Chapter 2
Essential Physics of Inertial Confinement Fusion (ICF)

In order to study plasma physics and its behavior for a source of driving fusion in a controlled thermonuclear reaction for purpose of generating energy, understanding of the fundamental knowledge of electromagnetic theory is essential. In this chapter, we introduce Maxwell equations and Coulomb's barrier or Tunnel effects for better understanding of plasma behavior for confinement purpose. The controlled thermonuclear reaction for generating clean energy that is confined magnetically or inertially requires some basic understanding of physics and mathematics rules and knowledge. We are mainly concerned with confinement of plasmas at terrestrial temperature, e.g., very hot plasmas, where primarily of interest is in application to controlled fusion research in magnetic confinement reactors such as tokomak or using high-power laser or high-energy particles for purpose of inertial confinement fusion. Dimensional analysis and self-similarity allow us to have better understanding of implosion and explosion process in case of lateral confinement approach. This chapter is walking through some of the essentials that one needs to know for the process of inertial confinement in particular as subject of this book, which are all about.

2.1 Introduction

Physics of laser-driven plasmas has a route that goes to day that the advent of the laser was a substantial confirmation of Einstein's derivation of Planck's radiation law, along with the discovery of quantum physics. The discovery of quantum physics opened a new door to huge industry of physics and quantum optics, which today is influencing our daily life in health to technical industry as well as in advanced defense by introducing new weapons to tomorrow's battle field, where high-energy lasers are dominating weapon systems [1].

© Springer International Publishing AG 2017

B. Zohuri, *Inertial Confinement Fusion Driven Thermonuclear Energy*,
DOI 10.1007/978-3-319-50907-5_2

Directed energy weapons are nothing new to mankind; historically the origin of such weapons began centuries ago, when the famous Greek mathematician, physicist, engineer, inventor, and astronomer Archimedes of Syracuse used different mirrors to collect sunbeams and focused them on the Roman fleet in order to destroy enemy ships with fire, driven by energy of the beams. This is known as the Archimedes Heat Ray. Archimedes may have used mirror acting collectively as a parabolic reflector to burn ships attacking Syracuse. The device was used to focus sunlight onto approaching ships, causing them to catch fire. Of course, the myth or reality of the Archimedes Heat Ray still is questionable. However, today with help of High-Energy Laser (HEL) technology existence, the myth is approaching to reality and battles of tomorrow will be fought with different weapons that have more lethal effects and faster delivery systems with more accurate focusing on target mechanisms.

The interaction of laser radiation or high-energy particle beam with matter has introduced us to a new technological domain that has opened the door to new basic physics rules. When the response of dielectric materials was extended to numerous nonlinearities known before with all kind of applications in electronics of optics and communication, a much stranger physics phenomena appeared at the very high intensities, where all materials are vaporized and ionized. This ionization pushed the laser interaction with matter to state of plasma, and the response of the generated plasmas exceeded all of the previously known curiosities. This chapter will cover the essential physics of such reaction that deals with behavior of plasma, introduced by this intense energy driven by laser or particle beams.

Plasma is generated about 50,000 °K, emitting ions of few eV energy, slightly higher intensities produced KeV ions. Not too long ago, scientists at different national laboratories demonstrate the nuclear fusion reactions were ignited, but highly nonthermal electron energy distributions were detected by X-rays and anomalous fast groups of ions appeared. Invention of Inertial Confinement Fusion (ICF) is a new way of creating clean source of energy, so long as we have oceans of water surrounding us however, this innovating approach for renewable source of energy has its own associated scientific and technological difficulties that require overcoming. For us to understand the nature of ICF and be able to deal with associated obstacles, we need to have the better understanding of essential physics that handles these obstacles.

2.2 General Concept of Electromagnetisms and Electrostatics

The subject of electricity is briefly touched upon for rest of this chapter to provide us with a fundamental of magnetism that we need in order to understand the science of plasma physics to go forward. We deal with the empirical concepts of charge and the force law between charges known as Coulomb's Law. However, we use the

mathematical tools of previous section to express this law in other or more powerful formulations and then extended to basics of plasma physics concept. The electric potential formulation and Gauss's Law are very important to the subsequent development of the subject. Electric charge is a fundamental and characteristic property of the microscopic particles that makes up matter. In fact, all atoms are composed of photons, neutrons, and electrons, and two of these particles bear charges. However, even charge particles, the powerful electrical forces associated with these particles are fairly well hidden in a macroscopic observation. The reason behind such statement exists because of nature of two kinds of charges existence, namely, *Positive* and *Negative* charges, and an ordinary piece of matter contains approximately equal amounts of each kind.

It is understood from experimental observation that charge can be neither created nor destroyed. The total charge of a closed system cannot change. From the macroscopic point of view, charges may be regrouped and combined in different ways; nevertheless, we may state that *net charge is conserved in a closed system* [2].

2.2.1 The Coulomb's Law

To establish the Coulomb's law we need to summarize in three following statements as

1. There are two and only two kinds of electric charge, now known as positive or negative.
2. Two point charges exert on each other forces that act along the line joining them and are inversely proportional to the square of the distance between them.
3. These forces are also proportional to the product of the charges, are repulsive for like charges, and attractive for unlike charges.

The last two statements, with the first as preamble, all together, are known as Coulomb's Law and for point charges may be concisely formulated in the vector notation as

$$\vec{F}_1 = C_u \frac{q_1 q_2}{r_{12}^2} \frac{\vec{r}_{12}}{r_{12}} \qquad \text{(Eq. 2.1a)}$$
$$\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$$

where $\vec{F}_1$ is the force on charge q_1, $\vec{r}_{12}$ is the vector to charge q_1 from charge q_2, r_{12} is the magnitude of vector $\vec{r}_{12}$, and C_u is a constant of proportionality about which is defined as to be equal to 1 in adoption with Gaussian system of units. Figure 1.6 will describe the vector $\vec{r}_{12}$ with respect to an arbitrary origin O.

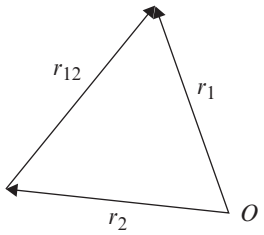


Fig. 2.1 Vector $\vec{r}_{12}$, extending between two points

In Fig. 2.1 vector $\vec{r}_{12}$ is extending from the point at the tip of vector $\vec{r}_2$ to the point at the tip of the vector $\vec{r}_1$ and clearly $\vec{r}_{12} = -\vec{r}_{21}$. Note that Coulomb's law applies to point charges and in macroscopic sense, a "point charge" is one whose spatial dimensions are very small compared with any other length pertinent to the problem under consideration and that is why we use the term "point charge" in this sense.

In the MKS system, Coulomb's law for the force between two point charges can thus be written as

$$\vec{F}_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}^2}\frac{\vec{r}_{12}}{r_{12}} \qquad \text{(Eq.2.1b)}$$

If more than two point charges are present, the mutual forces are determined by the repeated application of Eqs. 2.1a and 2.1b. In particular, if a system of N charges is considered, the force on the ith charge is given by

$$\vec{F}_1 = q_i \sum_{i \neq j}^{N} \frac{q_j}{4\pi\varepsilon_0 r_{ij}^3}\vec{r}_{ij} \qquad \text{(Eq.2.2)}$$
$$\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$$

where the summation on the right-hand side of Eq. 2.2 is extended over all of the charges except the ith. Equation 1.61 is the superposition principle for forces, which says that the total force acting on a body is the vector sum of the individual forces that act on it. Note that in MKS unit the value of Coulomb Constant $C = 9 \times 10^9 \mathrm{N\,m^2/C^3}$.

There are cases such as fully ionized plasma, where we may need to describe a charge distribution in terms of a *charge density function*, thus, it is defined as the limit of charge per unit volume as the volume becomes infinitesimal. However, care must be taken in applying this kind of description to atomic problems, since in such cases only a small number of electrons are involved, and the process of taking the limit is meaningless. Nevertheless, aside atomic case, we may proceed as though a segment of charges might be subdivided indefinitely, we thus, describe the charge distribution by means of point functions.

A *volume charge density* is defined by

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta q}{\Delta V} \qquad \text{(Eq. 2.3)}$$

and a *surface charge density* is defined by

$$\sigma = \lim_{\Delta S \to 0} \frac{\Delta q}{\Delta S} \qquad \text{(Eq. 2.4)}$$

From above statements and what has been said about point charge q, it is evident that ρ and σ are net charge, or excess charge, densities. It is worth to mention that in typical solid materials even a very large charge density ρ will involve a change in the local electron density of only about one part 10^9.

Now that we have some concept of point charge and established Eqs. 2.1a, 2.1b, and 2.2, we extend our knowledge to more general case. In this case, the charge is distributed through a volume V with density ρ, and on the surface S that bounds the volume V with a surface density σ, then the force exerted by this charge distribution on a point charge q located at $\vec{r}$ is obtained from Eq. 2.2 by replacing q_j with $\rho_j dv'_j$ or with $\sigma_j da'_j$ and processing to the limit as

$$\begin{aligned} \vec{F}_q = &\frac{q}{4\pi\varepsilon_0}\int_V \frac{\vec{r}-\vec{r}'}{\left|\vec{r}-\vec{r}'\right|^3}\rho(\vec{r}')dv' \\ &+ \frac{q}{4\pi\varepsilon_0}\int_S \frac{\vec{r}-\vec{r}'}{\left|\vec{r}-\vec{r}'\right|^3}\sigma(\vec{r}')da' \end{aligned} \qquad \text{(Eq. 2.5)}$$

The variable $\vec{r}'$ is used to locate a point within the charge distribution that is playing the role of the source point $\vec{r}_j$ in Eq. 2.2 [2].

Equations 2.2 and 2.5 provide a ready means for obtaining an expression for the electric field due to given distribution of charge as it is presented in Fig. 2.2 and electric field is discussed in the next section.

It may appear that the first integral in Eq. 1.64 will diverge if point $\vec{r}$ should fall inside the charge distribution, but that is not the case at all.

In Fig. 2.2, the vector $\vec{r}$ defines the observation point (i.e., field point), and $\vec{r}'$ ranges over the entire charge distribution, including point charges.

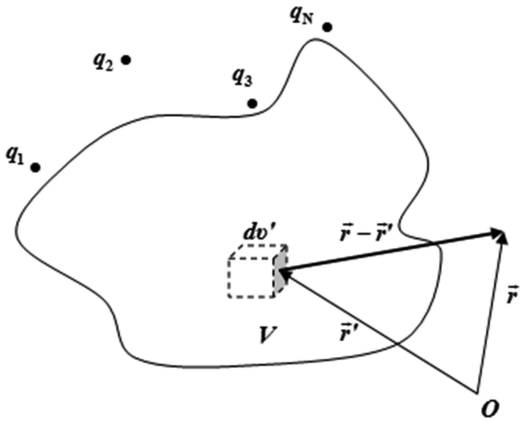


Fig. 2.2 Geometry of $\vec{r}$, $\vec{r}'$ and $\vec{r}-\vec{r}'$

2.2.2 The Electric Field

Our first attempt to seek the electric field is for point charge for the sake of simplicity. The *electric field* at a point is defined operationally as the limit of the force on a test charge placed at the point to the charge of the test charge and the limit being taken as the magnitude of the test charge goes to zero. The customary symbol for electric field in electromagnetic subject is $\vec{E}$ and not to be mistaken for energy presentation, which is the case by default. Thus, we can write

$$\vec{E} = \lim_{q \to 0} \frac{\vec{F}_q}{q} \qquad \text{(Eq. 2.6)}$$

The limiting process is included in the definition of electric field to ensure that the test charge does not affect the charge distribution that produces $\vec{E}$.

Using Fig. 2.2, we let the charge distribution consists of N point $q_1, q_2, \cdots, q_N$ located at the points $\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_N$, respectively, and a volume distribution of charge specified by the charge density $\rho(\vec{r}')$ in the volume V and a surface distribution characterized by the surface charge density $\sigma(\vec{r}')$ on the surface S. If a test charge q is located at the point $\vec{r}$, it experiences force $\vec{F}$ given by the following equation due to the given charge distribution:

$$\begin{aligned} \vec{F} = & \frac{q}{4\pi\varepsilon_0} \sum_{i=1}^{N} q_i \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|^3} \\ & + \frac{q}{4\pi\varepsilon_0} \int_V \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \rho(\vec{r}') d\upsilon' \\ & + \frac{q}{4\pi\varepsilon_0} \int_S \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \sigma(\vec{r}') da' \end{aligned} \qquad \text{(Eq. 2.7)}$$

In case of Eq. 2.7, the electric field at the point $\vec{r}$ is then the limit of the ratio of this force to the test charge q. Since the ratio is independent of q, the electric field at $\vec{r}$ is just

$$\begin{aligned} \vec{E}(\vec{r}) = & \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} q_i \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|^3} \\ & + \frac{1}{4\pi\varepsilon_0} \int_V \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \rho(\vec{r}') d\upsilon' \\ & + \frac{1}{4\pi\varepsilon_0} \int_S \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \sigma(\vec{r}') da' \end{aligned} \qquad \text{(Eq. 2.8)}$$

Equation 2.8 is very general and in most cases, one or more of the terms will not be needed.

In order to complete the electromagnetic foundation circle, we also quickly note the general form of the potential energy associated with an arbitrary conservative force $\vec{F}(\vec{r}')$ as the following form.

$$U(\vec{r}) = -\int_{\text{ref.}}^{\vec{r}} \vec{F}(\vec{r}') \cdot d\vec{r}' \qquad \text{(Eq. 2.9)}$$

where $U(\vec{r})$ is the potential energy at $\vec{r}$ relative to the reference point at which the potential energy is arbitrarily taken to be zero. Proof is left to the reader by referring to the book of Reitz et al. [2].

2.2.3 The Gauss's Law

One of the important relationships that exist between the integral of the normal component of the electric field over a closed surface and the total charge distribution enclosed by the surface is Gauss's Law. To investigate that briefly here, we look at the electric field $\vec{E}(\vec{r})$ for a point charge $\vec{r}\,q$ located at the origin we can write the following relation as before

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3} \qquad \text{(Eq. 2.10)}$$

Consider the surface integral of the normal component of this electric field over a closed surface such that shown in Fig. 2.3 that encloses the origin and, consequently, the charge q, then we can write

$$\oint_S \vec{E} \cdot \hat{n} da = \frac{q}{4\pi\varepsilon_0}\oint_s \frac{\vec{r} \cdot \hat{n}}{r^3} da \qquad \text{(Eq. 2.11)}$$

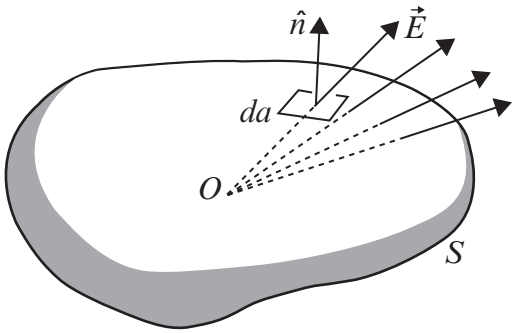


Fig. 2.3 An imaginary closed surface S including point charge at origin

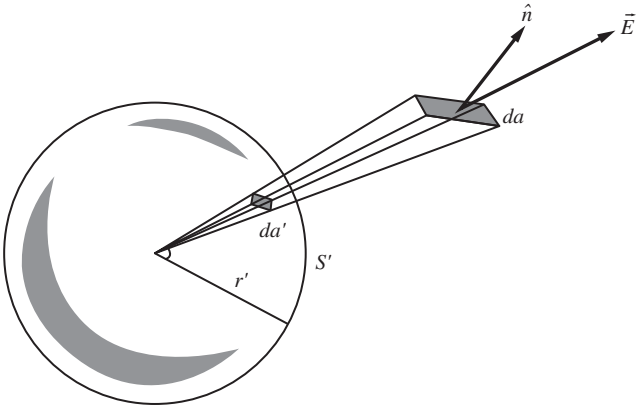


Fig. 2.4 Construction of the spherical surface $S^{'}$

The quantity $(\vec{r}/r) \cdot \hat{n}da$ is the projection of da on a plane perpendicular to $\vec{r}$. This projected area divided by r^2 is the solid angle subtended by da, which is written in $d\Omega$. It is clear from Fig. 2.4 that the solid angle subtended by the da is the same as the solid angle subtended by $da^{'}$, an element of the surface area of the sphere $S^{'}$ whose center is at origin and whose radius is $r^{'}$. It is then possible to write

$$\oint_s \frac{\vec{r} \cdot \hat{n}}{r^3} da = \oint_{S'} \frac{\vec{r}' \cdot \hat{n}}{r'^3} da' = 4\pi \qquad \text{(Eq. 2.12)}$$

which shows that as the following equation in the spherical case described above

$$\oint_S \vec{E} \cdot \hat{n} da = \frac{q}{4\pi\varepsilon_0}(4\pi)\frac{q}{\varepsilon_0} \qquad \text{(Eq. 2.13)}$$

Figure 2.4, illustrates the construction of the spherical surface $S^{'}$ as an aid to evaluation of the solid angle subtended by da. If q lies outside of S, it is clear from Fig. 1.10 that S can be divided into two areas, S_1 and S_2 each of which subtends the same solid angle at the charge q. For S_2, however, the direction of the normal is toward q, while for S_1 it is away from q.

More details can be found in reference by Reitz et al. [2], where readers need to go to; however, in case of several point charges $q_1, q_2, \ldots, q_N$ are enclosed by the surface S, then the total electric field is given by the first term of Eq. 2.8. Each charge subtends a full solid angle (4π); hence Eq. 2.13 becomes

$$\oint_S \vec{E} \cdot \hat{n} da = \frac{1}{\varepsilon_0} \sum_{i=1}^{N} q_i \qquad \text{(Eq. 2.14)}$$

The result in Eq. 2.14 can be readily generalized to the case of a continuous distribution of charge characterized by a charge density [2].

2.3 Solution of Electrostatic Problems

Briefly, we mention and write equations for the solution to an electrostatic problem, which is straightforward for the case in which the charge distribution is everywhere, specified, for then, as we have illustrated so far. The potential and electric field are given as an integral form over this charge distribution as

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{dq'}{|\vec{r} - \vec{r}'|} \qquad \text{(Eq. 2.15)}$$

$$\vec{E}(\vec{r}') = \frac{1}{4\pi\varepsilon_0} \int \frac{(\vec{r} - \vec{r}')dq'}{|\vec{r} - \vec{r}'|^3} \qquad \text{(Eq. 2.16)}$$

However, many of the problems that we encountered in real practice are not of this kind. If the charge distribution is not specified in advance, it may be necessary to determine the electric field *first*, before the charge distribution can be calculated.

2.3.1 Poisson's Equation

The only basic relationships we need here so far are developed in the preceding sections, thus for that matter, we first write the differential form of Gauss's law as

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0}\rho \qquad \text{(Eq. 2.17)}$$

Equation 2.17 in a purely electrostatic field $\vec{E}$ may be expressed as minus the gradient of the potential φ:

$$\vec{E} = -\vec{\nabla}\varphi \qquad \text{(Eq. 2.18)}$$

Combining Eqs. 2.17 and 2.18, we obtain the following relation as

$$\vec{\nabla} \cdot \vec{\nabla}\varphi = -\frac{\rho}{\varepsilon_0} \qquad \text{(Eq. 2.19a)}$$

Using vector identity as a single differential operator, $\vec{\nabla} \cdot \vec{\nabla}$ or ∇^2, which is called the *Laplacian*.

The Laplacian is a scalar differential operator and Eq. 2.18 is a differential equation that is known as *Poisson's Equation* and written as

$$\nabla^2\varphi = -\frac{\rho}{\varepsilon_0} \qquad \text{(Eq. 2.19b)}$$

The Laplace operator for Poisson Equation, in rectangular, cylindrical, and spherical coordinates is presented here as well.

2.3.1.1 Rectangular or Cartesian Coordinate

$$\nabla^2\varphi \equiv \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = -\frac{\rho}{\varepsilon_0} \qquad \text{(Eq. 2.20)}$$

2.3.1.2 Cylindrical Coordinate

$$\nabla^2\varphi \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2\varphi}{\partial\theta} + \frac{\partial^2\varphi}{\partial z^2} = -\frac{\rho}{\varepsilon_0} \qquad \text{(Eq. 2.21)}$$

2.3.1.3 Spherical Coordinate

$$\nabla^2\varphi \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2} = -\frac{\rho}{\varepsilon_0} \qquad \text{(Eq. 2.22)}$$

For the form of the Laplacian in other more complicated coordinated system, the reader is referred to the reference such as any vector analysis or advanced calculus books.

2.3.2 Laplace's Equation

Problems in electrostatic that are involving conductors, all the charges are either found on the surface of the conductors or in the form of fixed-point charges. In these cases, charge density ρ is zero at most points in space and in absence of charge density, the Poisson equation reduces to the simpler form as follows

$$\nabla^2\varphi = 0 \qquad \text{(Eq. 2.23)}$$

2.4 Electrostatic Energy

From then on, without further detailed discussion and proof of different aspects of electrostatic equation, we just write them down as basic knowledge and we leave details to the readers to refer themselves to various subject books out in the open market.

Therefore to go on with the subject in hand, we express that under static condition, the entire energy of the charge system exists as potential energy, and in this section we are particularly concerned with the potential energy that arises from electrical interaction of the charges, so-called *electrostatic energy U*.

We presented that the electrostatic energy U of a point charge is closely related to the electrostatic potential φ at the position of the point charge $\vec{r}$ per Eq. 2.9. In fact, if q is the magnitude of a particular point charge, then the work done by the force on the charge when it moves from position A to position B is given as

$$\begin{aligned} \text{Work} &= \int_A^B \vec{F} \cdot d\vec{l} = q \int_A^B \vec{E} \cdot d\vec{l} \\ &= -q \int_A^B \vec{\nabla}\varphi \cdot d\vec{l} = -q(\varphi_B - \varphi_A) \end{aligned} \qquad \text{(Eq. 2.24)}$$

Here $\vec{F}$ has been assumed to be only the electric force $q\vec{E}$ at each point along the path or the *total work* is finalized to

$$W = -q(\varphi_B - \varphi_A) \qquad \text{(Eq. 2.25)}$$

2.4.1 Potential Energy of a Group of Point Charges

The equation for potential energy of a group of point charges can be expressed as

$$U = \sum_{j=1}^{m} W_j = \sum_{j=1}^{m} \left(\sum_{k=1}^{j-1} \frac{q_j q_k}{4\pi\varepsilon_0 r_{jk}} \right) \qquad \text{(Eq. 2.26)}$$

or in summary Eq. 1.85 can be reduced to

$$U = \frac{1}{2} \sum_{j=1}^{m} \sum_{k=1}^{m}{}' \frac{q_j q_k}{4\pi\varepsilon_0 r_{jk}} \qquad \text{(Eq. 2.27)}$$

Note that on the second term of summation in Eq. 2.27, where the prime is the term $k = j$ is specifically needs to be excluded and Eq. 2.27 may be written in a somewhat different way by noting that the final value of the potential φ at the jth point charge due to the other charges of the system is

$$\varphi_j = \sum_{k=1}^{m}{}' \frac{q_k}{4\pi\varepsilon_0 r_{jk}} \qquad \text{(Eq. 2.28)}$$

Thus, the electrostatic energy of the system is given as

$$U = \frac{1}{2}\sum_{j=1}^{m} q_j \varphi_j \qquad \text{(Eq. 2.29)}$$

Proof of all the above equation is left to the readers.

2.4.2 Electrostatic Energy of a Charge Distribution

The electrostatic energy of an arbitrary charge distribution with volume density φ and surface density can be expressed based on assembled charge distribution by bringing in charge increments δq from a reference potential $\varphi_A = 0$. If the charge distribution is partly assembled and the potential at a particular point in the system is $\varphi'(x, y, z)$, then from Eq. 1.84, the work required to place δq at this point is written as

$$\delta W = \varphi'(x, y, z)\delta q \qquad \text{(Eq. 2.30)}$$

In this equation the charge increment δq may be added to a volume element located at (x, y, z), so that $\delta q = \delta\rho\Delta\upsilon$, or may be added to a surface element at the point in question, in which case $\delta q = \delta\rho\Delta a$. The total electrostatic energy of the assembled charge distribution is obtained by summing contributions of the form Eq. 2.30.

Let us assume that at any stage of the charging process, all charge densities will be at the same fraction of their final values and represented by symbol α, and if the final values of the charge densities are given by the function $\varphi(x, y, z)$ and $\sigma(x, y, z)$, then the charge densities at an arbitrary stage are $\alpha\varphi(x, y, z)$ and $\alpha\sigma(x, y, z)$. Furthermore, the increments in these densities are $\delta\rho = \varphi(x, y, z)d\alpha$ and $\delta\sigma = \sigma(x, y, z)d\alpha$, then the total electrostatic energy, which is obtained by summing Eq. 2.30, is given by

$$\begin{aligned} U = {} & \int_0^1 \delta d \int_V \varphi(x, y, z)\varphi'(x, y, z)d\upsilon \\ & + \int_0^1 \delta d \int_S \sigma(x, y, z)\varphi'(x, y, z)da \end{aligned} \qquad \text{(Eq. 2.31)}$$

However, since all charges are at the same fraction, α is readily done and yields as

$$U = \frac{1}{2}\int_V \rho(\vec{r})\varphi(\vec{r})d\upsilon + \frac{1}{2}\int_S \sigma(\vec{r})\varphi(\vec{r})da \qquad \text{(Eq. 2.32)}$$

This equation provides the desired result for the energy of a charge distribution. If all space is filled with a single dielectric medium except for certain conductors, the potential is then given by

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon}\int_V \frac{\varphi(\vec{r}')d\upsilon'}{|\vec{r}-\vec{r}'|} + \frac{1}{4\pi\varepsilon}\int_V \frac{\sigma(\vec{r}')da'}{|\vec{r}-\vec{r}'|} \qquad \text{(Eq. 2.33)}$$

Equations 2.32 and 2.33 are the generalization of Eqs. 2.28 and 2.29 for point charges. The latter can be recovered as a special case letting the following relationships as

$$\begin{aligned}\rho(\vec{r}) &= \sum_{j=1}^{m} q_j \delta(\vec{r}-\vec{r}_j) \\ \rho(\vec{r}') &= \sum_{k=1}^{m}{}' q_k \delta(\vec{r}-\vec{r}_k)\end{aligned} \qquad \text{(Eq. 2.34)}$$

where again, the prime on the second summation in Eq. 2.34 is indication of the term $k = j$ and is excluded when the double sum is constructed. Note that when ρ is a continuous distribution, the vanishing of the denominator in Eq. 2.33 does not cause the integral to diverge, and it is unnecessary to exclude the point $\vec{r}' = \vec{r}$.

The last integral involves, in part, integration over the surface of the conductor of interest; however, since a conductor is an equipotential region, each of these integrations may be done as

$$\frac{1}{2}\int_{\text{conductor } j} \sigma\varphi\, da = \frac{1}{2} Q_j \varphi_j \qquad \text{(Eq. 2.35)}$$

where Q_j is the charge on the jth conductor.

Equation 2.32 for *electrostatic energy of a charge distribution*, which includes conductor, then becomes as

$$U = \frac{1}{2}\int_V \rho\varphi\, d\upsilon + \frac{1}{2}\int_{S'} \sigma\varphi da + \frac{1}{2}\sum_j Q_j\varphi_j \qquad \text{(Eq. 2.36)}$$

where in Eq. 2.36, the last summation is over all conductors, and the surface integral is restricted to nonconducting surfaces.

Furthermore, in many practical problems of interest, all of the charges reside on the surfaces of conductor. In these circumstances, Eq. 2.36 reduces to the following form as

$$U = \frac{1}{2}\sum_j Q_j\varphi_j \qquad \text{(Eq. 2.37)}$$

Equation 2.37 is derived based on starting with uncharged macroscopic conductors that were gradually charged by bringing in charge increments. Thus, the energy is described by Eq. 2.37 including both interaction energy between different conductors and the self-energies of the charge on each individual conductors.

2.4.3 Forces and Torques

Thus far, we have developed to some extent a number of alternative procedures for calculating the electrostatic energy of a charge system. We now take an attempt to establish the force on one of the objects in the charge system that may be calculated from knowledge of this electrostatic energy.

If we dealing with an isolated system composed of conductors, point charges, and dielectrics and we make all of these items to make a small displacement under the influence of the electrical force $\vec{F}$ acting upon it. The work performed by the electrical force on the system in these circumstances is

$$dW = \vec{F} \cdot d\vec{r} = F_x dx + F_y d_y + F_z dz \qquad \text{(Eq. 2.38)}$$

Since we assume the system is isolated, this work is done at the expense of the electrostatic energy U. In other words, according to Eq. 2.24 we can write

$$dW = -dU \qquad \text{(Eq. 2.39)}$$

Combining Eqs. 2.38 and 2.39, the result is as follows:

$$-dU = F_x dx + F_y dy + F_z dz \qquad \text{(Eq. 2.40)}$$

and

$$\begin{aligned} F_x &= -\frac{\partial U}{\partial x} \\ F_y &= -\frac{\partial U}{\partial y} \\ F_z &= -\frac{\partial U}{\partial z} \end{aligned} \qquad \text{(Eq. 2.41)}$$

Therefore, sets of Eq. 2.41 indicate that in case $\vec{F}$ is a conservative force and $\vec{F} = -\vec{\nabla} U$. If the object under consideration is constrained to move in such a way that it rotates about an axis, then Eq. 2.38 may be replaced by the following equation as

$$dW = \vec{\tau} \cdot d\vec{\theta} \qquad \text{(Eq. 2.42)}$$

where $\vec{\tau}$ is the electrical torque and $d\vec{\theta}$ is the differential angular displacement. Writing $\vec{\tau}$ and $d\vec{\theta}$ in terms of their components, (τ_1, τ_2, τ_2) and $(d\theta_1, d\theta_2, d\theta_3)$, and combining Eqs. 1.98 and 2.42, we obtain the following relationships

$$\begin{aligned} \tau_1 &= -\frac{\partial U}{\partial \theta_1} \\ \tau_2 &= -\frac{\partial U}{\partial \theta_2} \\ \tau_3 &= -\frac{\partial U}{\partial \theta_3} \end{aligned} \qquad \text{(Eq. 2.43)}$$

This proves that our goal has been, achieved and we can write

$$\begin{cases} F_x = -\left(\frac{\partial U}{\partial x}\right)_Q \\ \tau_1 = -\left(\frac{\partial U}{\partial \theta_1}\right)_Q \end{cases} \qquad \text{(Eq. 2.44a)}$$

$$\begin{cases} F_y = -\left(\frac{\partial U}{\partial y}\right)_Q \\ \tau_2 = -\left(\frac{\partial U}{\partial \theta_2}\right)_Q \end{cases} \qquad \text{(Eq. 2.44b)}$$

$$\begin{cases} F_z = -\left(\frac{\partial U}{\partial x}\right)_Q \\ \tau_3 = -\left(\frac{\partial U}{\partial \theta_3}\right)_Q \end{cases} \qquad \text{(Eq. 2.44c)}$$

where the subscript Q has been added to denote that the system is isolated and hence, its total charge remains constant during the displacement $d\vec{r}$ or $d\vec{\theta}$.

Now we are at the stage that we need to talk electromagnetic force that is known as Lorentz force here.

The electromagnetic field exerts the following force (often called the Lorentz force) on charged particles:

$$\vec{F} = q\vec{E} + q\vec{\mathrm{v}} \times \vec{B} \qquad \text{(Eq. 2.45)}$$

where vector $\vec{F}$ is the force that a particle with charge q experiences, $\vec{E}$ is the electric field at the location of the particle, v is the velocity of the particle, and $\vec{B}$ is the magnetic field at the location of the particle.

The above equation illustrates that the Lorentz force is the sum of two vectors. One is the cross product of the velocity and magnetic field vectors. Based on the properties of the cross product, this produces a vector that is perpendicular to both the velocity and magnetic field vectors. The other vector is in the same direction as the electric field. The sum of these two vectors is the Lorentz force.

Therefore, in the absence of a magnetic field, the force is in the direction of the electric field, and the magnitude of the force is dependent on the value of the charge and the intensity of the electric field. In the absence of an electric field, the force is perpendicular to the velocity of the particle and the direction of the magnetic field. If both electric and magnetic fields are present, the Lorentz force is the sum of both of these vectors.

Therefore, in summary we can express that the classical theory of electrodynamics is built upon Maxwell's equations and the concepts of electromagnetic field, force, energy, and momentum, which are intimately tied together by Poynting's theorem and the Lorentz force law. Whereas Maxwell's macroscopic equations relate the electric and magnetic fields to their material sources (i.e., charge, current, polarization, and magnetization), Poynting's theorem governs the flow of electromagnetic energy and its exchange between fields and material media, while the Lorentz law regulates the back-and-forth transfer of momentum between the media and the fields. As it turns out, an alternative force law, first proposed in 1908 by Einstein and Laub, exists that is consistent with Maxwell's macroscopic equations and complies with the conservation laws as well as with the requirements of special relativity. While the Lorentz law requires the introduction of hidden energy and hidden momentum in situations where an electric field acts on a magnetic material, the Einstein–Laub formulation of electromagnetic force and torque does not invoke hidden entities under such circumstances. Moreover, the total force and the total torque exerted by electromagnetic fields on any given object turn out to be independent of whether force and torque densities are evaluated using the Lorentz law or in accordance with the Einstein–Laub formulas. Hidden entities aside, the two formulations differ only in their predicted force and torque distributions throughout material media. Such differences in distribution are occasionally measurable and could serve as a guide in deciding, which formulation, if either, corresponds to physical reality.

Furthermore, to have some general idea about Poynting's theorem, we can say that, in electrodynamics, Poynting's theorem is a statement of conservation of energy for the electromagnetic field. Moreover, it is in the form of a partial differential equation, due to the British physicist John Henry Poynting. Poynting's theorem is analogous to the work–energy theorem in classical mechanics, and mathematically similar to the continuity equation, because it relates the energy stored in the electromagnetic field to the work done on a charge distribution (i.e., an

electrically charged object) through energy flux. A detail of deriving this theorem is beyond the scope of this book and we leave to the readers to refer to some other classical electrodynamics books.

However, in general we can say this theorem is an energy balance and the following statement does apply:

The rate of energy transfer (per unit volume) from a region of space equals the rate of work done on a charge distribution plus the energy flux leaving that region.

A second statement can also explain the theorem: "The decrease in the electromagnetic energy per unit time in a certain volume is equal to the sum of work done by the field forces and the net outward flux per unit time."

Mathematically, the above statement can be expressed and is summarized in differential form as follows:

$$-\frac{\partial u}{\partial t} = \vec{\nabla} \cdot \vec{S} + \vec{J} \cdot \vec{E} \qquad \text{(Eq. 2.46)}$$

where $\vec{\nabla} \cdot \vec{S}$ is the divergence of Poynting vector or energy flow and $\vec{J} \cdot \vec{E}$ is the rate at which the fields do work on a charged object ($\vec{J}_{\mathrm{f}}$ is the free current density corresponding to the motion of charge, $\vec{E}$ is the electric field, and $\cdot$ is the dot product). The energy density u is given by

$$u = \frac{1}{2}\left(\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H}\right) \qquad \text{(Eq. 2.47)}$$

In this equation $\vec{D}$ is the electric displacement filed, $\vec{B}$ is the magnetic flux density, and $\vec{H}$ is the magnetic field strength. Since only some of the charges are free to move, and the $\vec{D}$ and $\vec{H}$ fields exclude the "bound" charges and currents in the charge distribution (by their definition), one obtains the free current density $\vec{J}_{\mathrm{f}}$ in the Poynting theorem, rather than the total current density $\vec{J}$.

The integral form of Poynting's theorem can be, established via utilization of divergence theorem expressed before as

$$-\frac{\partial}{\partial t}\int_V u\, dV = \oiint_{\partial V} \vec{S} \cdot d\vec{A} + \int_V \vec{J} \cdot \vec{E}\, dV \qquad \text{(Eq. 2.48)}$$

where ∂V is the boundary of a volume V and the shape of the volume is arbitrary, but fixed for the calculation.

In summary all of past couple sections in this chapter could be put in perspectives that are presented by Fig. 2.5, below

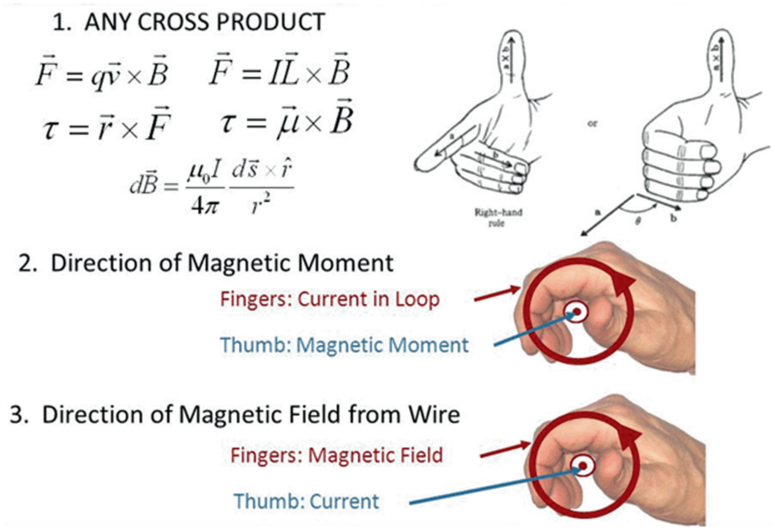


Fig. 2.5 Right-hand rule review

2.5 Maxwell's Equations

In order to understand physics of plasma and associated subject such as magneto-hydrodynamic equations that are known as MHD, in particular, encountering confinement of plasma as a way of driving fusion energy, we need to have some understanding of sets of equations that are known as Maxwell's Equations.

We are at the point and ready to introduce the keynote of Maxwell's electromagnetic theory as brief course and what is so-called *displacement current*. We shall now write all classical, i.e., nonquantum electromagnetic phenomena are governed by *Maxwell's equations*, which take the form as follows:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{Also known as Coulomb's Law} \tag{Eq. 2.49}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{Also known as Gauss's Law} \tag{Eq. 2.50}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{Also known as Faraday's Law} \tag{Eq. 2.51}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \text{Also known as Ampere's Law} \tag{Eq. 2.52}$$

All the quantities in the above equations defined as before. Here, $\vec{E}(\vec{r}, t)$, $\vec{B}(\vec{r}, t)$, $\rho(\vec{r}, t)$, and $\vec{J}(\vec{r}, t)$ represent the *electric field strength, the magnetic field strength*, the *electric charge density*, and the *electric current density*, respectively. Moreover

ε_0=8.8542 $\times$ 10^{-12} C^2 N^{-1} m^{-2} is the *electric permittivity of free space*, whereas $\mu_0 = 4\pi \times 10^{-7}$ N A^{-2} is the *magnetic permeability of free space*. As is well known, Eq. 2.49 is equivalent to *Coulomb's law* for the electric fields generated by point charges. Equation 2.50 is equivalent to the statement that magnetic monopoles do not exist, which implies that magnetic field lines can never begin or end. Equation 2.51 is equivalent to *Faraday's law of electromagnetic induction*. Finally, Eq. 2.52 is equivalent to the Biot–Savart's law for the magnetic fields generated by line currents and augmented by the induction of magnetic fields by changing electric fields.

Maxwell's equations are linear in nature. In other words, if $\rho \rightarrow \alpha\rho$ and $\vec{J} \rightarrow \alpha\vec{J}$, where α is an arbitrary spatial and temporal constant, then it is clear from Eqs. 1.108 to 1.111, that $\vec{E} \rightarrow \alpha\vec{E}$ and $\vec{B} \rightarrow \alpha\vec{B}$. The linearity of Maxwell's equations accounts for the well-known fact that the electric fields generated by point charges and as well as the magnetic fields generated by line currents are super imposable.

Taking the divergence of Eq. 1.108, and combining the resulting expression with Eq. 1.108, we obtain

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \qquad \text{(Eq. 2.53)}$$

In integral form, making use of the divergence theorem, this equation becomes

$$\frac{d}{dt}\int_V \rho \, dV + \int_S \vec{J} \cdot d\vec{S} = 0 \qquad \text{(Eq. 2.54)}$$

where V is a fixed volume bounded by a surface S. The volume integral represents the net electric charge contained within the volume, whereas the surface integral represents the outward flux of charge across the bounding surface. The previous equation, which states that the net rate of change of the charge contained within the volume V is equal to minus the net flux of charge across the bounding surface S is clearly a statement of the *conservation of electric charge*. Thus, Eq. 2.53 is the differential form of this conservation equation.

As is well known, a point electric q moving with velocity $\vec{v}$ in the presence of an electric filed $\vec{E}$ and a magnetic field $\vec{B}$ experiences a force that is known as Lorentz Force and was expressed by Eq. 2.45 as before. Likewise, a distributed charge density ρ and current density $\vec{J}$ experiences a force density that is given as

$$\vec{f} = \rho\vec{E} + \vec{J} \times \vec{B} \qquad \text{(Eq. 2.55)}$$

This is the extent of our presentation for the Maxwell's Equations within this book; further deviation of these equations can be found in any classical electrodynamics books [2].

2.6 Debye Length

Debye length is an important aspect of plasma physics and it is a quantity which is a measure of the shielding distance or thickness of the charged particle cloud also called sheath in plasma. One of the most significant properties of plasma is its tendency to maintain electrically neutral, that is, its tendency to balance positive (Ion) and negative (Electron) space charge in each macroscopic volume element. A slight imbalance in the space–charge densities gives rise to strong electrostatic forces that act, wherever possible, in the direction of restoring neutrality. On the other hand, if plasma is deliberately subjected to an external electric field, the space–charge densities will adjust themselves so that the major part of the plasma is shielded from the field.

To carry out this subject further, we can pay our attention to Poisson's equation and seek a solution for that equation in case of a point charge $+Q$ that is introduced into a plasma and thereby subjecting the plasma to an electric field for simplicity of analyses. Under these conditions, negative electrons existing in plasma, energetically have tendency to move closer to this positive charge favorably, whereas positive ions tend to move away from it. Under equilibrium conditions, the probability of finding a charged particle in a particular region of potential energy U is proportional to the Boltzmann factor as $\exp(-U/kT)$. Thus, the electron density n_e is given by the following equation as

$$n_e = n_0 \exp\left(e\frac{(\varphi - \varphi_0)}{kT}\right) \qquad \text{(Eq. 2.56a)}$$

For Eq. 2.56a, the following quantities are in order and are as follows:

φ=Is the local potential,
φ_0=Is the reference potential or in our case plasma potential,
T=Is the absolute temperature of the plasma,
k=Is the Boltzmann constant, and
n_0=Is the electron density in regions where $\varphi = \varphi_0$.

If n_0 is also the positive ion density in regions of potential φ_0, then positive ion density n_i is also given by the similar relation as Eq. 2.56b and that is

$$n_i = n_0 \exp\left(-e\frac{(\varphi - \varphi_0)}{kT}\right) \qquad \text{(Eq. 2.56b)}$$

Now that we have setup the initial conditions, first we attempt to derive Debye length by means of Poisson's Equation and then show its use in plasma physics and as criteria to identify a definition that plasmas fall into it.

A particular solution of Poisson's equation for potential φ is carried out here in the form of one-dimensional spherical symmetry around radius coordinate of r, and we start with the following differential equation as

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\varphi}{dr}\right) = -\frac{1}{\varepsilon_0}(n_i e - n_e e) = \frac{2n_0 e}{\varepsilon_0}\sinh\left(e\frac{(\varphi-\varphi_0)}{kT}\right) \qquad \text{(Eq. 2.57)}$$

The differential Eq. 2.57 is nonlinear, and hence must be integrated numerically. On the other hand, an approximate solution to Eq. 2.57, which is rigorous at high temperature plasma, is adequate for these purposes here. If $kT > e\varphi$, then $\sinh(e\varphi/kT) = e\varphi/kT$, and the differential Eq. 2.57 reduces to the following and simple form as

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\varphi}{dr}\right) = \frac{2n_0 e^2}{\varepsilon_0 kT}(\varphi-\varphi_0) \qquad \text{(Eq. 2.58)}$$

The solution to this equation is found to be (Readers can carry out the solution, as hint using Taylor series expansion for $|e\varphi/kT| \ll 1$ to drop the second order and higher terms off in expansion of $e\varphi/kT + \frac{1}{2}(e\varphi/kT)^2 + \cdots$)

$$\varphi = \varphi_0 + \frac{Q}{4\pi\varepsilon_0 r}\exp\left(-\frac{r}{h}\right) \qquad \text{(Eq. 2.59)}$$

where r is the distance from the point charge $+Q$, and λ_D, the Debye shielding distance or Debye length, is given by

$$\lambda_D = \left(\frac{\varepsilon_0 kT}{2n_0 e^2}\right) \qquad \text{(Eq. 2.60)}$$

Thus, the redistribution of electrons and ions in the gas is such as to screen out $+Q$ completely in a distance of a few λ_D.

The quantity λ_D as we said before is called the Debye length and is measure of the shielding or thickness of the charges cloud, which is also known as sheath. Note that as the density increases, λ_D decreases, as one would expect, since each layer of plasma contains more electrons. In addition, λ_D increases with increasing kT. Without thermal agitation, the charge cloud would collapse to an infinitely thin layer. Last but not least, it is the *electron* temperature which is used in the definition of λ_D and that is $T = T_e$, because the electrons being more mobile than their counterpart ions. In general, shielding do the moving so as to create a surplus or deficit of negative charge. Only in special situations is this not true. The following are a set of useful forms of Eq. 2.60

$$\lambda_D = 69(T_e/n)^{1/2} m \quad T_e \text{ in}^\circ\text{K} \qquad \text{(Eq. 2.61a)}$$

$$\lambda_D = 7430(T_e/n)^{1/2} m \quad kT_e \text{ in eV} \qquad \text{(Eq. 2.61b)}$$

2.7 Physics of Plasmas

An ionized gas is called a plasma if the Debye length, λ_D, is small compared with other physical dimensions of interest. This restriction is not severe so long as ionization of the gas is appreciable. Other conditions that will make an ionized gas to fall in the category of plasma can be described as the following statements.

One criterion for an ionized gas to be called plasma is that it needs to be dense enough that λ_D is much smaller than a dimensions L of a system and this dimension is much larger than λ_D, in other words, $\lambda_D \ll L$, then whenever local concentrations of charge arise or external potentials are introduced into the system, system could be a magnetron, klystrons.

The phenomenon of Debye shielding also occurs—in modified form—in single-species systems, such as the electron streams in Klystrons and Magnetrons or the proton beams in a Cyclotron. Under these situations, any local bunching of particles causes a large unshielded electric field unless the density is extremely low, which is more often is the case.

The Debye shielding picture that we have painted in above is valid only if there are enough particles in charge cloud or sheath. Thus, it is clear, if there is only one or two particles in the sheath region, Debye shielding would not be a statistically valid concept from viewpoint of electromagnetic physics. Using Eq. 2.56 in a general form, we can compute the number of N_D particles in a Debye sphere as

$$N_D = n\frac{4}{3}\pi\lambda_D^3 = 1.38 \times 10^6 T^{3/2}/n^{1/2} \quad T \text{ in}^\circ\text{K} \qquad \text{(Eq. 2.62)}$$

In addition to $\lambda_D \ll L$, "collective behavior" requires [3]

$$N_D \gg 1 \qquad \text{(Eq. 2.63)}$$

Furthermore, to qualify an ionized gas as plasma we can define more criteria. The two conditions in above were given so that an ionized gas must satisfy to be a plasma. A third condition has to do with collisions. The ionized gas in an airplane's jet exhaust, for example, does not qualify as a plasma because the charged particles collide so frequently with neutral atoms that their motion is controlled by ordinary hydrodynamic forces rather than by electromagnetic forces [3].

If ω is the frequency of typical plasma oscillations and τ is the mean time between collisions with neutral atoms, we require $\omega\tau > 1$ for the gas to behave like plasma rather than a neutral gas. Therefore, the three conditions a plasma must satisfy are therefore:

$$\begin{aligned} &1.\quad \lambda_D \ll L \\ &2.\quad N_D \gg 1 \\ &3.\quad \omega\tau > 1 \end{aligned} \qquad \text{(Eq. 2.64)}$$

As you can see, the three above conditions are necessary for an ionized gas to be, called plasma.

In addition, as part of physics of plasma, we look at plasma processes for purpose of our study of plasma confinement via Inertial Confinement Fusion (ICF). We see all objects through a medium, which could be interplanetary interstellar, or intergalactic and question arises that, how does this medium affect photons, what information we gain?. Doing so, we can focus on specific microphysical process by ignoring the effect of other matter. In fact, we can implicitly or explicitly assume propagation through a vacuum for most part and applications using our knowledge of electrodynamics.

When we introduced Maxwell's equations in above, we defined $\vec{D} = \varepsilon \vec{E}$ and $\vec{H} = \vec{B}/\mu$ to include the effects of matter, where ε is the dielectric constant and μ is the magnetic permeability. However, remember that Maxwell's equations explicitly include sources, in the form of charge density ρ and current density $\vec{J}$. If we do this consistently, for all charges and currents, whether or not they are in medium, then Maxwell's equations for electric field $\vec{E}$ and magnetic field $\vec{B}$ alone will work just fine.

From this point of view, then Maxwell's equations for a vacuum conditions work fine, as long as both free and bound charges are included. In that case, we can again think about the propagation of radiation, this time with more general form. Thus, we can assume a space and time variation of the form solution to Helmholtz equation as $\exp i\left(\vec{k} \cdot \vec{r} - \omega t\right)$ for propagation of radiation, where wave period in terms of wave frequency f is given as $\omega = 2\pi f$ and $k = 2\pi/\lambda$ in terms of wavelength λ. Another form of wave property k is written in terms of speed of light c in vacuum and ω as $k = \omega/c$. Helmholtz equation in z-direction within medium as empty or empty space, where there is no gravity (i.e., $g = 0$) can be written as

$$\frac{d^2 \vec{E}(z)}{dz^2} + (\omega/c)^2 \vec{E}(z) = 0 \qquad \text{(Eq. 2.65)}$$

Equation 2.65 is derived from the general form of a monochromatic wave equation, where the wave is characterized by a single frequency in the entire field of propagation using Maxwell's equations.

The methods of complex variable analysis provide a convenient way of implementing this procedure. For the time dependence of the field $\vec{E}(\vec{r}, t)$ is taken to be as $e^{-i\omega t}$, so that we can write the following mathematical relationship as

$$\vec{E}(\vec{r}, t) = \vec{E}(\vec{r}) e^{-i\omega t} \qquad \text{(Eq. 2.66)}$$

Note that the physical electric field is obtained by taking the real part of complex form of Eq. 2.66. Furthermore, $\vec{E}(\vec{r})$ is in general complex, so that the actual electric field is proportional to $\cos(\omega t + \phi)$, where ϕ is the phase of $\vec{E}(\vec{r})$. Thus using Eq. 2.66, we can write

$$\vec{E}(\vec{r},t) = \vec{E}(\vec{r})\mathrm{e}^{-i\omega t}\left\{\nabla^2\vec{E}(\vec{r}) + \omega^2\varepsilon\mu + i\omega g\mu\vec{E}(\vec{r})\right\} = 0 \qquad \text{(Eq. 2.67)}$$

Thus, using Eq. 2.67 in a medium of empty space, so that $g = 0, \varepsilon = \varepsilon_0, \mu = \mu_0$ and assuming one-dimensional form of $\vec{E}(\vec{r})$ that varies just in z-direction allow us to deduce the result of Eq. 2.66. In this case, we have written $\varepsilon_0\mu_0 = 1/c^2$ and ε_0 as well as μ_0 are dielectric constant and magnetic permeability in free space, respectively [2, 4].

Back to our discussion on plasma processes and considering the relation of $\exp i\left(\vec{k}\cdot\vec{r} - \omega t\right)$, then we can conclude that

$$\begin{cases} i\vec{k}\cdot\vec{E}(\vec{r}) = 4\pi\rho \\ i\vec{k}\cdot\vec{B}(\vec{r}) = 0 \\ i\vec{k}\times\vec{E}(\vec{r}) = i(\omega/c)\vec{B}(\vec{r}) \\ i\vec{k}\times\vec{B}(\vec{r}) = (4\pi/c)\vec{J} - ii(\omega/c)\vec{E}(\vec{r}) \end{cases} \qquad \text{(Eq. 2.68)}$$

Moreover, utilization of $\exp i\left(\vec{k}\cdot\vec{r} - \omega t\right)$ for propagation of radiation is totally justified because the Maxwell's equations are *linear*, i.e., there are no terms of the form E^2 or EB, thus every Fourier mode propagates independently. The wave number is designated as $\vec{k}$.

There are other physical theories (e.g., strong field general relativity) that are not linear, which means that these modes would mix and, thus it could not be considered independently in this way. The linearity of Maxwell's equations is also why we can get away with using complex numbers; the real and imaginary parts never mix, so they can be considered to yield independent solutions based on the original equations. However, in general this could be very difficult, if the medium is something arbitrary (air, water, glass). In our case, though, we are specifically interested in a plasma, which can be loosely defined as an ionized gas that is globally neutral. That means that all charges are mobile in principle. However, as we have done before, we will assume that the ions are basically stationary for our purposes and mainly serve to keep the plasma neutral. Another important simplifying assumption is that the plasma is isotropic (i.e., so ε and μ do not vary from point to point). Bear in your mind that in case of magnetic field, there is no significant external magnetic field, just because, that would violate the isotropy condition.

Now let us consider nonrelativistic electrons. A given electron follows Newton's law (i.e., Lorentz Equation with $\vec{B}(\vec{r}) = 0$) as

$$m\dot{\vec{\mathrm{v}}} = -e\vec{E}(\vec{r}) \qquad \text{(Eq. 2.69)}$$

Here, we have ignored the magnetic field as an internal impact and reason of this negligent is because the magnetic component of the force, that is an order of $\upsilon/c \to 0$, which is small if the motion is nonrelativistic. Given our assumption about the space and time variations of quantities, this means that:

$$\vec{\mathrm{v}} = e\vec{E}/i(\omega m) \tag{Eq. 2.70}$$

The current density is $\vec{\mathrm{j}} = -me\vec{\mathrm{v}}$, meaning that we get:

$$\vec{J} = \sigma\vec{E} \tag{Eq. 2.71}$$

where the conductivity is $\sigma = ine^2/(\omega m)$. This is Ohm's law; the current responds directly to the electric field. Note, however, that this statement requires isotropy.

From charge conservation, our propagation of radiation of $\exp i\left(\vec{k}\cdot\vec{r} - \omega t\right)$ assumption will provide the following:

$$-i\omega\rho + i\vec{k}\cdot\vec{J} = 0 \tag{Eq. 2.72}$$

so that

$$\rho = \omega^{-1}\vec{k}\cdot\vec{J} = \sigma\omega^{-1}\vec{k}\cdot\vec{E} \tag{Eq. 2.73}$$

If we define the dielectric constant by the following equations as

$$\varepsilon \equiv 1 - 4\pi\sigma/(i\omega) \tag{Eq. 2.74}$$

Note that this is real, since σ has an i in it, thus we get

$$\begin{cases} i\vec{k}\cdot\varepsilon\vec{E} = 0 \\ i\vec{k}\cdot\vec{B} = 0 \\ i\vec{k}\times\vec{E} = i(\omega/c)\vec{B} \\ i\vec{k}\times\vec{B} = -i(\omega/c)\varepsilon\vec{E} \end{cases} \tag{Eq. 2.75}$$

This looks just like the "source-free" vacuum equations we had before, except for ε. Indeed, arguing as before, we find that $\vec{k}$, $\vec{E}$, and $\vec{B}$ are mutually perpendicular. However, we find that

$$c^2k^2 = \varepsilon\omega^2 \tag{Eq. 2.76}$$

Since ε depends on ω, we no longer have the simple vacuum situation in which all frequencies travel at the same rate, the phase velocity is the same as the group velocity, and so on. Thus, the presence of a plasma introduces dispersion, where wave packets spread and there is effectively an index of refraction. If we substitute in expressions, we can rewrite the dielectric constant as

$$\varepsilon = 1 - \left(\frac{\omega_p}{\omega}\right)^2 \qquad \text{(Eq. 2.77)}$$

where $\omega_p = \sqrt{4\pi ne^2/m}$ is called the *plasma frequency*. Numerical value of plasma frequency is found to be $\omega_p = 5.63 \times 10^4 n^{1/2}\,s^{-1}$ if n is in cm^{-3}.

Now what we can ask ourselves is, from these definitions and dispersion relation, what does it tell us about propagation, when $\omega < \omega_p$? This means that $\vec{k}$ is imaginary and it is given as

$$\mathrm{Img}\left(\vec{k}\right) = (i/c)\sqrt{\omega_p^2 - \omega^2} \qquad \text{(Eq. 2.78)}$$

Now we can look at the condition for this dispersion relation, when propagation of radiation is below ω_p. This means that there is an exponential cutoff in the amplitude, with a distance scale of order $2\pi c/\omega_p$. Therefore, effective frequencies below ω_p cannot propagate in a plasma.

We can make two side notes from what we have learned so far and they are as follows:

1. One way to get quick intuition in a number of astrophysical situations is to have a number of characteristic quantities in mind. The plasma frequency is an example: if you have a plasma of a given number density and are considering propagation of electromagnetic waves, you should compare the frequency of the wave with the plasma frequency. If a magnetic field is involved, think of the cyclotron frequency. If a high-density plasma is relevant, think of the Fermi energy. Stuff like that. It helps you decide quickly what regime you are in and what processes are likely to be relevant.
2. Since σ is purely imaginary, Ohm's law means that there is a 90° phase shift between $\vec{j}$ and $\vec{E}$. Therefore, in a time-averaged sense, $\vec{j} \cdot \dot{E} = 0$ and there is no network done by the field in an isotropic plasma. That also means there is no dissipation, so below the plasma frequency you have a pure reflection. Thus, you can probe the ionosphere of the Earth by finding out when a wave of a given frequency is completely reflected. You can also communicate intercontinentally by bouncing low-frequency waves off of the ionosphere.

Back to our discussion of plasma processes, we know that electromagnetic radiation travels at a velocity different from c due to the presence of matter. The phase velocity, $v_{ph} = \omega/k$, is greater than the speed of light [2, 4]. However, the

physically relevant speed is the group velocity $v_{gr} = c\sqrt{1 - \left(\omega_p/\omega\right)^2}$, which is less than c and this is the speed at which wave energy or information travel. The frequency dependence means that there is dispersion in the propagation of light over a variety of frequencies. One especially useful application is to pulsars. Suppose a pulsar is a distance d away, then question comes to our mind is that, how long does it take for light of a given frequency to reach us? The answer for the time travel, t_p is given as follows:

$$t_p(\omega) = \int_0^d ds/v_{gr}(\omega) \qquad \text{(Eq. 2.79)}$$

Here s measures the line of sight distance to us. Plasma frequencies in the Interstellar Medium (ISM) are really low, usually 103 Hz or so, thus we can assume $\omega \gg \omega_p$ and therefore, we can write

$$v_{gr}^{-1} \approx \frac{1}{c}\left(1 + \frac{1\omega_p^2}{2\omega^2}\right) \qquad \text{(Eq. 2.80)}$$

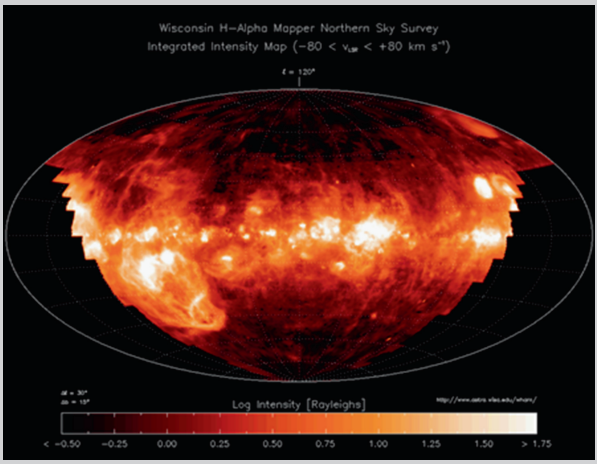
Therefore, the propagation time is

$$t_p(\omega) \approx \frac{d}{c} + \left(2c\omega^2\right)^{-1} \int_0^d \omega_p^2 ds \qquad \text{(Eq. 2.81)}$$

Interstellar Medium (ISM) Definition
In astronomy, the interstellar medium (ISM) is the matter that exists in the space between the star systems in a galaxy. This matter includes gas in ionic, atomic, and molecular form, as well as dust and cosmic rays. It fills interstellar space and blends smoothly into the surrounding intergalactic space. The energy that occupies the same volume, in the form of electromagnetic radiation, is the interstellar radiation field.

The interstellar medium is composed of multiple phases, distinguished by whether matter is ionic, atomic, or molecular, and the temperature as well as density of the matter. The interstellar medium is composed primarily of hydrogen followed by helium with trace amounts of carbon, oxygen, and nitrogen comparatively to hydrogen [5]. The thermal pressures of these phases are in rough equilibrium with one another. Magnetic fields and turbulent motions also provide pressure in the ISM and are typically more important dynamically than the thermal pressure is. The distribution of ionized hydrogen (known by astronomers as H II from old spectroscopic terminology) in the parts of the Galactic interstellar medium visible from the Earth's northern hemisphere as observed with the Wisconsin Ha Mapper (Haffner et al. 2003) [6] and it is shown in the following figure

(continued)



Wisconsin H-Alpha Mapper in Northern Sky Survey [6]

Equation 2.81 is the vacuum time (d/c) plus some extra terms. There is therefore a gradient in the time as a function of frequency, thus we can write

$$\frac{dt_{\mathrm{p}}(\omega)}{d\omega} = -\frac{4\pi e^2}{cm\omega^3}\mathbb{D} \qquad \text{(Eq. 2.82)}$$

where $\mathbb{D} \equiv \int_0^d n\,ds$ is the dispersion measure. In principle, this can be used to find the distance to a pulsar, given an estimate of the average number density of plasma in the ISM. In practice, the errors are pretty large, because the interstellar medium has a lot of small-scale structure. This is especially true in directions that have a lot of plasma, such as toward the galactic center.

For the above analysis to be valid as part of calculation, we assumed the absence of an external magnetic field $\vec{B}$; however, if we make the problem more complicated, we introduce this term into our analysis as part of plasma of our consideration and we observe what this term can do qualitatively. This means that the plasma and propagation in it can no longer be considered isotropic, since the magnetic field introduces a preferred direction. It also means that not all polarizations are equal in their propagation properties. This phenomenon in physics is called Faraday Effect or Faraday Rotation and it is a magneto-optical phenomenon and that is an interaction between light and a magnetic field in a medium. Faraday Effect causes a rotation of the plane of polarization, which is linearly proportional to the component of the magnetic field in the direction of propagation. Formally, it is a special case of gyro-electromagnetism obtained when the dielectric permittivity tensor is diagonal [7].

However, the linear polarized light that is seen to rotate in the Faraday effect can be seen as consisting of the superposition of a right- and a left-circularly polarized

beam (this superposition principle is fundamental in many branches of physics). We can look at the effects of each component (right- or left polarized) separately, and see what effect this has on the result.

In circularly polarized light, the direction of the electric field rotates at the frequency of the light, either clockwise or counterclockwise. In a material, this electric field causes a force on the charged particles comprising the material (because of their low mass, the electrons are most heavily affected). The motion thus affected will be circular, and circularly moving charges will create their own (magnetic) field in addition to the external magnetic field. There will thus be two different cases: the created field will be parallel to the external field for one (circular) polarization, and in the opposing direction for the other polarization direction, thus the net $\vec{B}$ field is enhanced in one direction and diminished in the opposite direction. These change the dynamics of the interaction for each beam and one of the beams will be slowed down more than the other, causing a phase difference between the left- and right-polarized beams. When the two beams are added after this phase shift, the result is again a linearly polarized beam, but with a rotation in the polarization direction.

The direction of polarization rotation depends on the properties of the material through which the light is shined. A full treatment would have to take into account the effect of the external and radiation-induced fields on the wave function of the electrons, and then calculate the effect of this change on the refractive index of the material for each polarization to see whether the right or left circular polarization is slowed down more. See Fig. 2.6, which is the schematic of Faraday rotation. This is for the rotation of polarization in matter, where there is a quantity $^\circ$ related to the properties of that matter. In our case, the angle $\mathcal{V}$ of rotation is proportional to the integral of the number density times the magnetic field strength along the propagation direction over the extent of the region.

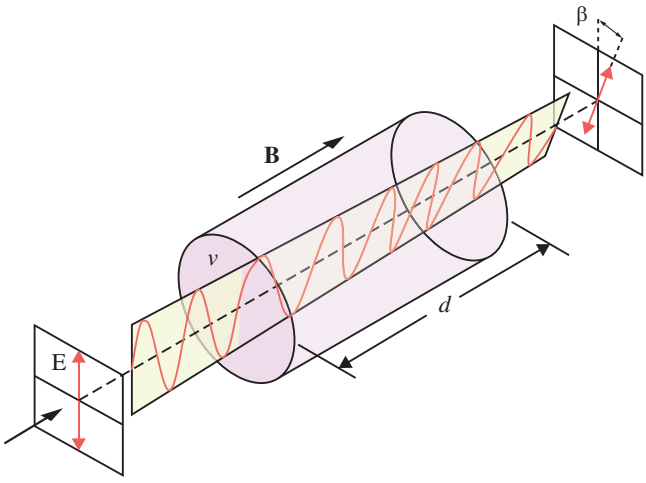


Fig. 2.6 Polarization rotation due to faraday effect

In order to deal with this complicated scenario, we can make some progress by thinking about the special case of propagation along the direction of the field and by considering only cold plasma (i.e., nonrelativistic motion). Also, let us assume that the magnitude of the external field is a lot greater than the magnitude of the fields of the propagating wave, so that the equation of motion becomes

$$m\frac{d\vec{\mathrm{v}}}{dt} = -e\vec{E} - \frac{e}{c}\vec{\mathrm{v}} \times \vec{B} \qquad \text{(Eq. 2.83)}$$

Equation 2.83 is the familiar equation of Lorentz in electromagnetic.

For propagation along a fixed direction, we only have to consider two polarization modes. Let us choose circular modes:

$$\vec{E}(t) = E_0 \mathrm{e}^{-i\omega t}(\widehat{\varepsilon}_1 \mp i\widehat{\varepsilon}_2) \qquad \text{(Eq. 2.84)}$$

where in this equation minus sign gives right circular, while plus sign is indication of left circular polarization [4].

If the wave propagates along $\vec{B} = B_0\widehat{\varepsilon}_3$, then by substituting we find that the steady-state velocity has the form of following equation:

$$\vec{\mathrm{v}}(\mathrm{t}) = \frac{-ie}{m(\omega \pm \omega_{\mathrm{B}})}\vec{E}(t) \qquad \text{(Eq. 2.85)}$$

where $\omega_0 = eB_0/mc$ is the cyclotron frequency for nonrelativistic particles. To see this explicitly, let us assume that $\vec{B} = B_0\widehat{\varepsilon}_3$, so that $\vec{B} \times \vec{E} = 0$. Note that $\widehat{\varepsilon}_1, \widehat{\varepsilon}_2$ and $\widehat{\varepsilon}_3$ are unit vectors. One more assumption that we can make is that $\vec{\mathrm{v}}(t) = C_1\vec{E}(t)$, where C_1 is constant quantity, thus our equation becomes

$$\begin{cases} C_1(-im\omega)E_0\mathrm{e}^{-i\omega t}(\widehat{\varepsilon}_1 \mp i\widehat{\varepsilon}_2) = -eE_0\mathrm{e}^{-i\omega t}(\widehat{\varepsilon}_1 \mp i\widehat{\varepsilon}_2) \\ \qquad\qquad\qquad\qquad -(e/c)C_1E_0\mathrm{e}^{-i\omega t}[(\widehat{\varepsilon}_1 \mp i\widehat{\varepsilon}_2) \times B_0\widehat{\varepsilon}_3] \\ C_1(im\omega)(\widehat{\varepsilon}_1 \mp i\widehat{\varepsilon}_2) = -e(\widehat{\varepsilon}_1 \mp i\widehat{\varepsilon}_2) - (e/c)C_1(-\widehat{\varepsilon}_2 \mp i\widehat{\varepsilon}_1)B_0 \\ (-C_1m\omega)\widehat{\varepsilon}_1 \mp (C_1m\omega)\widehat{\varepsilon}_2 = [-e \pm iC_1(eB_0/C)]\widehat{\varepsilon}_1 + \left[\pm ie + C_1\left(eB_0/c\right]\widehat{\varepsilon}_2\right. \end{cases} \qquad \text{(Eq. 2.86)}$$

Thus, we have two equations, one for the $\widehat{\varepsilon}_1$ direction and one for the $\widehat{\varepsilon}_2$ direction as

$$\begin{aligned} \widehat{\varepsilon}_1 &:\Rightarrow -eiC_1m\omega \pm iC_1(eB_0/c) = 0 \Rightarrow -e + C_1\left[im\omega \pm i\left(eB_0/c\right]\right. = 0 \\ \widehat{\varepsilon}_2 &:\Rightarrow \pm eiC_1m\omega + C_1(eB_0/c) = 0 \Rightarrow \pm ie + C_1[(eB_0/c) \pm m\omega] = 0 \end{aligned} \qquad \text{(Eq. 2.87)}$$

Both equations have the solution of the form $C_1 = ie/[m(\omega \pm \{eB_0/mc\})]$, which gives us our previous expression. However, the implications involved here comes about when the wave goes through the medium, since the speeds of different polarization are different, then there will be a net rotation of the polarization vector as the wave propagates through plasma. Thus, one can write this as an expression for the dielectric constant for right and left direction as

$$\varepsilon_{\mathrm{R,L}} = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega(\omega \pm \omega_{\mathrm{B}})} \qquad \text{(Eq. 2.88)}$$

In general, an electric field vector with wave number $\vec{k}$ traveling a distance d will rotate in phase by $\vec{k} \cdot d$. If the wave number is not constant along the path, then this must be integrated.

Further analysis of the equation reveals that for the field with very small strength, the rotation is also small and Eq. 2.88 satisfies this constraint as well. This just because small magnetic field B_0 means small ω_{B}, and hence a small difference between the polarizations. Now considering the above formula, what happens at extremely large B_0? The answer is that there also the difference is small, because ω_{B} in the denominator means that $\varepsilon \to 0$, when B_0 is large.

As it can be verified easily that in the common astrophysical limits $\omega \gg \omega_{\mathrm{P}}$ and $\omega \gg \omega_{\mathrm{B}}$, we have for angle of rotation the following relation as

$$\Delta\theta = \frac{2\pi e^3}{m^2 c^2 \omega^2} \int_0^d n B_{\parallel} ds \qquad \text{(Eq. 2.89)}$$

In Eq. 2.89 $B_{\parallel}$ is the component of $\vec{B}$ parallel to the line of sight. By practical application of such result, we first will be concerned with a uniform magnetic field, which means no change in direction or magnitude in the region, and in this case, what happens to the degree of polarization due to Faraday rotation? Nothing happens except it rotates. Additionally, what if the region has a tangled magnetic field and you observe a region that involves many tangles? Then, different parts are rotated by different amounts, so the net result is a decrease in the degree of polarization. This is sometimes used in observations of active galactic nuclei. Many times parts of the spectrum are thought to be due to synchrotron radiation, which you remember produces highly polarized light.

However, observed with low angular resolution, there is little polarization. Observations of higher angular resolution do give net polarization, so by comparing the two one can estimate the line of sight integrated in magnetic field. Similar methods are used to estimate the magnetic field strength in the interstellar medium or molecular clouds, when more direct spectroscopic information is unavailable.

Finally, we can just mention in passing one other effect of plasmas. Since the speed of electromagnetic waves is less than the vacuum speed of light, it becomes possible for particles to travel faster than the local speed of electromagnetic waves.

This produces effects similar to shocks in the atmosphere and generates Cerenkov radiation, which is bluish (i.e., it has a spectrum tipped toward high frequency and would actually look blue to the eye). Only particles traveling faster than c/n emit radiation, which has been used to detect neutrinos (if they scatter electrons in water, the electrons can move faster than c/n) and estimate the energies of cosmic rays by using materials with different indices of refraction.

2.8 Fluid Description of Plasma

Before paying our attention and departing for the actual derivation of the Magneto-Hydrodynamics (MHD) equation, which is the topic of our next section in this chapter, it is helpful to discuss briefly some general concepts of fluid dynamics.

Fluid equations are probably the most widely used equations for the description of inhomogeneous plasmas. While the phase fluid, which is, governed by the Boltzmann equation, represents a first example, many applications do not require the precise velocity distribution at any point in space.

Ordinary fluid equations for gases and plasmas would be obtained from the Boltzmann equation or can be derived using properties like the conservation of mass momentum and energy of the fluid. In the following chapter, we will derive a single set of ordinary fluid equations for a plasma and examine properties such an equilibrium and waves for these equations.

To further investigate the fluid aspect of plasma, we look at the equations of kinetic theory and taking a fundamental equation such as $f(\vec{r}, \vec{\mathrm{v}}, t)$ under consideration, which satisfies the Boltzmann equation as follows:

$$\frac{\partial f(\vec{r}, \vec{\mathrm{v}}, t)}{\partial t} + \vec{\mathrm{v}} \cdot \vec{\nabla} f(\vec{r}, \vec{\mathrm{v}}, t) + \frac{\vec{F}}{m} \cdot \frac{\partial f(\vec{r}, \vec{\mathrm{v}}, t)}{\partial \vec{\mathrm{v}}} = \left(\frac{\partial f(\vec{r}, \vec{\mathrm{v}}, t)}{\partial t} \right)_e \qquad \text{(Eq. 2.90)}$$

In Eq. 2.90, $\vec{F}$ is the force acting on the particles and $(\partial f(\vec{r}, \vec{\mathrm{v}}, t)/\partial t)_c$ is the time rate of change of $f(\vec{r}, \vec{\mathrm{v}}, t)$ due to collisions. The symbol $\vec{\nabla}$, as usual, for the gradient in (x, y, z) space. The symbol $\partial/\partial \vec{\mathrm{v}}$ or $\vec{\nabla}_{\vec{\mathrm{v}}}$ stands for the gradient in velocity space and it is written as

$$\frac{\partial}{\partial \vec{\mathrm{v}}} = \hat{x} \frac{\partial}{\partial \mathrm{v}_x} + \hat{y} \frac{\partial}{\partial \mathrm{v}_y} + \hat{z} \frac{\partial}{\partial \mathrm{v}_z} \qquad \text{(Eq. 2.91)}$$

The Boltzmann equation becomes more meaningful, if one should remember that function $f(\vec{r}, \vec{\mathrm{v}}, t)$ is a function of seven independent variables, which includes three for space (x, y, z), three for components of velocity $(\mathrm{v}_x, \mathrm{v}_y, \mathrm{v}_z)$, and the seventh one that accounts for time t, therefore we can expand Eq. 2.90 to all its seven variables and write down

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial \mathrm{v}_x}\frac{d\mathrm{v}_x}{dt} + \frac{\partial f}{\partial \mathrm{v}_y}\frac{d\mathrm{v}_x}{dt} + \frac{\partial f}{\partial \mathrm{v}_y}\frac{d\mathrm{v}_y}{dt} + \frac{\partial f}{\partial \mathrm{v}_z}\frac{d\mathrm{v}_z}{dt} \quad \text{(Eq. 2.92)}$$

Here, $\partial f/\partial t$ is the *explicit* dependence on time. The next three terms are just $\vec{\mathrm{v}} \cdot \vec{\nabla} f(\vec{r}, \vec{\mathrm{v}}, t)$. With the help of Newton's third law, we can write

$$m\frac{d\vec{\mathrm{v}}}{dt}\vec{F} \quad \text{(Eq. 2.93)}$$

As it can be seen from Eq. 2.93, the last three terms are recognized as $\left(\vec{F}/m\right) \cdot \left(\partial f/\partial \vec{\mathrm{v}}\right)$.

Additionally, the total derivative term presented by *df/dt* can be interpreted as the rate of change as seen in a frame moving with the particles. However, here we need to be concerned with particles to be moving in six-dimensional in space $(\vec{r}, \vec{\mathrm{v}})$, i.e., three in (x, y, z) direction and the associate three component of velocity (v_x, v_y, v_z) in their corresponding directions as well.

df/dt is the convective derivative in phase space and the Boltzmann Equation 2.90 simply says that *df/dt* is zero, unless there are collisions. That this should be true and it can be seen from the one-dimensional example shown in Fig. 2.7 here. Figure 2.7, illustrates, a group of points in phase space, representing the position and velocity coordinates of a group of particles. They all retain the same phase-space density as it moves with time.

Taking the Fig. 2.7 under consideration and assuming the group of particles in an infinitesimal element $dx\,d\mathrm{v}_x$ at point *A* all having velocity v_x and position x, then the density of particles in this phase space is just $f(x, \mathrm{v}_x)$. As the time passes, these particles will move to a different position in x because of their velocity v_x and will change their velocity due to the result of the force acting on them.

Since the forces depend on x and v_x only, all the particles at *A* will be accelerated the same amount. After a time t, all the particles will arrive at *B* will be the same as at *A*. If there exists any collisions, then the particles can be scattered and $f(\vec{r}, \vec{\mathrm{v}}, t)$

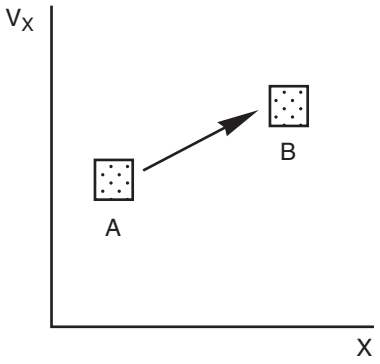


Fig. 2.7 Illustration of group points in phase space

can be changed by the term $(\partial f(\vec{r}, \vec{v}, t)/\partial t)_c$. In sufficiently hot plasma, collision can be neglected and furthermore, if the force $\vec{F}$ is entirely electromagnetic, Eq. 1.124 takes the speed form

$$\boxed{\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f + \frac{q}{m}\left(\vec{E} + \vec{v} \times \vec{B}\right) \cdot \frac{\partial f}{\partial \vec{v}} = 0} \qquad \text{(Eq. 2.94)}$$

Equation 2.94 is representing the Vlasov Equation and because of its comparative simplicity, this is the equation that is most commonly studied in kinetic theory. If there exist collisions with neutral atoms, then the collision term in Eq. 2.90 can be approximated to

$$\left(\frac{\partial f(\vec{r}, \vec{v}, t)}{\partial t}\right)_c = \frac{f_n(\vec{r}, \vec{v}, t) - f(\vec{r}, \vec{v}, t)}{\tau} \qquad \text{(Eq. 2.95)}$$

where $f_n(\vec{r}, \vec{v}, t)$ is the distribution function of the neutral atoms, and τ is a constant collision time. This equation is called a *Krook collision term*.

If the fluid equation of motion including collisions for any species is given by the following relation:

$$mn\frac{d\vec{v}}{dt} = mn\left[\frac{d\vec{v}}{dt} + \left(\vec{v} \cdot \vec{\nabla}\right)\vec{v}\right] = \pm en\vec{E} - \nabla\rho - mnv\vec{v} \qquad \text{(Eq. 2.96)}$$

where the sign $\pm$ is indication of the sign of the charge and v is generally called the collision frequency of plasma particles and is written as $v = n_n\overline{\sigma v}$, with σ being cross-sectional area and v is the particle velocity in a Maxwellian distribution and n_n is the number of neutral atoms per m^3 in slab of area A and thickness dx as illustrated in Fig. 2.8.

It is the kinetic generalization of the collision term in Eq. 2.96. When there are Coulomb collisions Eq. 2.90 can be approximated as

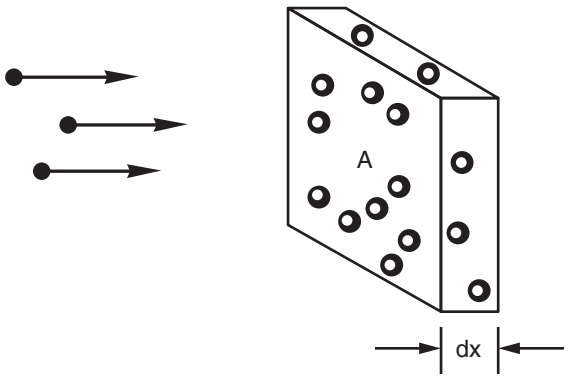


Fig. 2.8 Illustration of the definition of cross-section

$$\frac{df}{dt} = -\frac{\partial}{\partial \vec{v}} \cdot \left(f\langle \nabla \vec{v} \rangle\right) \frac{1}{2} \frac{\partial^2}{\partial \vec{v} \partial \vec{v}} : \left(f\langle \nabla \vec{v} \nabla \vec{v} \rangle\right) \quad \text{(Eq. 2.97)}$$

Equation 2.97 is called the Fokker–Planck Equation and it takes into account binary Coulomb collisions only [2].

2.9 Magneto-Hydrodynamics (MHD)

Magneto-hydrodynamics (MHD) describes the "slow" evolution of an electrically conducting fluid—most often a plasma consisting of electrons and protons (perhaps seasoned sparingly with other positive ions). In MHD, "slow" means evolution on time scales longer than those on which individual particles are important, or on which the electrons and ions might evolve independently of one another. Briefly, we can say that MHD falls in the following descriptions:

- MHD stands for magneto-hydrodynamics
- MHD is a simple, self-consistent fluid description of a fusion plasma
- Its main application involves the macroscopic equilibrium and stability of a plasma

Basically MHD can be described as a coupling of fluid dynamics equations with Maxwell's equations resulting in MHD equations and together these sets of equation are often used to describe the equilibrium state of the plasma. MHD can also be used to derive plasma waves, but it is considerably less accurate than the two-fluid equations we are familiar with and have used in our fluid mechanics knowledge.

Moreover, to define the plasma equilibrium and stability, we can categorize the definition into the following format as well and they are as follows:

- Why separate the macroscopic behavior into two pieces?
- Even though MHD is simple, it still involves nonlinear 3D + time equations
- This is tough to solve
- Separation simplifies the problem
- Equilibrium requires 2D nonlinear time independent
- Stability requires 3D+ time, but is linear
- This enormously simplifies the analysis

We need to understand the idea behind the plasma equilibrium, so it allows in case of Magnetic Confinement Fusion (MCF) to design a magnet system such that the p in steady state-force balance. So far, tokamak machines are the best design to demonstrate such equilibrium in plasma that we are looking for purpose of MCF. However, the spherical torus is another option and yet the Stellarator is another best option and each can provide force balance for a reasonably high plasma pressure.

Stability in plasma can be depicted in Fig. 2.9 and in general a plasma equilibrium may be stable or unstable. Naturally from both words expression we can tell

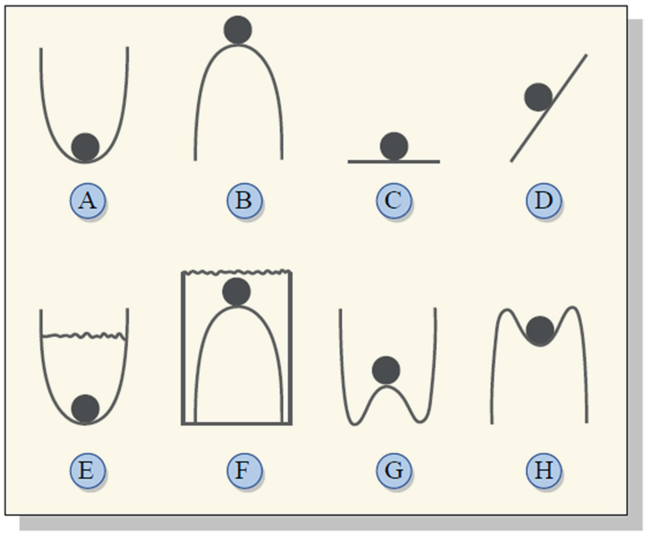


Fig. 2.9 Examples of stability

that stability is good and instability is bad in plasma confinement. However, effects of MHD instability can be summarized as follows:

- Usually disastrous
- Plasma moves and crashes into the wall
- No more fusion
- No more wall (in a reactor)
- This is known as a major disruption

The job of MHD is to find magnetic geometries that stably confine high-pressure plasmas and large amount of theoretical and computational work has been done and well tested in experiments. Although the claim is that some say there is nothing left to do in fusion MHD based on the fact that the theory is essentially complete and computational tools are readily available and used routinely in experiments.

Although there is some truth in this view, however, still there are major unsolved MHD problems that need attention.

Historically, the MHD equations have been used extensively by astrophysicists working in cosmic electrodynamics, by hydrodynamicists working on MHD energy conversion, and by fusion scientist and theorists working with complicated magnetic geometries.

Later we will talk about Rayleigh-Taylor Instabilities (RTI) in inertial confinement fusion targets, which is a growth rate in inertial confinement fusion (ICF) and it is so important and critical for determining the required driver energy. Many attempts have been made over the years to drive it analytically. The growth rates of

the acceleration and deceleration-phase Rayleigh–Taylor instability for imploding inertial confinement fusion target can be calculated analytically and numerically. In addition, many physicist and scientist who are doing research in target design for purpose of ICF investigate the effects of different physical parameters on RTI.

Calculations of the growth of Rayleigh–Taylor instabilities (1) in the ablator–pusher region may be caused by irregularities in an electron beam, and (2) in the pusher–fuel interface, a problem common to all inertial confinement fusion targets. For the first case, it can be found that long density gradient scale lengths and scattering of the beam by the target both stabilize the shorter wavelength instabilities, which would otherwise grow most rapidly of all. In the second case, it also can be found that moderately short wavelength instabilities may not degrade the target performance as much as has previously been supposed.

The determination of the instability growth rate is crucial to the success of inertial confinement fusion (ICF) because an excessive distortion of the front could lead to a severe degradation of the capsule performance with respect to the final core conditions by seeding the deceleration-phase Rayleigh–Taylor (RT) instability and preventing the onset of the ignition process. In later part of this book, we will discuss the growth rate under constant acceleration of fluid layers at the surface of ablation of the fusion of pellet target.

2.10 Physics of Dimensional Analysis Application in Inertial Confinement Fusion ICF

Dimensional Analysis is a method by which we deduce information about a phenomenon from the single premise that a phenomenon can be described by a dimensionally consistent equation of certain variables. The generality of the method is both its strength and its weakness. The result of a dimensional analysis of a problem is to reduce the number of variables in the problem, thereby gathering sufficient information from only a few experiments.

Dimensional analysis treats the general forms of equations that describe natural phenomena and its application is abounded in nearly all fields of engineering, particularly in fluid mechanics and in heat transfer theory.

The application of dimensional analysis to any particular phenomenon is based on the assumption that certain named variables are the independent variables of the problem, and that aside from the dependent variable all others are redundant or irrelevant. This initial steps of naming the variables—often requires and sometimes brings a philosophical insight into the natural phenomena that is being investigated.

The first step in modeling any physical phenomena is the identification of the relevant variables and then relating these variables via known physical laws. For sufficiently simple phenomena, we can usually construct a quantitative relationship among these variables from first principles; however, for many complex phenomena (which often occur in engineering applications) such an ab initio theory is often

difficult, if not impossible. In these situations, modeling methods are indispensable, and one of the most powerful modeling methods is dimensional analysis. We have probably encountered dimensional analysis in our previous physics courses when we were admonished to "check our units" to ensure that the left- and right-hand sides of an equation had the same units (so that our calculation of a force had the units of kg m/s^2, for example). In a sense, this is all there is to dimensional analysis, although "checking units" are certainly the most trivial example of dimensional analysis. Here we will use dimensional analysis to actually *solve* problems or at least infer valuable information about the solution.

According to Professor G. I. Barenblatt of University California at Berkeley, "many of those who have taught dimensional analysis (or have merely thought about how it should be taught) have realized that it has suffered an unfortunate fate. In fact, the idea on which dimensional analysis is based on, is very simple, and can be, understood by everybody: physical laws do not depend on arbitrarily chosen basic units of measurement. An important conclusion can be drawn from this simple idea, using a simple argument: the functions that express physical laws must possess a certain fundamental property, which in mathematics is called generalized homogeneity or symmetry. This property allows the number of arguments in these functions to be, reduced, thereby making it simpler to obtain them (by calculating them or determining them experimentally). This is, in fact, the entire content of dimensional analysis – there is nothing more to it."

The basic idea is the following: *physical laws do not depend upon arbitrariness in the choice of the basic units of measurement*. In other words, Newton's second law, $F = ma$, is true whether we choose to measure mass in kilograms, acceleration in meters per second squared, and force in Newton's, or whether we measure mass in slugs, acceleration in feet per second squared, and force in pounds. As a concrete example, consider the angular frequency of small oscillations of a point pendulum in small angle oscillation with length l and mass m:

$$\omega = \sqrt{\frac{g}{l}} \qquad \text{(Eq. 2.98)}$$

where g is the acceleration due to gravity, which is 9.8 m/s^2 on earth in the SI system of units. To derive Eq. 1.1, one usually needs to solve the differential equation which results from applying Newton's second law to the pendulum (do it!). See Appendices A and B for the analysis [8, 9]. Let us instead deduce (Eq. 2.98) from dimensional considerations alone. What can ω depend upon? It is reasonable to assume that the relevant variables are m, l, and g (it is hard to imagine others, at least for a point pendulum). Now suppose that we change the system of units so that the unit of mass is changed by a factor of M, the unit of length is changed by a factor of L, and the unit of time is changed by a factor of T. With this change of units, the units of frequency will change by a factor of g, the units of velocity will change by a factor of LT^{-1}, and the units of acceleration by a factor of LT^{-2}.

Therefore, the units of the quantity g/l will change by T^{-2}, and those of $(g/l)^{1/2}$ will change by T^{-1}. Consequently, the ratio

$$\Pi = \frac{\omega}{\sqrt{g/l}} \qquad \text{(Eq. 2.99)}$$

is invariant under a change of units; Π is called a *dimensionless number*. Since it does not depend upon the variables (m, g, l), it is in fact a constant. Therefore, from dimensional considerations alone we find that

$$\omega = \text{constant} \cdot \sqrt{g/l} \qquad \text{(Eq. 2.100)}$$

A few comments are in order:

1. The frequency is independent of the mass of the pendulum bob, a somewhat surprising conclusion to the uninitiated;
2. The constant cannot be determined from dimensional analysis alone.

These results are typical of dimensional analysis—we uncover often unexpected relations among the variables, while at the same time we fail to pin down numerical constants. Indeed, to fix the numerical constants we need a real *theory* of the phenomena in question, which goes beyond simple dimensional considerations.

Unites is a quantitative statement about an objective magnitude and is necessarily composed of two parts or factors: a number and a statement of the unit of measurement. The number is the mathematical ratio of the magnitude to that of the specific unit. Similarly, the ultimate end of all applied mathematics is the numerical evaluation, by the working of an arithmetical sum, of the magnitude of some physical quantity, which is inferred from the known magnitudes of others. Any physical quantity can be completely defined by a number and any arbitrarily valued unit, provided that the unit is exactly specified and relevant to the physical system. A collection of units for the measurement of physical quantities is known as a system of units, and, in such a system, the various units may be either arbitrarily defined or they may be made to depend in a simple way on other units. Per Prof. Barenblatt "The units for measuring physical quantities are divided into two categories: *fundamental units and derived units*." This means the following:

A class of phenomena (e.g., mechanics, i.e., the motion and equilibrium of bodies) is singled out for study. Certain quantities are listed and standard reference values for these quantities either natural or artificial are adopted as fundamental units; there is a certain amount of arbitrariness here. For example, when describing mechanical phenomena, we may adopt mass, length, and time standards as the fundamental units, even though it is also possible to adopt force, length, and time standards. However, these standards are insufficient for the description of, for example, heat transfer due to temperature. Additional standards also become necessary when studying electromagnetic phenomena, etc.

A set of fundamental units, which is sufficient for measuring the properties of the class of phenomena under consideration, is called a *system of units*. Until recently,

the CGS (Centimeter–Gram–Second) system, in which units for mass, length, and time are used as the basic units, and 1 gram (g) is adopted as the unit of mass, 1 centimeter (cm) is adopted as the unit of length, and 1 second (s) is adopted as the unit of time, has customarily been used. However, a system of units need not be *minimal* [2]. For example, one can use a system of units in which the unit of length is 1 cm, the unit of time is 1 s, and the unit of velocity is 1 knot (approximately 50 cm/s). However, in the case of this system, the velocity will not be numerically equal to the ratio of the distance traversed to the magnitude of the time interval in which this distance was traversed.

Classes of systems of units: In addition to the CGS system, there is a second system, in which a standard length of 1 km ($=10^5$ cm) is used as the unit of length. A standard mass of 1 metric ton ($=10^6$ g) is used as the unit mass, and finally, a standard time interval of 1 h ($=3600$ s) is used as the unit of time. These two systems of units have the following property in common:

- Standard quantities of the same physical nature (mass, length, and time) are used as fundamental units. Consequently, we say that these systems belong to the same ***class***. To generalize, a set of systems of units that differ only in the magnitude (but not in the physical nature) of the fundamental units is called a ***class of systems of units***.
- The system just mentioned and the cgs systems are members of the class in which standard lengths, masses, and the times are used as the fundamental units.

The corresponding units for an arbitrary system in this class are as follows:

$$\begin{aligned} \text{Unit of length} &= \text{cm} \quad L \\ \text{Unit of mass} &= \text{g} \quad M \\ \text{Unit of time} &= \text{s} \quad T \end{aligned} \qquad \text{(Eq. 2.101)}$$

where L, M, and T *are abstract positive numbers* that indicate the factors by which the fundamental units of length, mass, and time decrease in passing from the original system (in this case, the cgs system) to another system in the same class. This class is called the *LMT* class.

Note: The designation of a class of system of units is obtained by writing down, in consecutive order, the symbols for the quantities whose units are adopted as the fundamental units. These symbols simultaneously denote the factor by which the corresponding fundamental unit decreases upon passage from the original system to another system in the same class.

The SI (MKS) system has recently come into widespread use. This system, in which 1 m ($=100$ cm), is adopted as the unit of length, 1 kg ($=1000$ g) is adopted as the unit of mass, and 1 s is adopted as the unit of time, also belongs to the *LMT* class. Thus, when passing from the original system to the SI system, $M = 0.001$, $L = 0.01$, and $T = 1$.

Systems in the *LFT* class, where units for length, force, and time are chosen as the fundamental units, are also frequently used; the fundamental units for this class are as follows:

$$\begin{aligned} \text{Unit of length} &= \text{cm} \quad L \\ \text{Unit of force} &= \text{kgf} \quad M \\ \text{Unit of time} &= \text{s} \quad T \end{aligned} \qquad \text{(Eq. 2.102)}$$

The unit of force in the original system, the kilogram force (kgf), is the force that imparts an acceleration of 9.80665 m/s^2 to a mass equal to that of the standard kilogram.

Note: A change in the magnitudes of the fundamental units in the original system of units does not change the class of systems of units.

For example, the classes where the units of length, mass and time are given by

$$\begin{aligned} N(\chi_i, \tau)_p &= N(\chi_i, \tau)_m u_1^\alpha u_2^\beta \dots u_n^\eta \\ \left(n_{u1}^\alpha n_{u2}^\beta \dots n_{un}^\eta\right)_p &= \left(n_{u1}^\alpha n_{u2}^\beta \dots n_{un}^\eta\right)_m \\ N_{op} &= N_{om} \\ n_u &= u/u_0 \\ \frac{\left(u_1^\alpha u_2^\beta \dots u_n^\eta\right)_p}{\left(n_{10}^\alpha n_{20}^\beta \dots n_{n0}^\eta\right)_p} &= \frac{\left(n_1^\alpha n_2^\beta \dots n_n^\eta\right)_m}{\left(n_{10}^\alpha n_{20}^\beta \dots n_{n0}^\eta\right)_m} \\ k_u &= \frac{u(\chi_i, \tau)_p}{u(\chi_i, \tau)_m} = \frac{u_{op}}{u_{om}} = \text{constant} \end{aligned} \qquad \text{(Eq. 2.103)}$$

is the same as that defined in *LMT*. The only difference is that the numbers L, M and T for a given system of units (e.g., the SI system) will be different in the two representations of *LMT* class in the second representation, we obviously have $L = 1$, $M = 1$, and $T = 3600$.

Thus, in the investigation and study of implosion and explosion problem for pellet of fusion driven by high-energy laser or particles beams, understanding the method of similarity and self-similar with assist from dimensional analysis is very useful. These methods do represent classes of solutions to the hydrodynamic equations, which involves recognition of symmetry properties to overcome Rayleigh–Taylor instability (RTI) during ablation surface acceleration by the high-energy beams of laser or particles and can often be written in analytic form. In the mathematical form, the dimensional analysis derives in from of Pi-Theorem and the reader may refer to the books by Zohuri [8, 9].

2.10.1 Dimensional Analysis and Scaling Concept

Scaling is the branch of measurement that involves the construction of an instrument that associates qualitative constructs with quantitative metric units and the term describes a very simple situation. S.S. Stevens came up with the simplest and most straightforward definition of scaling. He said:

'Scaling is the assignment of objects to numbers according to a rule'

However, what does that mean?

Most physical magnitudes characterizing nanoscale systems differ enormously from those familiar in macro-scale systems. Estimate some of these magnitudes can be achieved, by applying scaling laws to the values for macro-scale systems. There are many different scaling laws. At one extreme, there are simple scaling laws that are easy to learn, easy to use, and very useful in everyday life. This has been true since Day One of modern science. Galileo presented several important scaling results in 1638 [10].

The existence of a power law relationship between certain variables y and$x$.

$$y = Ax^{\alpha} \qquad \text{(Eq. 2.104)}$$

where A and α are constant values. This type of relationship often can be seen in the mathematical modeling of various phenomena, not only in mechanical engineering and physics, but also in other science fields such as biology, economics, and other engineering discipline.

Distribution of Power Law is unique and has certain interesting features and graphically can be, presented as a log–log scale as a straight line. This can methodically be shown, if we take the base 10 of logarithm of Eq. 1.4 as follows:

$$\begin{cases} \log(y) = \log(Ax^{\alpha}) \\ \log(y) = \log A + \log x^{\alpha} \\ \text{Assume } \log A = B \text{ Than} \\ \log y = B + \alpha \log x \end{cases} \qquad \text{(Eq. 2.105)}$$

Last relationship in Eq. 1.5 has a general form of a linear function as presented by log$y$, and the slope of this linear logarithmic function is the exponential of power law α and it is known as *Hausdorff–Besicovitch* or *fractal dimension* [11].

Scaling analysis has its mathematical foundation in specifically the continuous symmetry group of uniform magnifications and contractions, which is known as Lie group theory. The properties of the latter group are useful when considering the operations involved when we change the units on the quantities that appear in dimensional equations [1].

Scaling laws reveal the fundamental property of phenomena, namely, self-similarity–repeating in time and/or space–that substantially simplifies the

mathematical modeling of the phenomena themselves. There are many books dealing with analysis of scaling and of the good one written by G. I. Barenblatt. This book begins from a nontraditional exposition of dimensional analysis, physical similarity theory, and general theory of scaling phenomena, using classical examples to demonstrate that the onset of scaling is not until the influence of initial and/or boundary conditions has disappeared but when the system is still far from equilibrium. Numerous examples from a diverse range of fields, including theoretical biology, fracture mechanics, atmospheric and oceanic phenomena, and flame propagation, are presented for which the ideas of scaling, intermediate asymptotic, self-similarity, and renormalization were of decisive value in modeling [12].

For example, when converting the length unit of centimeters to meters, all quantities expressed totally or partially in terms of length, units (heights, widths, velocities, accelerations, densities, etc.) experience either a uniform magnification or contraction. Knowingly then, all heights become smaller when expressed in terms of meters rather than centimeters, whereas all densities become larger.

Scalar analysis might not be very clear in connection between uniform magnifications and contractions in view of the fact that one is not changing units when one nondimensionalize a system of equations. Nondimensionalizing a quantity, it will involve dividing the quantity by another quantity or combination of quantities that should have same units.

2.10.2 Similarity and Estimating

The notion of similarity is familiar from geometry. Two triangles are said to be similar if all of their angles are equal, even if the sides of the two triangles are of different lengths. The two triangles have the same shape; the larger one is simply a scaled-up version of the smaller one. This notion can be generalized to include physical phenomena. This is important when *modeling* physical phenomena; for instance, testing a prototype of a plane with a scale model in a wind tunnel. The design of the model is dictated by dimensional analysis. Similarity is an extension of geometrical similarity. By definition, two systems are similar if their corresponding variables are proportional at corresponding locations and times. The famous of all and familiar similarity that one can even buy in today's market is Russian nested dolls. (See Sect. 10.1 of Chap. 2 for more details).

A **Matryoshka doll** or a **Russian nested doll** (often incorrectly referred to as a **Babushka doll**–babushka means "grandmother" in Russian) is a set of dolls of decreasing sizes placed one inside the other. "Matryoshka" (Матрёшка) is a derivative of the Russian female first name "Matryona," which was a very popular name among peasants in old Russia. The name "Matryona" in turn is related to the Latin root "mater" and means "mother," so the name is closely connected with motherhood and in turn, the doll has come to symbolize fertility [8, 9].

A set of matryoshkas consists of a wooden figure, which can be pulled apart to reveal another figure of the same sort inside. It has, in turn, another figure inside,

Fig. 2.10 Russian nested dolls

and so on. The number of nested figures is usually five or more. The shape is mostly cylindrical, rounded at the top for the head and tapered toward the bottom, but little else; the dolls have no hands (except those that are painted). Traditionally the outer layer is a woman, dressed in a sarafan. Inside, it contains other figures that may be of both genders, usually ending in a baby that does not open. The artistry is in the painting of each doll, which can be extremely elaborate. See Fig. 2.10

Return to the mathematical statement of the Π Theorem. We can identify the following dimensionless parameters [8, 9]:

$$\Pi = \frac{a_n}{a_1^p \ldots a_k^r} \quad \Pi_1 = \frac{a_{k+1}}{a_1^{p_{k+1}} \ldots a_k^{r_{k+1}}} \qquad \text{(Eq. 2.106)}$$

and so on, such that, it can be written as

$$\Pi = \Phi(\Pi_1, \ldots, \Pi_{n-k}) \qquad \text{(Eq. 2.107)}$$

The parameters (Π, Π_1, $\ldots, \Pi_{n-k}$) are known as *similarity parameters*. Now if two physical phenomena are similar, they will be described by the same function Φ. Denote the similarity parameters of the model and the prototype by the superscripts m and p, respectively. Then if the two are similar, their similarity parameters are equal:

$$\Pi_1^{(p)} = \Pi_1^{(m)}, \ldots, \Pi_{n-k}^{(p)} = \Pi_{n-k}^{(m)} \qquad \text{(Eq. 2.108)}$$

So that

$$\Pi^{(p)} = \Pi^{(m)} \qquad \text{(Eq. 2.109)}$$

Therefore, in order to have an accurate physical model of a prototype, we must first identify all of the similarity parameters and then ensure that they are equal for the model and the prototype.

2.10.3 Self-Similarity

Now that we are here, the question is what is self-similarity?. Simply we can use the answer that is given in Wikipedia and it seems a good description of it.

"In mathematics, a self-similar object is exactly or approximately similar to a part of itself (i.e. the whole has the same shape as one or more of the parts). Many objects in the real world, such as coastlines, are statistically self-similar: parts of them show the same statistical properties at many scales. Self-similarity is a typical property of fractals.

Scale invariance is an exact form of self-similarity where at any magnification there is a smaller piece of the object that is similar to the whole. For instance, a side of the Koch snowflake is both symmetrical and scale invariant; it can be continually magnified 3× without changing shape."

Therefore, in a simple form, self-similar means that the form of the solutions is scaled invariant (*temporally* and *spatially*). Dealing with astrophysical hydrodynamics problem, mostly supernova and strong shock phenomenon we encounter, '*similar*' or '*self-similar*' solutions and using '*similarity methods*.'

Self-similarity means that a structure, or a process, and a part of it appear to be the same when compared. A self-similar structure is infinite and it is not differentiable in any point.

In physics and mathematics, scale invariance is a feature of objects or laws that do not change if length scales (or energy scales) are multiplied by a common factor. The technical term for this transformation is a dilatation (also known as dilation), and the dilatations can also form part of a larger conformal symmetry.

- In mathematics, scale invariance usually refers to an invariance of individual functions or curves. A closely related concept is self-similarity, where a function or curve is invariant under a discrete subset of the dilatations. It is also possible for the probability distributions of random processes to display this kind of scale invariance or self-similarity.
- In classical field theory, scale invariance most commonly applies to the invariance of a whole theory under dilatations. Such theories typically describe classical physical processes with no characteristic length scale.
- In quantum field theory, scale invariance has an interpretation in terms of particle physics. In a scale-invariant theory, the strength of particle interactions does not depend on the energy of the particles involved.

- In statistical mechanics, scale invariance is a feature of phase transitions. The key observation is that near a phase transition or critical point, fluctuations occur at all length scales, and thus one should look for an explicitly scale-invariant theory to describe the phenomena. Such theories are scale-invariant statistical field theories, and are formally very similar to scale-invariant quantum field theories.
- Universality is the observation that widely different microscopic systems can display the same behavior at a phase transition. Thus, phase transitions in many different systems may be described by the same underlying scale-invariant theory.
- In general, dimensionless quantities are scale invariant. The analogous concept in statistics is standardized moments, which are scale-invariant statistics of a variable, while the unstandardized moments are not.

The self-similarity can be grouped to the following general categories as follows [13]:

1. **Approximate self-similarity:** means that the object does not display perfect self-similarity. For example, a coastline is a self-similar object, a natural fractal, but it does not have perfect self-similarity. A map of a coastline consists of bays and headlands, but when magnified, the coastline is not identical but statistically the average proportions of bays and headlands remain the same no matter the scale. It is not only natural fractals, which display approximate self-similarity but the Mandelbrot set is another example. Identical pictures do not appear straight away, but when magnified, smaller examples will appear at all levels of magnification [14, 15].
2. **Statistical self-similarity:** means that the degree of complexity repeats at different scales instead of geometric patterns. Many natural objects are statistically self-similar whereas artificial fractals are geometrically self-similar.
3. **Geometrical similarity:** is a property of the space–time metric, whereas physical similarity is a property of the matter fields. The classical shapes of geometry do not have this property; a circle if on a large enough scale will look like a straight line. This is why people believed that the world was a flat pancake; the earth just looks that way to humans [15–18].

One well-known example of self-similarity and scale invariance is fractals, patterns that form of smaller objects that look the same when magnified. Many natural forms, such as coastlines, fault and joint systems, folds, layering, topographic features, turbulent water flows, drainage patterns, clouds, trees, leaves, bacteria cultures [16], blood vessels, broccoli, roots, lungs, and even universe, etc., look alike on many scales [17].

Let us see what experts such as Barenblatt [18] are saying about self-similarity and how they describe it.

Although, in general, self-similarity may be expressed in several different ways it is often manifested mathematically as a power function $y = ax^{\beta}$, which obeys the homogeneity relation $y(\lambda x) = \lambda^{\beta} y(x)$, where λ is a (positive) scale factor, and β is a scaling exponent. Functions that satisfy this relation are said to be scaling functions

while processes or objects that are described by such functions are said to exhibit scaling behavior. With this, the terms scaling, scale invariance, and self-similarity are often used as interchangeable terms. There are numerous examples of power relationships between geological variables (Turcotte) [10], though the ranges of reported scaling behavior are often less than one order of magnitude. Actually, this is not surprising, as scaling behavior in nature is always limited between internal (small) and external (large) scales introduced by the driving mechanisms or by structural properties. A good example may be found in turbulence where classical Kolmogorov's scaling (Monin and Yaglom [19]; Frisch [20]) is constrained by viscosity at small scales and by the flow size at large scales. With limited data, such constraints introduce unavoidable uncertainties in the identification of true scaling behavior or scaling regions.

Self-similarity is a special condition of a single system. A system is said to be self-similar if there exists a separable variable of the principal equations and initial and boundary conditions of the system. The separable variable is called a similarity variable. Similarity variables are valuable in the solution of special partial differential equations with special initial and boundary conditions. Solutions of the diffusion equation and the *Prandtl* boundary-layer equations are classical examples of the application of similarity variables [21].

Self-similar solutions provide some of the greatest simplifications to one-dimensional flows. Self-similarity allows the reduction of the partial differential equations, which contain two independent variables (space and time), into a set of ordinary differential equations (ODEs), where the single independent variable is a combination of space and time. The ODEs are then relatively easy to solve numerically or even analytically in some cases. They describe the asymptotic behavior of one-dimensional flow in a variety of circumstances. Typically, they are far away from the initial conditions and provided that the boundary conditions contain no spatial scale.

Some exceptions apply. For example, self-similarity can prevail in exponential density gradient in planar geometry.

Whether or not a system is self-similar is not obvious, and the discovery of similarity variables may be a tedious process. Two approaches may be followed. The first one starts with the initial and boundary conditions. The second one starts with the principal equations. The first approach is simpler, if it is known or assumed similarity variables apply. The second approach may reveal a more general class of separable variables, which may or may not satisfy specified initial and boundary conditions.

A typical initial or boundary condition of self-similar systems is that

$$f(a, y) = f(x, b) \qquad \text{(Eq. 2.110)}$$

where x, y may be either coordinates or time. Equation 2.110 may be satisfied, in some cases, by a similarity variable of the form

$$\zeta = x^m y^n \qquad \text{(Eq. 2.111)}$$

Besides geometrical similarity, the first one to recognize a coherent structure in a physical phenomenon was Fourier with his study of the heat propagation. Then, mostly the fluid dynamists of the late nineteenth to the beginning twentieth century recognize the idea of physical similarity between different experiments and the possibility of comparing their results after the introduction of properly chosen dimensionless quantities. From these works, it emerges the concept of Dimensional Analysis with Π theorem. At the same time, reduced models are used in engineering. Invariance under similarity transformation and/or under time translation is quite common properties in the equations modeling the physical world. They allow partial or total integration and lead to much simpler equations, which eventually can be solved numerically with a much reduced numerical effort (decrease in the dimension of the phase space or the parameters space, or decrease in the number of independent variables). Nevertheless, similarity transformations and time translations put constraints on the initial conditions which can be treated although they often point out these initial conditions or the critical parameters for which the nature of the solution changes. Embedding these concepts in the physical frame of rescaling can permit to precise the nature of these Self-Similar Solutions (SSS) and give information on their possible asymptotic nature. In that case, the knowledge of the physicists complements nicely the more rigorous mathematical treatment [8, 9].

2.10.4 General Results of Similarity

If the general requirements of similarity are satisfied, the solution of the nondimensional equations will be the same for a prototype and its model. Therefore, the result of similarity is that corresponding, dependent, nondimensional variables are equal at corresponding points. Mathematically, the result of similarity is that

$$n_D(\chi_i, \tau)_p = n_D(\chi_i, \tau)_m \qquad \text{(Eq. 2.112)}$$

where n_D is a dependent nondimensional variable.

2.10.5 Principles of Similarity

By combining the requirements and results of similarity, the principles of similarity are

$$N_{om} = N_{op} \quad \text{and} \quad n_u(\chi_i, \tau)_m = n_u(\chi_i, \tau)_p \qquad \text{(Eq. 2.113)}$$

where

N_{om} = model reference similarity number.
N_{op} = prototype reference similarity number.
u = dimensional variable.
n = nondimensional variable.
χ_i = nondimensional coordinate.
τ = nondimensional time.

$n_u(\chi_i, \tau)_m = n_u(\chi_i, \tau)_p$ includes geometrical requirements.

To derive a corollary to the principle of similarity, consider a local similarity number

$$N(\chi_i, \tau) = u_1^{\alpha} u_2^{\beta} \ldots .u_m^{\eta} \qquad \text{(Eq. 2.114)}$$

By combining Eq. 2.114 with Eq. 2.114, we get Eq. 2.115 as follows

$$\left(n_{u1}^{\alpha} n_{u2}^{\beta} \ldots .n_{un}^{\eta}\right)_p = \left(n_{u1}^{\alpha} n_{u2}^{\beta} \ldots .n_{un}^{\eta}\right)_m \qquad \text{(Eq. 2.115)}$$

Since $n_u = u/u_0$,

$$\frac{\left(u_1^{\alpha} u_2^{\beta} \ldots .u_n^{\eta}\right)_p}{\left(n_{10}^{\alpha} n_{20}^{\beta} \ldots n_{n0}^{\eta}\right)_p} = \frac{\left(n_1^{\alpha} n_2^{\beta} \ldots n_n^{\eta}\right)_m}{\left(n_{10}^{\alpha} n_{20}^{\beta} \ldots n_{n0}^{\eta}\right)_m} \qquad \text{(Eq. 2.116)}$$

however, since $N_{op} = N_{om}$, the denominators are equal, and the numerators

$$N(\chi_i, \tau)_p = N(\chi_i, \tau)_m \qquad \text{(Eq. 2.117)}$$

Therefore, a corollary to the principle of similarity is that corresponding local similarity numbers are equal at corresponding points of similar systems. Local similarity numbers do not involve reference variables and are useful in some applications.

2.10.6 Self-Similarity Solutions of the First and Second Kind

We have learned so far that two geometrical objects are called similar if they both have the same shape. The second object may be obtained from the first by the result of a uniform scaling (enlarging or shrinking). Also from all demonstrations and example presented in above section, we have established for certain engineering and physics problems that we cannot find a close analytical solution, therefore dimensional analysis is a good tool to use. Meanwhile we observed how extensions

of this tool such as scaling and similarity as well as self-similarity have great influence to establish a solution for these non linear problems. Now we need to take one step beyond where we are by defining different types of self-similarities in particular when we are dealing with typical gas dynamic and fluid mechanics where non-linear ordinary or partial differential equations present themselves. For example, in gas dynamics two types of self-similar process, termed as self-similar motions of the first kind and the second kind, have been considered by scientist such as Zel'dovich and Raizer [19] and G. I. Barenblatt [22]. Taylor's explosion problem and one-dimensional centered rarefaction waves are typical scenarios of the flows of this kind, while emergence of strong shock near the surface of star, Sakurai [17], Sachdev and Ashraf [23], and converging cylindrical and spherical shocks Guderley [24], are examples of the flows of the second kind.

Zel'dovich and Raizer [19] are suggesting that the solutions of the first type possess the property that the similarity exponent α and the exponent of t and R in all scales are determined either by dimensional considerations or from the conservation laws [19]. They also describe under these circumstances the exponents are simple rational fractions with integral numerators and denominators. They explain the problem of this type always contain two parameters with independent dimensions, which means there is a type of self-similar solution in which the exponents are determined by the boundary conditions and may be set arbitrary within certain limits. Although the exponents in such solutions are not simple rational fractions in general, the solutions are to be considered as the first type, because the two independent parameters exist and the exponents are determinable in advance. These parameters are used to construct a parameter a whose dimensions contain the primary type units, which is mass and the other two parameters of length and time and it is designated by A. With latter parameter, A it is possible to construct a dimensionless combination, the similarity variable $\xi = r/At^{\alpha}$. The dimensions of the parameter A are given in terms of length and time as $LT^{-\alpha}$ and are determined by the similarity exponent a. Examples of these types were well presented in above (i.e., Taylor Explosion problem) [25].

Taylor [25, 26], Von Neumann, [27] and Sedov [28], known as the Sedov–Taylor solution describe an explosion in which a strong shock wave propagates into cold surroundings whose density (i.e., gas is assumed ideal and the density is for preshocked status) profile decreases as a power law $\rho \propto r^{-k}$ where r being varying radius of shock expansion from point blast. They all used the conservation of energy approach to obtain the scaling of the shock radius as a function of time (see Sect. 8.1 of Chap. 1). That is why such solutions are called *First Kind* (or *Type*) solutions. Yet in contrast looking at Guderley [14] (also see the discussion in Zel'dovich and Raizer) [12] on implosion problem where they also found a self-similar solution describing imploding shock waves in a constant density environment, energy consideration cannot be used to deduce the scaling of the shock radius as a function of time. Instead, the scaling of the radius as a function of time must be found by forcing that the solution pass through a singular point of equation and that is why such solutions are considered or called self-similar solutions of the *Second Kind* (or *Type*). Therefore, it is safe to say that second-type solutions do not obey

global conservation laws. In reality, the true problem therefore cannot be completely described by a second-type self-similar solution. Those describe only part of the flow, in some region of interest whereas other regions deviate from the solutions. So in order to prevent any influences on the self-similar part, a sonic point, where the equations are singular, must separate the non-self-similar parts from it. This requirement replaces the energy conservation as a means of reducing the scaling of Lorentz factor with radius, i.e., finding m (see Waxman and Shavarts) [29] for a discussion of the nonrelativistic case and (Best and Sari) [30] for the relativistic case. Reference 27 shows that if the density falls fast enough ($k > 3$), energy considerations give the wrong scaling. Same reference also showed that solution should be of the second type for $k > 3.26$. A good discussion around the first- and second- type self-similar solution of implosions and explosions containing ultra-relativistic shocks is given by Re'me Sari [31].

In self-similar problem of the second kind, the exponent α cannot be found from dimensional considerations or from the conservation laws without solving the equations. In this case, the determination of the similarity exponent requires that the ordinary differential equations for the reduced functions be integrated. Examples of self-similar motions of the second kind are the problems of an imploding shock wave and of an impulsive load [8].

Solutions of specific problems of the second kind show that in all these cases the initial conditions of the problem contain only one-dimensional parameter with the unit of mass but lacking parameter A. This condition eliminates the possibility of determining the number α from the dimensions of A, which means it would not be possible to construct the dimensionless combination $\xi = r/At^{\alpha}$. However, the dimensions of this parameter α are not dictated by the initial conditions of the problem but rather are found from the solution of the equations [19]

In summary, many engineering problems are too complex to find a mathematically closed form of solution for them. In such cases, a type of analysis, which involves the dimensions of the quantities entering the problem, may be useful. This is as we have describing and shown in different examples called *Dimensional Analysis*. Uses and applications for dimensional analysis include the following:

- To reduce the number of variables to be studied or plotted
- In planning experiments
- In designing engineering models to be studied and in interpreting model data
- To emphasize the relative importance of parameters entering a problem
- To enable units of measurement to be changed from one system to another

The last of these is common, although relatively trivial, application.

In general, dimensional analysis is any mathematical operation, which involves units or dimensions.

Finally, as far as similar and self-similar definition is concerned, we can put it into the following perspective [8].

Note: Blandford and McKee [32] are using notation of m and Γ as Lorentz factor of the shocked fluid and they show for their analysis of an adiabatic blast wave where they argue an approximate adiabatic similarity solution as part of suggested blast wave variables and is appropriate choice of similarity for well-known Sedov–Taylor similarity for a non-relativistic as follows:

$$\xi = (1 - r/R)\Gamma^2 \geq 0$$

where R being the radius of blast from the center. If the total energy contained in shocked fluid remains constant with t representing time for shock traveling at some characteristic velocity, then

$$\Gamma^2 \propto t^{-3}$$

If we consider the more general case, then we can show the above equation in the following form.

$$\Gamma^2 \propto t^{-m}, \quad m > -1$$

This allows us to treat the case when the energy is supplied continuously at a rate proportional to a power of the time.

Their solution is valid when the density of the external medium into which the shock wave propagates varies with the distance r from the origin as r^{-k}, for $k < 4$. These are first-type self-similar solutions in which the shock Lorentz factor Γ varies as $\Gamma^2 \propto t^{-m}$, where $m = 3 - k$ to ensure energy conservation.

Best and Sari [30] show new second-type self-similar solutions, valid for $k > 5 - 3\sqrt{3} \approx 4.13$, in their paper. In these types of solution, Γ varies as $\Gamma^2 \propto t^{-m}$, with $m = \left(3 - 2\sqrt{3}\right)k - 4\left(5 - 3\sqrt{3}\right)$ so that the shock accelerates and the fraction of the flow energy contained in the vicinity of the shock decreases with time.

We need to have the better understanding of similar and self-similar methods and their definition in subject of dimensional analysis. Once we have these methods defined properly, then we can extend it to motion of a medium in particular from self-similarity point of view. In addition, we are able to deal with complexity of partial differential equations of conservation laws, such as mass conservation law, the momentum conservation law and finally the energy conservation law of non-linear type both in Eulerian and in Lagrangian schemes using all three coordinates systems that we are familiar with. These coordinates are, i.e., cartesian, cylindrical, and spherical coordinate systems. Further this allows us to have better understanding of what is the self-similarity of first and second kind, and their definitions, what are the differences between them, as well as where and how they get applied to our physics and mathematics problems in hand. Few of these examples that we can mention here are self-similar motion of a gas with central

symmetry, both sudden explosion (Taylor) [5, 6] and sudden implosion (Guderley) [7] problems. The first one is considered self-similarity of first kind while the lateral is considered as self-similarity of second kind. Through these understandings, we can have better grasp of gas dynamics differential equations and their properties in a medium. In addition, the analysis of such differential equations for a gas motion with central symmetry becomes much easier, by utilizing self-similar method.

Self-similar motion of a medium is one in which the parameters that are characterizing the state and motion of the medium vary in a way as the time varies, the spatial distribution of any of these parameters remains similar to itself. However, the scale characterizing this perturbation/distribution can also vary with time in accordance with definite rules. In other words, if the variation of any of the above parameters with time are specified at a given point in space, then the variation of these parameters with time will remain, the same at other points lying on a definite line or surface, providing, that the scale of given parameter and the value of the time are suitably changed [9].

The analytical conditions for self-similar motion lead to one or more relations between the independent variables, defining functions, which play the role of new independent variables using dimensional analysis and self-similarity approach [8]. This approach follows that, in case of self-similar motion, the number of independent variables in the fundamental systems of equations is correspondingly, reduced. This technique, considerably, simplifies the complex and nonlinear partial differential equations to sets of ordinary differential equations. Thus, sometimes, this makes it possible to obtain several analytical solutions describing, for example, the self-similar motion of the medium, As it was said, in the case of two independent variables, and sometimes even in the case of three independent variables, the fundamental system of equations becomes a system of ordinary rather than partial differential equations [9].

Applications of self-similar approach can be seen to all unsteady self-similar motions with symmetry, all steady plane motions and certain axial symmetrical motions as well. These types of approaches have solved problems of Self-Similarity of First Kind [5, 6] and Second Kind [7] in past, where complex partial differential equations of conservations law are described by systems of ordinary differential equations. Investigation of most important of modern gas dynamics motions or plasma physics such as laser-driven pellet for fusion confinement via self-similar methods enables us to produce very useful conclusions by solving the conservation law equations in them, using self-similarity model. To be concerned about more general types of motion of the medium also allows us to develop and establish laws of motion in various cases of practical interest. They may include the propagation of strong shock waves in case of explosion and implosion events, propagation of soliton waves, and the reflection of shock waves are few examples that can fall into category of self-similarity methods. To further have better understanding of subject similarity and self-similarity requires knowledge of fundamental equation of gas dynamics, where we can investigate a compressible liquid or gas. Therefore, next few sections of this chapter are, allocated to this matter and related thermodynamics aspect of state of medium equations. For this, we also need to understand the

difference between compressible and incompressible flows. In addition, the detail analyses of similarity can be found in the book by this author, so we do not have to repeat the same information here [8].

2.11 Physics of Implosion and Explosion in ICF: Self-Similarity Methods

In order to consider this problem and possible integration of it into physics of inertial confinement fusion (ICF) mechanism, we need to have some fundamental understanding of self-similar motion of spherical symmetry in particular. Applying the self-similar method for motions of spherical, cylindrical, and plane waves in a gas, was understood by many scientist and researcher in the past. Here we look at in one-dimensional motion of a fluid to determine this motion whose characteristics depend only on a single geometrical coordinate (i.e., r in case spherical and cylindrical shape) and on time t.

As it has been stated in previous two chapters of this book, Sedov [8], Guderley [1], Taylor [2] and others have tackled this problem independent of each other within various closed time. Basically, what they have shown is that in a one-dimensional motions, which are produced by spherical, cylindrical, and planar waves, the method of dimensional analysis and similarity theory lead the problem of nonlinear to an exact solutions for problems unsteady motion of a compressible fluid. These types of approaches by finding the exact solutions might be helpful to confirm the accuracy of various approximated solutions of the problem in fluid dynamics.

For this matter, we consider the characterization of the problems that can be solved by dimensional analysis and similarity methods; we can consider suitable functions and characteristic parameters describing the one-dimensional motion within Eulerian system. From this condition viewpoint the main suitable/desired function has variables of velocity υ, density ρ, and pressure p and the characteristic parameters as we stated are the linear coordinate r and time t. This assumption is involving another characteristic among the ones in above as a constant a, with the dimension that at least contains the symbol of mass M. Therefore, putting constant characteristic of a and its dimension perspective without loss of generality, we can write it as

$$[a] = ML^k T^s \qquad \text{(Eq. 2.118)}$$

Thus, for the unknown functions of velocity, density, and pressure, we can establish the following relationships as

$$\upsilon = \frac{r}{t}V \quad \rho = \frac{a}{r^{k+3}t^s}R \quad p = \frac{a}{r^{k+1}t^{s+2}}P \qquad \text{(Eq. 2.119)}$$

where, V, R, and P are abstract quantities and therefore, they depend only on nondimensional combinations including r, t, and other parameters involved in the problem of interest in hand. Generally speaking, these characteristics are functions of two dimensional variables; however, if among the characteristic parameters, in addition to a, there is one more individual constant b with dimension independent of a. In general, there can be many characteristic constants, but their dimension has to depend on a and b with possible independent dimensions with fixed exponent of k, s, m, and n that can be integral, fractional, or transcendental numbers [33].

However, the actual determination of these exponents in a particular problem of interest is connected with the setup formulation of the problem and properties of unknown solutions, which always exceed the limits of dimensional theory. See Chapter one of the book by Zohuri [8].

Given the preceding text, since the dimension constant characteristic a is depending on symbol of mass M, then without again, loss of generality, we can always present the constant b so that its dimension will not contain the element of mass symbol M as

$$[b] = L^m T^n \qquad \text{(Eq. 2.120)}$$

In this case, $r^m t^n/b$ will lead to only nondimensional combination, which for $m \neq 0$ can be replaced by the variable λ asfollows:

$$\lambda = \frac{r}{b^{1/m} t^{\delta}} \quad \text{where} \quad \delta = -\frac{n}{m} \qquad \text{(Eq. 2.121)}$$

However, if $m = 0$, then V, R, and P will be dependent only on time t, where in that case velocity υ is proportional to r. The corresponding particular motions are studied by Sedov [8] in detail;, he also shows in addition to the variable parameter λ, the solution can also depend on number of constant abstract parameters. He assumes that among characteristic parameters of the problem, in addition to r and time t. there are only two constants with independent dimensions. With this content in mind, then partial differential equations, which are satisfied by the velocity, density, and pressure in the unsteady one-dimensional motion of an incompressible fluid, can be replaced by a set of ordinary differential equations for the quantities V, R, and P. Solutions of these ordinary differential equations either, can be, obtained in exact closed form or approximated by means of numerical integration. Suck kinds of motions are called self-similar types, and we now formulate problem of explosion and implosion, which can easily be solved by the method of self-similar.

Considering the continuity equation of motion and energy in an ideal gas medium in absence of heat conductivity as follows

$$\frac{\partial \upsilon}{\partial t} + \upsilon \frac{\partial \upsilon}{\partial r} + \frac{1}{\rho} \frac{\partial \rho}{\partial r} = 0 \qquad \text{(Eq. 2.122)}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho \upsilon}{\partial r} + (v-1)\frac{\rho \upsilon}{r} = 0 \qquad \text{(Eq. 2.123)}$$

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^{\gamma}}\right) + \upsilon \frac{\partial}{\partial r}\left(\frac{p}{\rho^{\gamma}}\right) = 0 \qquad \text{(Eq. 2.124)}$$

These sets of equation are very similar to sets of equations of conservation of momentum and mass energy, but written in different form, where again, γ is adiabatic index, and $v = 1$ is for the planar motion for ideal gas, $v = 2$ for the cylindrical, and $v = 3$ for the spherical case (see Chap. 3 of the book by Zohuri) [8]. Applying the arbitrary quantities V, R, and P from Eq. 3–18, we can easily find that $k = 3$, $s = 2$ and in case of general relativity theory there are two other fundamental constants such as speed of light c and the gravitational constant f come to play. In this case $a = f$ and arbitrary quantities V, R, and P are dependent only on a quantity$\lambda = r/ct$.

A self-similar method can be in place to solve the new sets of equation based on the functions of V, R, and P so in case of strong shock in one-dimensional spherical coordinate system moving outward we are solving Taylor's Problem and for the shock going inward we are solving Guderley's Problem. Sedov [8] has shown the algebraic integrals solution for self-similar motions in detail by and for strong shock, we have the following form by introducing a new variable z as function of V in the form of $z(V)$ that results from relation of $z = \gamma P/R$, where it is formulated from $\Re T = (r^2/\gamma t^2)z$. Here T is the temperature and $\Re$ is the gas constant.

$$\frac{z}{R^{\gamma-1}} = C_1\left[R(V-1) + \frac{C_2}{\lambda^{v-\omega}}\right]^{[\omega(\gamma-1)]/[v-\omega]}\frac{1}{\lambda^2} \qquad \text{(Eq. 2.125)}$$

where $\omega = k + 3$ and C_1, C_2. are arbitrary constant of integration. It is obvious that variables z and V as well as function of $z(V)$ are independent of indexes k, s, and m, but are well determined by the type of self-similar motion of first or second kind that falls into explosion and implosion problem, respectively. Sedov [28] shows different plots of adiabatic integral paths for different conditions of point $O(z, V)$ is z and V v plane, where an asymptotic formulation induced from Eq. 2.125, based on ω either being, negative ($\omega < 0$), positive ($\omega > 0$) or being equal to zero ($\omega = 0$). These asymptotic sets of formulation in case of $O(z = 0, V = 0)$ are given as

$$z = CV^2 \quad \lambda = \frac{C_1}{V} \quad \text{and} \quad z = \frac{\gamma}{\omega}V \quad \lambda = \frac{C_1}{\sqrt{V}} \qquad \text{(Eq. 2.126)}$$

Note that for *non-self-similar motions*, different curves in $z - V$ plane, correspond to the gas motion at different instants. On the other hand, for *self-similar motions*, the field of gas motion in $z - V$ plane at different instants or for different points or particles corresponds with same curve on the adiabatic integral curve, which is corresponding to the plot of ordinary differential equations (ODEs) for the shock conditions under self-similar motions [8].

From the formulation of self-similar motion of these ODEs, it follows that the shock coordinate r in form of $r = \lambda bt^{\alpha}$ and variable $\lambda = r/bt^{\alpha}$ at the shock are functions of time t and characteristic dimensional constant a and b.

In particular cases, the following situation is possible: The gas motion is self-similar but the motion of boundaries of shock waves is determined by supplementary constants. Thus, the shock coordinate r depends not only on a, b, and t, but also on other dimensional constants. In these cases, the formula for $\lambda = r/bt^{\delta}$ approaching to constant value λ_0 at the shock is not true, thus, in correspondence with the assumed definitions, such motions, considered as a whole, will be called non-self-similar, although self-similarity is violated only on the boundary. Moreover, a nondimensional combination cannot be formed from the three quantities on a, b, and t, therefore for the discontinuity surface, we have the following conditions [8]:

$$\lambda = \lambda_0 = \text{constant} \quad r = \lambda_0 bt^{\alpha} \qquad \text{(Eq. 2.127)}$$

Consequently, in $z - V$ plane a fixed point corresponds to the shocks for self-similar motions with fixed values of variables λ, R, z, P, and V. Furthermore, for the value of shock velocity c, a formula of the following form always may be written as

$$c = \frac{dr}{dt} = \alpha \cdot \frac{r}{t} \qquad \text{(Eq. 2.128)}$$

Analyses of Eq. 2.128, it is obvious for self-similar motions, α is constant. For $r > 0$ and $t > 0$, the velocities of phase propagation are directed outward and away from center when $\alpha > 0$. Therefore, for $\alpha > 0$, the shock waves are divergent and thus, for $\alpha < 0$, the shock waves are directed inward and they are convergent and the velocity of phase motion decreases. If $r > 0$, the time t increases, but $t < 0$, then we have the reverse character behavior of the motion of shock waves. Figure 2.11 shows depiction of divergence and convergence of such shock waves behavior and characteristics along with adiabatic compression or rarefaction arises in front of the core.

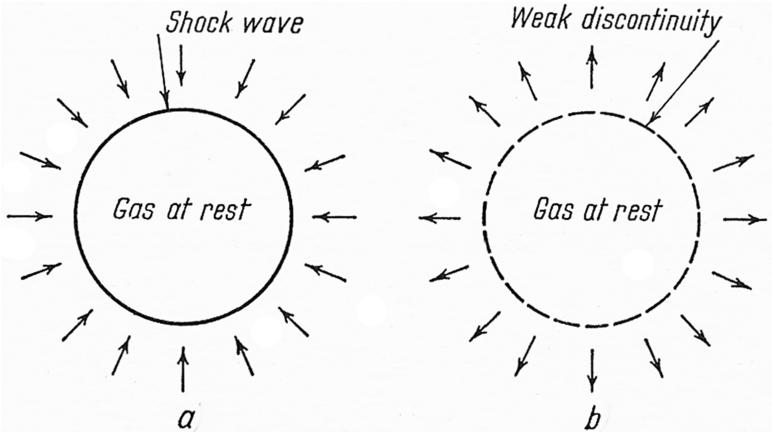


Fig. 2.11 Depiction of motion (**a**) implosion and (**b**) explosion characteristics

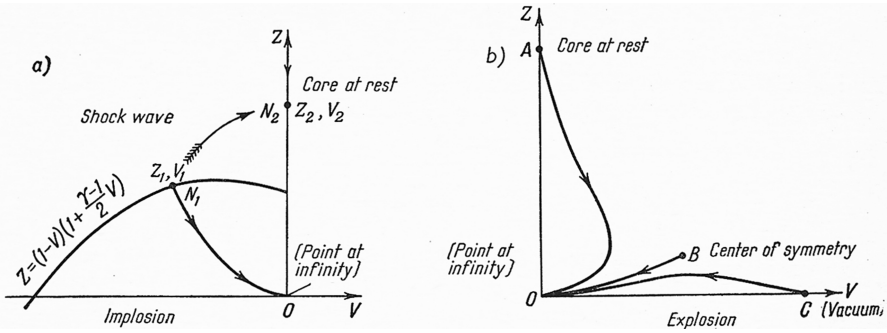


Fig. 2.12 Depiction of integral curves corresponding to (**a**) implosion at a point and (**b**) explosion from a point

On parabola, equation of $z=(\alpha-V)^2$ phase velocities is equal to the speed of sound, thus above this parabola, the velocities are subsonic and below it, they are subsonic. In the general case of non-self-similar motions, the abstract quantity α is a certain function of time t.

For case of explosion and implosion problem at a point where the corresponding point $O(z,V)$ is placed at infinity as $O(z=0,V=1)$, when initial velocity, density, and pressure is uniform everywhere as it is depicted in Fig. 3.10, then $\omega=0$ and $\alpha=1$, the appropriate field of the integral curves in the $z-V$ plane is depicted in Fig. 2.12.

However, for points at infinity, corresponding to strong implosion or explosion, the asymptotic formulas near the point $O(z,V)$ are given as

$$z=CV^2 \quad \lambda=\frac{C_1}{V} \qquad \text{(Eq. 2.129)}$$

Sedov [28] extensively has provided the interpretations of curves in both plots of Fig. 2.12.

2.12 Self-Similarity and Sedov–Taylor Problem

The mathematical formulation of the problem of the nuclear explosion and the estimation of its mechanical and physical effects on the surroundings was itself a challenging task. There was hardly any literature on this subject. Therefore, some of the best minds in applied mathematics and physics were made to put their heads together to unravel this topic. This gave a great fillip to nonlinear science, which has since made great strides and which now permeates and influences all sciences—pure and applied.

The explosion problem in a perfect gas could be considered for the case when initial velocity, density, and pressure are assumed to be uniform. Many authors have studied the motion of diverging spherical and cylindrical shock waves in a perfect gas, for a homogeneous and symmetrical form [2–7].

The diverging spherical shock waves for Trinity explosion of fission atomic bomb was studied by Taylor (1950a) [25] and Sedov (1969) [28] are few examples that are mentioned here, although there were other authors that independently did similar study. This is the class of solutions known as self-similar solutions of the first kind. Taylor (1950a) [25] demonstrated the existence of the self-similar solutions for a shock wave propagating in the vicinity of the center of divergence.

Mathematically, the continuous flow behind the shock is governed by the nonisentropic equations of gas dynamics, which must be solved subject to the so-called Rankine–Hugoniot conditions at the shock and the symmetry condition at the center requiring that the particle velocity there is zero. Along the shock trajectory, the theory of shocks imposes more boundary conditions than are appropriate to the given system. This overdetermined data, however, leads to the finding of the shock trajectory, which itself is unknown factor. This, in this sense, constitutes a free boundary value problem. In this simplest model, the role of heat conduction is ignored. Taylor (1950) [25, 26] made some highly intuitive physical statements about this phenomenon [3].

For example, he observed that the explosion forces most of the air within the shock front into a thin shell just inside the front (see Fig. 2.13). This, is subject of discussion for this section, forms the basis of an analytic theory of blast waves in an exponential atmosphere by Laumbach and Probestein (1969) [4]. Taylor (1950) [3]

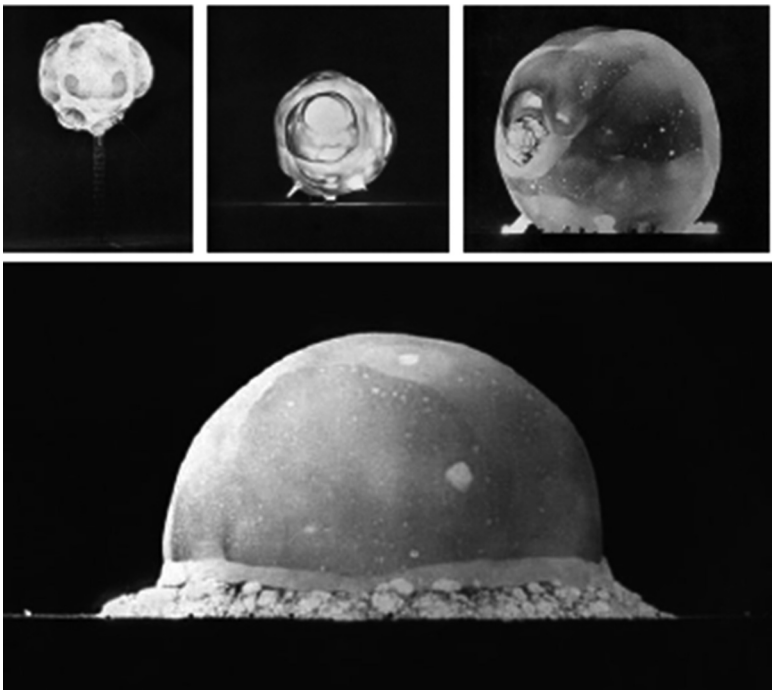


Fig. 2.13 Trinity nuclear test, New Mexico on July 16 1945

also observed that as the front expands, the maximum pressure decreases till at about 10 atmospheres: the analysis under the assumption of an infinitely strong shock ceases to hold.

On July 16 1945, the first atomic bomb ever was detonated in New Mexico. The pictures shown in Fig. 2.13 were released and published in Life Magazine. The energy of the blast, however, was highly classified and it was kept secret. The story goes that Geoffrey Ingram Taylor, the British physicist, used dimensional analysis to estimate the latter energy from the data available in the pictures. The analysis is presented here to guide you through Taylor's analysis and formulation of the similarity solution was derived entirely from physical arguments.

What is seen in the pictures is a spherical shock wave separating the undisturbed air from the region affected by the explosion. As usual, a dimensional analysis is simplified by some educated guess. Taylor's analysis is based on the following assumptions:

- The explosion itself is so rapid that the only relevant characteristic of the bomb is the amount of energy E that it releases. The duration of the explosion is irrelevant,
- The shock wave propagation is so quick that it can be modeled as an adiabatic process, characterized by adiabatic exponent (i.e., adiabatic index) γ,
- The pressure generated by the shock is much larger than the atmospheric pressure, so that the latter should not be accounted for in the analysis. Only the density of the air ρ_0 matters.

Based on these simplifying assumptions, use dimensional analysis to find the way in which the radius R of the shock wave increases, with time t, and it can be seen as function of $R = f(t, E, \rho, \gamma)$, and this function can be established from classical mechanics theory of kinetic energy. By using dimensional arguments, he wrote the similarity form of the solution in Eulerian coordinates in terms of the similarity variables r/R, where R, the radius of the shock, was found to be proportional to $t^{2/5}$; he did not use any sophisticated transformation theory of nonlinear Partial Differential Equations (PDEs). Taylor reduced the system of nonlinear PDEs to nonlinear Ordinary Differential Equations (ODEs) and numerically solved the latter, subject to the strong shock conditions (appropriately transformed) and the requirement of spherical symmetry, namely, that the particle velocity at the center of the explosion must be zero. He also used the conservation of total energy, E, behind the shock to derive the shock trajectory. The constant $B = E/\rho_0 A^2$, which appears in the shock law $R = Bt^{2/5}$, involves the nondimensional form of energy and was found from the numerical solution; it varies with adiabatic index γ the ratio of specific heats as $\gamma = C_p/C_v$, where C_p and C_v are specific heat at constant pressure and volume, respectively.

The general solution of the problem is as follows [8, 9]:

$$f(\gamma) = \frac{\rho R^5}{E t^2} \qquad \text{(Eq. 2.130)}$$

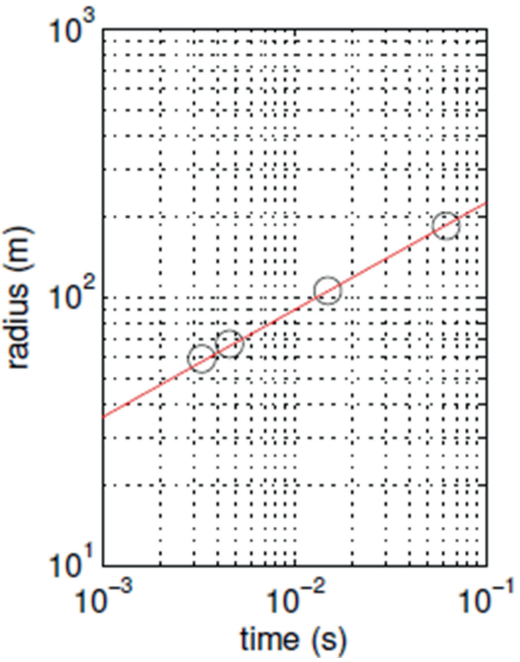


Fig. 2.14 Comparison of the data from the released trinity test pictures (*small circles*) and $R = t^{2/5}$ power law (*red line*)

Table 2.1 Time-dependent radius of the trinity shock wave as determined from the released pictures

Time (ms) t	3.3	4.6	16	62
Radius (m) R	59	67	100	185

Equation 2.130 predicts a propagation of the shock according to a $R \simeq t^{2/5}$ power law, as it is, stated before. This law is extremely well, followed by the data as it can be, seen in Fig. 2.14.

The general solution of Taylor's problem in terms of energy releases from nuclear fission explosion is then given as follows:

$$E = \left(R^5 \rho_0\right)/t^2 \qquad \text{(Eq. 2.131)}$$

Table 2.1, presents the values of R as a function of t, determined from the released pictures of the Trinity test explosion in 1947.

In one frame $r = 100$ m at a time of $t = 0.016$ s after the explosion and air density at that altitude was $\rho = 1.1 - 1.2\,\text{kg/m}^3$. See Table 2.1 and Fig. 1.8a. Substituting in these values in Eq. 2.131 gives an estimated energy release of $E = 4 \times 10^{13}$ J, which is equivalent to 1000 tons of TNT explosion of about 4.2×10^{12} J, which is indication of Trinity fission test bomb had a yield of 10 kilotons TNT based on the above calculation. The actual test bomb yield was 18–22 kilotons. Even closer values can be obtained from other frames.

See http://en.wikipedia.org/wiki/Nuclear_weapon_yield

Still this estimate is remarkably close to the reality, given the crudeness of our analysis. Taylor (1950) [2] carefully has shown the numerical solution and noticed that the particle velocity distribution behind the shock as a function of similarity variable was quite close to linear as it is depicted in Fig. 2.14, particularly near the center of the blast. His assumption was toward particle velocity to form a solution, which the sum of a linear term and nonlinear correction term in the similarity variables, then he was able to explicitly determine this term by making use of the governing equations and the Rankine–Hugoniot conditions. This enabled him to find an approximate closed form solution of the entire problem, which was in error in comparison with the numerical solution by less than 5%.

As it is stated in above, Taylor (1950b) [26] in his second publication was able to check power law of $R \approx t^{2/5}$ to show comparison with the shock trajectory that was obtained, experimentally from the Trinity fission bomb explosion in New Mexico. The agreement of the two various values for adiabatic index $\gamma = C_p/C_v$ was remarkably good.

In this comparison, photographs were used to measure the velocity of the rise of the slowing center of the heated volume. This velocity was found to be 35 m/s. The hemispherical explosive ball behaves like a large bubble in water until the hot air suffers turbulent mixing with the surrounding cold air. The vertical velocity of this 'equivalent' bubble was computed from this analysis and was found to be 35 m/s [1].

While, Taylor (1950) [25, 26] was quite aware of the advantages of a Lagrangian approach to the problem, he was rather skeptical of its practicality since, as he remarked, that would introduce great complexity, and, in general, solutions can only be derived by using step-by-step "numerical integration" of the full system of nonlinear PDEs. Actually, as a particle crosses the shock, it has an adiabatic relationship between pressure and density corresponding with the entropy, which is endowed upon it by the shock wave during its passage past it. This naturally suggests a Lagrangian approach wherein the Lagrangian coordinate is defined as one which retains its value along the particle path. Indeed, this matter was raised much later again by Hayes (1968) [34] who tried to contradict the suggestion by Zel'dovich and Raizer (1967) [19] that the Lagrangian formulation is as convenient as the Eulerian, even more so for the problems of blast wave type. He argued that the basic differential equation to be solved numerically is in a nonanalytic form in the Lagrangian formulation and would therefore pose difficulties, a view in agreement with "Taylor's apprehension" [24].

Further analysis of this matter could be found in the classical text by Sachdev [35]. In the same reference, equations involving shock wave for exact solutions of spherically symmetric flows in Eulerian coordinates are, presented with details. The exact solution is for one-dimensional gas dynamic equation, which shade light on the structure of the solutions, the blast wave being one class of solutions of these equations, where self-similarity approach has taken place. He also has suggested even exact solutions of gas dynamic equations in Lagrangian coordinates, where the approach is quite distinct and applies to all geometries—planar, cylindrical, and spherical. The basic idea behind this approach is to use the single second-order

nonlinear partial differential equation governing the Eulerian coordinate with the Lagrangian coordinate enthalpy h and time t as independent variables.

These solutions depend upon an arbitrary function, which is related to the entropy distribution in the gas. Applications of isentropic and nonisentropic solutions include flows with shocks of finite and infinite strength and vacuum fronts. This presentation is as follows:

$$h = \int_{y(0,t)}^{y(h,t)} r^{n-1}\rho(r,t)dr \quad n = 1,2,3 \qquad \text{(Eq. 2.132)}$$

where $y(h,t)$ is the radius of the particle with Lagrangian coordinate h at time, and $n = 1\,,2\,,3$ for planar, cylindrical, and spherical symmetry, respectively. In the latter two cases, y represents the distance from the axis and center of symmetry, respectively. However, for complete details of this approach, refer to Sachdev book [3].

As a final note for Taylor's Explosion Problem, as a warning and drawback, you should remember mathematical functions only take dimensionless arguments. This is shown by power series expansions:

$$\begin{aligned} f(\xi) &= \mathrm{e}^{\xi} \\ &= 1 + \xi + \frac{1}{2}\xi^2 + \ldots \end{aligned} \qquad \text{(Eq. 2.133)}$$

In this case, the leading term is obviously dimensionless, and all terms added to it must be also. In general, a function has terms of many different orders, which must be dimensionless to add up.

- Some ratios of variables and their derivatives can lead to ambiguous cases. Like the ideal pendulum that is presented in Appendix A of this book as

$$\omega_{\mathrm{B}} = \sqrt{\frac{g}{\theta}\frac{d\theta}{dz}} \qquad \text{(Eq. 2.134)}$$

  This equation is dimensionally correct for *any* substitution for θ.
- Derivatives and ratios are indistinguishable to a dimensional analysis, since g/z has the same dimensions as dg/dz
- Dimensional analysis is an aid to insight, thus it cannot completely describe the physics.

In conclusion, the creation and performance of shock waves have been the focus of study by many engineers and scientists working in topics related to continuum physics. Shock waves—either in their weak form (acoustic waves) or their moderate to stronger form—play an important part in scientific and engineering calculations whether their existence is desirable or not.

2.13 Self-Similarity and Guderley Problem

The study of converging spherical and cylindrical shock waves in a homogeneous and symmetrical mode is of importance due to its applications in the field of nuclear engineering such as controlled thermonuclear fusion, cavitations, and blast waves. Although, Guderly (1949) [24] was the first author among the others demonstrating such study in a perfect gas situation, similar technique was used in nuclear fission bomb fabrications both during the Manhattan project and later on in design of super bomb for thermonuclear fusion process.

The creation and performance of shock waves have been the focus of study by many engineers and scientists working in topics related to continuum physics. Shock waves, either in their weak form (acoustic waves) or their moderate to stronger form, play an important part in scientific and engineering calculations whether their existence is desirable or not. For example, in the case of gas pipelines, a sudden valve closure or opening (or any other blockage or leak) creates a response signal in the form of shock or expansion waves, whose speeds depend on the aero-thermodynamic state of the gas. The change in properties behind such waves should be taken into consideration for designing the pipeline as well as the surrounding installations (for safety considerations)

For internal combustion engines, the sudden opening and closing of valves create a continuous stream of shock or expansion waves, interacting and moving down the muffler as well as other ducts. This stream has to be controlled and optimized for environmental protection, Matsumora (1993) [36].

In the field of interior ballistics of guns, the existence of shock waves ahead of and behind the projectile is an unavoidable side effect to contend with. The designs are made to divert the blast and reduce its noise level, Phan (1991) [37]. Another important military application of cylindrical converging shock waves, are the generating partly converging and partly advancing shock waves in shaped charges for armor piercing. In this case, the important usage of cylindrical converging shock waves lies in production of localized high gas pressure and enthalpies. Theoretically, area convergence is expected to strengthen the shocks, thus producing infinitely dense amounts of energy at the center of convergence (point of collapse).

The common factor here is the need for accumulating great amounts of energy in virtually point-size domains. This fact probably explains why the technical steps required to create converging shocks have their inherent difficulties.

The converging shock waves via a powerful spherical and cylindrical compression in the neighborhood of the center of the sphere and of the cylinder axis, that originally was studied by Guderley is the first examples of a class of self-similar solutions of second kind. However, Stanyukovich (1969) [12] first developed an approximate method for obtaining the similarity exponent analytically.

For the case of cylindrical converging shock waves, one of the main issues faced in establishing the physical process is the shock stability. This is defined as the ability of the generated shocks to retain their required symmetric shapes if subjected to perturbations due to geometrical or physical irregularities, which is

inevitable in practical considerations. Unlike plane shocks, which retain their shape due to transverse waves [13–16], two contradictory processes affect cylindrical waves: stabilizing effect due to the transverse waves, and the increase in shock speed associated with the reduction in the frontal area. Therefore, the measure of stability for converging shocks should aim at minimizing the ratio between the magnitude of unavoidable perturbations and the mean value of the shock radius.

The need to stabilize the shock for as long as possible requires an efficient method for simulating the shock performance throughout the implosion process. In this respect, research activities have been diversified according to the available theoretical and technical facilities.

The 'Classical Guderley Problem' [9] is considering an infinitely strong, symmetric, and homogeneous shock wave focusing on either cenetr (or point) of spherical geometry or axial of cylindrical geometry shape. Although, he did not discuss the source of generating the shock, for solving this classical problem, however, the initial state of the gas into which the shock wave is propagating is well defined and described and denoted by sets of Eq. 2.138, in one-dimensional Eulerian space of *r*-coordinate (i.e., *Spherical* and *Cylindrical* geometry). His assumption for the perfect gas was under perfect inviscid gas conditions. The inviscid flow is a schematic representation of the motion of mobile media such as gaseous or liquid, and as well as solids under the rapid action of high pressures, which is the main theoretical model for many fields of modern technology.

Guderley demonstrated that strong cylindrical converging shock waves propagate according to a power law relation as described below, when approaching the center and that their Mach numbers reach infinite values at the point of collapse. However, this is not possible in reality due to the effect of viscosity and heat conduction. Lighthill [17], Butler [10, 18], Stanyukoouvich [12], and Whitman [10] conducted subsequent studies under the same assumptions.

The sets of Eulerian Conservation Equations for inviscid flow or gas conditions are as follows and they are valid if the viscosity and thermal conductivity of the fluid or gas is ignored

$$\rho \frac{d\vec{\boldsymbol{u}}}{dt} = \rho \vec{\boldsymbol{F}} - \text{grad}\, p \quad \text{Conservation of Momentum} \qquad \text{(Eq. 2.135)}$$

$$\frac{1}{\rho}\frac{d\rho}{dt} = -\text{div}\, \vec{\boldsymbol{u}} \quad \text{Conservation of Mass} \qquad \text{(Eq. 2.136)}$$

$$\rho \frac{d}{dt}\left(e + \frac{u^2}{2}\right) = \rho \vec{\boldsymbol{F}} \vec{\boldsymbol{u}} - \text{div}\, p\vec{\boldsymbol{u}} + \rho q \quad \text{Conservation of Energy} \qquad \text{(Eq. 2.137)}$$

In all these three sets of equation, variables and parameters expressed in them are defined as follows:

$\vec{\boldsymbol{u}}$ = Velocity of gas in vector form
p = Pressure quantity
ρ = Density of fluid or gas
e = Specific internal energy

The above four elements are measures at the point within a fluid or gas, where they are continuous in that space.

Equation 2.135 also known as *Eider's Equation* relates fluid particle acceleration within an element of volume of that fluid to an external body force $\vec{F}$ and the pressure force applied on the side of the neighboring fluid particles. This equation is generalized form of Newton's second law, which is in classical mechanics we know it as the conservation of momentum as applied to the motion of fluid particles.

Equation 2.136 is basically expressing the law of mass conservation, which is indicating that the rate of change of density of a fluid particle is equal, with the sign reversed, to the rate of change of volume.

Equation 2.137 is expression for the law of energy conservation, which is describing change in the internal energy e and kinetic energy $\frac{1}{2}\mathrm{v}^2$ of a fluid or gas particle as a result of an action of impressed mass forces $\vec{F}$ and surface forces (i.e., pressure p), and to an inflow of heat with intensity q from an external source.

Denoting physical flow variables in the unshocked region by the subscript 0 that is depicted in Fig. 2.15, the initial state is then expressed as

$$\begin{cases} u_0(r,t) = 0 \\ \rho_0(r,t) = \text{constant} \\ p_0(r,t) = 0 \end{cases} \qquad \text{(Eq. 2.138)}$$

where r denotes position ($r \geq 0$), t time in interval of ($-\infty < t < 0$) for the converging shock wave mode and for the interval of ($0 < t < +\infty$) for the reflected shock wave mode, while u velocity, ρ mass density, and p material pressure.

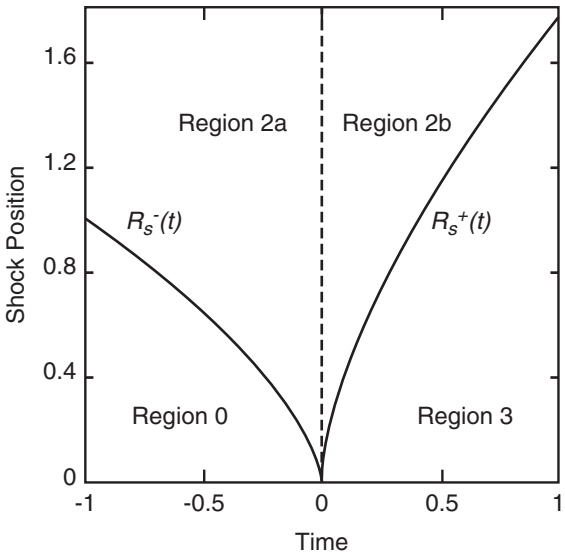


Fig. 2.15 Notional representation of converging and reflecting shock trajectory

Note that in Fig. 2.15, $R_s^-(t)$ is designation for converging shock wave trajectory, while $R_s^+(t)$ is trajectory for reflecting shock wave, and space–time regions are 0, 2a, 2b, and 3.

The basic sets of conservation equations (Eqs. 2.135 through 2.137) of mass, momentum, and energy governing adiabatic flow for Guderley problem, where we have smooth flow free of viscosity, heat conduction, radiation, and body forces, the one-dimensional Eulerian equations that are describing fluid motion at all continuous (i.e., non-shock) are expressed as

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial r} + (m-1)\frac{\rho u}{r} = 0 \qquad \text{(Eq. 2.139)}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\gamma}\left[\frac{a^2}{\rho}\frac{\partial \rho}{\partial r} + 2a\frac{\partial a}{\partial r}\right] = 0 \qquad \text{(Eq. 2.140)}$$

$$\frac{\partial a}{\partial t} + u\frac{\partial a}{\partial r} + (\gamma - 1)\left[\frac{\partial u}{\partial r} + \frac{(m-1)u}{r}\right] = 0 \qquad \text{(Eq. 2.141)}$$

where a is expressing the local speed of sound, and defined through the pressure and density by

$$a^2 \equiv \gamma\frac{p}{\rho} \qquad \text{(Eq. 2.142)}$$

Here, we are considering only a polytropic gas with the incomplete equation of state is given by the following equation known as Mie–Gruneisen type as

$$p(\rho, e) \equiv \rho e \Gamma\left(\frac{\rho}{\rho_0}\right) = (\gamma - 1)\rho e \qquad \text{(Eq. 2.143)}$$

Again, symbol of e is the specific internal energy and γ denotes the adiabatic index of fluid or gas in the range of $1 < \gamma < \infty$, and m the space dimension $m = 1, 2, 3$ for one-dimensional planar, cylindrical, or spherical geometries of symmetry and homogeneous of the shock wave. The symbol of $\Gamma(\rho/\rho_0)$ is the Gruneisen coefficient and for a perfect gas is a constant and is equal to $\gamma - 1$ [38].

Note that Eqs. 2.139 through 2.141 are not valid globally, though shock jump conditions are available to connect the preshock and postshock flow field. In particular, since the converging shock wave is assumed to be infinitely strong limit to Rankine–Hugoniot jump conditions may be used to connect the flow just behind and ahead of the shock front and they can be written as

$$\begin{cases} \dfrac{\rho_{2a}}{\rho_0} = \dfrac{\gamma+1}{\gamma-1} \\ u_{2a} = \dfrac{2}{\gamma-1} R_s^-(t) \\ p_{2a} = \dfrac{2}{\gamma-1}\rho_0 \left[R_s^-(t)\right]^2 \end{cases} \qquad \text{(Eq. 2.144)}$$

Equation 2.144 sets are valid for all value of $t < 0$ where the convergent mode and subscript $2a$ as per Fig. 2.15 denotes the state just behind the converging shock along with symbol of $R_s^-(t)$ is presentation of converging shock [38].

After shock focus and subsequent reflection about the axis or point of symmetry as well as analogous to reflection from a rigid wall in one-dimensional planar symmetry, these equation cease to be valid. Ramsey, et al. [19] show the detail analysis of their novel approach to the Guderley [24] solution and argue the case for $t > 0$.

Theoretical handling of the one-dimensional form of governing equations continues to this day, with the introduction of new equations of state or constitutive relations to simulate shock dynamics in more, complicated physical situations, or in other types of continua (including real gases).

Some other techniques were developed by Chester-Chisnell-Whitham, which is known as (C-C-W) [39–41] theory and along with Whitham's Easy-shock theory [23, 24], provided researchers with a new graphically operable tool to simulate shock dynamics.

For the CCW theory, shocks are considered as discontinuities between continuously varying sections of fluids. The continuous fluid sections were solved for by using the modified set of quasi-1-dimensional Euler equations in its characteristics form, while the variations across wave fronts are governed by the Rankine–Hugoniot relations.

The ultimate result was a new governing equation relating the local duct area (at shock location) to the local shock Mach number, incorporating the newly, defined Chester function, named after its inventor. Although the quasi-1D derivation was meant to deal with flows in ducts with varying cross-sections, the simplified set of equations was used for handling the cylindrical and spherical converging shocks (where symmetry ensures one-dimensionality) [42, 43].

This method of solution was, extended to multiple dimensions using the Ray shock theory deduced by Whitham [23, 24]. Based on concepts from geometrical acoustics, the method employs successive shock contours and their orthogonal trajectories (rays) as curvilinear coordinate lines. It was assumed that no lateral flow takes place across the ray lines, meaning that the rays coincide with streamlines at the shock location. The geometrical compatibility requirements lead to one differential equation relating the shock Mach number, M, and the ray-tube area, A, for each tube. A second relationship between the two quantities is deduced using the C-C-W theory for the motion of a shock wave down a tube of varying cross-section.

The resulting equations are of hyperbolic nature and a solution can be deduced using the method of characteristics, which also describes the motion of lateral waves on the shock front. These are interpreted as the intersection of acoustic waves with the shock front, and the case where these waves break is termed a "shock–shock" (which is well visualized in the case of Mach reflection) [44].

More details can be found in reference by El-Mallah [44] in his thesis. In summary, the problem of Collapse of a Spherical or Cylindrical Cavity and Converging Shock Wave from a Spherical or Cylindrical as result of implosion are considered to be self-similar solutions and it can be shown to be unstable for most ranges of γ.

A related problem is that of converging shock waves, which also possesses a similarity solution. Indeed, self-similar solutions of both these problems belong to the class called the "second kind" (Zel'dovich and Raizer (1967)) [19] for which dimensional analysis or group properties of the PDEs do not fully determine the self-similar form of the problem; they require a global solution of an eigenvalue problem for the reduced system of ODEs. Typically, for this class of problems the exponent in the definition of the similarity variable turns out, in general, to be an irrational number.

For the converging shock problem, which was first studied by Guderley (1942) [24], this exponent in the similarity variable $\xi = rt^{-\alpha}$ was found to be 0.717 for the spherical converging shock for $\gamma = 1.4$. Several other investigators later refined this value and good discussion is given by Sachdev [23].

References

1. B. Zohuri, *Directed Energy Weapons, Physics of High Energy Lasers (HEL)* (Springer, 2016)
2. J.R. Reitz, F.J. Milford, R.W. Christy, *Foundations of Electromagnetic Theory*, 4th edn. (Pearson, Addison Wesley, San Francisco, 2009)
3. F. Chen, *Introduction to Plasma Physics and Controlled Fusion*, 3rd edn. (Springer, 2016)
4. B. Zohuri, *Directed Energy Weapons: Physics of High Energy Lasers (HEL)*. Appendix F: Short Course in Electromagnetic and Appendix G: Short Course in Optics (Springer, 2016)
5. E. Herdst, Chemistry in the interstellar medium. Annu. Rev. Phys. Chem. (1995)
6. L.M. Haffner, R.J. Reynolds, S.L. Tufte, G.J. Madsen, K.P. Jaehnig, J.W. Percival, The Wisconsin Ha Mapper Northern sky survey. Astrophys. J. Suppl. **145**(2), 405 (2003)
7. E. Prati, Propagation in gyro-electromagnetic guiding systems. J. Electr. Wav. Appl. **17**(8), 1177–1196 (2003)
8. B. Zohuri, *Dimensional Analysis Beyond the Pi Theorem*, 1st edn. (Springer, 2017)
9. B. Zohuri, *Dimensional Analysis and Self-Similarity Methods for Engineers and Scientists*, 1st edn. (Springer, 2015)
10. G. Galilei, Discorsi e Dimostrazioni Matematiche *intorno à due nuoue scienze* Attenenti alla Mecanica & i Movimenti Locali (1638)
11. H. Schlichting, *Boundary Layer Theory*, 4th edn. (McGraw-Hill Book Company, New York, 1960)
12. I. Proudman, J.R.A. Pearson, J. Fluid. Mech. **2**, 237 (1957)
13. T. Komulainen, Helsinki University of Technology, Laboratory of Process Control and Automation. tiina.komulainen@hut.fi

14. J. Sylvan Katz, The self-similar science system. Res. Policy **28**, 501–517 (1999)
15. C. Judd, Fractals C Self-similarity. http://www.bath.ac.uk/~ma0cmj/FractalContents.html. Accessed 16 Mar 2003
16. S. Yadegari, Self-similarity. http://www-crca.ucsd.edu/~syadegar/MasterThesis/node25.html. Accessed 16 Mar 2003
17. B.J. Carr, A.A. Coley, Self-similarity in general relativity. http://users.math.uni-potsdam.de/~oeitner/QUELLEN/ZUMCHAOS/selfsim1.htm. Accessed 16 Mar 2003
18. G.I. Barenblatt, *Scaling Phenomena in Fluid Mechanics*, 1st edn. (Cambridge University Press, Cambridge, 1994)
19. Y.B. Zel'dovich, Y.P. Raizer, *Physics of Shock Waves and High-Temperature Hydrodynamics Phenomena* (Dover, New York, 2002)
20. H.H. Olsen, Bukingham's Pi Theorem. www.math.ntnu.no/~hanche/notes/buckingham/buckingham-a4.pdf
21. V. Skglund, *Similitude: Theory and Applications* (International Textbook Company, Scranton, 1967)
22. G.I. Barenblatt, *'Scaling' Cambridge Texts in Applied Mathematics* (2006)
23. P.L. Sachdev, S. Ashraf, Strong shock with radiation near the surface of a star. Phys. Fluids **14**, 2107 (1971a)
24. G. Guderley, Starke kugelige und zylindrische Verdichtungsstosse in der Nahe des Kugelmittelpunktes bzw. der Zylinderachse. Luftfahrt-Forsch **19**, 302–312 (1942)
25. G.I. Taylor, The formation of a blast wave by a very intense explosion. I. Theoretical discussion. Proc. R. Soc. A **201**, 159–174 (1950a)
26. G.I. Taylor, The formation of a blast wave by a very intense explosion. II. The atomic explosion of 1945. Proc. R. Soc. A **201**, 175–186 (1950b)
27. J. von Neumann, Blast Waves Los Alamos Science Laboratory Technical Series, Los Alamos, NM, vol 7 (1947)
28. L. Sedov, *Similarity and Dimensional Methods in Mechanics*, Chap. IV. (Academic, New York, 1969)
29. E. Waxman, D. Shvarts, Second-type self-similar solutions to the strong explosion problem. Phys. Fluids A **5**, 1035 (1993)
30. P. Best, R. Sari, Second-type self-similar solutions to the ultra-relativistic strong explosion problem. Phys. Fluids. **12**, 3029 (2000)
31. R. Sari, First and second type self-similar solutions of implosions and explosions containing ultra relativistic shocks. Phys. Fluids **18** (2006)
32. C.L. Dym, *Principle of Mathematical Modeling*, 2nd edn. (Elsevier, 2004)
33. F.R. Giordano, W.P. Fox, S.B. Horton, M.D. Weir, *A First Course in Mathematical Modeling*, 4th edn. (Brooks Cole Publication, 2008)
34. W.D. Hayes, The propagation upward of the shock wave from a strong explosion in the atmosphere. J. Fluid. Mech. **32**, 317 (1968)
35. P.L. Sachdev, *Shock Waves and Explosions* (Chapman & Hall/CRC, 2004)
36. S. Matsumora, O. Onodera, I.L. Takayama, Noise Induced by Weak Shock Waves in Automobile Exhaust Systems (Effects of Viscosity and Back Pressure), in *Proc- of the 19th Int. Symp. on Shock Waves and Shock Tubes*, Marseille, France, vol 3 (1993), pp. 367–372
37. K.C. Phan, On the Performance of Blast Deflectors and Impulse Attenuators, in *Proc. of the 18th Int. Symp. on Shock Waves and Shock Tubes*, Sendai, Japan (1991), pp. 927–934
38. S.D. Ramsey, J.R. Kamm, J.H. Bolstad, The Guderley problem revised. Int. J. Comput. Fluid Dyn. **26**(2), 79–99 (2012)
39. W. Chester, The propagation of shock waves in a channel of non-uniform width. Q. J. Mech. Appl. Math. **6**, 440–452 (1953)
40. R.F. Chisnell, The motion of a shock wave in a channel, with applications to cylindrical and spherical shock waves. J. Fluid. Mech. **4**, 286–298 (1958)
41. G.B. Whitham, On the propagation of shock waves through regions of nonuniform area of flow. J. Fluid. Mech. **4**(Part 1), 37–360 (1958)

42. Fong, K., Ahlborn, B., “Stability of converging shock waves”, Phys. Fluids, 22(3), 4 16–42 1 (1979)
43. J.H. Gardner, D.L. Book, I.B. Bernstein, Stability of imploding shocks in the CCW approximation. J. Fluid. Mech. **1**(14), 41–58 (1982)
44. M. El-Mallah, Experimental and numerical study of the bleed effect on the propagation of strong plane and converging cylindrical shock waves. The Department of Mechanical Engineering, Presented in partial fulfillment of the requirements for the degree of Doctor of Philosophy at Concordia University Montreal, Quebec, Canada, May 1997

http://www.springer.com/978-3-319-50906-8

Inertial Confinement Fusion Driven Thermonuclear Energy

Zohuri, B.

2017, XVI, 313 p. 107 illus., 53 illus. in color., Hardcover
ISBN: 978-3-319-50906-8