

# Contents

## Part I The General Theory

<b>1</b>	<b>Probabilistic Background</b>	3
1.1	Markov Chains	3
1.2	Random Walks in a Quarter Plane	4
1.3	Functional Equations for the Invariant Measure	6
<b>2</b>	<b>Foundations of the Analytic Approach</b>	9
2.1	Fundamental Notions and Definitions	9
2.1.1	Covering Manifolds	10
2.1.2	Algebraic Functions	12
2.1.3	Elements of Galois Theory	13
2.1.4	Universal Covering and Uniformization	15
2.1.5	Abelian Differentials and Divisors	15
2.2	Restricting the Equation to an Algebraic Curve	16
2.2.1	First Insight (Algebraic Functions)	16
2.2.2	Second Insight (Algebraic Curve)	17
2.2.3	Third Insight (Factorization)	18
2.2.4	Fourth Insight (Riemann Surfaces)	19
2.3	The Algebraic Curve $Q(x, y) = 0$	20
2.3.1	Branches of the Algebraic Functions on the Unit Circle	23
2.3.2	Branch Points	25
2.4	Galois Automorphisms and the Group of the Random Walk	29
2.4.1	Construction of the Automorphisms $\hat{\xi}$ and $\hat{\eta}$ on $S$	31
2.5	Reduction of the Main Equation to the Riemann Torus	32

<b>3</b>	<b>Analytic Continuation of the Unknown Functions in the Genus 1 Case</b>	37
3.1	Lifting the Fundamental Equation onto the Universal Covering	37
3.1.1	Lifting of the Branch Points	39
3.1.2	Lifting of the Automorphisms on the Universal Covering	39
3.2	Analytic Continuation	41
3.3	More About Uniformization	45
<b>4</b>	<b>The Case of a Finite Group</b>	55
4.1	Conditions for $\mathcal{H}$ to be Finite	55
4.1.1	Criterion for Groups of Order 4	61
4.1.2	Criterion for Groups of Order 6	62
4.1.3	Criterion for Groups of Order $4m$	65
4.1.4	Criterion for Groups of Order $4m - 2$	71
4.2	Further General Results	72
4.2.1	A Theorem About $\delta^s$	72
4.2.2	Form of the General Criterion	73
4.3	On Some Symmetric Quantities of the $\wp$ Function	75
4.4	Examples	77
4.4.1	$\mathcal{H}$ of Order 4	77
4.4.2	$\mathcal{H}$ of Order 6	78
4.5	Various Comments	78
4.6	Rational Solutions	79
4.6.1	The Case $N(f) \neq 1$	81
4.6.2	The Case $N(f) = 1$	83
4.7	Algebraic Solutions	91
4.7.1	The Case $N(f) = 1$	91
4.7.2	The Case $N(f) \neq 1$	96
4.8	Final Form of the General Solution	99
4.9	The Problem of the Poles and Examples	103
4.9.1	Rational Solutions	104
4.10	An Example of an Algebraic Solution by Flatto and Hahn	114
4.11	Two Queues in Tandem	117
<b>5</b>	<b>Solution in the Case of an Arbitrary Group</b>	119
5.1	Informal Reduction to a Riemann–Hilbert–Carleman BVP	119
5.2	Introduction to BVPs in the Complex Plane	121
5.2.1	A Bit of History	121
5.2.2	The Sokhotski–Plemelj Formulae	122
5.2.3	The Riemann Boundary Value Problem for a Closed Contour	123

5.2.4	The Riemann BVP for an Open Contour. . . . .	126
5.2.5	The Riemann–Carleman Problem with a Shift. . . . .	128
5.3	Further Properties of the Branches Defined by $Q(x, y) = 0$ . . . .	135
5.4	Index and Solution of the BVP (5.1.5). . . . .	145
5.5	Complements . . . . .	151
5.5.1	Analytic Continuation . . . . .	151
5.5.2	Computation of $w$ . . . . .	151
<b>6</b>	<b>The Genus 0 Case</b> . . . . .	<b>155</b>
6.1	Properties of the Branches . . . . .	155
6.2	Case 1: $p_{01} = p_{-1,0} = p_{-1,1} = 0$ . . . . .	157
6.3	Case 3: $p_{11} = p_{10} = p_{01} = 0$ . . . . .	158
6.4	Case 4: $p_{-1,0} = p_{0,-1} = p_{-1,-1} = 0$ . . . . .	161
6.4.1	Integral Equation . . . . .	161
6.4.2	Series Representation. . . . .	162
6.4.3	Uniformization. . . . .	162
6.4.4	Setting a Boundary Value Problem (BVP) . . . . .	164
6.5	Case 5: $M_x = M_y = 0$ . . . . .	164
6.5.1	Playing with the Uniformization . . . . .	169
<b>7</b>	<b>Criterion for the Finiteness of the Group in the Genus 0 Case</b> . . . .	<b>171</b>
7.1	The Main Theorem. . . . .	171
7.2	Proof of Part (a) of Theorem 7.1.1 . . . . .	172
7.2.1	Limit Conformal Gluing When Passing from Genus 1 to Genus 0 . . . . .	173
7.2.2	Limit of the Uniformization When Passing from Genus 1 to Genus 0. . . . .	176
7.2.3	Second Form of the Criterion in the Zero Drift Case . . . . .	179
7.2.4	Another Proof of the Criterion. . . . .	180
7.3	Proof of Part (b) of Theorem 7.1.1 . . . . .	181
7.3.1	Miscellaneous Remarks . . . . .	181
<b>8</b>	<b>Miscellanea</b> . . . . .	<b>183</b>
8.1	On Explicit Solutions . . . . .	183
8.2	Asymptotics . . . . .	184
8.2.1	Large Deviations . . . . .	184
8.2.2	Martin Boundary . . . . .	186
8.2.3	Random Walks Absorbed on the Axes . . . . .	187
8.3	Generalized Problems and Analytic Continuation . . . . .	187
8.3.1	Arbitrary Finite Jumps. . . . .	187
8.3.2	Space Inhomogeneity. . . . .	188
8.3.3	Dimension $\geq 3$ . . . . .	189

8.4	Transient Behavior and Laplace Transforms . . . . .	189
8.4.1	Sojourn Time in a Jackson Network with Overtaking . . . . .	189
8.4.2	Diffusion Process . . . . .	191
8.5	Outside Probability . . . . .	191
 <b>Part II Applications to Queueing Systems and Analytic Combinatorics</b>		
<b>9</b>	<b>A Two-Coupled Processor Model . . . . .</b>	<b>195</b>
9.1	Model and Equations . . . . .	195
9.2	Reduction to a Boundary Value Problem on a Circle . . . . .	197
9.3	The Case $pq = \mu_1\mu_2$ : Solutions with Elliptic Integrals . . . . .	198
9.4	The Case $pq \neq \mu_1\mu_2$ . . . . .	199
9.4.1	$p + q = 0$ : Rational Solutions . . . . .	199
<b>10</b>	<b>Joining the Shorter of Two Queues: Reduction to a Generalized BVP . . . . .</b>	<b>201</b>
10.1	Model and Historical Remarks . . . . .	201
10.2	Equations . . . . .	202
10.2.1	Reduction of the Number of Unknown Functions . . . . .	203
10.3	Meromorphic Continuation to the Complex Plane . . . . .	204
10.4	A Fredholm Integral Equation for $Q(x)$ When $\alpha \neq \beta$ . . . . .	208
10.4.1	Complete Solution of System (10.2.4) . . . . .	216
10.4.2	Explicit Integral Forms for Equal Service Rates ( $\alpha = \beta$ ) . . . . .	216
10.5	A Functional Equation for $G_1(y)$ . . . . .	217
10.5.1	A Non-standard BVP for $G_1(y)$ . . . . .	218
10.5.2	Poles and Residues of $G_1(y)$ : Miscellaneous Issues . . . . .	218
<b>11</b>	<b>Counting Lattice Walks in the Quarter Plane . . . . .</b>	<b>221</b>
11.1	A Functional Equation for a Tri-Variate CGF . . . . .	222
11.2	Group Classification of the 79 Main Random Walks . . . . .	224
11.3	Holonomy and Algebraicity of the Generating Functions . . . . .	225
11.4	The Nature of the Counting Generating Functions . . . . .	227
11.5	Some Exact Asymptotics . . . . .	229
11.5.1	The Simple Random Walk . . . . .	230
11.6	A General Approach for Walks with Small Steps and Eight Possible Neighbors . . . . .	234
11.6.1	Outline of the Arguments . . . . .	234
11.6.2	Reduction to a BVP . . . . .	235
11.6.3	Solution of the BVP . . . . .	236

11.6.4	Computation of $F(0, 0, z)$ , $F(1, 0, z)$ , $F(0, 1, z)$ , $F(1, 1, z)$ . . . . .	236
11.6.5	Groups of Order 4 . . . . .	237
11.6.6	On the Singularities of the Generating Functions . . . . .	238
<b>References</b> . . . . .		243
<b>Index</b> . . . . .		247

Random Walks in the Quarter Plane  
Algebraic Methods, Boundary Value Problems,  
Applications to Queueing Systems and Analytic  
Combinatorics

Fayolle, G.; Iasnogorodski, R.; Malyshev, V.

2017, XVII, 248 p. 17 illus., Hardcover

ISBN: 978-3-319-50928-0