
- (h) How do the results for $e = 0.25$ compare to those for $e = 0$ and $e = 0.20$. Verify that the case $e = 0$ corresponds to a formulation where it is assumed that energy is conserved.

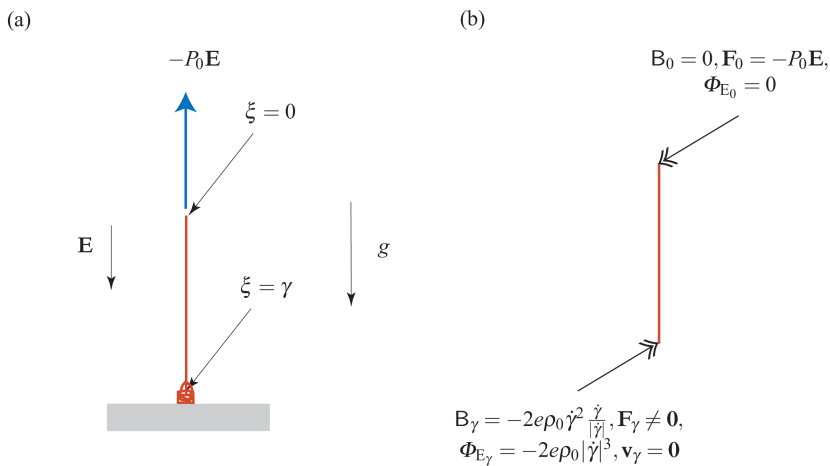


Fig. 2.23 Raising a chain from a stationary heap. (a) Schematic of the string and (b) schematic of the supplies at the shock $\xi = \gamma$ and the terminal point $\xi = 0$.

Exercise 2.3: As shown in Figure 2.23, an inextensible string of mass density ρ_0 per unit length is being pulled from a quiescent heap by a force $-P_0\mathbf{E}$ applied at $\xi = 0$.²⁸ At time $t = 0$, the length of the string above the heap is ℓ_0 and is at rest when the force is applied.

- (a) Show that the tangent vector \mathbf{r}' and velocity vector \mathbf{v} of any point $\xi \in (0, \gamma)$ on the string have the representations

$$\mathbf{r}' = \mathbf{E}, \quad \mathbf{v} = v\mathbf{E} = -\dot{\gamma}\mathbf{E}. \quad (2.131)$$

- (b) Using an appropriate jump condition, show that the contact force at $\xi = 0$ is $\mathbf{n}(0, t) = P_0\mathbf{E}$.
- (c) Show that the contact force immediately above the heap has the representation

$$\mathbf{n}(\gamma^-, t) = P_0\mathbf{E} - \rho_0\gamma(\dot{\gamma} + g)\mathbf{E}. \quad (2.132)$$

²⁸ Versions of this classic problem are discussed in several textbooks (cf. [174, Problem 8, Page 238], [236, Problem 526], and [237, Sample Problem 4/10]). We highly recommend reading [142] and [237, Sample Problem 4/10] for additional perspectives on the present problem.

- (d) With the help of the appropriate jump conditions and assuming that $B_\gamma = -2e\rho_0\dot{\gamma}^2\frac{\dot{\gamma}}{|\dot{\gamma}|}$, show that

$$\begin{aligned} \mathbf{n}(\gamma^-, t) - \rho_0\dot{\gamma}^2\mathbf{E} &= \mathbf{F}_\gamma, \\ \frac{\rho_0}{2}\dot{\gamma}^2 - n(\gamma^-, t) &= -2e\rho_0\dot{\gamma}^2\frac{\dot{\gamma}}{|\dot{\gamma}|}. \end{aligned} \quad (2.133)$$

- (e) Show that the differential equation governing the string is

$$\rho_0\gamma(\dot{\gamma} + g) + \frac{\rho_0}{2}\left(1 + 4e\frac{\dot{\gamma}}{|\dot{\gamma}|}\right)\dot{\gamma}^2 = P_0. \quad (2.134)$$

- (f) After non-dimensionalizing (2.134), construct a phase portrait of the resulting equation of motion for a given value of $P_0/\rho_0 g \ell_0$ and a value of $e \neq 0$ of your choice.²⁹ Your phase portrait should have an equilibrium point where P_0 balances the weight of a section of the string.
- (g) Show that the total energy E of the string has the following representation:

$$E = \frac{\rho_0\gamma}{2}(\dot{\gamma}^2 + g\gamma) - P_0\gamma. \quad (2.135)$$

Using this identity, show that $\dot{E} = \Phi_{E_\gamma}$.

- (h) Show that the assumption that the tension in the string immediately above the heap vanishes corresponds to the choice $e = -0.25$ and implies that a force from the heap pushes upwards on the string at $\xi = \gamma$ and the shock at the heap supplies energy to the string.

Exercise 2.4: With the help of results from Exercise 2.3, solve the following problem from Jeans' classic textbook [174, Problem 8, Page 238]: "A uniform chain is coiled in a heap on a horizontal plane, and a man takes hold of one end and raises it uniformly with a velocity v . Show that when his hand is at a height x from the plane, the pressure on his hand is equal to the weight of a length $x + \frac{v^2}{g}$ of the chain."

Exercise 2.5: With the help of results from Section 2.6, solve the following problem from Jeans' classic textbook [174, Problem 17, Page 252]: "A heavy, perfectly flexible uniform string hanging vertically with its lowest point at a height h above an inelastic horizontal plane is suddenly allowed to fall on to the plane. Show that the pressure on the table when a length x of the string has fallen on to the table is $(3x + 2h)mg$."

Exercise 2.6: The following problem is adapted from the textbooks of Lamb [193, Exercise 4, Page 149] and Tait and Steele [339, Section 316]. Suppose a flexible and uniform chain of length 2ℓ hangs over a smooth pulley and let x be the length of

²⁹ It should not be surprising given the earlier comments on the classic formulations of these problems that the choice $e = 0.25$ corresponds to the case considered in [236, Problem 526] and [237, Sample Problem 4/10].

cable hanging on one side at time t . Show that the differential equation governing the motion of the string is

$$2\rho_0\ell(\ddot{x} - g) = 2\rho_0gx. \quad (2.136)$$

If the chain starts at rest with $x = \ell_0 + \varepsilon$, where ε is a small positive number, then how long does it take for the chain to leave the pulley? Show that the velocity of the chain when it leaves the pulley is $\sqrt{g\ell}$.

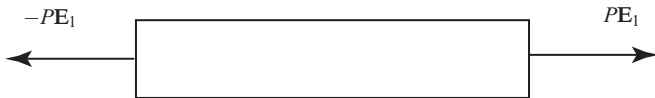


Fig. 2.24 Schematic of a bar which has an undeformed length ℓ and is loaded at its ends by equal and opposite forces $\pm PE_1$. The bar is composed of a material whose strain energy function is prescribed by Eqn. (2.137).

Exercise 2.7: This problem is inspired by the works of studies on phase transformations in bars by Abeyaratne and Knowles [5] and Ericksen [98]. The exercise can be considered as an extension to Exercises 1.8, 1.9, and 1.10 and the aim of the exercise is to show how multiple solutions to this simple problem can be present. We refer the reader to [5] for further analyses, including stability and propagation of $\gamma(t)$, and discussions of related works including [80, 202, 203, 354], on this intriguing well-studied example.

As shown in Figure 2.24, consider a bar of length ℓ in its undeformed state which is then loaded by terminal forces $\pm PE_1$. The bar is modeled by an elastic string with a strain energy function:

$$\rho_0\psi = \alpha_1 \left(\frac{1}{4}(\mu - \alpha_2)^4 + \mu \left(1 + \alpha_2 - \frac{\mu}{2} \right) \right) + \alpha_1\alpha_3 \log \left(\mu - 1 + \frac{1}{\mu} \right), \quad (2.137)$$

where α_1 , α_2 , and α_3 are constants. As discussed in Exercise 1.9, we choose $\alpha_1 > 0$, $\alpha_2 \approx 2.32472$, and $\alpha_3 \geq 0$. The contact force n associated with the strain energy function is (cf. Eqn. (1.148))

$$n = \alpha_1 \left((\mu - \alpha_2)^3 + 1 + \alpha_2 - \mu \right) + \alpha_1\alpha_3 \left(\frac{\mu^2 - 1}{\mu^3 + \mu - \mu^2} \right). \quad (2.138)$$

Referring to Figure 2.25, each of three regions of the function $n = n(\mu)$ are known as phases. There are three such phases, I, II, and III, for the strain energy function (2.137).

- (a) Assuming a static uniaxial solution to the boundary-value problem where $\mathbf{r} = x\mathbf{E}_1$, establish an expression for the material contact force C in the bar as a function of the stretch μ .

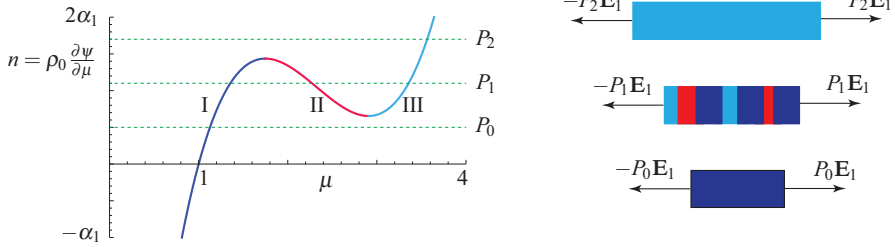


Fig. 2.25 The force n as a function of μ for the strain energy (2.137) and associated solutions to a boundary-value problem for a terminally loaded bar. For the examples shown, $\alpha_2 \approx 2.32472$, $\alpha_3 = 0.1$, $P_0 < 0.654306\alpha_1$, $P_1 \in (0.654306\alpha_1, 1.43586\alpha_1)$, and $P_2 > 1.43586\alpha_1$.

- (b) Numerically compute the graph of C as a function of n for $\alpha_2 \approx 2.32472$ and various values of α_3 . It is helpful to note the direction of increasing μ along the branches of this graph.³⁰
- (c) Suppose that $\alpha_2 \approx 2.32472$, $\alpha_3 = 0.1$, and $P = \alpha_1$. Show that solutions to the boundary-value problem for phase I with $\mu = 1.30595$, phase II with $\mu = 2.37068$, and phase III with $\mu = 3.30683$ are possible. At the interface $\xi = \gamma$ between these phases show that a singular supply of material momentum is typically present (i.e., $B_\gamma \neq 0$).³¹
- (d) Argue that, depending on the magnitude of P , the stretch in the bar is either uniquely determined or an infinite number of solutions are possible.

³⁰ The results shown in Figure 1.20 might be of assistance with these computations.

³¹ The relationship between B_γ and the driving force f is discussed in Exercise 1.7 on Page 45.

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