Chapter 1
Probabilistic Background

1.1 Markov Chains

For this brief section, we only provide the minimum amount of information necessary; for further details, readers who are not probabilists may refer to any standard textbook on countable Markov chains (MCs) in discrete time.

A discrete time homogeneous Markov chain with a denumerable state space $\mathcal{A}$ is defined by the stochastic matrix

$$\mathbf{P} = \left\| p_{\alpha\beta} \right\|, \quad \alpha, \beta \in \mathcal{A},$$

such that

$$p_{\alpha\beta} \geq 0, \quad \sum_{\beta} p_{\alpha\beta} = 1, \quad \forall \alpha \in \mathcal{A}.$$

The matrix elements of $\mathbf{P}^n$ will be denoted by $p_{\alpha\beta}^{(n)}$.

Definition 1.1.1 An MC is called irreducible if, for any ordered pair α, β, there exits an m, depending on (α, β), such that

$$p_{\alpha\beta}^{(m)} \neq 0.$$

In addition, an irreducible MC is called *aperiodic* if, for some $\alpha, \beta \in \mathcal{A}$, the set $\{n : p_{\alpha\beta}^{(n)} \neq 0\}$ has a greatest common divisor equal to 1. It follows that the same property is true for all ordered pairs α, β.

Definition 1.1.2 An irreducible aperiodic MC is called *ergodic* if, and only if, the equation

$$\pi \mathbf{P} = \pi, \tag{1.1.1}$$

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where π is the row vector $\pi = (\pi_\alpha, \alpha \in \mathcal{A})$, has a unique ℓ_1-solution up to a multiplicative factor, which can be chosen so that

$$\sum_\alpha \pi_\alpha = 1, \quad \text{together with } \pi_\alpha > 0.$$

The π_α's are called *stationary probabilities* (see [15]). The random variable representing the position of the chain at time n will be written X_n and X will denote the random variable with distribution π.

1.2 Random Walks in a Quarter Plane

The class of MCs we shall mainly consider in this book are called *maximally space homogeneous random walks*. They are characterized by the following three properties.

P1 The state space is $\mathcal{A} = \mathbb{Z}_+^2 = \{(i, j) : i, j \geq 0 \text{ are integers}\}$.

P2 *(Maximal space homogeneity)* $\mathbb{Z}_+^2$ is supposed to be represented as the union of a finite number of non-intersecting classes

$$\mathbb{Z}_+^2 = \bigcup_r S_r \,. \tag{1.2.1}$$

Moreover, for each r and for all $\alpha \in S_r$ such that

$$p_{\alpha,\alpha+(i,j)} \neq 0, \quad \alpha + (i, j) \in \mathbb{Z}_+^2 \,,$$

$p_{\alpha,\alpha+(i,j)}$ does not depend on α, and can therefore be denoted by ${}_r p_{ij}$. Throughout most of the book, the classes S_r will have the following structure:

$$\mathbb{Z}_+^2 = S \cup S' \cup S'' \cup \{(0, 0)\} \tag{1.2.2}$$

where

$$\begin{cases} S &= \{(i, j) : i, j > 0\}, \\ S' &= \{(i, 0) : i > 0\}, \\ S'' &= \{(0, j) : j > 0\}. \end{cases}$$

The *internal* parts S' and S'' are called respectively the x-axis and y-axis. In this case, the probabilities ${}_r p_{i,j}$ will be simply written p_{ij}, p'_{ij}, p''_{ij}, p^0_{ij}, according to their respective regions S, S', S'', and $\{(0, 0)\}$. It is worth noting, nevertheless, that in Sect. 1.3 a more general partition of the state space is considered.

The last property deals with the boundedness of the jumps, which will be assumed unless otherwise stated (see e.g., Chaps. 5 and 6).

P3 *(Boundedness of the jumps)* For any $\alpha \in S_r$,

$$p_{\alpha\beta} = 0, \text{ unless } -d_r^- \leq (\beta - \alpha)_i \leq d_r^+,$$

for some constants $0 \leq d_r^{\pm} < \infty$, where $(\beta - \alpha)_i$ is the i-th coordinate of the vector $\beta - \alpha, i = 1, 2$. In addition, the next important assumption will hold throughout this book

$$d_r^{\pm} = 1, \quad \text{for the class } S_r = S.$$

The ergodicity conditions for the random walk $\mathcal{L}$ can be given in terms of the mean jump vectors

$$\begin{cases} \overrightarrow{\mathbf{M}} &= (M_x, M_y) = \left(\sum i p_{ij}, \sum j p_{ij}\right) \\ \overrightarrow{\mathbf{M}}' &= (M'_x, M'_y) = \left(\sum i p'_{ij}, \sum j p'_{ij}\right) \\ \overrightarrow{\mathbf{M}}'' &= (M''_x, M''_y) = \left(\sum i p''_{ij}, \sum j p''_j\right). \end{cases} \tag{1.2.3}$$

We shall consider only *irreducible aperiodic random walks.*

Theorem 1.2.1 *When* $\overrightarrow{\mathbf{M}} \neq 0$*, the random walk is ergodic if, and only if, one of the following three conditions holds:*

1. $\begin{cases} M_x < 0, \ M_y < 0\,, \\ M_x M'_y - M_y M'_x < 0\,, \\ M_y M''_x - M_x M''_y < 0; \end{cases}$
2. $M_x < 0, \quad M_y \geq 0, \quad M_y M''_x - M_x M''_y < 0;$
3. $M_x \geq 0, \quad M_y < 0, \quad M_x M'_y - M_y M'_x < 0.$

■

A probabilistic proof of this theorem exists in [36]. A new and purely analytic proof is presented in Chap. 5 and the analysis of the case $\overrightarrow{\mathbf{M}} = 0$ is carried out in detail in Chap. 6.

As stated in the general introduction, this monograph intends to provide a methodology of an essentially analytic nature for constructing and effectively computing the invariant measures associated with the random walks introduced in the present section. In fact, it is worth emphasizing that all these methods can also be employed (up to some additional technicalities) to analyze the transient behavior of the random walk, and to solve explicitly *Kolmogorov's classical equations*, which describe the time-evolution of the semigroup associated to a Markov process, see, e.g., [15].

1.3 Functional Equations for the Invariant Measure

We derive here the fundamental functional equations to be used throughout the book. It seems useful to present them in a more general situation, which means that for now we do not assume any *boundedness of the jumps*. To that end, consider the MC

$$X_n = \left(X_n^1, \dots, X_n^k\right), \quad n \geq 0, \quad X_n \in \mathbb{Z}_+^k,$$

with state space

$$\mathbb{Z}_+^k = \{z = (z_1, \dots, z_k) : z_i \geq 0, \quad i = 1, \dots, k\}$$

which is partitioned into a finite number of classes

$$\mathbb{Z}_+^k = \bigcup S_r,$$

so that the following assumption holds: *two states belong to the same class S_r if, and only if, the probability distributions P_r of the jumps from these states are the same*. The corresponding probability densities are the ${}_r p_{ij}$ introduced previously in Sect. 1.2.

Let us define the vector of complex variables

$$u = (u_1, \dots, u_k), \quad u_i \in \mathbb{C}, \quad |u_i| = 1, \quad i = 1, \dots, k,$$

and the jump generating functions

$$P_r(u) = E[u^{(X_{n+1} - X_n)} / X_n = z], \quad z \in S_r, \tag{1.3.1}$$

with the standard notation

$$u^z = \prod_{i=1}^{k} u_i^{z_i}.$$

Since by our assumptions $P_r(u)$ does not depend on z, Kolmogorov's equations take the form

$$E[u^{X_{n+1}}] = E\left[u^{X_n} u^{X_{n+1} - X_n}\right] = \sum_r E\left[u^{X_n} 1\!\!1_{\{X_n \in S_r\}}\right] P_r(u). \tag{1.3.2}$$

To account for the stationary case, we introduce the generating functions

$$\pi_r(u) = E\left[u^X 1\!\!1_{\{X \in S_r\}}\right] = \sum_{z \in S_r} \pi_z u^z, \tag{1.3.3}$$

where π_z denotes the stationary probability of being in state z. Taking the limit $n \to \infty$ in (1.3.2) and using (1.3.3), we get the basic equation

$$\sum_r [1 - P_r(u)]\,\pi_r(u) = 0. \tag{1.3.4}$$

Note that when the jumps are bounded from below, (1.3.4) is defined for all $u = (u_1, \ldots, u_k)$, $u_i \in \mathbb{C}$, $|u_i| \le 1$, $i = 1, \ldots, k$. Since we shall mainly consider the case $k = 2$, it will be convenient to rewrite (1.3.4) in a more explicit way, by means of the notation below, which will be ubiquitous throughout the book.

$$\begin{cases} \pi(x, y) &= \displaystyle\sum_{i,j=1}^{\infty} \pi_{ij} x^{i-1} y^{j-1}, \\ \pi(x) &= \displaystyle\sum_{i\ge 1} \pi_{i0} x^{i-1}, \\ \widetilde{\pi}(y) &= \displaystyle\sum_{j\ge 1} \pi_{0j} y^{j-1}, \\ Q(x, y) &= xy\Big(\displaystyle\sum_{i,j} p_{ij} x^i y^j - 1\Big), \\ q(x, y) &= x\Big(\displaystyle\sum_{i\ge -1}\sum_{j\ge 0} p'_{ij} x^i y^j - 1\Big) \\ \widetilde{q}(x, y) &= y\Big(\displaystyle\sum_{i\ge 0}\sum_{j\ge -1} p''_{ij} x^i y^j - 1\Big), \\ q_0(x, y) &= \Big(\displaystyle\sum_{i,j} p^0_{ij} x^i y^j - 1\Big), \end{cases} \tag{1.3.5}$$

where we have set $p^0_{ij} \stackrel{\text{def}}{=} p_{(0,0),(i,j)}$.

Now Eq. (1.3.4) takes the fundamental form

$$\boxed{-Q(x, y)\pi(x, y) = q(x, y)\pi(x) + \widetilde{q}(x, y)\widetilde{\pi}(y) + \pi_{00} q_0(x, y)}\,. \tag{1.3.6}$$

When property **P3** holds, it is immediate to check that the functions $Q, q, \widetilde{q}$ and q_0 introduced in (1.3.5) are polynomials in x, y. In addition, $\pi(x, y), \pi(x), \pi(y)$ have to be holomorphic in the region $|x|, |y| < 1$. Thus, the analysis of the invariant measure of the random walk amounts to solving the functional Eq. (1.3.6), in agreement with the next theorem.

Theorem 1.3.1 *For the irreducible aperiodic random walk to be ergodic, it is necessary and sufficient that there exist $\pi(x, y), \pi(x), \pi(y)$ holomorphic in $|x|, |y| < 1$, and a constant π_{00}, satisfying the fundamental Eq. (1.3.6) together with the ℓ_1-condition*

$$\sum_{i,j=0}^{\infty} \left|\pi_{ij}\right| < \infty. \tag{1.3.7}$$

In this case these functions are unique. ■

Theorem 1.3.1 proceeds directly from the material given in Definition 1.1.2, asserting existence and uniqueness of a finite invariant measure for irreducible ergodic Markov chains. We shall look for solutions of (1.3.6) from the following point of view:

Find functions $\pi(x, y)$, $\pi(x)$, $\widetilde{\pi}(y)$, *satisfying* (1.3.6), *holomorphic in* $\mathcal{D} \times \mathcal{D}$ *and continuous in* $\overline{\mathcal{D}} \times \overline{\mathcal{D}}$, *where*

$$\mathcal{D} \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z| < 1\} \quad \text{and} \quad \overline{\mathcal{D}} \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z| \leq 1\}.$$

The main idea consists in working on the variety $Q(x, y) = 0$, $(x, y) \in \overline{\mathcal{D}} \times \overline{\mathcal{D}}$ and the content of this book shows, by means of various approaches, that this is sufficient to obtain all the aforementioned functions.

Remark 1.3.2 *A priori*, finding a solution $\pi(x, y)$, holomorphic in $\mathcal{D} \times \mathcal{D}$ and continuous in $\overline{\mathcal{D}} \times \overline{\mathcal{D}}$, does not imply the ℓ_1-condition (1.3.7), as it emerges from the theory of functions of several complex variables (see for instance [10]). Furthermore, supposing that the system is not ergodic, we will see that a solution of (1.3.6) exists, holomorphic in $\mathcal{D}_a \times \mathcal{D}_a$, with $a < 1$, where $\mathcal{D}_a$ is the disc $\mathcal{D}_a \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z| < a\}$.

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