

Contents

1	Tensor Products	1
1.1	Multilinear Maps	1
1.1.1	Multilinear Maps Between Free Modules	1
1.1.2	Universal Multilinear Map	3
1.2	Tensor Product of Modules	4
1.2.1	Existence of Tensor Product	5
1.2.2	Linear Maps as Tensors	7
1.2.3	Tensor Products of Abelian Groups	9
1.3	Commutativity, Associativity, and Distributivity Isomorphisms	10
1.4	Tensor Product of Linear Maps	13
1.5	Tensor Product of Modules Presented by Generators and Relations	15
	Problems for Independent Solution to Chapter 1	17
2	Tensor Algebras	21
2.1	Free Associative Algebra of a Vector Space	21
2.2	Contractions	22
2.2.1	Complete Contraction	22
2.2.2	Partial Contractions	23
2.2.3	Linear Support and Rank of a Tensor	25
2.3	Quotient Algebras of a Tensor Algebra	26
2.3.1	Symmetric Algebra of a Vector Space	26
2.3.2	Symmetric Multilinear Maps	27
2.3.3	The Exterior Algebra of a Vector Space	29
2.3.4	Alternating Multilinear Maps	30
2.4	Symmetric and Alternating Tensors	31
2.4.1	Symmetrization and Alternation	32
2.4.2	Standard Bases	33
2.5	Polarization of Polynomials	35
2.5.1	Evaluation of Polynomials on Vectors	36
2.5.2	Combinatorial Formula for Complete Polarization	37

2.5.3	Duality	38
2.5.4	Derivative of a Polynomial Along a Vector	38
2.5.5	Polars and Tangents of Projective Hypersurfaces	40
2.5.6	Linear Support of a Homogeneous Polynomial	43
2.6	Polarization of Grassmannian Polynomials	45
2.6.1	Duality	45
2.6.2	Partial Derivatives in an Exterior Algebra	46
2.6.3	Linear Support of a Homogeneous Grassmannian Polynomial	47
2.6.4	Grassmannian Varieties and the Plücker Embedding	49
2.6.5	The Grassmannian as an Orbit Space	49
	Problems for Independent Solution to Chapter 2	51
3	Symmetric Functions	57
3.1	Symmetric and Sign Alternating Polynomials	57
3.2	Elementary Symmetric Polynomials	60
3.3	Complete Symmetric Polynomials	61
3.4	Newton's Sums of Powers	62
3.4.1	Generating Function for the p_k	62
3.4.2	Transition from e_k and h_k to p_k	63
3.5	Giambelli's Formula	65
3.6	Pieri's Formula	67
3.7	The Ring of Symmetric Functions	69
	Problems for Independent Solution to Chapter 3	71
4	Calculus of Arrays, Tableaux, and Diagrams	75
4.1	Arrays	75
4.1.1	Notation and Terminology	75
4.1.2	Vertical Operations	76
4.1.3	Commutation Lemma	77
4.2	Condensing	79
4.2.1	Condensed Arrays	79
4.2.2	Bidense Arrays and Young Diagrams	80
4.2.3	Young Tableaux	81
4.2.4	Yamanouchi Words	82
4.2.5	Fiber Product Theorem	83
4.3	Action of the Symmetric Group on DU-Sets	86
4.3.1	DU-Sets and DU-Orbits	86
4.3.2	Action of $S_m = \text{Aut}(J)$	86
4.4	Combinatorial Schur Polynomials	88
4.5	The Littlewood–Richardson Rule	91
4.5.1	The Jacobi–Trudi Identity	93
4.5.2	Transition from e_λ and h_λ to s_λ	93
4.6	The Inner Product on Λ	95
	Problems for Independent Solution to Chapter 4	96

5	Basic Notions of Representation Theory	99
5.1	Representations of a Set of Operators	99
5.1.1	Associative Envelope	99
5.1.2	Decomposability and (Semi)/Simplicity	100
5.1.3	Homomorphisms of Representations	103
5.2	Representations of Associative Algebras	104
5.2.1	Double Centralizer Theorem	104
5.2.2	Digression: Modules Over Noncommutative Rings	106
5.3	Isotypic Components	107
5.4	Representations of Groups	109
5.4.1	Direct Sums and Tensor Constructions	109
5.4.2	Representations of Finite Abelian Groups	111
5.4.3	Reynolds Operator	113
5.5	Group Algebras	114
5.5.1	Center of a Group Algebra	115
5.5.2	Isotypic Decomposition of a Finite Group Algebra	115
5.6	Schur Representations of General Linear Groups	121
5.6.1	Action of $GL(V) \times S_n$ on $V^{\otimes n}$	122
5.6.2	The Schur–Weyl Correspondence	124
	Problems for Independent Solution to Chapter 5	124
6	Representations of Finite Groups in Greater Detail	131
6.1	Orthogonal Decomposition of a Group Algebra	131
6.1.1	Invariant Scalar Product and Plancherel’s Formula	131
6.1.2	Irreducible Idempotents	133
6.2	Characters	134
6.2.1	Definition, Properties, and Examples of Computation	134
6.2.2	The Fourier Transform	137
6.2.3	Ring of Representations	140
6.3	Induced and Coinduced Representations	141
6.3.1	Restricted and Induced Modules Over Associative Algebras	141
6.3.2	Induced Representations of Groups	142
6.3.3	The Structure of Induced Representations	143
6.3.4	Coinduced Representations	146
	Problems for Independent Solution to Chapter 6	148
7	Representations of Symmetric Groups	151
7.1	Action of S_n on Filled Young Diagrams	151
7.1.1	Row and Column Subgroups Associated with a Filling	151
7.1.2	Young Symmetrizers $s_T = r_T \cdot c_T$	153
7.1.3	Young Symmetrizers $s'_T = c_T \cdot r_T$	155
7.2	Modules of Tabloids	157

7.3	Specht Modules	159
7.3.1	Description and Irreducibility	159
7.3.2	Standard Basis Numbered by Young Tableaux	160
7.4	Representation Ring of Symmetric Groups	161
7.4.1	Littlewood–Richardson Product	162
7.4.2	Scalar Product on \mathfrak{R}	163
7.4.3	The Isometric Isomorphism $\mathfrak{R} \simeq \Lambda$	164
7.4.4	Dimensions of Irreducible Representations	168
	Problems for Independent Solution to Chapter 7	170
8	\mathfrak{sl}_2-Modules	173
8.1	Lie Algebras	173
8.1.1	Universal Enveloping Algebra	173
8.1.2	Representations of Lie Algebras	174
8.2	Finite-Dimensional Simple \mathfrak{sl}_2 -Modules	176
8.3	Semisimplicity of Finite-Dimensional \mathfrak{sl}_2 -Modules	179
	Problems for Independent Solution to Chapter 8	183
9	Categories and Functors	187
9.1	Categories	187
9.1.1	Objects and Morphisms	187
9.1.2	Mono-, Epi-, and Isomorphisms	189
9.1.3	Reversing of Arrows	190
9.2	Functors	191
9.2.1	Covariant Functors	191
9.2.2	Presheaves	192
9.2.3	The Functors Hom	195
9.3	Natural Transformations	197
9.3.1	Equivalence of Categories	198
9.4	Representable Functors	200
9.4.1	Definitions via Universal Properties	203
9.5	Adjoint Functors	205
9.5.1	Tensor Products Versus Hom Functors	206
9.6	Limits of Diagrams	213
9.6.1	(Co) completeness	217
9.6.2	Filtered Diagrams	218
9.6.3	Functorial Properties of (Co) limits	219
	Problems for Independent Solution to Chapter 9	222
10	Extensions of Commutative Rings	227
10.1	Integral Elements	227
10.1.1	Definition and Properties of Integral Elements	227
10.1.2	Algebraic Integers	230
10.1.3	Normal Rings	231
10.2	Applications to Representation Theory	232
10.3	Algebraic Elements in Algebras	234

10.4	Transcendence Generators	236
	Problems for Independent Solution to Chapter 10	239
11	Affine Algebraic Geometry	241
11.1	Systems of Polynomial Equations	241
11.2	Affine Algebraic–Geometric Dictionary	243
11.2.1	Coordinate Algebra	243
11.2.2	Maximal Spectrum	244
11.2.3	Pullback Homomorphisms	246
11.3	Zariski Topology	250
11.3.1	Irreducible Components	251
11.4	Rational Functions	253
11.4.1	The Structure Sheaf	254
11.4.2	Principal Open Sets as Affine Algebraic Varieties	255
11.5	Geometric Properties of Algebra Homomorphisms	256
11.5.1	Closed Immersions	257
11.5.2	Dominant Morphisms	257
11.5.3	Finite Morphisms	258
11.5.4	Normal Varieties	259
	Problems for Independent Solution to Chapter 11	261
12	Algebraic Manifolds	265
12.1	Definitions and Examples	265
12.1.1	Structure Sheaf and Regular Morphisms	268
12.1.2	Closed Submanifolds	268
12.1.3	Families of Manifolds	269
12.1.4	Separated Manifolds	269
12.1.5	Rational Maps	271
12.2	Projective Varieties	272
12.3	Resultant Systems	274
12.3.1	Resultant of Two Binary Forms	276
12.4	Closeness of Projective Morphisms	278
12.4.1	Finite Projections	279
12.5	Dimension of an Algebraic Manifold	281
12.5.1	Dimensions of Subvarieties	283
12.5.2	Dimensions of Fibers of Regular Maps	285
12.6	Dimensions of Projective Varieties	286
	Problems for Independent Solution to Chapter 12	290
13	Algebraic Field Extensions	295
13.1	Finite Extensions	295
13.1.1	Primitive Extensions	296
13.1.2	Separability	297
13.2	Extensions of Homomorphisms	300
13.3	Splitting Fields and Algebraic Closures	302
13.4	Normal Extensions	304

13.5	Compositum	306
13.6	Automorphisms of Fields and the Galois Correspondence	307
	Problems for Independent Solution to Chapter 13	311
14	Examples of Galois Groups	315
14.1	Straightedge and Compass Constructions	315
14.1.1	Effect of Accessory Irrationalities	318
14.2	Galois Groups of Polynomials	319
14.2.1	Galois Resolution	321
14.2.2	Reduction of Coefficients	322
14.3	Galois Groups of Cyclotomic Fields	323
14.3.1	Frobenius Elements	324
14.4	Cyclic Extensions	326
14.5	Solvable Extensions	328
14.5.1	Generic Polynomial of Degree n	331
14.5.2	Solvability of Particular Polynomials	332
	Problems for Independent Solution to Chapter 14	333
	Hints to Some Exercises	335
	References	355
	Index	357

Algebra II

Textbook for Students of Mathematics

Gorodentsev, A.L.

2017, XV, 370 p. 155 illus., 2 illus. in color., Hardcover

ISBN: 978-3-319-50852-8