Chapter 2
Singular Cohomology

In this chapter we give a short introduction to singular cohomology. Many properties are only sketched, as this theory is considerably better known than de Rham cohomology, for example.

2.1 Relative Cohomology

Let X be a topological space. Sometimes, if indicated, X will be the underlying topological space of an analytic or algebraic variety, also denoted by X. To avoid technicalities, X will always be assumed to be a locally compact, Hausdorff space, and satisfying the second countability axiom. In particular, it is paracompact.

From now on, let $\mathcal{F}$ be a sheaf of abelian groups on X and consider sheaf cohomology $H^i(X, \mathcal{F})$ from Sect. 1.4. Mostly, we will consider the case of the constant sheaf $\mathcal{F} = \mathbb{Z}$. All statements also hold with $\mathbb{Z}$ replaced by $\mathbb{Q}$ or $\mathbb{C}$.

Definition 2.1.1. (*Relative Cohomology*) For $A \subset X$ a closed subset, $U = X \setminus A$ the open complement, and $i : A \hookrightarrow X$ and $j : U \hookrightarrow X$ be the inclusion maps. We define *relative cohomology* as

$$H^i(X, A; \mathbb{Z}) := H^i(X, j_!\mathbb{Z}),$$

where $j_!$ is the extension by zero, i.e., the sheafification of the presheaf

$$V \mapsto \begin{cases} \mathbb{Z} & V \subset U, \\ 0 & \text{else.} \end{cases}$$

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Convention 2.1.2. If X is an algebraic variety defined over a field k contained in $\mathbb{C}$ and $A \subset X$ a closed subvariety defined over k, we abbreviate

$$H^i(X, A; \mathbb{Z}) = H^i(X^{\mathrm{an}}, A^{\mathrm{an}}; \mathbb{Z})$$

where X^{an} and A^{an} are the analytifications of $X \times_k \mathbb{C}$ and $A \times_k \mathbb{C}$, respectively.

Remark 2.1.3. (Functoriality and homotopy invariance) The association

$$(X, A) \mapsto H^i(X, A; \mathbb{Z})$$

is a contravariant functor from pairs of topological spaces to abelian groups. In particular, for every continuous map $f : (X, A) \to (X', A')$ of pairs, i.e., satisfying $f(A) \subset A'$, one has a homomorphism $f^* : H^i(X', A'; \mathbb{Z}) \to H^i(X, A; \mathbb{Z})$. Given two homotopic maps f and g, the homomorphisms f^*, g^* are equal. As a consequence, if two pairs (X, A) and (X', A') are homotopy equivalent, then the cohomology groups $H^i(X', A'; \mathbb{Z})$ and $H^i(X, A; \mathbb{Z})$ are isomorphic.

Proposition 2.1.4. *There is a long exact sequence*

$$\cdots \to H^i(X, A; \mathbb{Z}) \to H^i(X, \mathbb{Z}) \to H^i(A, \mathbb{Z}) \xrightarrow{\delta} H^{i+1}(X, A; \mathbb{Z}) \to \cdots$$

Proof. This follows from the exact sequence of sheaves

$$0 \to j_!\mathbb{Z} \to \mathbb{Z} \to i_*\mathbb{Z} \to 0.$$

□

Note that by our definition of cones, see Sect. 1.3, one has a quasi-isomorphism $j_!\mathbb{Z} = \mathrm{Cone}(\mathbb{Z} \to i_*\mathbb{Z})[-1]$. For Nori motives we need a version for triples, which can be proved using iterated cones by the same method:

Corollary 2.1.5. *Let $X \supset A \supset B$ be a triple of relative closed subsets. Then there is a long exact sequence*

$$\cdots \to H^i(X, A; \mathbb{Z}) \to H^i(X, B; \mathbb{Z}) \to H^i(A, B; \mathbb{Z}) \xrightarrow{\delta} H^{i+1}(X, A; \mathbb{Z}) \to \cdots$$

Here, δ is the connecting homomorphism, which in the cone picture is explained in Sect. 1.3.

Proposition 2.1.6. (Mayer–Vietoris) *Let $\{U, V\}$ be an open cover of X. Let $A \subset X$ be closed. Then there is a natural long exact sequence*

$$\begin{aligned}\cdots \to H^i(X, A; \mathbb{Z}) &\to H^i(U, U \cap A; \mathbb{Z}) \oplus H^i(V, V \cap A; \mathbb{Z}) \\ &\to H^i(U \cap V, U \cap V \cap A; \mathbb{Z}) \to H^{i+1}(X, A; \mathbb{Z}) \to \cdots\end{aligned}$$

Proof. Pairs (U, V) of open subsets form an excisive couple in the sense of [Spa66, p. 188], and therefore the Mayer–Vietoris property holds by [Spa66, pp. 189–190]. □

Theorem 2.1.7. (Proper base change) *Let $\pi : X \to Y$ be proper, i.e., the preimage of a compact subset is compact. Let $\mathcal{F}$ be a sheaf on X. Then the stalk in $y \in Y$ is computed as*

$$(R^i\pi_*\mathcal{F})_y = H^i(X_y, \mathcal{F}|_{X_y}).$$

Proof. See [KS90, Proposition 2.6.7]. As π is proper, we have $R\pi_* = R\pi_!$. □

Now we list some properties of the sheaf cohomology of algebraic varieties over a field $k \hookrightarrow \mathbb{C}$. As usual, we will not distinguish in notation between a variety X and the topological space of the analytification X^{an}. The first property is:

Proposition 2.1.8. (Excision, or abstract blow-up) *Let $f : (X', D') \to (X, D)$ be a proper, surjective morphism of algebraic varieties over $\mathbb{C}$, which induces an isomorphism $f : X' \setminus D' \to X \setminus D$. Then*

$$f^* : H^*(X, D; \mathbb{Z}) \cong H^*(X', D'; \mathbb{Z}).$$

Proof. This fact goes back to A. Aeppli [Aep57]. It is a special case of proper base change: let $j : U \to X$ be the complement of D and $j' : U \to X'$ its inclusion into X'. For all $x \in X$, we have

$$(R^i f_* j'_!\mathbb{Z})_x = H^i(X_x, j'_!\mathbb{Z}|_{X'_x}).$$

For $x \in U$, the fibre is one point. It has no higher cohomology. For $x \in D$, the restriction of $j'_!\mathbb{Z}$ to X'_x is zero. Together this means

$$Rf_*j'_!\mathbb{Z} = j_!\mathbb{Z}.$$

The statement then follows from the Leray spectral sequence [Spa66]. □

We will later prove a slightly more general proper base change theorem for singular cohomology, see Theorem 2.5.12.

The second property is:

Proposition 2.1.9. (Gysin isomorphism, localisation, weak purity) *Let X be an irreducible variety of dimension n over k, and Z a closed subvariety of pure codimension r. Then there is an exact sequence*

$$\cdots \to H^i_Z(X, \mathbb{Z}) \to H^i(X, \mathbb{Z}) \to H^i(X \setminus Z, \mathbb{Z}) \to H^{i+1}_Z(X, \mathbb{Z}) \to \cdots$$

where $H^i_Z(X, \mathbb{Z})$ is cohomology with supports in Z, defined as the cohomology of $\mathrm{Cone}(\mathbb{Z} \to Rj_*\mathbb{Z})[-1]$ *for the open immersion $j : X \setminus Z \to X$.*

If, moreover, X and Z are both smooth, then one has the Gysin isomorphism

$$H^i_Z(X, \mathbb{Z}) \cong H^{i-2r}(Z, \mathbb{Z}).$$

In particular, one has weak purity:

$$H^i_Z(X, \mathbb{Z}) = 0 \text{ for } i < 2r,$$

and $H^{2r}_Z(X, \mathbb{Z}) = H^0(Z, \mathbb{Z})$ *is free of rank equal to the number of components of* Z.

Proof. A modern presentation of such properties of cohomology theories is given in [Pan03, Sect. 2]. It contains other examples of cohomology theories and an axiomatic treatment with more general properties. □

2.2 Singular (Co)homology

Let X be a topological space satisfying the same general assumptions as in Sect. 2.1. The definition of singular homology and cohomology uses topological simplexes.

Definition 2.2.1. The *topological n-simplex* Δ_n is defined as

$$\Delta_n := \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1,\ t_i \geq 0 \right\}.$$

Δ_n has natural codimension one faces defined by $t_i = 0$.

Singular (co)homology is defined by looking at all possible continuous maps from simplices to X.

Definition 2.2.2. A *singular n-simplex* is a continuous map

$$f : \Delta_n \to X.$$

In the case where X is a differentiable manifold, a singular simplex f is called *differentiable* if the map f can be extended to a $\mathcal{C}^\infty$-map from a neighbourhood of $\Delta_n \subset \mathbb{R}^{n+1}$ to X. The *group of singular n-chains* is the free abelian group

$$S_n(X) := \mathbb{Z}[f : \Delta_n \to X \mid f \text{ singular } n\text{-simplex}].$$

In a similar way, we denote by $S^\infty_n(X)$ the free abelian *group of differentiable singular n-chains* by requiring that all f are differentiable. The boundary map $\partial_n : S_n(X) \to S_{n-1}(X)$ is defined as

$$\partial_n(f) := \sum_{i=0}^{n} (-1)^i f|_{t_i=0}.$$

It respects the subgroups $S_n^\infty(X)$. The *group of singular n-cochains* is the free abelian group

$$S^n(X) := \mathrm{Hom}_{\mathbb{Z}}(S_n(X), \mathbb{Z}).$$

The *group of differentiable singular n-cochains* is the free abelian group

$$S_\infty^n(X) := \mathrm{Hom}_{\mathbb{Z}}(S_n^\infty(X), \mathbb{Z}).$$

The adjoint of ∂_{n+1} defines the boundary map

$$d_n : S_{(\infty)}^n(X) \to S_{(\infty)}^{n+1}(X).$$

Lemma 2.2.3. *One has $\partial_{n-1}\partial_n = 0$ and $d_{n+1}d_n = 0$, i.e., the groups $S_\bullet(X)$ and $S^\bullet(X)$ define complexes of abelian groups.*

The proof is left to the reader as an exercise.

Definition 2.2.4. Singular homology and cohomology with values in $\mathbb{Z}$ are defined as

$$H^i_{\mathrm{sing}}(X, \mathbb{Z}) := H^i(S^\bullet(X), d_\bullet), \quad H_i^{\mathrm{sing}}(X, \mathbb{Z}) := H_i(S_\bullet(X), \partial_\bullet).$$

In a similar way, we define (for X a manifold) the differentiable singular (co)homology as

$$H^i_{\mathrm{sing},\infty}(X, \mathbb{Z}) := H^i(S_\infty^\bullet(X), d_\bullet), \quad H_i^{\mathrm{sing},\infty}(X, \mathbb{Z}) := H_i(S_\bullet^\infty(X), \partial_\bullet).$$

Theorem 2.2.5. *Assume that X is a locally contractible topological space, i.e., every neighbourhood of every point contains an open contractible neighbourhood. In this case, singular cohomology $H^i_{\mathrm{sing}}(X, \mathbb{Z})$ agrees with sheaf cohomology $H^i(X, \mathbb{Z})$ with coefficients in $\mathbb{Z}$. If Y is a differentiable manifold, differentiable singular (co)homology agrees with singular (co)homology.*

Proof. Let $\mathcal{S}^n$ be the sheaf associated to the presheaf $U \mapsto S^n(U)$. One shows that $\mathbb{Z} \to \mathcal{S}^\bullet$ is a fine resolution of the constant sheaf $\mathbb{Z}$ [War83, p. 196]. The proof uses that X is locally contractible [War83, p. 194]. If X is a manifold, differentiable cochains also define a fine resolution [War83, p. 196]. Therefore, the inclusion of complexes $S_\bullet^\infty(X) \hookrightarrow S_\bullet(X)$ induces isomorphisms

$$H^i_{\mathrm{sing},\infty}(X, \mathbb{Z}) \cong H^i_{\mathrm{sing}}(X, \mathbb{Z}) \text{ and } H_i^{\mathrm{sing},\infty}(X, \mathbb{Z}) \cong H_i^{\mathrm{sing}}(X, \mathbb{Z})\,.$$

□

Of course, topological manifolds satisfy the assumption of the theorem.

2.3 Simplicial Cohomology

In this section, we want to introduce simplicial (co)homology and its relation to singular (co)homology. Simplicial (co)homology can be defined for topological spaces with an underlying combinatorial structure.

The literature contains various notions of such spaces. In increasing order of generality, these are: geometric and abstract simplicial complexes, Δ-complexes (sometimes also called semi-simplicial complexes), and topological realisations of simplicial sets. A good reference with a discussion of various definitions is the book by Hatcher [Hat02], or the introductory paper [Fri12] by Friedman. We will only look at finitely generated spaces.

By construction, such spaces are built from topological simplices Δ_n in various dimensions n.

Definition 2.3.1. A *geometric n-simplex* is the convex hull of $n+1$ points $v_0, \ldots, v_n$ in some Euclidean space $\mathbb{R}^N$, such that $v_i - v_0$ are linearly independent for $i = 1, \ldots, n$. The *standard (ordered) n-simplex Δ_n* is the convex hull of the standard basis $e_0, \ldots, e_n$ of $\mathbb{R}^{n+1}$.
A *finite geometric simplicial complex $X \subset \mathbb{R}^N$* is the collection of finitely many geometric simplices in $\mathbb{R}^N$, such that

- Every face of a simplex of X is again a simplex of X (i.e., contained in X).
- The intersection of two simplices of X is a face of each of them and contained in X.

Using this definition, a finite geometric simplicial complex X induces a topological space also denoted by X, which is a topological quotient of the finite set of geometric simplices of X which are glued along common faces, see [Fri12, Sect. 2] or [Hat02, Sect. 2.1]. It can be built up inductively by adjoining simplices of increasing dimensions. The topological space X, i.e., the union of all faces, is not distinguished in notation from the collection X. The restriction to finitely many simplices is not necessary in this definition, but it is enough for our purposes. Geometric simplicial complexes arise more generally in geometric situations in the triangulations of real manifolds or algebraic varieties defined over $\mathbb{C}$.

Example 2.3.2. An example is the case of an analytic space X^{an} where X is an algebraic variety defined over $\mathbb{R}$. There one can always find a semi-algebraic triangulation by a result of Lojasiewicz, cf. Hironaka [Hir75, p. 170] and Proposition 2.6.9. See Sect. 2.6.2 for the notion of a semi-algebraic triangulation.

A little bit more general is the notion of an abstract simplicial complex:

Definition 2.3.3. A *finite abstract simplicial complex X* consists of a finite set of vertices X^0 together with—for each integer n—a set X^n of subsets of $n+1$ points in X^0. Subsets of $k+1$ elements in a set of $n+1$ elements in X^0, i.e., k-dimensional faces of n-dimensional faces of X, are contained in X^k. A simplicial complex X is called *ordered* if there is a chosen ordering on X^0.

Every finite geometric simplicial complex is an abstract finite simplicial complex and can be ordered. Vice versa, one can associate to an abstract simplicial complex a geometric one up to homeomorphism, by associating to each point in X^n an n-simplex and gluing these sets along common faces. Thus, we will only speak of simplicial complexes. The natural morphisms $f : X \to Y$ in the category of (abstract, finite) simplicial complexes are the simplicial maps which take the vertices in X^0 to vertices in Y^0 and every k-face of X to a k-face of Y under this map [Fri12, Sect. 2.2]. A similar definition of morphisms applies to ordered simplicial complexes.

Example 2.3.4. A *tetrahedron* $X = \partial\Delta_3$ is a geometric simplicial complex with four vertices (0-simplices), six non-degenerate edges (1-simplices), and four non-degenerate faces (2-simplices).

The *torus* T^2 has a well-known minimal triangulation with 14 vertices, 21 edges and 7 faces (triangles). The graph formed by the edges and vertices is called the *Heawood graph*. It divides the torus into 7 mutually adjacent regions.

Remark 2.3.5. There is also the slightly more abstract notion of a Δ-complex, which is intermediate between simplicial complexes and simplicial sets, see [Fri12, Sect. 2.4], [Hat02, Sect. 2.1]. Every Δ-complex is homeomorphic to a simplicial complex [Hat02, Sect. 2.1].

Even more generally, one can think of a simplicial space as the topological realisation of a finite simplicial set: Let $X_\bullet$ be a finite simplicial set in the sense of Remark 1.5.5. Then one has the face maps

$$\partial_i : X_n \to X_{n-1},$$

and the degeneracy maps

$$s_i : X_n \to X_{n+1}.$$

Every finite simplicial set gives rise to a topological space $|X_\bullet|$.

Definition 2.3.6. The *topological realisation* $|X_\bullet|$ of $X_\bullet$ is defined as

$$|X_\bullet| := \coprod_{n=0}^{\infty} X_n \times \Delta_n / \sim,$$

where each X_n carries the discrete topology, Δ_n is the topological n-simplex, and the equivalence relation is given by the two relations

$$(x, \partial_i(y)) \sim (\partial_i(x), y), \quad x \in X_{n-1},\ y \in \Delta_n,$$
$$(x', s_i(y')) \sim (s_i(x'), y'), \quad x' \in X_n,\ y \in \Delta_{n-1}.$$

(Note that we denote the face and degeneracy maps for the n-simplex by the same letters ∂_i, s_i.)

There is no essential difference between working with finite simplicial complexes or realisations of finite simplicial sets:

Proposition 2.3.7. *Let X be a finite simplicial complex. Then there is a finite simplicial set $X_\bullet$ associated to it by adding degeneracies. The spaces $|X_\bullet|$ and X are homeomorphic.*

Proof. See [Fri12, Example 3.3], [Hat02, Appendix A]. □

Remark 2.3.8. For a finite simplicial set $X_\bullet$, it is known that the realisation $|X_\bullet|$ is a compactly generated CW-complex [Hat02, Appendix A]. In fact, every finite CW-complex is homotopy equivalent to a finite simplicial complex of the same dimension by [Hat02, Theorem 2C.5].

The skeletal filtration from Remark 1.5.5 defines a filtration of $|X_\bullet|$

$$|\mathrm{sq}_0 X_\bullet| \subseteq |\mathrm{sq}_1 X_\bullet| \subseteq \cdots \subseteq |\mathrm{sq}_N X_\bullet| = |X_\bullet|$$

by closed subspaces, if X_n is degenerate for $n > N$.

There is finite number of simplices in each degree n. Associated to each of them is a continuous map $\sigma : \Delta_n \to |X_\bullet|$. We denote the free abelian group of all such σ of degree n by $C_n^\Delta(X_\bullet)$ and the maps

$$\partial_n : C_n^\Delta(X_\bullet) \to C_{n-1}^\Delta(X_\bullet)$$

are given by alternating restriction maps to faces, as in the case of singular homology. Note that the vertices of each simplex are ordered, so that this is well-defined.

Definition 2.3.9. Simplicial homology of the topological space $X = |X_\bullet|$ is defined as

$$H_n^{\mathrm{simpl}}(X, \mathbb{Z}) := H_n(C_*^\Delta(X_\bullet), \partial_*),$$

and simplicial cohomology as

$$H^n_{\mathrm{simpl}}(X, \mathbb{Z}) := H^n(C^*_\Delta(X_\bullet), d_*),$$

where $C^n_\Delta(X_\bullet) = \mathrm{Hom}(C_n^\Delta(X_\bullet), \mathbb{Z})$ and d_n is adjoint to ∂_n.

This definition does not depend on the representation of a topological space X as the topological realisation of a simplicial set, since one can prove that simplicial (co)homology coincides with singular (co)homology:

Theorem 2.3.10. *Singular and simplicial (co)homology of X are equal.*

Proof. (For homology only.) The chain of closed subsets

$$|\mathrm{sq}_0 X_\bullet| \subseteq |\mathrm{sq}_1 X_\bullet| \subseteq \cdots \subseteq |\mathrm{sq}_N X_\bullet| = |X_\bullet|$$

gives rise to long exact sequences of simplicial homology groups

$$\cdots \to H_n^{\mathrm{simpl}}(|\mathrm{sq}_{n-1}X_\bullet|, \mathbb{Z}) \to H_n^{\mathrm{simpl}}(|\mathrm{sq}_n X_\bullet|, \mathbb{Z}) \\ \to H_n^{\mathrm{simpl}}(|\mathrm{sq}_{n-1}X_\bullet|, |\mathrm{sq}_n X_\bullet|; \mathbb{Z}) \to \cdots$$

A similar sequence holds for singular homology, and there is a canonical map $C_n^{\Delta}(X) \to C_n(X)$ from simplicial to singular chains. The result is then proved by induction on n. We use that the relative complex $C_n^{\Delta}(|\mathrm{sq}_{n-1}X_\bullet|, |\mathrm{sq}_n X_\bullet|)$ has zero differential and is a free abelian group of rank equal to the cardinality of X_n. Therefore, the assertion follows by computing that the singular (co)homology of Δ_n is given by $H^i(\Delta_n, \mathbb{Z}) = \mathbb{Z}$ for $i = 0$ and zero otherwise. □

In a similar way, one can define the simplicial (co)homology of a pair $(X, D) = (|X_\bullet|, |D_\bullet|)$, where $D_\bullet \subset X_\bullet$ is a simplicial subobject. The associated chain complex is given by the quotient complex $C_*^{\Delta}(X_\bullet)/C_*^{\Delta}(D_\bullet)$. The same proof will then show that the singular and simplicial (co)homology of pairs coincide.

Example 2.3.11. For the tetrahedron $X = \partial\Delta_3$, a computation shows that $H_i(X, \mathbb{Z}) = \mathbb{Z}$ for $i = 0, 2$ and zero otherwise. This was a priori clear, since X is topologically a sphere.

For the torus T^2, one computes $H_1(T^2, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$, and $H_0(T^2, \mathbb{Z}) = H_2(T^2, \mathbb{Z}) = \mathbb{Z}$. Both are obvious, as T^2 is topologically a product $S^1 \times S^1$.

In the special case when X is the topological space underlying the analytic space attached to an affine algebraic variety X over $\mathbb{C}$, or more generally a Stein manifold, one can show:

Theorem 2.3.12. (Artin vanishing) *Let X be an affine variety over $\mathbb{C}$ of dimension n. Then $H^q(X^{\mathrm{an}}, \mathbb{Z}) = 0$ for $q > n$. In fact, X^{an} is homotopy equivalent to the topological realisation of a finite simplicial set where all non-degenerate simplices are of dimension at most n.*

Proof. The proof was first given by Andreotti and Fraenkel [AF59] for Stein manifolds. For Stein spaces, i.e., allowing singularities, this is a theorem of Kaup, Narasimhan and Hamm, see [Ham83, Satz 1] and the correction in [Ham86]. An algebraic proof was given by M. Artin [Art73, Corollaire 3.5, tome 3]. □

The choice of such a triangulation implies the choice of a skeletal filtration.

Corollary 2.3.13. (Good topological filtration) *Let X be an affine variety over $\mathbb{C}$ of dimension n. Then there is a filtration of X^{an} given by*

$$X^{an} = X_n \supset X_{n-1} \supset \cdots \supset X_0$$

where the pairs (X_i, X_{i-1}) have only cohomology in degree i. There is an induced chain complex of abelian groups

$$\cdots \to H^i(X_i, X_{i-1}; \mathbb{Z}) \xrightarrow{\delta_i} H^{i+1}(X_{i+1}, X_i; \mathbb{Z}) \to \cdots$$

which computes the cohomology of X. The maps δ_i are coboundary maps in the long exact sequences associated to the triples $X_{i-1} \subset X_i \subset X_{i+1}$.

Proof. The existence of the filtration follows from Theorem 2.3.12. The rest of the statements are shown (in the dual homological version) in [Hat02, Theorem 2.35]. □

Remark 2.3.14. The Basic Lemma of Nori and Beilinson, see Theorem 2.5.7, shows that there is even an algebraic variant of this topological skeletal filtration.

Corollary 2.3.15. (Artin vanishing for relative cohomology) *Let X be an affine variety of dimension n and $Z \subset X$ a closed subvariety. Then*

$$H^i(X, Z; \mathbb{Z}) = 0 \text{ for } i > n.$$

Proof. Consider the long exact sequence for relative cohomology and use Artin vanishing for X and Z from Theorem 2.3.12. □

The following theorem is strongly related to the Artin vanishing theorem.

Theorem 2.3.16. (Lefschetz hyperplane theorem) *Let $\bar{X} \subset \mathbb{P}^N_{\mathbb{C}}$ be a smooth integral projective variety of dimension n, and $H \subset \bar{X}$ a transversal hyperplane section. Then the inclusion $H \subset \bar{X}$ is $(n-1)$-connected. In particular, one has $H^q(\bar{X}, H; \mathbb{Z}) = 0$ for $q \leq n-1$.*

Proof. By [AF59, Theorem 2], the map $H^q(\bar{X}, \mathbb{Z}) \to H^q(H, \mathbb{Z})$ is bijective for $q < n-1$ and injective for $q = n-1$. □

This also implies an analogous statement in the affine case.

Corollary 2.3.17. *Let $\bar{X}$ be a smooth projective integral variety of dimension n, and $H, H' \subset \bar{X}$ transversal hyperplane sections which are also transversal to each other. Let $X = \bar{X} \setminus H$ and $Z = X \cap H'$. Then $H^q(X, Z; \mathbb{Z})$ vanishes for $q \leq n-1$.*

Proof. By comparing the Gysin sequences of Proposition 2.1.9 for the smooth pairs $(\bar{X}, H)$ and $(H, H' \cap H)$, we also obtain a Gysin sequence in relative cohomology:

$$\cdots \to H^{q-2}(H, H' \cap H; \mathbb{Z}) \to H^q(\bar{X}, H; \mathbb{Z}) \to H^q(X, Z; \mathbb{Z}) \\ \to H^{q-1}(H, H' \cap H; \mathbb{Z}) \to \dots$$

The Lefschetz hyperplane Theorem 2.3.16 says that the q-th cohomology groups of $(\bar{X}, H)$ and $(H, H \cap H')$ vanish for $q \leq n-1$ and $q \leq n-2$, respectively. Hence the cohomology of (X, Z) vanishes for $q \leq n-1$. □

2.4 The Künneth Formula and Poincaré Duality

Assume that we are given two topological spaces X and Y, and two closed subsets $i : A \hookrightarrow X$, and $i' : C \hookrightarrow Y$. By the above, using the inclusions $j : X \setminus A \hookrightarrow X$, and $j' : Y \setminus C \hookrightarrow Y$, we have

$$H^*(X, A; \mathbb{Z}) = H^*(X, j_!\mathbb{Z}),$$

and

$$H^*(Y, C; \mathbb{Z}) = H^*(Y, j'_!\mathbb{Z}).$$

The relative cohomology group

$$H^*(X \times Y, X \times C \cup A \times Y; \mathbb{Z})$$

can by definition be computed as $H^*(X \times Y, \tilde{j}_!\mathbb{Z})$, where

$$\tilde{j} : (X \times Y) \setminus (X \times C \cup A \times Y) \hookrightarrow X \times Y$$

is the inclusion map. One has $\tilde{j}_! = j_! \boxtimes j'_!$ where $\boxtimes$ denotes the external tensor product of sheaves. Hence, we have a natural exterior product map

$$H^i(X, A; \mathbb{Z}) \otimes H^j(Y, C; \mathbb{Z}) \xrightarrow{\times} H^{i+j}(X \times Y, X \times C \cup A \times Y; \mathbb{Z}).$$

This is related to the so-called *Künneth formula*:

Theorem 2.4.1. (Künneth formula for pairs) *Let $A \subset X$ and $C \subset Y$ be closed subsets. The exterior product map induces a natural isomorphism*

$$\bigoplus_{i+j=n} H^i(X, A; \mathbb{Q}) \otimes H^j(Y, C; \mathbb{Q}) \xrightarrow{\cong} H^n(X \times Y, X \times C \cup A \times Y; \mathbb{Q}).$$

The same result holds with $\mathbb{Z}$-coefficients, provided all cohomology groups of (X, A) and (Y, C) in all degrees are free.

Proof. Using the sheaves of singular cochains, see the proof of Theorem 2.2.5, one has fine resolutions $j_!\mathbb{Z} \to F^\bullet$ on X, and $j'_!\mathbb{Z} \to G^\bullet$ on Y. The exterior tensor product $F^\bullet \boxtimes G^\bullet$ is thus a fine resolution of $\tilde{j}_!\mathbb{Z} = j_!\mathbb{Z} \boxtimes j'_!\mathbb{Z}$. Here one uses that the tensor product of fine sheaves is fine [War83, p. 193]. The cohomology of the tensor product complex $F^\bullet \otimes G^\bullet$ induces a short exact sequence

$$0 \to \bigoplus_{i+j=n} H^i(X, A; \mathbb{Z}) \otimes H^j(Y, C; \mathbb{Z}) \to H^n(X \times Y, X \times C \cup A \times Y; \mathbb{Z})$$
$$\to \bigoplus_{i+j=n+1} \mathrm{Tor}_1^{\mathbb{Z}}(H^i(X, A; \mathbb{Z}), H^j(Y, C; \mathbb{Z})) \to 0$$

by [God58, Théorème 5.5.1] or [Wei94, Theorem 3.6.3]. If all cohomology groups are free, the last term vanishes. □

The following is a standard consequence of the definition of the Künneth isomorphism for complexes of abelian groups:

Proposition 2.4.2. *The Künneth isomorphism of Theorem 2.4.1 is associative and graded commutative.*

In later constructions, we will need a certain compatibility of the exterior product with coboundary maps.

Proposition 2.4.3. *Assume that $X \supset A \supset B$ and $Y \supset C$ are closed subsets. The diagram involving coboundary maps for the triples $X \supset A \supset B$ and $X \times Y \supset X \times C \cup A \times Y \supset X \times C \cup B \times Y$ combined with the excision isomorphism*

$$\begin{array}{ccc}
H^i(A, B; \mathbb{Z}) \otimes H^j(Y, C; \mathbb{Z}) & \longrightarrow & H^{i+j}(A \times Y, A \times C \cup B \times Y; \mathbb{Z}) \\
{\scriptstyle \delta\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \delta} \\
H^{i+1}(X, A; \mathbb{Z}) \otimes H^j(Y, C; \mathbb{Z}) & \longrightarrow & H^{i+j+1}(X \times Y, X \times C \cup A \times Y; \mathbb{Z})
\end{array}$$

commutes up to a sign $(-1)^j$. The diagram

$$\begin{array}{ccc}
H^i(Y, C; \mathbb{Z}) \otimes H^j(A, B; \mathbb{Z}) & \longrightarrow & H^{i+j}(Y \times A, Y \times B \cup C \times A; \mathbb{Z}) \\
{\scriptstyle \mathrm{id}\otimes\delta}\downarrow & & \downarrow{\scriptstyle \delta} \\
H^i(Y, C; \mathbb{Z}) \otimes H^{j+1}(X, A; \mathbb{Z}) & \longrightarrow & H^{i+j+1}(Y \times X, Y \times A \cup C \times X; \mathbb{Z})
\end{array}$$

commutes (without a sign).

Proof. We indicate the argument, without going into full details. Let $G^\bullet$ be a complex computing $H^*(Y, C; \mathbb{Z})$. Let $F_1^\bullet$ and $F_2^\bullet$ be complexes computing $H^*(A, B; \mathbb{Z})$ and $H^*(X, A; \mathbb{Z})$. Let $K_1^\bullet$ and $K_2^\bullet$ be the complexes computing cohomology of the corresponding product varieties. The cup product is induced from maps of complexes $F_i^\bullet \otimes G^\bullet \to K_i^\bullet$. In order to get compatibility with the boundary map, we have to consider the diagram of the form

$$\begin{array}{ccc}
F_1^\bullet \otimes G^\bullet & \longrightarrow & K_1^\bullet \\
\downarrow & & \downarrow \\
(F_2^\bullet[1]) \otimes G^\bullet & \longrightarrow & K_2^\bullet[1]
\end{array}$$

However, by Lemma 1.3.6, the complexes $(F_2^\bullet[1]) \otimes G^\bullet$ and $(F_2^\bullet \otimes G^\bullet)[1]$ are not equal. We need to introduce the sign $(-1)^j$ in bidegree (i, j) to make the identification and get a commutative diagram.

The argument for the second type of boundary map is the same, but does not need the introduction of signs by Lemma 1.3.6. □

Assume now that $X = Y$ and $A = C$. Then, $j_!\mathbb{Z}$ has an algebra structure, and we obtain the *cup product* maps:

$$H^i(X, A; \mathbb{Z}) \otimes H^j(X, A; \mathbb{Z}) \longrightarrow H^{i+j}(X, A; \mathbb{Z})$$

via the multiplication maps

$$H^{i+j}(X \times X, \tilde{j}_!\mathbb{Z}) \to H^{i+j}(X, j_!\mathbb{Z}),$$

induced by

$$\tilde{j}_! = j_! \boxtimes j_! \to j_!.$$

In the case where $A = \emptyset$, the cup product induces Poincaré duality:

Proposition 2.4.4. (Poincaré duality*) Let X be a compact, orientable topological manifold of dimension m. Then the cup product pairing*

$$H^i(X, \mathbb{Q}) \times H^{m-i}(X, \mathbb{Q}) \longrightarrow H^m(X, \mathbb{Q}) \cong \mathbb{Q}$$

is non-degenerate. With $\mathbb{Z}$-coefficients, the map

$$H^i(X, \mathbb{Z})/\text{torsion} \times H^{m-i}(X, \mathbb{Z})/\text{torsion} \longrightarrow H^m(X, \mathbb{Z}) \cong \mathbb{Z}$$

is non-degenerate.

Proof. We will give a proof of a slightly more general statement in the algebraic situation below. A proof of the stated theorem can be found in [GH78, p. 53], although it is stated in a homological version. There it is shown that $H^{2n}(X, \mathbb{Z})$ is torsion-free of rank one, and the cup-product pairing is unimodular modulo torsion, using simplicial cohomology, and the relation between Poincaré duality and the dual cell decomposition. □

We will now prove a relative version in the algebraic case. It implies the version above in the case where $A = B = \emptyset$. By abuse of notation, we again do not distinguish between an algebraic variety over $\mathbb{C}$ and its underlying topological space.

Theorem 2.4.5. (Poincaré duality for algebraic pairs) *Let X be a smooth and proper complex variety of dimension n and $A, B \subset X$ two normal crossing divisors, such that $A \cup B$ is also a normal crossing divisor. Then there is a non-degenerate duality pairing*

$$H^d(X \setminus A, B \setminus (A \cap B); \mathbb{Q}) \times H^{2n-d}(X \setminus B, A \setminus (A \cap B); \mathbb{Q}) \longrightarrow H^{2n}(X, \mathbb{Q}) \cong \mathbb{Q}.$$

Again, with $\mathbb{Z}$-coefficients this is true modulo torsion.

Proof. We give a sheaf theoretic proof using Verdier duality and some formulas and ideas of Beilinson (see [Beĭ87]). Look at the commutative diagram:

$$\begin{tikzcd} U = X \setminus (A \cup B) \arrow[r, "\ell_U"] \arrow[d, "\kappa_U"'] & X \setminus A \arrow[d, "\kappa"] \\ X \setminus B \arrow[r, "\ell"] & X. \end{tikzcd}$$

Note that the involved morphisms are affine. Assuming $A \cup B$ is a normal crossing divisor, we want to show first that the natural map

$$\ell_! R\kappa_{U*}\mathbb{Q}_U \longrightarrow R\kappa_* \ell_{U!}\mathbb{Q}_U,$$

extending $\mathrm{id} : \mathbb{Q}_U \to \mathbb{Q}_U$, is an isomorphism. This is a local computation. We look without loss of generality at a neighbourhood of an intersection point $x \in A \cap B$ (in the analytic topology), since the computation at other points is even easier. Hence, we may choose a polydisk neighbourhood D in X around x such that D decomposes as

$$D = D_A \times D_B$$

and such that

$$A \cap D = A_0 \times D_B, \quad B \cap D = D_A \times B_0$$

for some suitable topological spaces A_0, B_0. Using the same symbols for the maps as in the above diagram, the situation looks locally like

$$\begin{tikzcd} (D_A \setminus A_0) \times (D_B \setminus B_0) \arrow[r, "\ell_U"] \arrow[d, "\kappa_U"'] & (D_A \setminus A_0) \times D_B \arrow[d, "\kappa"] \\ D_A \times (D_B \setminus B_0) \arrow[r, "\ell"] & D = D_A \times D_B. \end{tikzcd}$$

Using the Künneth formula, one concludes that both sides $\ell_! R\kappa_{U*}\mathbb{Q}_U$ and $R\kappa_* \ell_{U!}\mathbb{Q}_U$ are isomorphic to

$$R\kappa_{U*}\mathbb{Q}_{D_A \setminus A_0} \boxtimes \ell_! \mathbb{Q}_{D_B \setminus B_0}$$

near the point x, and the natural map provides an isomorphism.

Now, one has

$$H^d(X \setminus A, B \setminus (A \cap B); \mathbb{Q}) = H^d(X, \ell_! \kappa_{U*}\mathbb{Q}_U),$$

(using that the maps involved are affine and hence their higher direct image functors exact), and

$$H^{2n-d}(X \setminus B, A \setminus (A \cap B); \mathbb{Q}) = H^{2n-d}(X, \kappa_! \ell_{U*} \mathbb{Q}_U).$$

We have to show that there is a perfect pairing

$$H^d(X \setminus A, B \setminus (A \cap B); \mathbb{Q}) \times H^{2n-d}(X \setminus B, A \setminus (A \cap B); \mathbb{Q}) \to \mathbb{Q}.$$

However, by Verdier duality, we have a perfect pairing

$$\begin{aligned}
H^{2n-d}(X \setminus B, A \setminus (A \cap B); \mathbb{Q})^\vee &= H^{2n-d}(X, \kappa_! \ell_{U*} \mathbb{Q}_U)^\vee \\
&\cong H^{-d}(X, \kappa_! \ell_{U*} \mathbb{D}\mathbb{Q}_U) \\
&\cong H^{-d}(X, \mathbb{D}(\kappa_* \ell_{U!} \mathbb{Q}_U)) \\
&\cong H^d(X, \kappa_* \ell_{U!} \mathbb{Q}_U) \\
&\cong H^d(X, \ell_! \kappa_{U*} \mathbb{Q}_U) \\
&= H^d(X \setminus A, B \setminus (A \cap B)); \mathbb{Q}).
\end{aligned}$$

In this computation, $\mathbb{D}$ is Verdier's duality operator on the derived category of constructible sheaves in the analytic topology.

The statement on integral cohomology follows again by unimodularity of the cup-product pairing. □

Remark 2.4.6. The normal crossing condition is necessary, as one can see in the example of $X = \mathbb{P}^2$, where A consists of two distinct lines meeting in a point, and B a line different from A going through the same point.

2.5 The Basic Lemma

In this section we prove the basic lemma of Nori [Nor00, Nora, Nor02], a topological result, which was also known to Beilinson [Beĭ87] and Vilonen (unpublished). Let $k \subset \mathbb{C}$ be a subfield. Beilinson's proof works more generally in positive characteristics, as we will see below.

2.5.1 Formulations of the Basic Lemma

Convention 2.5.1. We fix an embedding $k \hookrightarrow \mathbb{C}$. All sheaves and all cohomology groups in the following section are to be understood in the analytic topology on $X(\mathbb{C})$.

Theorem 2.5.2. (Basic Lemma I) *Let $k \subset \mathbb{C}$. Let X be an affine variety over k of dimension n and $W \subset X$ be a Zariski closed subset with $\dim(W) < n$. Then there exists a Zariski closed subset $Z \subset X$ defined over k with $\dim(Z) < n$ such that Z contains W and*

$$H^q(X, Z; \mathbb{Z}) = 0, \text{ for } q \neq n$$

and, moreover, the cohomology group $H^n(X, Z; \mathbb{Z})$ is a free $\mathbb{Z}$-module.

We formulate the Lemma for coefficients in $\mathbb{Z}$, but by the universal coefficient theorem [Wei94, Theorem 3.6.4] it will hold with other coefficients as well.

Example 2.5.3. There is an example where there is an easy way to obtain Z. Assume that X is of the form $\bar{X} \setminus H$ for some smooth projective $\bar{X}$ and a transversal hyperplane section $H \subset \bar{X}$ (with respect to a fixed embedding of $\bar{X}$ into a projective space) and $W = \emptyset$. Then choose another transversal hyperplane section $H' \subset \bar{X}$ also meeting H transversally and put $Z := H' \cap X$. It follows from Corollary 2.3.17 of the Lefschetz hyperplane theorem that $H^q(X, Z; \mathbb{Z}) = 0$ for $q \leq n - 1$. On the other hand, cohomology vanishes for $q > n$ by Artin vanishing, see Corollary 2.3.15, because X is affine. This argument will be generalised in two of the proofs below.

An inductive application of this Basic Lemma starting with the case $W = \emptyset$ yields a filtration of X by closed subsets

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset.$$

As in Corollary 2.3.13, this filtration induces a complex of free $\mathbb{Z}$-modules

$$\cdots \xrightarrow{\delta_{i-1}} H^i(X_i, X_{i-1}; \mathbb{Z}) \xrightarrow{\delta_i} H^{i+1}(X_{i+1}, X_i; \mathbb{Z}) \xrightarrow{\delta_{i+1}} \cdots ,$$

where the maps $\delta_\bullet$ arise from the coboundary in the long exact sequence associated to the triples $X_{i+1} \supset X_i \supset X_{i-1}$, computing the cohomology of X.

Remark 2.5.4. This means that we can understand this filtration as an algebraic analogue of the skeletal filtration of (the topological realisation) of a simplicial set, see Corollary 2.3.13. Note that the filtration is not only algebraic, but even defined over the base field k.

The Basic Lemma is deduced from the following variant, which was also known to Beilinson [Beĭ87]. To state it, we need the notion of a (weakly) constructible sheaf, which omits the finite generation condition for the stalks of constructible sheaves. This is often useful.

Definition 2.5.5. A sheaf of abelian groups on a variety X over k is *weakly constructible* if there is a decomposition of X into a disjoint union of finitely many Zariski locally closed subsets Y_i defined over k, and such that the restriction of F to Y_i is locally constant. It is called *constructible* if, in addition, the stalks of F are finitely generated abelian groups. We call such a decomposition a *stratification* if in addition all strata $S = Y_i$ are smooth and connected.

Remark 2.5.6. This combination of sheaves in the analytic topology together with strata algebraic and defined over k is usually not discussed in the literature. In fact, the formalism works in the same way as with algebraic strata over $\mathbb{C}$. What we need are enough Whitney stratifications algebraic over k. That this is possible can be deduced from [Tei82, Théorème 1.2, p. 455] (characterisation of Whitney stratifications) and [Tei82, Proposition 2.1] (Whitney stratifications are generic).

Theorem 2.5.7. (Basic Lemma II) *Let X be an affine variety over k of dimension n and F be a weakly constructible sheaf on X. Then there exists a Zariski open subset $j : U \hookrightarrow X$ such that the following three properties hold:*

1. $\dim(X \setminus U) < n$.
2. $H^q(X, F') = 0$ *for $q \neq n$, where $F' := j_! j^* F \subset F$.*
3. *If F is constructible then $H^n(X, F')$ is finitely generated.*
4. *If the stalks of F are torsion-free, then $H^n(X, F')$ is torsion-free.*

In order to relate the two versions of the Basic Lemma, we will also need some basic facts about sheaf cohomology. If $j : U \hookrightarrow X$ is a Zariski open subset with closed complement $i : W \hookrightarrow X$ and F a sheaf of abelian groups on X, then there is an exact sequence of sheaves

$$0 \to j_! j^* F \to F \to i_* i^* F \to 0.$$

In addition, for the constant sheaf $F = \mathbb{Z}$ on X, one has $H^q(X, j_! j^* F) = H^q(X, W; \mathbb{Z})$ and $H^q(X, i_* i^* F) = H^q(W, \mathbb{Z})$, see Sect. 2.1.

Version II of the Lemma implies version I. Let $V = X \setminus W$ with open immersion $h : V \hookrightarrow X$, and the sheaf $F = h_! h^* \mathbb{Z}$ on X. Version II for F gives an open subset $\ell : U \hookrightarrow X$ such that the sheaf $F' = \ell_! \ell^* F$ has non-vanishing cohomology only in degree n. Let $W' = X \setminus U$. Since F was zero on W, we have that F' is zero on $Z := W \cup W'$ and it is the constant sheaf on $X \setminus Z$, i.e., $F' = j_! j^* F$ for $j : X \setminus Z \hookrightarrow X$. In particular, F' computes the relative cohomology $H^q(X, Z; \mathbb{Z})$ and it vanishes for $q \neq n$. Freeness follows from property (3) and (4). $\square$

We will give two proofs of the Basic Lemma II in Sects. 2.5.3 and 2.5.4 below.

2.5.2 Direct Proof of Basic Lemma I

We start by giving a direct proof of Basic Lemma I. It was given by Nori in the unpublished notes [Nora]. Close inspection shows that it is actually a variant of Beilinson's argument in this very special case.

Lemma 2.5.8. *Let X be affine and $W \subset X$ closed. Then there exist*

1. *$\tilde{X}$ smooth projective;*

2. *$D_0, D_\infty \subset \tilde{X}$ closed such that $D_0 \cup D_\infty$ is a simple normal crossings divisor and $\tilde{X} \setminus D_0$ is affine;*
3. *$\pi : \tilde{X} \setminus D_\infty \to X$ proper surjective, an isomorphism outside of D_0 such that $Y := \pi(D_0 \setminus D_\infty \cap D_0)$ contains W and $\pi^{-1}(Y) = D_0 \setminus D_\infty \cap D_0$.*

Proof. By enlarging W, we may assume without loss of generality that $X \setminus W$ is smooth. Let $\bar{X}$ be a projective closure of X and $\bar{W}$ the closure of W in $\bar{X}$. By resolution of singularities, there is $\tilde{X} \to X$ proper surjective and an isomorphism above $X \setminus W$ such that $\tilde{X}$ is smooth. Let $D_\infty \subset \tilde{X}$ be the complement of the preimage of X. Let $\tilde{W}$ be the closure of the preimage of W. By resolution of singularities, we can also assume that $\tilde{W} \cup D_\infty$ is a divisor with normal crossings.

Note that $\bar{X}$ and hence also $\tilde{X}$ are projective. We choose a generic hyperplane $\tilde{H}$ such that $\tilde{W} \cup D_\infty \cup \tilde{H}$ is a divisor with normal crossings on $\tilde{X}$. This is possible as the ground field k is infinite and the condition is satisfied in a non-empty Zariski open subset of the space of hyperplane sections. We put $D_0 = \tilde{H} \cup \tilde{W}$. As $\tilde{H}$ is a hyperplane section, it is an ample divisor. Therefore, $D_0 = \tilde{H} \cup \tilde{W}$ is the support of the ample divisor $\tilde{H} + m\tilde{W}$ for m sufficiently large [Har77, Exercise II 7.5b]. Hence $\tilde{X} \setminus D_0$ is affine, as the complement of an ample divisor in a projective variety is affine. □

Proof of Basic Lemma I. We prove the Basic Lemma for cohomology with coefficients in a field K. We use the varieties constructed in the last lemma. We claim that Y has the right properties. We have $Y \supset W$. From Artin vanishing, see Corollary 2.3.15, we immediately have vanishing of $H^i(X, Y; K)$ for $i > n$.

By excision, see Proposition 2.1.8

$$H^i(X, Y; K) = H^i(\tilde{X} \setminus D_\infty, D_0 \setminus (D_0 \cap D_\infty); K).$$

By Poincaré duality for pairs, see Theorem 2.4.5, it is dual to

$$H^{2n-i}(\tilde{X} \setminus D_0, D_\infty \setminus (D_0 \cap D_\infty); K).$$

The variety $\tilde{X} \setminus D_0$ is affine. Hence, by Artin vanishing, the cohomology group $H^i(X, Y; K)$ vanishes for all $i \neq n$ and any coefficient field K.

It remains to treat the case of integral coefficients. Let i be the smallest index such that $H^i(X, Y; \mathbb{Z})$ is non-zero. By relative Artin vanishing for $\mathbb{Z}$-coefficients, see Corollary 2.3.15, we have $i \leq n$.

If $i < n$, then the group $H^i(X, Y; \mathbb{Z})$ has to be torsion because the cohomology vanishes with $\mathbb{Q}$-coefficients. The short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{F}_p \to 0$$

induces an exact sequence

$$0 \to H^{i-1}(X, Y; \mathbb{F}_p) \to H^i(X, Y; \mathbb{Z}) \xrightarrow{p} H^i(X, Y; \mathbb{Z})$$

which implies that $H^{i-1}(X, Y; \mathbb{F}_p)$ is non-trivial for the occurring torsion primes. This contradicts the vanishing for $K = \mathbb{F}_p$. Hence $i = n$. The same argument shows that $H^n(X, Y; \mathbb{Z})$ is torsion-free. □

2.5.3 Nori's Proof of Basic Lemma II

We now present the proof of the stronger Basic Lemma II published by Nori in [Nor02].

We start with a couple of lemmas on weakly constructible sheaves.

Lemma 2.5.9. *Let* $0 \to F_1 \to F_2 \to F_3 \to 0$ *be a short exact sequence of sheaves on* X *with* F_1, F_3 *(weakly) constructible. Then* F_2 *is (weakly) constructible.*

Proof. By assumption, there are stratifications of X such that F_1 and F_3 become locally constant, respectively. We take a common refinement. We replace X by one of the strata and are now in the situation that F_1 and F_3 are locally constant on a smooth connected variety. Then F_2 is also locally constant. Indeed, by passing to a suitable open cover (in the analytic topology), F_1 and F_3 even become constant. We restrict to a contractible open U, which exists because X^{an} is locally contractible. If $V \subset U$ is an inclusion of an open connected subset, then the restrictions $F_1(U) \to F_1(V)$ and $F_3(U) \to F_3(V)$ are isomorphisms. This implies the same statement for F_2, because $H^1(U, F_1) = 0$, as constant sheaves do not have higher cohomology on contractible sets. □

Lemma 2.5.10. *The notion of (weak) constructibility is stable under* $j_!$ *for* j *an open immersion and* π_* *for* π *finite.*

Proof. The assertion for $j_!$ is obvious. The same holds for i_* in the case of closed immersions.

Now assume $\pi : X \to Y$ is finite and in addition étale. Let F be (weakly) constructible on X. Let $X_0, \dots, X_n \subset X$ be a stratification such that $F|_{X_i}$ is locally constant. Let Y_i be the image of X_i. These are locally closed subvarieties of Y because π is closed and open. We refine them into a stratification of Y. As π is finite étale, it is locally in the analytic topology of the form $I \times B$ with I finite and $B \subset Y(\mathbb{C})$ an open set in the analytic topology. Obviously $\pi_* F|_B$ is locally constant on the strata we have defined.

Now let π be finite. As we have already discussed closed immersions, it suffices to assume that π is surjective. There is an open dense subscheme $U \subset Y$ such that π is étale above U. Let $U' = \pi^{-1}(U)$, $Z = Y \setminus U$ and $Z' = X \setminus U'$. We consider the exact sequence on X

$$0 \to j_{U'!} j_{U'}^* F \to F \to i_{Z'*} i_{Z'}^* F \to 0.$$

As π is finite, the functor π_* is exact and hence

$$0 \to \pi_* j_{U'!} j_{U'}^* F \to \pi_* F \to \pi_* i_{Z'*} i_{Z'}^* F \to 0.$$

By Lemma 2.5.9, it suffices to consider the outer terms. We have

$$\pi_* j_{U'!} j_{U'}^* F = j_{U!} \pi|_{U'*} j_{U'}^* F,$$

and this is (weakly) constructible by the étale case and the assertion on open immersions. We also have

$$\pi_* i_{Z'*} i_{Z'}^* F = i_{Z*} \pi|_{Z'*} i_{Z'}^* F,$$

and this is (weakly) constructible by noetherian induction and the case of closed immersions. □

Nori's proof of Basic Lemma II. The argument will show a more precise version of property (3) and (4): there exists a finite subset $E \subset U(\mathbb{C})$ such that $H^{\dim(X)}(X, F')$ is isomorphic to a direct sum $\bigoplus_x F_x$ of stalks of F at points of E.

Let $n := \dim(X)$. In the first step, we reduce to $X = \mathbb{A}^n$. We use Noether normalisation to obtain a finite morphism $\pi : X \to \mathbb{A}^n$. By Lemma 2.5.10, the sheaf $\pi_* F$ is (weakly) constructible.

Let then $v : V \hookrightarrow \mathbb{A}^n$ be a Zariski open set with the property that $F' := v_! v^* \pi_* F$ satisfies the Basic Lemma II on $\mathbb{A}^n$. Let $U := \pi^{-1}(V) \overset{j}{\hookrightarrow} X$ be the preimage in X. One has an isomorphism of sheaves:

$$\pi_* j_! j^* F \cong v_! v^* \pi_* F.$$

Therefore, $H^q(X, j_! j^* F) \cong H^q(\mathbb{A}^n, v_! v^* \pi_* F)$ for all q and the latter vanishes for $q < n$. The formula for the n-th cohomology on $\mathbb{A}^n$ implies the one on X.

So let us now assume that F is weakly constructible on $X = \mathbb{A}^n$. We argue by induction on n and all F. The case $n = 0$ is trivial.

By replacing F by $j_! j^* F$ for an appropriate open $j : U \to \mathbb{A}^n$, we may assume that F is locally constant on U and that $\mathbb{A}^n \setminus U = V(f)$. By Noether normalisation or its proof, there is a surjective projection map $\pi : \mathbb{A}^n \to \mathbb{A}^{n-1}$ such that $\pi|_{V(f)} : V(f) \to \mathbb{A}^{n-1}$ is surjective and finite.

We will see in Lemma 2.5.11 that $R^q \pi_* F = 0$ for $q \neq 1$ and $R^1 \pi_* F$ is weakly constructible. The Leray spectral sequence now gives that

$$H^q(\mathbb{A}^n, F) = H^{q-1}(\mathbb{A}^{n-1}, R^1 \pi_* F).$$

In the induction procedure, we apply the Basic Lemma II to $R^1 \pi_* F$ on $\mathbb{A}^{n-1}$. By induction, there exists a Zariski open $h : V \hookrightarrow \mathbb{A}^{n-1}$ such that $h_! h^* R^1 \pi_* F$ has cohomology only in degree $n - 1$. Let $U := \pi^{-1}(V)$ and $j : U \hookrightarrow \mathbb{A}^n$ be the inclusion. Then $j_! j^* F$ has cohomology only in degree n. The explicit description of cohomology in degree n follows from the description of the stalks of $R^1 \pi_* F$ in the proof of Lemma 2.5.11. □

Lemma 2.5.11. *Let $\pi : \mathbb{A}^n \to \mathbb{A}^{n-1}$ be a coordinate projection. Let $V(f) \subset \mathbb{A}^n$ such that $\pi|_{V(f)}$ is finite surjective. Let F on $\mathbb{A}^n$ be locally constant on $U = \mathbb{A}^n \setminus V(f)$ and vanish on $V(f)$.*

Then $R^q\pi_*F = 0$ for $q \neq 1$ and $R^1\pi_*F$ is weakly constructible. Moreover, for every $y \in \mathbb{A}^{n-1}(\mathbb{C})$ there is a finite set $E \subset \pi^{-1}(y)$ such that $(R^1\pi_*F)_y = \bigoplus_{e\in E} F_e$.

Proof. This is a standard fact, but Nori gives a direct proof.

The stalk of $R^q\pi_*F$ at $y \in \mathbb{A}^{n-1}$ is given by $H^q(\{y\} \times \mathbb{A}^1, F|_{\{y\}\times\mathbb{A}^1})$ by the variation of proper base change in Theorem 2.5.12 below.

Let, more generally, G be a sheaf on $\mathbb{A}^1$ which is locally constant outside a finite, non-empty set S where it vanishes. Let T be a finite embedded tree in $\mathbb{A}^1(\mathbb{C}) = \mathbb{C}$ with vertex set S. Then the restriction map to the tree defines a retraction isomorphism $H^q(\mathbb{A}^1, G) \cong H^q(T, G_T)$ for all $q \geq 0$. Using Čech cohomology, we can compute $H^q(T, G_T)$: for each vertex $v \in S$, let U_v be the open star of all outgoing half open edges at the vertex v. Then U_a and U_b only intersect if the vertices a and b have a common edge $e = e(a, b)$. The intersection $U_a \cap U_b$ is contractible and contains the center $t(e)$ of the edge e. There are no triple intersections. Hence $H^q(T, G_T) = 0$ for $q \geq 2$. We have $G(U_s) = 0$ because G is zero on S, locally constant away from S and U_s is simply connected. Therefore also $H^0(T, G_T) = 0$ and $H^1(T, G_T)$ is isomorphic to $\bigoplus_e G_{t(e)}$.

This implies already that $R^q\pi_*F = 0$ for $q \neq 1$.

To show that $R^1\pi_*F$ is weakly constructible means to show that it is locally constant on some stratification. We see that the stalks $(R^1\pi_*F)_y$ depend only on the set of points in $\{y\} \times \mathbb{A}^1 = \pi^{-1}(y)$ where $F|_{\{y\}\times\mathbb{A}^1}$ vanishes. But the sets of points where the vanishing set has the same degree (cardinality) defines a suitable stratification. Note that the stratification only depends on the branching behaviour of $V(f) \to \mathbb{A}^{n-1}$, hence the stratification is algebraic and defined over k. □

Theorem 2.5.12. (Variation of Proper Base Change) *Let $\pi : X \to Y$ be a continuous map between locally compact, locally contractible topological spaces which is a fibre bundle and let G be a sheaf on X. Assume $W \subset X$ is closed and such that G is locally constant on $X \setminus W$ and π restricted to W is proper. Then $(R^q\pi_*G)_y \cong H^q(\pi^{-1}(y), G_{\pi^{-1}(y)})$ for all q and all $y \in Y$.*

Proof. The statement is local on Y, so we may assume that $X = T \times Y$ is a product with π the projection. Since Y is locally compact and locally contractible, we may assume that Y is compact by passing to a compact neighbourhood of y. As $W \to Y$ is proper, this implies that W is compact. By enlarging W, we may assume that $W = K \times Y$ is a product of compact sets for some compact subset $K \subset T$. Since Y is locally contractible, we replace Y by a contractible neighbourhood. (We may lose compactness, but this does not matter any more.) Let $i : K \times Y \to X$ be the inclusion and $j : (T \setminus K) \times Y \to X$ the complement.

Look at the exact sequence

$$0 \to j_! G_{(T\setminus K)\times Y} \to G \to i_* G_{K\times Y} \to 0.$$

The result holds for $G_{K\times Y}$ by the usual proper base change, see [KS90, Proposition 2.5.2].

Since Y is contractible, we may assume that $G_{(T\setminus K)\times Y}$ is the pull-back of the constant sheaf on $T\setminus K$. Now the result for $j_!G_{(T\setminus K)\times Y}$ follows from the Künneth formula. □

2.5.4 Beilinson's Proof of Basic Lemma II

We follow Beilinson [Beĭ87, Proof 3.3.1], who even proves a more general result. Note that Beilinson works in the setting of étale sheaves, independent of the characteristic of the ground field. We have translated it to weakly constructible sheaves. The argument is intrinsically about perverse sheaves, and the perverse t-structure, even though we have downplayed their use as far as possible. For an extremely short introduction, see Sect. 2.5.5.

Let X be affine and reduced of dimension n over a field $k\subset\mathbb{C}$. Let F be a (weakly) constructible sheaf on X. We choose a projective compactification $\kappa: X\hookrightarrow\bar{X}$ such that κ is an affine morphism. Let W be a divisor on X such that F is a locally constant sheaf on $X\setminus W$ and $X\setminus W$ is smooth. Let $h: X\setminus W\hookrightarrow X$ be the open immersion. Then define $M:=h_!h^*F$.

Let $\bar{H}\subset\bar{X}$ be a generic hyperplane. We will see in the proof of Lemma 2.5.13 below what the conditions on $\bar{H}$ are. Let $H=X\cap\bar{H}$ be the corresponding hyperplane in X.

We denote by $V=\bar{X}\setminus\bar{H}$ the complement and by $\ell: V\hookrightarrow\bar{X}$ the open inclusion. Furthermore, let $\kappa_V: V\cap X\hookrightarrow V$ and $\ell_X: V\cap X\hookrightarrow X$ be the open inclusion maps, and $i:\bar{H}\hookrightarrow\bar{X}$ and $i_X: H\hookrightarrow X$ the closed immersions. We set $U:=X\setminus(W\cup H)$ and consider the open inclusion $j: U\hookrightarrow X$ with complement $Z=W\cup H$. Let $M_{V\cap X}$ be the restriction of M to $V\cap X$. Summarising, we have a commutative diagram

$$\begin{tikzcd} & U \arrow[d, "j"] & \\ V\cap X \arrow[r, "\ell_X"] \arrow[d, "\kappa_V"'] & X \arrow[d, "\kappa"] & H \arrow[l, "i_X"'] \arrow[d, "\tilde{\kappa}"] \\ V \arrow[r, "\ell"] & \bar{X} & \bar{H}. \arrow[l, "i"'] \end{tikzcd}$$

Lemma 2.5.13. *For generic $\bar{H}$ in the above set-up, there is an isomorphism*

$$\ell_!\ell^*R\kappa_*M \xrightarrow{\cong} R\kappa_!\ell_{X*}M_{V\cap X}$$

extending naturally $\mathrm{id}: M_{V\cap X}\to M_{V\cap X}$.

Proof. We consider the map of distinguished triangles

$$\begin{tikzcd} \ell_!\ell^* R\kappa_* M \arrow[r] \arrow[d] & R\kappa_* M \arrow[r] \arrow[d, "\mathrm{id}"'] & i_* i^* R\kappa_* M \arrow[d] \\ R\kappa_* \ell_{X!} M_{V\cap X} \arrow[r] & R\kappa_* M \arrow[r] & i_* R\tilde{\kappa}_* i_X^* M \end{tikzcd}$$

The existence of the arrows follows from standard adjunctions together with proper base change in the simple formulas $\kappa^*\ell_! \cong \ell_{X!}\kappa_V^*$ and $\kappa^* i_* \cong i_{X*}\tilde{\kappa}^*$, respectively.

Hence it is sufficient to prove that

$$i^* R\kappa_* M \xrightarrow{\cong} R\tilde{\kappa}_* i_X^* M. \tag{2.1}$$

To prove this, we make a base change to the universal hyperplane section. In detail: Let $\mathbb{P}$ be the space of hyperplanes in $\bar{X}$. Let

$$\overline{\mathcal{H}}_{\mathbb{P}} \to \mathbb{P}$$

be the universal family. It comes with a natural map

$$i_{\mathbb{P}} : \overline{\mathcal{H}}_{\mathbb{P}} \to \bar{X}.$$

By [Gro, p. 9] and [Jou83, Théorème 6.10] there is a dense Zariski open subset $T \subset \mathbb{P}$ such that the induced map

$$i_T : \overline{\mathcal{H}}_T \hookrightarrow \bar{X} \times T \longrightarrow \bar{X}$$

is smooth. Let $\mathcal{H}_T$ be the preimage of X.

We apply a smooth base change in the square

$$\begin{tikzcd} \mathcal{H}_T \arrow[r, "i_{X,T}"] \arrow[d, "\tilde{\kappa}_T"'] & X \arrow[d, "\kappa"] \\ \overline{\mathcal{H}}_T \arrow[r, "i_T"] & \bar{X} \end{tikzcd}$$

and obtain a quasi-isomorphism

$$i_T^* R\kappa_* M \xrightarrow{\cong} R\tilde{\kappa}_{T*} i_{X,T}^* M$$

of complexes of sheaves on $\overline{\mathcal{H}}_T$.

We specialise to some $t \in T(k)$ and get a hyperplane $t : \bar{H} \subset \overline{\mathcal{H}}_T$ to which we restrict. The left-hand side turns into $i^* R\kappa_* M$.

We apply the Generic Base Change Theorem 2.5.14 to $\bar{\kappa}_T$ over the base T and $\mathcal{G} = i^*_{X,T}M$. Hence after shrinking T further, the right-hand side turns into

$$t^* R\bar{\kappa}_{T*} i^*_{X,T} M \cong R\bar{\kappa}_* t^*_X i^*_{X,T} M \cong R\bar{\kappa} i^*_X M.$$

Putting these equations together, we have verified Eq. 2.1. □

Proof of Basic Lemma II. We keep the notation fixed at the beginning of the present Sect. 2.5.4. Let $\bar{H} \subset \bar{X}$ be a generic hyperplane in the sense of Lemma 2.5.13.

By Artin vanishing for constructible sheaves (see Theorem 2.5.23), the group $H^i(X, j_! j^* F)$ vanishes for $i > n$. It remains to show that $H^i(X, j_! j^* F)$ vanishes for $i < n$. We obviously have $j_! j^* F \cong \ell_{X!} M_{V \cap X}$. Therefore,

$$\begin{aligned} H^i(X, j_! j^* F) &\cong H^i(X, \ell_{X!} M_{V\cap X}) \\ &\cong H^i(\bar{X}, R\kappa_* \ell_{X!} M_{V\cap X}) \\ &\cong H^i(\bar{X}, \ell_! \ell^* R\kappa_* M) \qquad \text{by 2.5.13} \\ &= H^i_c(V, (R\kappa_* M)_V). \end{aligned}$$

The last group vanishes for $i < n$ by Artin's vanishing Theorem 2.5.23 for compact supports once we have checked that $R\kappa_* M_V[n]$ is perverse for the middle perversity, see Definition 2.5.21. Recall that $M = h_! h^* F$. The restriction $F|_{X\setminus W}$ is a locally constant sheaf and $X\setminus W$ smooth. Hence $F|_{X\setminus W}[n]$ is perverse. Both h and κ are affine, hence the same is true for $R\kappa_* h_! F|_{X\setminus W}[n]$ by Theorem 2.5.23 (3).

If, in addition, F is constructible, then by the same theorem, the complex $R\kappa_* h_! F|_{X\setminus W}[n]$ is in $D^{\geq 0}_c(X)$. Hence our cohomology with compact support is also finitely generated.

If the stalks of F are torsion-free, then by the same theorem $R\kappa_* h_! F|_{X\setminus W}$ is in ${}^+D^{\geq n}_{wc}(X)$. Hence $H^n_c(X, R\kappa_* h_! F_{X\setminus W})$ is torsion-free as well. □

Theorem 2.5.14. (Generic Base Change) *Let S be a separated scheme of finite type over k and $f : X \to Y$ a morphism of separated S-schemes of finite type over S. Let $\mathcal{F}$ be a (weakly) constructible sheaf on X. Then there is a dense open subset $U \subset S$ such that:*

1. *over U, the sheaves $R^i f_* \mathcal{F}$ are (weakly) constructible and vanish for almost all i;*
2. *the formation of $R^i f_* \mathcal{F}$ is compatible with any base change $S' \to U \subset S$.*

This is the analogue of [Del77, Théorème 1.9 in Sect. Theorem finitude], which is for constructible étale sheaves in the étale setting.

Proof. The case $S = Y$ was treated by Arapura, see [Ara13, Theorem 3.1.10]. We explain the reduction to this case, using the same arguments as in the étale case.

All schemes can be assumed reduced.

Using Nagata, we can factor f as a composition of an open immersion and a proper map. The assertion holds for the latter by the proper base change theorem, hence it suffices to consider open immersions.

As the question is local on Y, we may assume that it is affine over S. We can then cover X by affines. Using the hypercohomology spectral sequence for the covering, we may reduce to the case when X is affine. In this case (X and Y affine, f an open immersion) we argue by induction on the dimension of the generic fibre of $X \to S$.

If $n = 0$, then, at least after shrinking S, we are in the situation where f is the inclusion of a connected component and the assertion is trivial.

We now assume the case $n-1$. We embed Y into $\mathbb{A}^m_S$ and consider the coordinate projections $p_i : Y \to \mathbb{A}^1_S$. We apply the inductive hypothesis to the map f over $\mathbb{A}^1_S$. Hence there is an open dense $U_i \subset \mathbb{A}^1_S$ such that the conclusion is valid over $p^{-1}U_i$. Hence the conclusion is valid over their union, i.e., outside a closed subvariety $Y_1 \subset Y$ finite over S. By shrinking S, we may assume that it is finite étale.

We fix the notation in the resulting diagram as follows:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "a"'] & Y \arrow[d, "b"] & Y_1 \arrow[l, "i"'] \arrow[dl, "b_1"] \\ & S & \end{tikzcd}$$

Let j be the open complement of i. We have checked that $j^*Rf_*\mathcal{G}$ is (weakly) constructible and compatible with any base change. We apply Rb_* to the triangle defined by the sequence

$$j_!j^*Rf_*\mathcal{G} \to Rf_*\mathcal{G} \to i_*i^*Rf_*\mathcal{G}$$

and obtain

$$Rb_*j_!j^*Rf_*\mathcal{G} \to Ra_*\mathcal{G} \to b_{1*}i^*Rf_*\mathcal{G}.$$

The first two terms are (possibly after shrinking S) (weakly) constructible by the previous considerations and the case $S = Y$. We also obtain that they are compatible with any base change. Hence the same is true for the third term. As b_1 is finite étale this also implies that $i^*Rf_*\mathcal{G}$ is (weakly) constructible and compatible with base change. Indeed, this follows because a direct sum of sheaves is constant if and only if every summand is constant. The same is true for $j_!j^*Rf_*\mathcal{G}$ by the previous considerations and base change for $j_!$. Hence the conclusion also holds for the middle term of the first triangle and we are done. □

2.5.5 *Perverse Sheaves and Artin Vanishing*

We clarify the setting used in Beilinson's proof of the Basic Lemma II above. Our aim is to formulate and prove the version of Artin vanishing that we need. Note that the notion of a perverse sheaf and the perverse t-structure is *not* needed for this purpose. We choose to explain the notion anyway because this is the real story behind the story. For a complete introduction to the theory of perverse sheaves, see the original reference [BBD82] by Beilinson, Bernstein and Deligne. For the more specific aspects we refer to Schürmann's monograph [Sch03].

Definition 2.5.15. ([BBD82, Définition 1.3.1]) Let D be a triangulated category. A *t-structure* on D consists of a pair $(D^{\leq 0}, D^{\geq 0})$ of full subcategories such that

1. $D^{\leq -1} := D^{\leq 0}[1] \subset D^{\leq 0}$, $D^{\geq 1} := D^{\geq 0}[-1] \subset D^{\geq 0}$,
2. $\mathrm{Hom}_D(X, Y) = 0$ for all $X \in D^{\leq 0}$, $Y \in D^{\geq 1}$,
3. for any object $X \in D$ there is a distinguished triangle

$$X^{\leq 0} \to X \to X^{\geq 1} \to X^{\leq 0}[1]$$

with $X^{\leq 0} \in D^{\leq 0}$, $X^{\geq 1} \in D^{\geq 1}$.

We call $\mathcal{A} = D^{\leq 0} \cap D^{\geq 0}$ the *heart* of the t-structure. For $n \in \mathbb{Z}$ we put

$$D^{\leq n} = D^{\leq 0}[-n], D^{\geq n} = D^{\geq 0}[-n].$$

Example 2.5.16. Let $\mathcal{A}$ be an abelian catgegory and $D = D(\mathcal{A})$ its derived category. We put $D^{\leq 0}$ and $D^{\geq 0}$ the subcategory with objects concentrated in non-positive and non-negative degrees, respectively. This is a t-structure with heart $\mathcal{A}$. Indeed, the axioms mimic the properties of this example.

Example 2.5.17. ([BBD82, Sect. 3.3], [Sch03, Example 6.0.2. 3, p. 378]) Let $D(\mathbb{Z})$ be the derived category of abelian groups. Let ${}^+D^{\leq 0}$ be the subcatgory of complexes $K^\bullet$ such that $H^i(K^\bullet)$ vanishes for $i \geq 2$ and is torsion for $i = 1$. Let ${}^+D^{\geq 0}$ be the subcategory of complexes $K^\bullet$ such that $H^i(K^\bullet)$ vanishes for $i < 0$ and is torsion-free for $i = 0$. Then $({}^+D^{\leq 0}, {}^+D^{\geq 0})$ is a t-structure, because $\mathrm{Hom}(T, F) = 0$ for any torsion group T and F torsion-free.

Theorem 2.5.18. ([BBD82, Théorème 1.3.6]) *The heart of a t-structure is an abelian category.*

Probably the best-known non-trivial example is the following:

Example 2.5.19. ([BBD82, Sects. 2.1 and 2.2]) Let $\pi : X \to \mathbb{C}$ be an algebraic variety. Let $\mathcal{S}(X^{\mathrm{an}}, \mathbb{Z})$ be the category of abelian sheaves on X^{an} and let

$$D^b_c(X, \mathbb{Z}) \subset D(\mathcal{S}(X^{\mathrm{an}}, \mathbb{Q}))$$

be the subcategory of complexes whose cohomology objects are all constructible, see Definition 2.5.5, and almost all zero. Then we obtain a t-structure as follows:

- The full subcategory $D^{\leq 0}(X)$ is given by the complexes $\mathcal{F}^\bullet$ such that there is a stratification of X such that for the inclusion $i_S : S \to X$ of a stratum the sheaves $H^i i_S^* \mathcal{F}^\bullet$ are locally constant and vanish for $i > -\dim_{\mathbb{C}} S$.

- The full subcategory $D^{\geq 0}(X)$ is given by the complexes $\mathcal{F}^\bullet$ such that there is a stratification of X such that for the inclusion $i_S : S \to X$ of a stratum the sheaves $H^i i_S^! \mathcal{F}^\bullet$ are locally constant and vanish for $i < -\dim_{\mathbb{C}} S$.

It goes by the name of the t-structure for the *middle perversity*. Its heart is called the category of *perverse sheaves* (for the middle perversity). If X is smooth, then a locally constant sheaf of finitely generated abelian groups viewed as a complex concentrated in degree $-\dim X$ is a perverse sheaf.

Recall from Definition 2.5.5 that the strata of a stratification are assumed algebraic and in addition smooth and connected.

We have been working in a more general setting: Let $k \subset \mathbb{C}$ be a subfield, X an algebraic variety over k. Let $\mathcal{S}(X^{\mathrm{an}}, \mathbb{Z})$ be the category of sheaves of abelian groups on X^{an}.

Definition 2.5.20. Let X and $\mathcal{S}(X^{\mathrm{an}}, \mathbb{Z})$ be as just defined.

1. Let
$$D_{wc}(X, \mathbb{Z}) \subset D(\mathcal{S}(X^{\mathrm{an}}, \mathbb{Z}))$$
be the full subcategory of complexes such that there is a stratification of X by locally closed algebraic subvarieties over k such that the cohomology sheaves are weakly constructible with respect to this stratification, see Definition 2.5.5.
2. Let
$$D^b_c(X, \mathbb{Z}) \subset D(\mathcal{S}(X^{\mathrm{an}}, \mathbb{Z}))$$
be the full subcategory of complexes whose cohomology objects are constructible and almost all zero.

Note that the condition on objects of $D_{wc}(X, \mathbb{Z})$ is stronger than the assumption that all cohomology sheaves are weakly constructible.

The six functor formalism is available in these settings by [Sch03, Propositions 4.0.2 on p. 214 and 6.0.1 on p. 379]. The necessary properties of the stratifications by algebraic subvarieties over k hold, see Remark 2.5.6. It turns out that there are two choices of t-structure for the middle perversity on $D_{wc}(X, \mathbb{Z})$ and $D^b_c(X, \mathbb{Z})$, the standard one and one based on Example 2.5.17.

Definition 2.5.21. 1. Let $D^{\leq 0}_{wc}(X)$ and $D^{\geq 0}_{wc}(X)$ be the subcategories of $D_{wc}(X, \mathbb{Z})$ defined by the same condition as in Example 2.5.19 but with strata defined over k.
2. Let ${}^+D^{\leq 0}_{wc}(X)$ be the full subcategory of $D_{wc}(X, \mathbb{Z})$ that contains the complexes $\mathcal{F}^\bullet$ such that there is a sufficiently fine stratification of X by locally closed algebraic strata such that for the inclusion $i_S : S \to X$ of a stratum the sheaves $H^i i_S^* \mathcal{F}^\bullet$ are locally constant and for some (and hence every) point $x \in S$ with inclusion $i_x : s \to S$
$$i_x^* i_S^* \mathcal{F}^\bullet[-\dim_{\mathbb{C}} S] \in {}^+D^{\leq 0}(\mathbb{Z}).$$

3. Let ${}^+D^{\geq 0}_{wc}(X)$ be the full subcategory of $D_{wc}(X, \mathbb{Z})$ that contains the complexes $\mathcal{F}^\bullet$ such that there is a sufficiently fine stratification of X by locally closed algebraic strata such that for the inclusion $i_S : S \to X$ of a stratum the sheaves $H^i i^!_S \mathcal{F}^\bullet$ are locally constant and for some (and hence every) point $x \in S$ with inclusion $i_x : x \to S$

$$i^*_x i^!_S \mathcal{F}^\bullet[-\dim_{\mathbb{C}} S] \in {}^+D^{\geq 0}(\mathbb{Z}).$$

4. Let $D^{\leq 0}_c(X) = D^{\leq 0}_{wc}(X) \cap D^b_c(X, \mathbb{Z})(X)$ and analoguously for the other cases.

In any of these settings, we call the intersection ${}^?D^{\leq 0}_?(X) \cap {}^?D^{\geq 0}_?(X)$ the category of *perverse sheaves*.

Remark 2.5.22. It is not hard to deduce from the stability results of Schürmann in [Sch03, Sect. 6.0.1] and the methods of [BBD82, Chap. 2 and Sect. 3.3] that the pairs $({}^?D^{\leq 0}_?(X), {}^?D^{\geq 0}_?(X))$ define a t-structure in each of the four cases above. However, we are not going to give details because we are not aware of a readibly available reference and we do not need these facts.

If X is an algebraic variety over k, and $j : X \hookrightarrow \bar{X}$ an arbitrary compactification, then cohomology with supports with coefficients in a weakly constructible sheaf $\mathcal{G}$ is defined by

$$H^i_c(X, \mathcal{G}) := H^i(\bar{X}, j_! \mathcal{G}).$$

It follows from proper base change that this is independent of the choice of compactification.

Theorem 2.5.23. (Schürmann, Artin vanishing for weakly constructible sheaves) *Let X be a variety over $k \subset \mathbb{C}$.*

1. *Let X be affine of dimension n. Let $\mathcal{G}$ be weakly constructible on X. Then $H^q(X, \mathcal{G}) = 0$ for $q > n$;*
2. *Let X be affine of dimension n. Let $\mathcal{F}^\bullet$ be a perverse sheaf on X. Then $H^q_c(X, \mathcal{F}^\bullet) = 0$ for $q < 0$.*
   More precisely, if $\mathcal{F}^\bullet$ is an object of the category $D^{\geq 0}_{wc}(X)$, or $D^{\geq 0}_c(X)$, or ${}^+D^{\geq 0}_{wc}(X)$, or ${}^+D^{\geq 0}_c(X)$, then the complex $R\Gamma_c(X, \mathcal{F}^\bullet)$ computing cohomology with compact support also belongs to $D^{\geq 0}_{wc}(pt)$, or $D^{\geq 0}_c(pt)$, or ${}^+D^{\geq 0}_{wc}(pt)$, or ${}^+D^{\geq 0}_c(pt)$, respectively. This means it vanishes in negative degrees, or is bounded with finitely generated cohomology, or also has torsion-free H^0, or all of this together, respectively.
3. *Let X be a variety over k, $g : U \to X$ an affine open immersion and $\mathcal{F}_\bullet$ a perverse sheaf on U. Then both $g_! \mathcal{F}^\bullet$ and $Rg_* \mathcal{F}_\bullet$ are perverse on X.*

The word perverse refers to any of the four settings of Definition 2.5.21.

Proof. The first two statements are [Sch03, Corollary 6.0.4, p. 391]. Note that a weakly constructible sheaf lies in ${}^m D^{\leq n}(X)$ in the notation of loc. cit.

The last statement combines the vanishing results for affine morphisms [Sch03, Theorem 6.0.4, p. 409] with the standard vanishing for all compactifiable morphisms [Sch03, Corollary 6.0.5, p. 397] for a morphism of relative dimension 0.

The way the theory in [Sch03] is set up, it holds relative to a choice of a suitable subcategory B of the subcategory of the derived category of abelian groups, e.g. $B = {}^+D^{\leq 0}_?$ or $B = {}^?D^{\leq 0}_?$, see [Sch03, Example 6.0.2, p. 388]. Hence we get all versions of Artin vanishing in parallel. □

Example 2.5.24. Let X be a variety over k and let $j : U \subset X$ a smooth open subvariety, equidimensional of dimension d. Assume that j is affine. Let $\mathcal{F}$ be a locally constant sheaf of U^{an}. We consider

$$j_!\mathcal{F}[d],\ Rj_*\mathcal{F}[d].$$

1. These complexes are in $D^{\leq 0}_{wc}(X) \cap D^{\geq 0}_{wc}(X)$.
2. If the stalks of $\mathcal{F}$ are finitely generated, then these complexes are even in $D^{\leq 0}_c(X) \cap D^{\geq 0}_c(X)$.
3. If the stalks are torsion-free, these complexes are in ${}^+D^{\leq 0}_{wc}(X) \cap {}^+D^{\geq 0}_{wc}(X)$.
4. If the stalks are finitely generated and torsion-free, then these complexes are even in ${}^+D^{\leq 0}_c(X) \cap {}^+D^{\geq 0}_c(X)$.

Proof. We have $\mathcal{F}[d] \in {}^+D^{\leq 0}_?(U) \cap {}^+D^{\geq 0}_?(U)$. We then apply Theorem 2.5.23. □

2.6 Triangulation of Algebraic Varieties

If X is a variety defined over $\mathbb{Q}$, we may ask whether any singular homology class $\gamma \in H_*^{\mathrm{sing}}(X^{\mathrm{an}}; \mathbb{Q})$ can be represented by an object described by polynomials. This is indeed the case. For a precise statement we need several definitions. The result will be formulated in Proposition 2.6.9.

This section follows closely the Diploma thesis of Benjamin Friedrich, see [Fri04]. The results are due to him.

Let $\mathbb{K} \subset \mathbb{R}$ be a subfield. We are mostly interested in the cases $\mathbb{K} = \mathbb{Q}$ and $\mathbb{K} = \widetilde{\mathbb{Q}}$ where $\widetilde{\mathbb{Q}}$ is the integral closure of $\mathbb{Q}$ in $\mathbb{R}$. Note that $\widetilde{\mathbb{Q}}$ is a field.

In this section, we use X to denote a variety over $\widetilde{\mathbb{Q}}$, and X^{an} for the associated analytic space over $\mathbb{C}$ (cf. Sect. 1.2.1).

2.6.1 Semi-algebraic Sets

Definition 2.6.1. ([Hir75, Definition 1.1, p. 166]) Let $\mathbb{K} \subset \mathbb{R}$ be a subfield. A subset of $\mathbb{R}^n$ is said to be $\mathbb{K}$-*semi-algebraic* if it is of the form

$$\{\underline{x} \in \mathbb{R}^n \,|\, f(\underline{x}) \geq 0\}$$

for some polynomial $f \in \mathbb{K}[x_1, \ldots, x_n]$, or can be obtained from sets of this form in a finite number of steps, where each step consists of one of the following basic operations:

1. complementary set,
2. finite intersection,
3. finite union.

A $\mathbb{K}$-semi-algebraic set is called *bounded* if it is bounded as a subset of $\mathbb{R}^n$.

As the name suggests, any algebraic set should in particular be $\widetilde{\mathbb{Q}}$-semi-algebraic. We also need a definition for maps:

Definition 2.6.2. ($\mathbb{K}$-*semi-algebraic map* [Hir75, p. 168]) Let $\mathbb{K} \subset \mathbb{R}$ be a subfield. A continuous map f between $\mathbb{K}$-semi-algebraic sets $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ is said to be $\mathbb{K}$-*semi-algebraic* if its graph

$$\Gamma_f := \big\{(a, f(a)) \,|\, a \in A\big\} \subseteq A \times B \subseteq \mathbb{R}^{n+m}$$

is $\mathbb{K}$-semi-algebraic.

Example 2.6.3. Any polynomial map

$$\begin{aligned} f : A &\longrightarrow B \\ (a_1, \ldots, a_n) &\mapsto (f_1(a_1, \ldots, a_n), \ldots, f_m(a_1, \ldots, a_n)) \end{aligned}$$

between $\mathbb{K}$-semi-algebraic sets $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ with $f_i \in \mathbb{K}[x_1, \ldots, x_n]$ for $i = 1, \ldots, m$ is $\mathbb{K}$-semi-algebraic, since it is continuous and its graph $\Gamma_f \subseteq \mathbb{R}^{n+m}$ is cut out from $A \times B$ by the polynomials

$$y_i - f_i(x_1, \ldots, x_n) \in \widetilde{\mathbb{Q}}[x_1, \ldots, x_n, y_1, \ldots, y_m] \quad \text{for} \quad i = 1, \ldots, m. \tag{2.2}$$

We can even allow f to be a rational map with rational component functions

$$f_i \in \mathbb{K}(x_1, \ldots, x_n), \quad i = 1, \ldots, m$$

as long as none of the denominators of the f_i vanish at a point of A. The argument remains the same except that the expression (2.2) has to be multiplied by the denominator of f_i.

Fact 2.6.4. (Tarski–Seidenberg) *The image (respectively preimage) of a $\widetilde{\mathbb{Q}}$-semi-algebraic set under a $\widetilde{\mathbb{Q}}$-semi-algebraic map is again $\widetilde{\mathbb{Q}}$-semi-algebraic. The same holds for the image of a $\mathbb{Q}$-semi-algebraic set under a $\mathbb{Q}$-semi-algebraic map.*

Proof. Historically, this was first observed by Tarski. A proof over $\mathbb{R}$ can be found in [Hir75, Proposition II, p. 167]. A proof over $\mathbb{Q}$, or any extension such as for example $\widetilde{\mathbb{Q}}$, can be found in [Sei54, Theorem 3, p. 370] or [BCR98, Theorem 1.4.2 and Corollary 1.4.7]. □

The Tarski–Seidenberg theorem is related to the principle of *quantifier elimination*, see [BCR98, Proposition 5.2.2].

Throughout the theory, it does not matter whether we work with $\widetilde{\mathbb{Q}}$-coefficients or $\mathbb{Q}$-coefficients. The proof of the following result was suggested to us by C. Scheiderer.

Proposition 2.6.5. *Let $G \subset \mathbb{R}^n$ be a $\widetilde{\mathbb{Q}}$-semi-algebraic set. Then G is even $\mathbb{Q}$-semi-algebraic. More precisely, the defining inequalities in $\mathbb{R}^n$ can be chosen with $\mathbb{Q}$-coefficients.*

Proof. Assume that G is defined by inequalities $h_i \leq 0$ for $h_i \in \widetilde{\mathbb{Q}}[x_1, \ldots, x_n]$ for $i = 1, \ldots, m$. The coefficients are already contained in a field $K \subset \mathbb{R}$ which is finite over $\mathbb{Q}$. Let u be a primitive element of K with $f \in \mathbb{Q}[y]$ a minimal polynomial. Write the polynomials as $h_i(x_1, \ldots, x_n) = H_i(x_1, \ldots, x_n, u)$ with $H_i \in \mathbb{Q}[x_1, \ldots, x_n, u]$. Choose rational numbers $a, b \in \mathbb{Q}$ such that u is the only root of f between a and b. Then G can be described by

$$G = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \exists y \text{ with } f(y) = 0 \text{ and } a < y < b \text{ such that } H_i(x_1, \ldots, x_n, y) \leq 0 \; \forall i = 1, \ldots, m\}.$$

Hence G is the image of the $\mathbb{Q}$-semi-algebraic set

$$\tilde{G} = \{(x_1, \ldots, x_n, y) \in \mathbb{R}^{n+1} \mid f(y) = 0 \text{ and } a < y < b \text{ and } H_i(x_1, \ldots, x_n, y) \leq 0 \; \forall i = 1, \ldots, m\}$$

under the projection to the first n coordinates. By the $\mathbb{Q}$-version of Fact 2.6.4 this implies that G is defined by polynomial equations with rational coefficients. □

As the terminology suggests, algebraic varieties are semi-algebraic. Indeed, this is even true for the associated complex analytic space.

Lemma 2.6.6. *Let X be a quasi-projective algebraic variety defined over $\widetilde{\mathbb{Q}}$ (or $\mathbb{Q}$). Then we can regard the complex analytic space X^{an} associated to the base change $X_{\mathbb{C}} = X \times_{\widetilde{\mathbb{Q}}} \mathbb{C}$ (or $X_{\mathbb{C}} = X \times_{\mathbb{Q}} \mathbb{C}$) as a bounded $\widetilde{\mathbb{Q}}$-semi-algebraic subset (or $\mathbb{Q}$-semi-algebraic subset).*

$$X^{an} \subseteq \mathbb{R}^N \tag{2.3}$$

for some N. Moreover, if $f : X \to Y$ is a morphism of varieties defined over $\widetilde{\mathbb{Q}}$, we can consider $f^{an} : X^{an} \to Y^{an}$ as a $\widetilde{\mathbb{Q}}$-semi-algebraic map with respect to these embeddings.

Remark 2.6.7. We will mostly consider the case when X is affine. Then $X \subset \mathbb{C}^n$ is defined by polynomial equations with coefficients in $\tilde{\mathbb{Q}}$. We identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and rewrite the equations for the real and imaginary part. Hence X is obviously $\tilde{\mathbb{Q}}$-semi-algebraic. In the lemma, we will show in addition that X can be embedded as a *bounded* $\tilde{\mathbb{Q}}$-semi-algebraic set.

Proof of Lemma 2.6.6. The case for $\mathbb{Q}$ follows from the case $\tilde{\mathbb{Q}}$ as the two notions agree. Alternatively, the proof given below works without changes over other fields than $\tilde{\mathbb{Q}}$.

First step $X = \mathbb{P}^n_{\tilde{\mathbb{Q}}}$: Consider

- $\mathbb{P}^n_{\mathbb{C}} = (\mathbb{P}^n_{\tilde{\mathbb{Q}}} \times_{\tilde{\mathbb{Q}}} \mathbb{C})^{\mathrm{an}}$ with homogeneous coordinates $x_0, \dots, x_n$, which we split as $x_m = a_m + ib_m$ with $a_m, b_m \in \mathbb{R}$ the real and imaginary parts, and
- $\mathbb{R}^N$, $N = 2(n+1)^2$, with coordinates $\{y_{kl}, z_{kl}\}_{k,l=0,\dots,n}$.

We define an explicit map

$$\begin{aligned}\psi : \mathbb{P}^n_{\mathbb{C}} &\longrightarrow \mathbb{R}^N \\ [x_0 : \dots : x_n] &\mapsto \Bigg(\dots, \underbrace{\frac{\operatorname{Re} x_k\overline{x}_l}{\sum_{m=0}^n |x_m|^2}}_{y_{kl}}, \underbrace{\frac{\operatorname{Im} x_k\overline{x}_l}{\sum_{m=0}^n |x_m|^2}}_{z_{kl}}, \dots\Bigg) \\ [a_0 + ib_0 : \dots : a_n + ib_n] &\mapsto \Bigg(\dots, \underbrace{\frac{a_ka_l + b_kb_l}{\sum_{m=0}^n (a_m^2 + b_m^2)}}_{y_{kl}}, \underbrace{\frac{b_ka_l - a_kb_l}{\sum_{m=0}^n (a_m^2 + b_m^2)}}_{z_{kl}}, \dots\Bigg).\end{aligned}$$

We can understand this map as a section of a natural fibre bundle on $\mathbb{P}^n_{\mathbb{C}}$. Its total space is given by the set E of hermitian $(n+1) \times (n+1)$-matrices of rank 1. The map

$$\phi : E \to \mathbb{P}^n_{\mathbb{C}}$$

takes a linear map M to its image in $\mathbb{C}^{n+1}$. We get a section of ϕ by mapping a 1-dimensional subspace L of $\mathbb{C}^{n+1}$ to the matrix of the orthogonal projection from $\mathbb{C}^{n+1}$ to L with respect to the standard hermitian product on $\mathbb{C}^{n+1}$. We can describe this section in coordinates. Let $(x_0, \dots, x_n) \in \mathbb{C}^{n+1}$ be a vector of length 1. Then an elementary computation shows that $M = (x_i\bar{x}_j)_{i,j}$ is the hermitian projector to the line $L = \mathbb{C}(x_0, \dots, x_n)$. Writing the real and imaginary part of the matrix M separately gives us precisely the formula for ψ. In particular, ψ is injective.

Therefore, we can consider $\mathbb{P}^n_{\mathbb{C}}$ via ψ as a subset of $\mathbb{R}^N$. It is obvious from the explicit formula that it takes values in the unit sphere $S^{N-1} \subset \mathbb{R}^N$, hence it is bounded. We claim that $\psi(\mathbb{P}^n_{\mathbb{C}})$ is also $\tilde{\mathbb{Q}}$-semi-algebraic. The composition of the projection

$$\begin{aligned}\pi : \mathbb{R}^{2(n+1)} \setminus \{(0, \dots, 0)\} &\longrightarrow \mathbb{P}^n_{\mathbb{C}} \\ (a_0, b_0, \dots, a_n, b_n) &\mapsto [a_0 + ib_0 : \dots : a_n + ib_n]\end{aligned}$$

with the map ψ is a polynomial map, hence it is $\widetilde{\mathbb{Q}}$-semi-algebraic by Example 2.6.3. Thus

$$\operatorname{Im}(\psi \circ \pi) = \operatorname{Im} \psi \subseteq \mathbb{R}^N$$

is $\widetilde{\mathbb{Q}}$-semi-algebraic by Fact 2.6.4.

Second step (*zero set of a polynomial*): We use the notation

$$\begin{aligned} V(g) &:= \{\underline{x} \in \mathbb{P}^n_{\mathbb{C}} \mid g(\underline{x}) = 0\} \quad \text{for} \quad g \in \mathbb{C}[x_0, \ldots, x_n] \text{ homogeneous,} \quad \text{and} \\ W(h) &:= \{\underline{t} \in \mathbb{R}^N \mid h(\underline{t}) = 0\} \quad \text{for} \quad h \in \mathbb{C}[y_{00}, \ldots, z_{nn}]. \end{aligned}$$

Let $X^{\text{an}} = V(g)$ for some homogeneous $g \in \widetilde{\mathbb{Q}}[x_0, \ldots, x_n]$. Then $\psi(X^{\text{an}}) \subseteq \mathbb{R}^N$ is a $\widetilde{\mathbb{Q}}$-semi-algebraic subset, as a little calculation shows. Setting for $k = 0, \ldots, n$

$$\begin{aligned} g_k &:= \text{``}g(\underline{x}\,\overline{x}_k)\text{''} \\ &= g(x_0\overline{x}_k, \ldots, x_n\overline{x}_k) \\ &= g\big((a_0a_k + b_0b_k) + i(b_0a_k - a_0b_k), \ldots, (a_na_k + b_nb_k) + i(b_na_k - a_nb_k)\big), \end{aligned}$$

where $x_j = a_j + ib_j$ for $j = 0, \ldots, n$, and

$$h_k := g(y_{0k} + iz_{0k}, \ldots, y_{nk} + iz_{nk}),$$

we obtain

$$\begin{aligned} \psi(X^{\text{an}}) &= \psi(V(g)) \\ &= \bigcap_{k=0}^{n} \psi(V(g_k)) \\ &= \bigcap_{k=0}^{n} \psi(\mathbb{P}^n_{\mathbb{C}}) \cap W(h_k) \\ &= \bigcap_{k=0}^{n} \psi(\mathbb{P}^n_{\mathbb{C}}) \cap W(\operatorname{Re} h_k) \cap W(\operatorname{Im} h_k). \end{aligned}$$

Final step: We can choose an embedding

$$X \subseteq \mathbb{P}^n_{\widetilde{\mathbb{Q}}},$$

thus getting

$$X^{\text{an}} \subseteq \mathbb{P}^n_{\mathbb{C}}.$$

Since X is a locally closed subvariety of $\mathbb{P}^n_{\widetilde{\mathbb{Q}}}$, the space X^{an} can be expressed in terms of subvarieties of the form $V(g)$ with $g \in \widetilde{\mathbb{Q}}[x_0, \dots, x_n]$, using only the following basic operations

1. complementary set,
2. finite intersection,
3. finite union.

Now $\widetilde{\mathbb{Q}}$-semi-algebraic sets are stable under these operations as well, hence the first assertion is proved.

Second assertion: The first part of the lemma provides us with $\widetilde{\mathbb{Q}}$-semi-algebraic inclusions

$$\psi : X^{\mathrm{an}} \subseteq \mathbb{P}^n_{\mathbb{C}} \subseteq \mathbb{R}^N,$$
$$\phi : Y^{\mathrm{an}} \subseteq \mathbb{P}^m_{\mathbb{C}} \subseteq \mathbb{R}^M.$$

We use the complex coordinates $\underline{x} = [x_0 : \dots : x_n]$ and $\underline{u} = [u_0 : \dots : u_m]$ on $\mathbb{P}^n_{\mathbb{C}}$ and $\mathbb{P}^m_{\mathbb{C}}$, respectively, and the real coordinates $(y_{00}, z_{00}, \dots, y_{nn}, z_{nn})$ and $(v_{00}, w_{00}, \dots, v_{mm}, w_{mm})$ on $\mathbb{R}^N$ and $\mathbb{R}^M$, respectively. We use the notation

$$V(g) := \{(\underline{x}, \underline{u}) \in \mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^m_{\mathbb{C}} \mid g(\underline{x}, \underline{u}) = 0\}$$

for $g \in \mathbb{C}[x_0, \dots, x_n, u_0, \dots, u_m]$ homogeneous in both $\underline{x}$ and $\underline{u}$, and

$$W(h) := \{\underline{t} \in \mathbb{R}^{N+M} \mid h(\underline{t}) = 0\}$$

for $h \in \mathbb{C}[y_{00}, \dots, z_{nn}, v_{00}, \dots, w_{mm}]$. Let $\{U_i\}$ be a finite open affine covering of X such that $f(U_i)$ satisfies

- $f(U_i)$ does not meet the hyperplane $\{u_j = 0\} \subset \mathbb{P}^m_{\widetilde{\mathbb{Q}}}$ for some j, and
- $f(U_i)$ is contained in an open affine subset V_i of Y.

This is always possible, since we can start with the open covering $Y \cap \{u_j \neq 0\}$ of Y, take a subordinate open affine covering $\{V_{i'}\}$, and then choose a finite open affine covering $\{U_i\}$ subordinate to $\{f^{-1}(V_{i'})\}$. Now each of the maps

$$f_i := f^{\mathrm{an}}{}_{|U_i} : U_i^{\mathrm{an}} \longrightarrow Y^{\mathrm{an}}$$

has image contained in V_i^{an} and does not meet the hyperplane $\{\underline{u} \in \mathbb{P}^m_{\mathbb{C}} \mid u_j = 0\}$ for an appropriate j. Being associated to an algebraic map between affine varieties, this map is rational

$$f_i : \underline{x} \mapsto \left[\frac{g_0'(\underline{x})}{g_0''(\underline{x})} : \dots : \frac{g_{j-1}'(\underline{x})}{g_{j-1}''(\underline{x})} : \underset{j}{1} : \frac{g_{j+1}'(\underline{x})}{g_{j+1}''(\underline{x})} : \dots : \frac{g_m'(\underline{x})}{g_m''(\underline{x})} \right],$$

with $g'_k, g''_k \in \widetilde{\mathbb{Q}}[x_0, \ldots, x_n]$, $k = 0, \ldots, \widehat{j}, \ldots, m$. Since the graph $\Gamma_{f^{\text{an}}}$ of f^{an} is the finite union of the graphs Γ_{f_i} of the f_i, it is sufficient to prove that $(\psi \times \phi)(\Gamma_{f_i})$ is a $\widetilde{\mathbb{Q}}$-semi-algebraic subset of $\mathbb{R}^{N+M}$. Now

$$\Gamma_{f_i} = \left(U_i^{\text{an}} \times V_i^{\text{an}}\right) \cap \bigcap_{\substack{k=0\\k\neq j}}^{n} V\left(\frac{y_k}{y_j} - \frac{g'_k(\underline{x})}{g''_k(\underline{x})}\right)$$

$$= \left(U_i^{\text{an}} \times V_i^{\text{an}}\right) \cap \bigcap_{\substack{k=0\\k\neq j}}^{n} V(y_k g''_k(\underline{x}) - y_j g'_k(\underline{x})),$$

so all we have to deal with is

$$V(y_k g''_k(\underline{x}) - y_j g'_k(\underline{x})).$$

Again a little calculation is necessary. Setting

$$\begin{aligned}
g_{pq} &:= \text{“}u_k \overline{u}_q g''_k(\underline{x}\,\overline{x}_p) - u_j \overline{u}_q g'_k(\underline{x}\,\overline{x}_p)\text{”} \\
&= u_k \overline{u}_q g''_k(x_0 \overline{x}_p, \ldots, x_n \overline{x}_p) - u_j \overline{u}_q g'_k(x_0 \overline{x}_p, \ldots, x_n \overline{x}_p) \\
&= \big((c_k c_q + d_k d_q) + i(d_k c_q - c_k d_q)\big) \\
&\quad\; g''_k\big((a_0 a_p + b_0 b_p) + i(b_0 a_p - a_0 b_p), \ldots, (a_n a_p + b_n b_p) + i(b_n a_p - a_n b_p)\big) \\
&\quad - \big((c_j c_q + d_j d_q) + i(d_j c_q - c_j d_q)\big) \\
&\quad\; g'_k\big((a_0 a_p + b_0 b_p) + i(b_0 a_p - a_0 b_p), \ldots, (a_n a_p + b_n b_p) + i(b_n a_p - a_n b_p)\big),
\end{aligned}$$

where $x_l = a_l + i b_l$ for $l = 0, \ldots, n$, $u_l = c_l + i d_l$ for $l = 0, \ldots, m$, and

$$\begin{aligned}
h_{pq} := \;& (v_{kq} + i w_{kq}) g''_k(y_{0p} + i z_{0p}, \ldots, y_{np} + i z_{np}) - \\
& (v_{jq} + i w_{jq}) g'_k(y_{0p} + i z_{0p}, \ldots, y_{np} + i z_{np}),
\end{aligned}$$

we obtain

$$\begin{aligned}
&(\psi \times \phi)\left(V\big(y_k g''_k(\underline{x}) - y_j g'_k(\underline{x})\big)\right) \\
&\qquad = \bigcap_{p=0}^{n} \bigcap_{q=0}^{m} (\psi \times \phi)(V(g_{pq})) \\
&\qquad = \bigcap_{p=0}^{n} \bigcap_{q=0}^{m} (\psi \times \phi)(U_i^{\text{an}} \times V_j^{\text{an}}) \cap W(h_{pq}) \\
&\qquad = \bigcap_{p=0}^{n} \bigcap_{q=0}^{m} (\psi \times \phi)(U_i^{\text{an}} \times V_j^{\text{an}}) \cap W(\operatorname{Re} h_{pq}) \cap W(\operatorname{Im} h_{pq}).
\end{aligned}$$

□

2.6.2 Semi-algebraic Singular Chains

We need further prerequisites in order to state the promised Proposition 2.6.9.

Definition 2.6.8. ([Hir75, p. 168]) By an *open simplex* $\triangle^\circ$ we mean the interior of a simplex (i.e., the convex hull of $r+1$ points in $\mathbb{R}^n$ which span an r-dimensional subspace). For convenience, a point is considered as an open simplex as well.

The notation $\triangle_d$ will be reserved for the *closed standard simplex* spanned by the standard basis

$$\{e_i = (0, \ldots, 0, \underset{i}{1}, 0, \ldots, 0) \mid i = 1, \ldots, d+1\}$$

of $\mathbb{R}^{d+1}$.

Consider the following data $(*)$:

- X a variety defined over $\widetilde{\mathbb{Q}}$,
- D a divisor in X with normal crossings, and
- $\gamma \in H_p^{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q})$, $p \in \mathbb{N}_0$.

As before, we have denoted by X^{an} and D^{an} the complex analytic space associated to the base change $X_{\mathbb{C}} = X \times_{\widetilde{\mathbb{Q}}} \mathbb{C}$ and $D_{\mathbb{C}} = D \times_{\widetilde{\mathbb{Q}}} \mathbb{C}$), respectively.

By Lemma 2.6.6, we may consider both X^{an} and D^{an} as bounded $\widetilde{\mathbb{Q}}$-semi-algebraic subsets of $\mathbb{R}^N$.

We are now able to formulate the main result of Sect. 2.6.

Proposition 2.6.9. *With data $(*)$ as above, we can find a representative of γ that is a rational linear combination of $\widetilde{\mathbb{Q}}$-semi-algebraic singular simplices.*

The proof of this proposition relies on the following proposition due to Lojasiewicz, which has been written down by Hironaka.

Proposition 2.6.10. ([Hir75, p. 170]) *For $\{X_i\}$ a finite system of bounded $\widetilde{\mathbb{Q}}$-semi-algebraic sets in $\mathbb{R}^n$, there exists a simplicial decomposition*

$$\mathbb{R}^n = \coprod_j \triangle^\circ{}_j$$

by open simplices $\triangle^\circ{}_j$ of dimensions $d(j)$ and a $\widetilde{\mathbb{Q}}$-semi-algebraic automorphism

$$\kappa : \mathbb{R}^n \to \mathbb{R}^n$$

such that each X_i is a finite union of some of the $\kappa(\triangle^\circ{}_j)$.

Note 2.6.11. Although Hironaka considers $\mathbb{R}$-semi-algebraic sets, we can safely replace $\mathbb{R}$ by $\widetilde{\mathbb{Q}}$ in this result (including the fact he cites from [Sei54]). The only problem that could possibly arise concerns a "good direction lemma":

Lemma 2.6.12. (Good direction lemma for $\mathbb{R}$, [Hir75, p. 172], [KB32, Theorem 5.I, p. 242])
Let Z be an $\mathbb{R}$-semi-algebraic subset of $\mathbb{R}^n$, which is nowhere dense. A direction $v \in \mathbb{P}^{n-1}_{\mathbb{R}}(\mathbb{R})$ is called good *if any line l in $\mathbb{R}^n$ parallel to v meets Z in a discrete (possibly empty) set of points; otherwise v is called* bad. *Then the set $B(Z)$ of bad directions is a Baire category set in $\mathbb{P}^{n-1}_{\mathbb{R}}(\mathbb{R})$.*

This immediately gives good directions $v \in \mathbb{P}^{n-1}_{\mathbb{R}}(\mathbb{R}) \setminus B(Z)$, but not necessarily $v \in \mathbb{P}^{n-1}_{\widetilde{\mathbb{Q}}}(\widetilde{\mathbb{Q}}) \setminus B(Z)$. However, in Remark 2.1 of [Hir75], which follows directly after the lemma, the following statement is made: If Z is compact, then $B(Z)$ is closed in $\mathbb{P}^{n-1}_{\mathbb{R}}(\mathbb{R})$. In particular, $\mathbb{P}^{n-1}_{\widetilde{\mathbb{Q}}}(\widetilde{\mathbb{Q}}) \setminus B(Z)$ will be non-empty. Since we only consider *bounded* $\widetilde{\mathbb{Q}}$-semi-algebraic sets Z', we may take $Z := \overline{Z'}$ (which is compact by Heine–Borel), and thus find a good direction $v \in \mathbb{P}^{n-1}_{\widetilde{\mathbb{Q}}}(\widetilde{\mathbb{Q}}) \setminus B(Z')$ using $B(Z') \subseteq B(Z)$. Hence:

Lemma 2.6.13. (Good direction lemma for $\widetilde{\mathbb{Q}}$) *Let Z' be a bounded $\widetilde{\mathbb{Q}}$-semi-algebraic subset of $\mathbb{R}^n$ which is nowhere dense. Then the set $\mathbb{P}^{n-1}_{\widetilde{\mathbb{Q}}}(\widetilde{\mathbb{Q}}) \setminus B(Z)$ of good directions is non-empty.*

Proof of Proposition 2.6.9. Applying Proposition 2.6.10 to the two-element system of $\widetilde{\mathbb{Q}}$-semi-algebraic sets $X^{\mathrm{an}}, D^{\mathrm{an}} \subseteq \mathbb{R}^N$, we obtain a $\widetilde{\mathbb{Q}}$-semi-algebraic decomposition

$$\mathbb{R}^N = \coprod_j \triangle_j^\circ$$

of $\mathbb{R}^N$ by open simplices $\triangle_j^\circ$ and a $\widetilde{\mathbb{Q}}$-semi-algebraic automorphism

$$\kappa : \mathbb{R}^N \to \mathbb{R}^N.$$

We write $\triangle_j$ for the closure of $\triangle_j^\circ$. The sets

$$K := \{\triangle_j^\circ \mid \kappa(\triangle_j^\circ) \subseteq X^{\mathrm{an}}\} \text{ and } L := \{\triangle_j^\circ \mid \kappa(\triangle_j^\circ) \subseteq D^{\mathrm{an}}\}$$

can be thought of as finite simplicial complexes, but built out of open instead of closed simplices. We define their *geometric realisations*

$$|K| := \bigcup_{\triangle^\circ{}_j \in K} \triangle_j^\circ \text{ and } |L| := \bigcup_{\triangle^\circ{}_j \in L} \triangle_j^\circ.$$

Then Proposition 2.6.10 states that κ maps the pair of topological spaces $(|K|, |L|)$ homeomorphically to $(X^{\mathrm{an}}, D^{\mathrm{an}})$.

Easy case: If X is complete, so is $X_{\mathbb{C}}$ by [Har77, Corollary II.4.8c, p. 102], hence X^{an} and D^{an} will be compact by [Har77, Appendix B.1, p. 439]. In this situation,

$$\overline{K} := \{\triangle_j \mid \kappa(\triangle_j) \subseteq X^{\mathrm{an}}\} \text{ and } \overline{L} := \{\triangle_j \mid \kappa(\triangle_j) \subseteq D^{\mathrm{an}}\}$$

are (ordinary) simplicial complexes (see Definition 2.3.3), whose geometric realisations coincide with those of K and L, respectively. In particular,

$$\begin{aligned} H_*^{\mathrm{simpl}}(\overline{K}, \overline{L}; \mathbb{Q}) &\cong H_*^{\mathrm{sing}}(\left|\overline{K}\right|, \left|\overline{L}\right|; \mathbb{Q}) \\ &\cong H_*^{\mathrm{sing}}(|K|, |L|; \mathbb{Q}) \\ &\cong H_*^{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q}). \end{aligned} \tag{2.4}$$

Here $H_*^{\mathrm{simpl}}(\overline{K}, \overline{L}; \mathbb{Q})$ denotes simplicial homology, of course.

We write $\gamma_{\mathrm{simpl}} \in H_p^{\mathrm{simpl}}(\overline{K}, \overline{L}; \mathbb{Q})$ and $\gamma_{\mathrm{sing}} \in H_p^{\mathrm{sing}}(\left|\overline{K}\right|, \left|\overline{L}\right|; \mathbb{Q})$ for the image of γ under this isomorphism. Any representative Γ_{simpl} of γ_{simpl} is a rational linear combination

$$\Gamma_{\mathrm{simpl}} = \sum_j a_j \triangle_j, \quad a_j \in \mathbb{Q}$$

of closed simplices $\triangle_j \in \overline{K}$. We orient them according the global orientation of X^{an}. We can choose orientation-preserving affine-linear maps of the standard simplex $\triangle_p$ to $\triangle_j$

$$\sigma_j : \triangle_p \longrightarrow \triangle_j \quad \text{for} \quad \triangle_j \in \Gamma_{\mathrm{simpl}}.$$

These maps yield a representative

$$\Gamma_{\mathrm{sing}} := \sum_j a_j \sigma_j$$

of γ_{sing}. Composing with κ yields $\Gamma := \kappa_* \Gamma_{\mathrm{sing}} \in \gamma$, where Γ has the desired properties.

In the *general case*, we perform a barycentric subdivision $\mathcal{B}$ on K twice (once is not enough) and define $|K|$ and $|L|$ not as the "closure" of K and L, but as follows (see Fig. 2.1)

$$\begin{aligned} \overline{K} &:= \{\triangle \mid \triangle^\circ \in \mathcal{B}^2(K) \text{ and } \triangle \subseteq |K|\}, \\ \overline{L} &:= \{\triangle \mid \triangle^\circ \in \mathcal{B}^2(K) \text{ and } \triangle \subseteq |L|\}. \end{aligned} \tag{2.5}$$

The point is that the pair of topological spaces $(\left|\overline{K}\right|, \left|\overline{L}\right|)$ is a strong deformation retract of $(|K|, |L|)$. Assuming this, we see that in the general case with $\overline{K}, \overline{L}$ defined as in (2.5), the isomorphism (2.4) still holds and we can proceed as in the easy case to prove the proposition.

We define the retraction map

$$\rho : (|K| \times [0,1], |L| \times [0,1]) \to (\left|\overline{K}\right|, \left|\overline{L}\right|)$$

as follows: Let $\triangle_j^\circ \in K$ be an open simplex which is not contained in the boundary of any other simplex of K and set

$$\mathit{inner} := \triangle_j \cap \overline{K}, \qquad \mathit{outer} := \triangle_j \setminus \overline{K}.$$

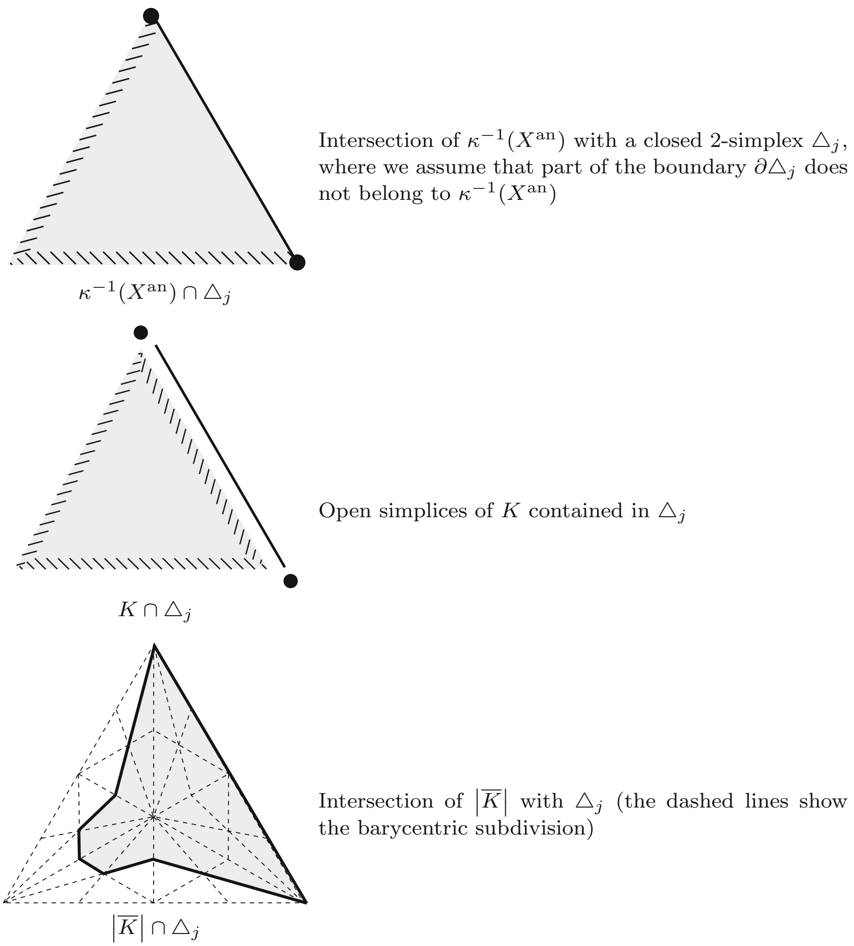

Fig. 2.1 Definition of $\overline{K}$

Note that *inner* is closed. For any point $p \in \textit{outer}$ the ray $\overrightarrow{c\,p}$ from the center c of $\triangle_j^\circ$ through p "leaves" the set *inner* at a point q_p, i.e., $\overrightarrow{c\,p} \cap \textit{inner}$ equals the line segment $c\,q_p$; see Fig. 2.2. The map

$$\rho_j : \triangle_j \times [0,1] \to \triangle_j$$
$$(p,t) \mapsto \begin{cases} p \ \text{ if } \ p \in \textit{inner}, \\ q_p + t \cdot (p - q_p) \ \text{if} \ p \in \textit{outer} \end{cases}$$

retracts $\triangle_j$ onto *inner*.

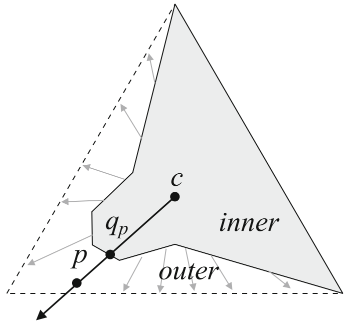

Fig. 2.2 Definition of q_p

Now these maps ρ_j glue together to give the desired homotopy ρ. □

We want to state one of the intermediate results of this proof explicitly:

Corollary 2.6.14. *Let X and D be as above. Then the pair of topological spaces (X^{an}, D^{an}) is homotopy equivalent to a pair of (realisations of) simplicial complexes $(|X^{\mathrm{simpl}}|, |D^{\mathrm{simpl}}|)$.*

2.7 Singular Cohomology via the h′-Topology

In order to give a simple description of the period isomorphism for singular varieties, we are going to need a more sophisticated description of singular cohomology.

We work in the category An of complex analytic spaces with morphisms given by holomorphic maps.

Definition 2.7.1. Let X be a complex analytic space. The h′-*topology* on the category $(\mathrm{An}/X)_{\mathrm{h}'}$ of complex analytic spaces over X is the smallest Grothendieck topology such that the following are covering maps:

1. proper surjective morphisms;
2. open covers.

If $\mathcal{F}$ is a presheaf of An/X we denote by $\mathcal{F}_{\mathrm{h}'}$ its sheafification in the h′-topology.

Remark 2.7.2. This definition is inspired by Voevodsky's h-topology on the category of schemes, see Sect. 3.2. We are not sure if it is the correct analogue in the analytic setting. However, it is good enough for our purposes.

Lemma 2.7.3. *For* $Y \in \mathrm{An}$ *let* $\mathbb{C}_Y$ *be the (ordinary) sheaf associated to the constant presheaf* $\mathbb{C}$. *Then*

$$Y \mapsto \mathbb{C}_Y(Y)$$

is an h′*-sheaf on* An.

Proof. We have to check the sheaf condition for the generators of the topology. By assumption, it is satisfied for open covers. Let $\tilde{Y} \to Y$ be a proper surjective morphism. Without loss of generality, we can assume that Y is connected. Let $Y = \bigcup_{j\in J} Y_j$ be the decomposition into irreducible components. Let $\tilde{Y}_i$ for $i \in I$ be the collection of connected components of $\tilde{Y}$. The index set is at most countable. For each i the image of $\tilde{Y}_i$ in Y is closed. An irreducible analytic space cannot be covered by countably many proper closed subspaces, hence for every irreducible component Y_j of Y we can choose an index $i(j)$ such that Y_j is contained in the image of $\tilde{Y}_{i(j)}$. Then

$$\tilde{Y} \times_Y \tilde{Y} = \bigcup_{i,i'\in I} \tilde{Y}_i \times_Y \tilde{Y}_{i'}.$$

We have to compute the kernel of

$$\prod_{i\in I} \mathbb{C}(\tilde{Y}_i) \to \prod_{i,i'} \mathbb{C}(\tilde{Y}_i \times_Y \tilde{Y}_{i'})$$

via the difference of the two natural restriction maps. We have $\mathbb{C}(\tilde{Y}_i) = \mathbb{C}$. Let $a = (a_i)_{i\in I}$ be in the kernel. Comparing the complex numbers a_i and $a_{i'}$ in $\mathbb{C}(\tilde{Y}_i \times_Y \tilde{Y}_j)$ we see that they agree unless $\tilde{Y}_i \times_Y \tilde{Y}_{i'}$ is empty. If the image of $\tilde{Y}_i$ meets the irreducible component Y_i, then $a_i = a_{i(j)}$ for the distinguished index chosen above. In particular, $a_{i(j)} = a_{i(j')}$ if $Y_j \cap Y_{j'} \neq \emptyset$. As Y is connected, this implies that all a_i are the same. Hence the kernel is just the one copy of $\mathbb{C} = \mathbb{C}(Y)$. □

Proposition 2.7.4. *Let X be an analytic space and $i : Z \subset X$ a closed subspace. Then there is a morphism of sites $\rho : (\mathrm{An}/X)_{\mathrm{h}'} \to X$. It induces an isomorphism*

$$H^i_{\mathrm{sing}}(X, Z; \mathbb{C}) \to H^i_{\mathrm{h}'}((\mathrm{An}/X)_{\mathrm{h}'}, \ker(\mathbb{C}_{\mathrm{h}'} \to i_*\mathbb{C}_{\mathrm{h}'}))$$

compatible with long exact sequences and products.

Remark 2.7.5. This statement and the following proof can be extended to more general sheaves $\mathcal{F}$ on An.

The argument uses the notion of a hypercover, see Definition 1.5.8.

Proof. We first treat the absolute case with $Z = \emptyset$. We use the theory of cohomological descent as developed in [SD72]. Singular cohomology satisfies cohomological descent for open covers. Proper base change, see Theorem 2.7.6, implies cohomological descent for proper surjective maps. Hence it satisfies cohomological descent for h′-covers. This implies that singular cohomology can be computed as a direct limit

$$\lim_{\mathfrak{X}_\bullet} \mathbb{C}(\mathfrak{X}_\bullet),$$

where $\mathfrak{X}_\bullet$ runs through all h$'$-hypercovers. On the other hand, the same limit computes h$'$-cohomology, see Proposition 1.6.9. For the general case, recall that we have a short exact sequence

$$0 \to j_!\mathbb{C} \to \mathbb{C} \to i_*\mathbb{C} \to 0$$

of sheaves on X. Its pull-back to An/X maps naturally to the short exact sequence

$$0 \to \ker(\mathbb{C}_{\mathrm{h}'} \to i_*\mathbb{C}_{\mathrm{h}'}) \to \mathbb{C}_{\mathrm{h}'} \to i_*\mathbb{C}_{\mathrm{h}'} \to 0.$$

This reduces the comparison in the relative case to the absolute case once we have shown that $Ri_*\mathbb{C}_{\mathrm{h}'} = i_*\mathbb{C}_{\mathrm{h}'}$. The sheaf $R^n i_*\mathbb{C}_{\mathrm{h}'}$ is given by the h$'$-sheafification of the presheaf

$$X' \mapsto H^n_{\mathrm{h}'}(Z \times_X X', \mathbb{C}_{\mathrm{h}'}) = H^n_{\mathrm{sing}}(Z \times_X X', \mathbb{C})$$

for $X' \to X$ in An/X. By resolution of singularities for analytic spaces we may assume that X' is smooth and $Z' = X' \times_X Z$ is a divisor with normal crossings. By passing to an open cover, we may assume that Z' is an open ball in a union of coordinate hyperplanes, in particular contractible. Hence, its singular cohomology is trivial. This implies that $R^n i_*\mathbb{C}_{\mathrm{h}'} = 0$ for $n \geq 1$. $\square$

Theorem 2.7.6. (Descent for proper hypercoverings) *Let $D \subset X$ be a closed subvariety and $D_\bullet \to D$ a proper hypercovering (see Definition 1.5.8), such that there is a commutative diagram*

$$\begin{tikzcd} D_\bullet \arrow[r] \arrow[d] & X_\bullet \arrow[d] \\ D \arrow[r] & X. \end{tikzcd}$$

Then one has cohomological descent for singular cohomology:

$$H^*(X, D; \mathbb{Z}) = H^*\left(\mathrm{Cone}(\mathrm{Tot}(X_\bullet) \to \mathrm{Tot}(D_\bullet))[-1]; \mathbb{Z}\right).$$

Here, $\mathrm{Tot}(-)$ *denotes the total complex in* $\mathbb{Z}[\mathrm{Var}]$ *associated to the corresponding simplicial variety, see Definition 1.5.11.*

Proof. The relative case follows from the absolute case. The essential ingredient is proper base change, which allows us to reduce to the case where X is a point. The statement then becomes a completely combinatorial assertion on contractibility of simplicial sets. The results are summed up in [Del74b] (5.3.5). For a complete reference, see [SD72], in particular Corollaire 4.1.6. $\square$

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