Chapter 2
Betti's Theorem

This chapter is devoted to the formulation of influence functions (see Fig. 2.1) with *Betti's theorem*.

2.1 Basics

The work done by one load on the displacement due to a second load is equal to the work done by the second load on the displacement due to the first

$$W_{1,2} = W_{2,1}\,. \tag{2.1}$$

This is *Betti's theorem*. It is based on Green's second identity

$$\mathscr{B}(w, \hat{w}) = \mathscr{G}(w, \hat{w}) - \mathscr{G}(\hat{w}, w) = 0 - 0 = 0\,. \tag{2.2}$$

In the case of a beam, this is the expression

$$\begin{aligned}\mathscr{B}(w_1, w_2) = &\underbrace{\int_0^l EI\, w_1^{IV}\, w_2\, dx + [V_1\, w_2 - M_1\, w_2{}']_0^l}_{W_{1,2}} \\ &- \underbrace{[V_2\, w_1 - M_2\, w_1']_0^l - \int_0^l w_1\, EI\, w_2^{IV}\, dx}_{W_{2,1}} = 0\,.\end{aligned} \tag{2.3}$$

The original version of this chapter was revised: See the "Chapter Note" section at the end of this chapter for details. The erratum to this chapter is available at 10.1007/978-3-319-51222-8_9.

F. Hartmann and P. Jahn, *Statics and Influence Functions—From a Modern Perspective*, DOI 10.1007/978-3-319-51222-8_2

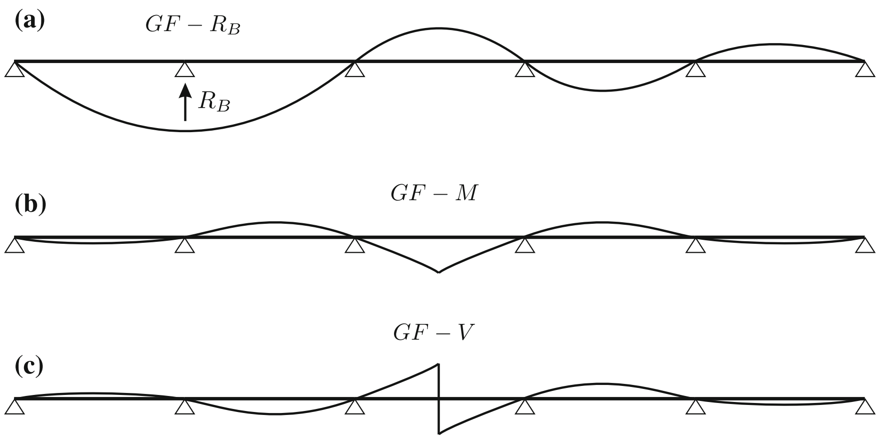


Fig. 2.1 Continuous beam and typical influence functions (Green's functions)

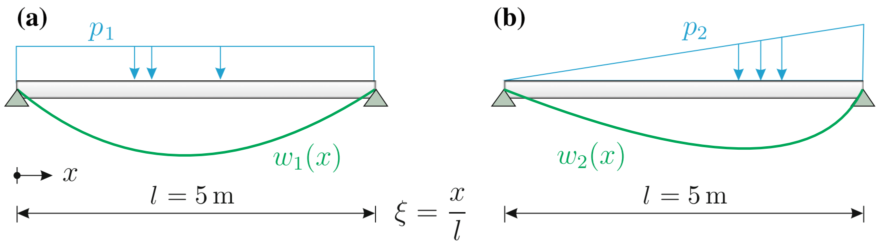


Fig. 2.2 Betti's theorem

This is the result when we start with the work integral

$$\int_0^l EI\, w_1^{IV}(x)\, w_2(x)\, dx \tag{2.4}$$

and shift via integration by parts the derivatives in four steps from w_1 to w_2.

Differential equations where this maneuver leads to the same equation as the original equation are called *self-adjoint*. All linear differential equations of even order are self-adjoint.

A differential equation such as $u' = p$ is called *skew adjoint*, because you have to multiply the result with (-1) to achieve congruence with the original expression

$$\int_0^l u'\,\hat{u}\, dx = [u\,\hat{u}]_0^l - \int_0^l u\,\hat{u}'\, dx\,. \tag{2.5}$$

Integration by parts is therefore, if you will, a 'skew-symmetric' operation.

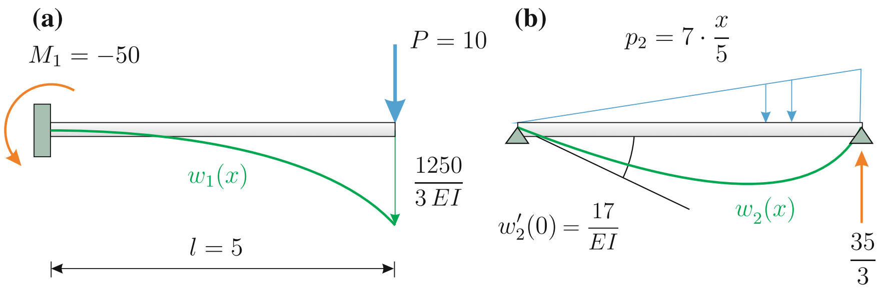


Fig. 2.3 Application of *Betti's theorem* to two systems with different support conditions

Example

For a first numerical test, we consider the two beams in Fig. 2.2, identical in length and in the supports, but each carrying a different load

$$p_1 = 10 \qquad w_1(x) = \frac{10 \cdot 5^4}{24\,EI}(\xi - 2\,\xi^3 + \xi^4) \qquad \xi = \frac{x}{l} \tag{2.6}$$

$$p_2 = 7\,\xi \qquad w_2(x) = \frac{7 \cdot 5^3\,x}{360\,EI}(7 - 10\,\xi^2 + 3\,\xi^4)\,, \tag{2.7}$$

and we find that the reciprocal exterior work is indeed the same

$$\begin{aligned} \mathscr{B}(w_1, w_2) &= \int_0^l p_1(x)\,w_2(x)\,dx - \int_0^l p_2(x)\,w_1(x)\,dx \\ &= \frac{1}{EI} \cdot 911.46 - \frac{1}{EI} \cdot 911.46 = 0\,. \end{aligned} \tag{2.8}$$

In the literature, it is required that each system is in equilibrium for *Betti's theorem* to hold true. This restriction is necessary because the authors do not start with Green's second identity (2.3), but instead, they jump directly to (2.8) and this equation is then only correct if the two curves satisfy the differential equations $EI\,w_1^{IV} = p_1$ and $EI\,w_2^{IV} = p_2$ and the boundary conditions as well.

Betti's theorem holds also true when the support conditions of the two beams are not of identical type, as in Fig. 2.3. Even in this situation, the work $W_{1,2}$

$$W_{1,2} = M_1(0)\,w_2'(0) = -50 \cdot \frac{17}{EI} = -850 \cdot \frac{1}{EI} \tag{2.9}$$

is the same as $W_{2,1}$

$$\begin{aligned} W_{2,1} &= \int_0^l p_2\,w_1(x)\,dx - B_2\,w_1(l) \\ &= \int_0^5 7 \cdot \frac{x}{5} \cdot \left(25\,x^2 - \frac{10}{6}\,x^3\right)\frac{1}{EI}\,dx - \frac{35}{3} \cdot \frac{1250}{3\,EI} \end{aligned}$$

$$= 4010.42 \cdot \frac{1}{EI} - 4861.1 \cdot \frac{1}{EI} = -850 \cdot \frac{1}{EI}\,. \tag{2.10}$$

2.2 Influence Functions for Displacement Terms

The influence function $G_0(y, x)$ for the displacement at a point x is the shape of the structure when a force $P = 1$ pushes the source point x in the direction of the displacement.

The reaction of the structure to such a point load is symmetric, $G_0(y, x) = G_0(x, y)$. Whether the force acts at the point x and we observe the displacement at a point y, or whether the force acts at y and we observe what happens at the point x, is the same (*Maxwell's theorem*).

For our purposes, it will be convenient to choose x as the source point and y as the location of the force.

A distributed load p can be resolved into a series of small point loads

$$dP(y) = p(y)\,dy \tag{2.11}$$

each of which lets the deflection at the source point grow by an amount

$$dw = G_0(y, x)\,dP(y) \tag{2.12}$$

and so the total deflection is the sum over all dw

$$w(x) = \int_0^l dw = \int_0^l G_0(y, x)\,p(y)\,dy\,. \tag{2.13}$$

If the load happens to be just one single force P acting at a point y, then this expression reduces of course to a simple multiplication

$$w(x) = G_0(y, x) \cdot P\,. \tag{2.14}$$

We write the influence functions with a capital G because in mathematics influence functions are called *Green's functions* and since the deflection $w(x)$ is a zero-order derivative, we add a subscript 0 to its influence function G.

Strictly speaking, we must distinguish between the *kernel* and the influence function. Green's function $G_0(y, x)$ is the kernel, and the integral (2.13) is the influence function, but often we call also the kernels influence functions—which they are if the load happens to be a point load.

The kernels for the higher-order derivatives of a beam

$$w'(x) = \frac{d}{dx}\,w(x) \qquad M(x) = -EI\,\frac{d^2}{dx^2}\,w(x) \qquad V(x) = -EI\,\frac{d^3}{dx^3}\,w(x)\,, \tag{2.15}$$

are

$$G_0(y,x) \qquad = \text{influence function for } w(x) \tag{2.16a}$$

$$G_1(y,x) = \frac{d}{dx}\,G_0(y,x) \qquad = \text{influence function for } w'(x) \tag{2.16b}$$

$$G_2(y,x) = -EI\,\frac{d^2}{dx^2}\,G_0(y,x) \quad = \text{influence function for } M(x) \tag{2.16c}$$

$$G_3(y,x) = -EI\,\frac{d^3}{dx^3}\,G_0(y,x) \quad = \text{influence function for } V(x)\,. \tag{2.16d}$$

This cascade-like pattern will also shine through in FE-analysis because in the FE-context the higher-order influence functions are finite difference approximations based on G_0^h, the FE-influence function for $w(x)$.

2.2.1 Derivation of an Influence Function

Technically speaking, the influence function (2.13) is obtained as follows: We apply at the source point x a single force $P = 1$ (see Fig. 2.4a), find the deflection curve $G_0(y,x)$, and formulate Green's second identity with the two functions $G_0(y,x)$ and $w(y)$.

This cannot be done in one step because the third derivative ($\simeq V$) of the influence function jumps at the source point. This means we integrate from the left end up to the source point x, step over the source point, and continue the integration beyond the source point

$$\mathscr{B}(G_0, w) = \mathscr{B}(G_0^L, w)_{(0,x)} + \mathscr{B}(G_0^R, w)_{(x,l)}\,. \tag{2.17}$$

At both ends of the beam, w and M are zero and so the result of this maneuver is

$$\begin{aligned}
\mathscr{B}(G_0, w) &= \mathscr{B}(G_0^L, w)_{(0,x)} + \mathscr{B}(G_0^R, w)_{(x,l)} \\
&= V_0^L(x)\,w(x) - M_0^L(x)\,w'(x) - \int_0^x G_0^L(y,x)\,p(y)\,dy \\
&\quad - V_0^R(x)\,w(x) + M_0^R(x)\,w'(x) - \int_x^l G_0^R(y,x)\,p(y)\,dy \\
&= \underbrace{(V_0^L(x) - V_0^R(x))}_{=1}\,w(x) - \underbrace{(M_0^L(x) - M_0^R(x))}_{=0}\,w'(x) \\
&\quad - \int_0^l G_0(y,x)\,p(y)\,dy \\
&= 1 \cdot w(x) - \int_0^l G_0(y,x)\,p(y)\,dy = 0
\end{aligned} \tag{2.18}$$

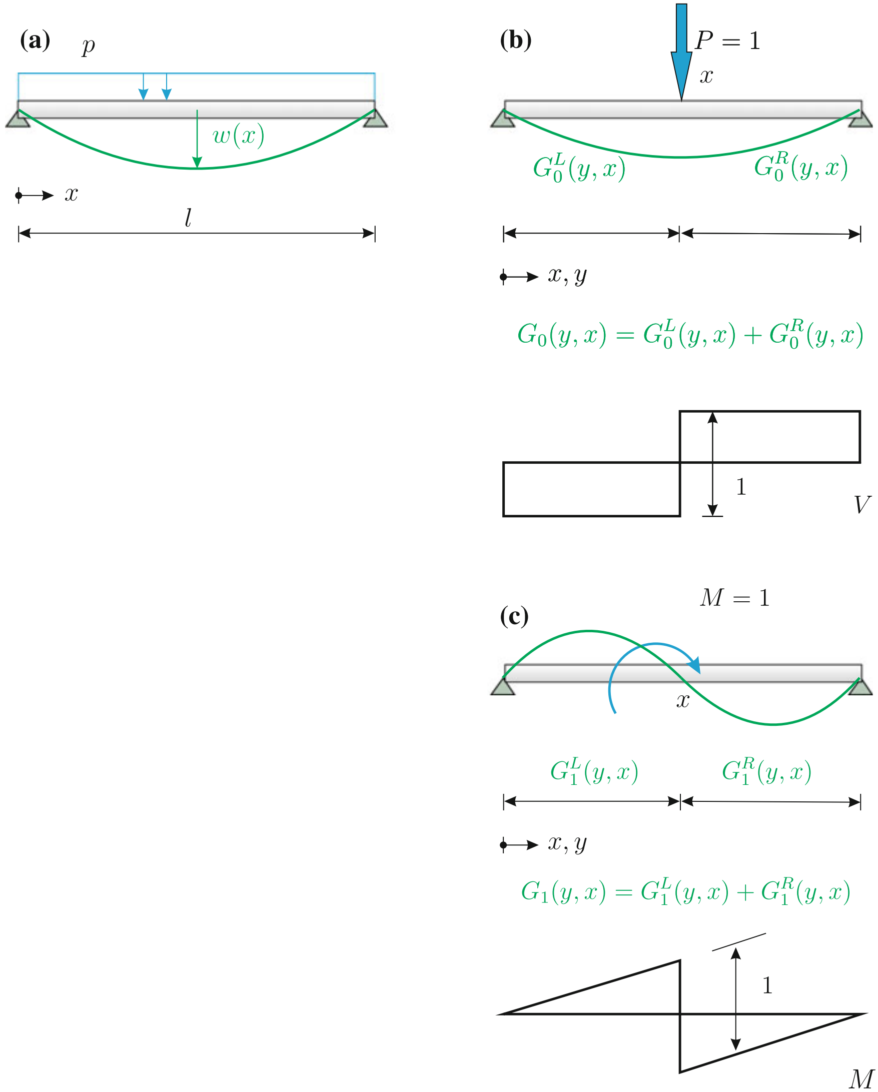


Fig. 2.4 Beam and application of *Betti's theorem*

or

$$1 \cdot w(x) = \int_0^l G_0(y,x)\, p(y)\, dy\,, \tag{2.19}$$

which is the influence function for $w(x)$.

2.2.1.1 Influence Function for $w'(x)$

The influence function $G_1(y, x)$ for the slope $w'(x)$ is obtained in the same manner. In this case, we apply a single moment $M = 1$ at the source point x, split the deflection curve into two parts G_1^L and G_1^R, and formulate with both parts *Betti's theorem* (see Fig. 2.4c),

$$\mathscr{B}(G_1, w) = \mathscr{B}(G_1^L, w)_{(0,x)} + \mathscr{B}(G_1^R, w)_{(x,l)} = 0 + 0\,. \tag{2.20}$$

The discontinuity of the bending moment at the source point makes that at the interface, the term

$$(M_L(x, x) - M_R(x, x))\, w'(x) = 1 \cdot w'(x) \tag{2.21}$$

is left over and placing this term on one side alone the result is the influence function

$$1 \cdot w'(x) = \int_0^l G_1(y, x)\, p(y)\, dy\,. \tag{2.22}$$

2.2.1.2 Influence Function for the Longitudinal Displacement $u(x)$

Green's second identity (*Betti's theorem*) of the differential equation $-EA\, u''(x) = p(x)$ is

$$\begin{aligned}\mathscr{B}(u, \hat{u}) = \int_0^l -EA\, u''(x)\, \hat{u}(x)\, dx + [N\, \hat{u}]_0^l \\ - [u\hat{N}]_0^l - \int_0^l u(x)\, (-EA\, \hat{u}''(x))\, dx = 0\,,\end{aligned} \tag{2.23}$$

and so when we apply a single horizontal force $P = 1$ at a source point x (see Fig. 2.5), we obtain

$$\begin{aligned}\mathscr{B}(G_0, u) &= \mathscr{B}(G_0^L, u)_{(0,x)} + \mathscr{B}(G_0^R, u)_{(x,l)} = 0 + 0 \\ &= N_0^L(x)\, u(x) - \int_0^x G_0^L(y, x)\, p(y)\, dy \\ &- N_0^R(x)\, u(x) - \int_x^l G_0^R(y, x)\, p(y)\, dy \\ &= \underbrace{(N_0^L(x) - N_0^R(x))}_{=1}\, u(x) - \int_0^l G_0(y, x)\, p(y)\, dy\end{aligned} \tag{2.24}$$

or

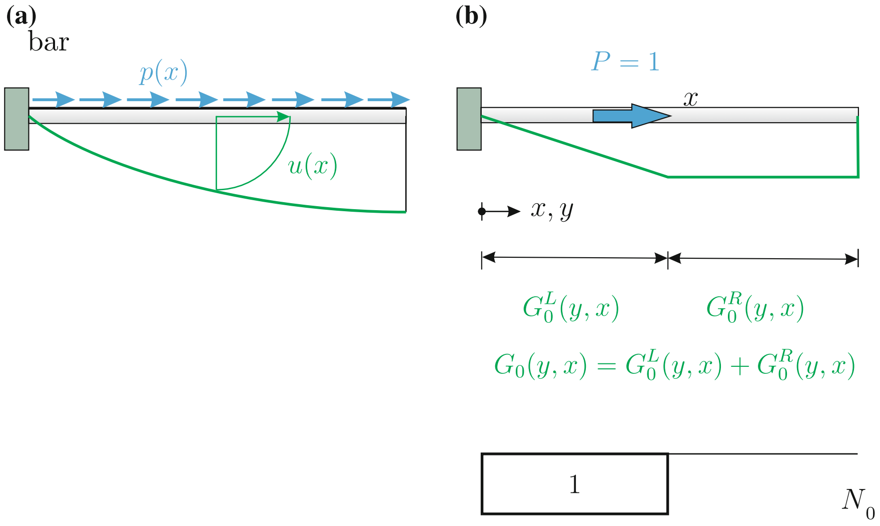


Fig. 2.5 Bar and application of *Betti's theorem*

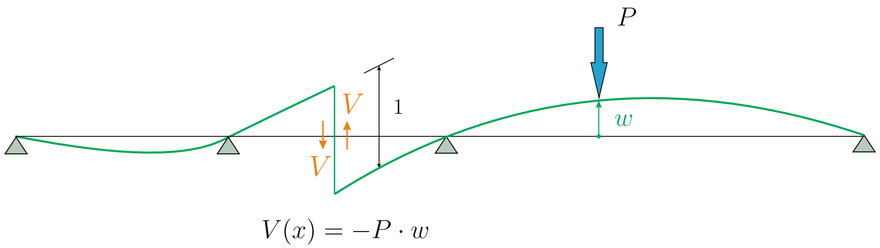


Fig. 2.6 An influence function can be compared to a seesaw

$$1 \cdot u(x) = \int_0^l G_0(y, x)\, p(y)\, dy\,. \tag{2.25}$$

2.3 Influence Functions for Force Terms

Influence functions for force terms require two steps:

- Installation of a hinge corresponding to the force term
- A spread of the hinge by one unit (see Fig. 2.6).

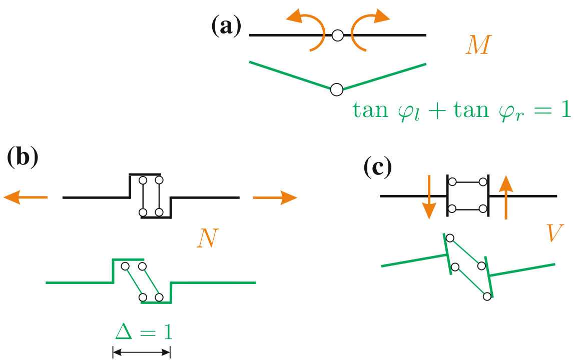


Fig. 2.7 Hinges make it possible to generate influence functions, **a** M-hinge, **b** N-hinge, and **c** V-hinge

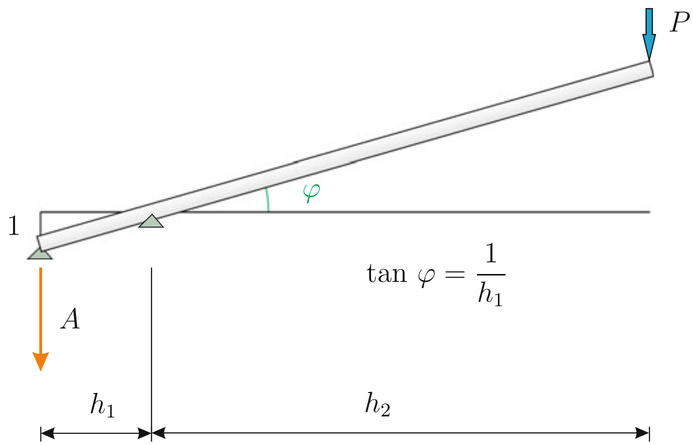


Fig. 2.8 Archimedes' lever

Which influence function requires which hinge is explained by the *Müller–Breslau principle* (see Fig. 2.7).

When we install a hinge in a statically determinate structure, the structure becomes a mechanism, and then, no effort is needed to spread the hinge; the resulting deformation is composed of straight-line segments.

In statically indeterminate structures, the same maneuver requires some effort. To size the necessary action, we first apply on each side of the hinge a force or moment $X = \pm 1$ and we watch by how much the hinge spreads and then we scale the pair $\pm X$ so that the spread is exactly one unit.

Archimedes knew that when he removes the left support of a cantilever as in Fig. 2.8 and pushes the lever one unit of displacement down that the work done by the support reaction A and the force P must be zero

$$W_{1,2} = A \cdot 1 - P\,h_2\,\tan\,\varphi = 0\,, \tag{2.26}$$

and so he found for A the value

$$A = P\,h_2\,\tan\,\varphi = P \cdot \frac{h_2}{h_1}\,. \tag{2.27}$$

All influence functions for force terms resemble such 'seesaws' (see Fig. 2.6).

Statics is not static but 'kinematic.'

To each internal force belongs a certain hinge and the movements of the structure induced by a unit spread of the hinge, the echo of the structure so to speak, determines the magnitude of the internal force.

Remark 2.1 There is no need to be specific about the scale of the unit displacement. Whether we push the left end down by 1 cm or 1 m is irrelevant because at the end what counts is the *ratio* of the displacement of the load, here $h_2 \cdot \tan\,\varphi$, and the triggering displacement 1

$$1 \cdot A = P\,h_2\,\tan\,\varphi \qquad \Rightarrow \qquad A = P\,\frac{h_2\,\tan\,\varphi}{1} \tag{2.28}$$

and a quotient such as a/b is the same in any unit.

2.3.1 Influence Function for $N(x)$

The influence function $G_1(y, x)$ for a normal force $N(x)$ in a bar is the reaction of the bar to a dislocation of one unit length at the source point x (see Fig. 2.9b),

$$G_1(x_-) - G_1(x_+) = 1\,. \tag{2.29}$$

It consists of two parts, G_1^L and G_1^R, and correspondingly, we must formulate *Betti's theorem* in two parts as well

$$\begin{aligned} \mathscr{B}(G_1, u) &= \mathscr{B}(G_1^L, u)_{(0,x)} + \mathscr{B}(G_1^R, u)_{(x,l)} = 0 + 0 \\ &= [\ldots]_0^x - \int_0^x G_1^L(y, x)\,p(y)\,dy + [\ldots]_x^l - \int_x^l G_1^R(y, x)\,p(y)\,dy\,. \end{aligned} \tag{2.30}$$

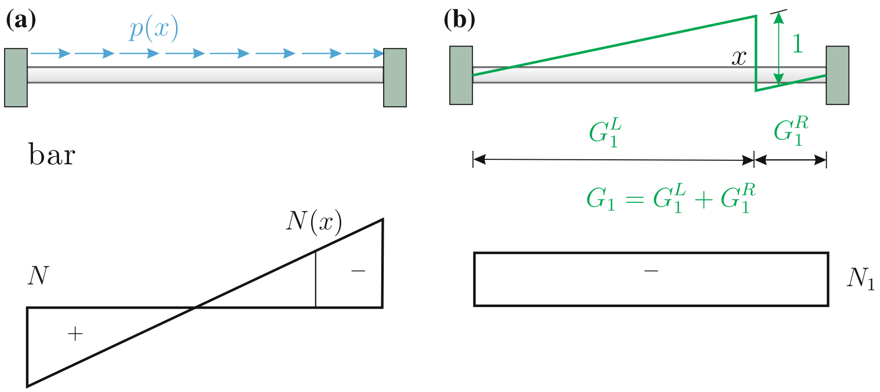


Fig. 2.9 Computing the influence function for the normal force $N(x)$

Because of the zero boundary values

$$u(0) = u(l) = 0 \qquad G_1(0, x) = G_1(l, x) = 0 \tag{2.31}$$

no work is done at the two ends and we only need to focus on the work done at the source point x.

The normal force N_1 belonging to G_1 is continuous at the source point, because the slope of G_1 is the same on both sides (see Fig. 2.9b), and since also $u(x)$ is continuous at x, the work of the two normal forces $\pm N_1(x)$ is zero

$$\underbrace{N_1(x_-)\, u(x)}_{left} - \underbrace{N_1(x_+)\, u(x)}_{right} = (N_1(x_-) - N_1(x_+))\, u(x) = 0\,, \tag{2.32}$$

and consequently, *Betti's theorem* reduces to

$$\begin{aligned} \mathscr{B}(G_1, u) &= N(x)(G_1(x_-) - G_1(x_+)) - \int_0^l G_1(y, x)\, p(y)\, dy \\ &= N(x) \cdot 1 - \int_0^l G_1(y, x)\, p(y)\, dy = 0 \end{aligned} \tag{2.33}$$

or

$$1 \cdot N(x) = \int_0^l G_1(y, x)\, p(y)\, dy\,. \tag{2.34}$$

2.3.2 Influence Function for $M(x)$

To set the stage for the influence function of $M(x)$, we install a hinge at the source point x, spread it by one unit

$$G_2'(x_-) - G_2'(x_+) = 1\,, \tag{2.35}$$

and formulate with both parts, G_2^L and G_2^R, *Betti's theorem*

$$\mathscr{B}(G_2, w) = \mathscr{B}(G_2^L, w)_{(0,x)} + \mathscr{B}(G_2^L, w)_{(x,l)} = 0 + 0\,. \tag{2.36}$$

The work done at the beam ends is zero and of the work done at the source point x, the interface between the two halves, only the term

$$G_2'(x_-)\,M(x) - G_2'(x_+)\,M(x) = 1 \cdot M(x) \tag{2.37}$$

is left over and so it follows that

$$1 \cdot M(x) = \int_0^l G_2(y, x)\,p(y)\,dy\,. \tag{2.38}$$

Influence Function for *V(x)*

In this case, we install a shear hinge (see Fig. 2.10e), which we spread by one unit, (-1),

$$G_3(x_-) - G_3(x_+) = 1\,, \tag{2.39}$$

and we formulate with both parts G_3^L and G_3^R *Betti's theorem*

$$\mathscr{B}(G_3, w) = \mathscr{B}(G_3^L, w)_{(0,x)} + \mathscr{B}(G_3^R, w)_{(x,l)} = 0 + 0\,. \tag{2.40}$$

At the interface, at the source point x, all terms cancel up to

$$G_3(x_-)\,V(x) - G_3(x_+)\,V(x) = 1 \cdot V(x) \tag{2.41}$$

and so the equation $W_{1,2} = 0$ provides the result

$$1 \cdot V(x) = \int_0^l G_3(y, x)\,p(y)\,dy\,. \tag{2.42}$$

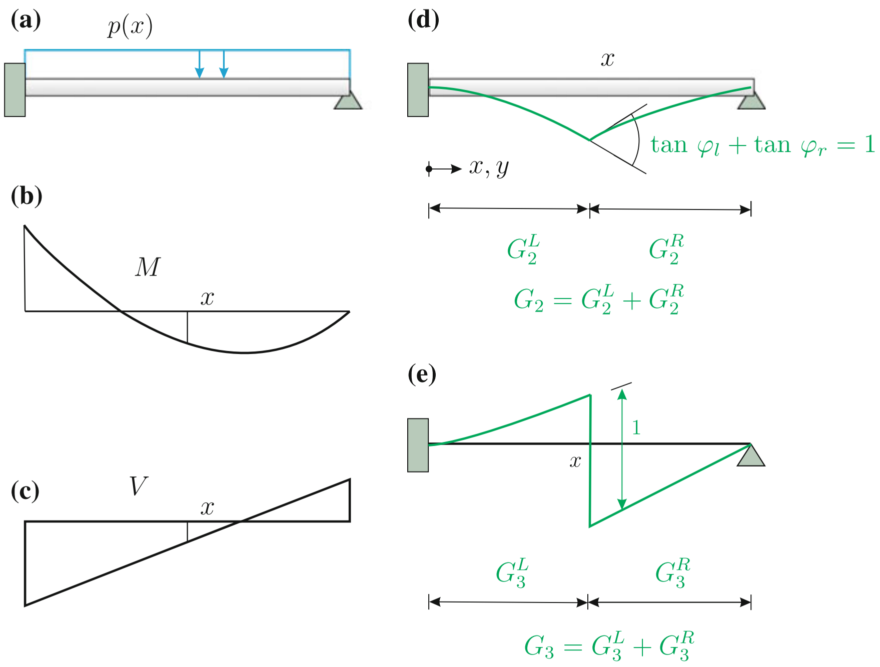


Fig. 2.10 Influence functions for $M(x)$ and $V(x)$

2.3.3 Influence Functions for Higher-Order Derivatives

Theoretically, you only need to know the influence function for the zero-order displacement and the influence functions for the higher-order derivatives are simply obtained by differentiating this influence function with regard to the source point x (see Fig. 2.11).

$$u(x) = \int_0^l G_0(y, x)\, p(y)\, dy \qquad \rightarrow \qquad N(x) = \int_0^l EA \frac{d}{dx} G_0(y, x)\, p(y)\, dy\,, \tag{2.43}$$

but this counts as differentiating an integral with respect to a parameter and this requires some care.

2.3.4 Moments Differentiate Influence Functions

The moment that acts at the support y in Fig. 2.12 can be replaced by two single forces $P = \pm M/\Delta y$ a distance Δy apart. So if $G_0(y, l)$ is the influence function for

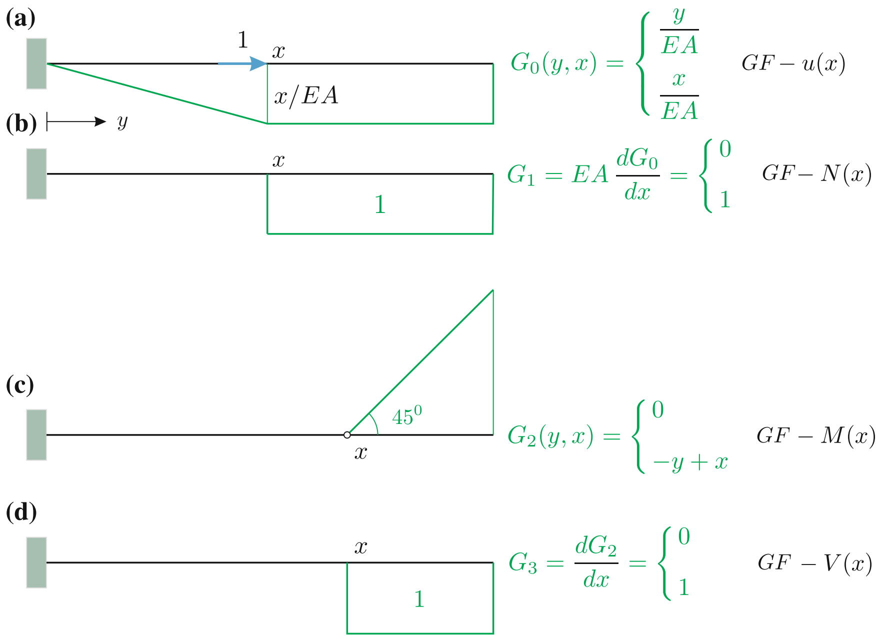


Fig. 2.11 Derivation of influence functions by differentiation

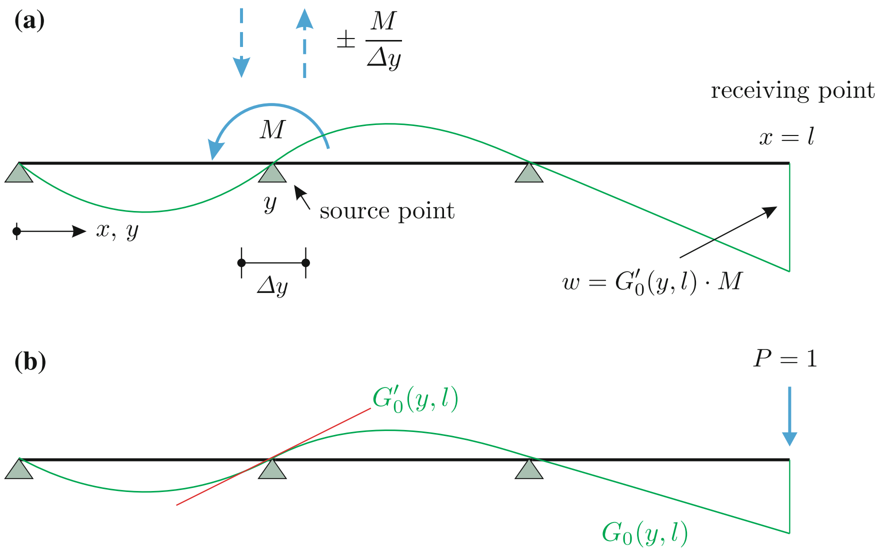


Fig. 2.12 The influence of a moment is proportional to the slope of the influence function at the place of M, **a** deflection caused by the moment and **b** influence function for deflection of the beam end

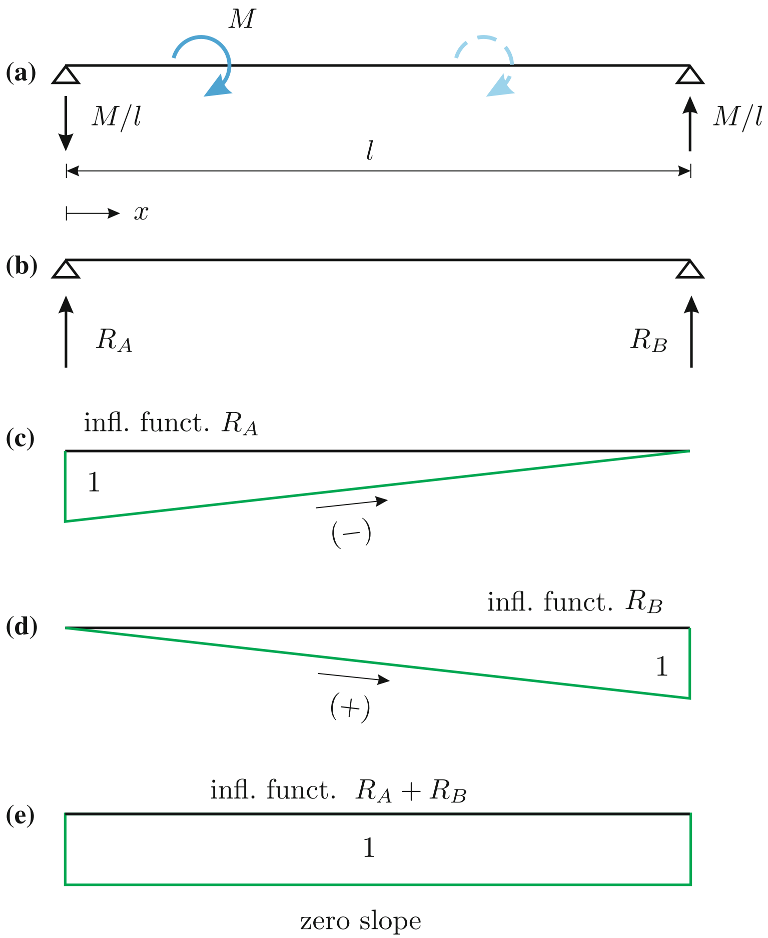


Fig. 2.13 Hinged beam, **a** loading and support reactions, **b** positive support reactions, **c** influence function for R_A with negative slope $-1/l$, **d** influence function for R_B with positive slope $1/l$, and **e** influence function for $R_A + R_B$ with zero slope

the deflection at the end $x = l$ of the cantilever beam, then

$$w(l) = \lim_{\Delta y \to 0} (G_0(y + 0.5\, \Delta y, l) - G_0(y - 0.5\, \Delta y, l)) \cdot \frac{M}{\Delta y} = \frac{d}{dy} G_0(y, l) \cdot M \tag{2.44}$$

is the deflection due to the moment M and this effectively means that the moment differentiates the influence function, and its effect is proportional to the slope of the influence function at the load point y.

The beam in Fig. 2.13 carries a single moment M. The influence functions for the two support reactions R_A and R_B are linear functions with a constant slope ($\pm 1/l$), and therefore, the moment can act at any point of the beam—it makes no difference. And because the influence function for the sum of the two support reactions has zero slope (see Fig. 2.13e), the sum $R_A + R_B = 0$ is always zero.

2.4 Statically Determinate Structures

When we install a hinge in a statically determinate structure, the structure (or parts thereof) becomes a mechanism which means that the influence functions for force terms are piecewise linear in such a structure (see Figs. 2.14 and 2.15).

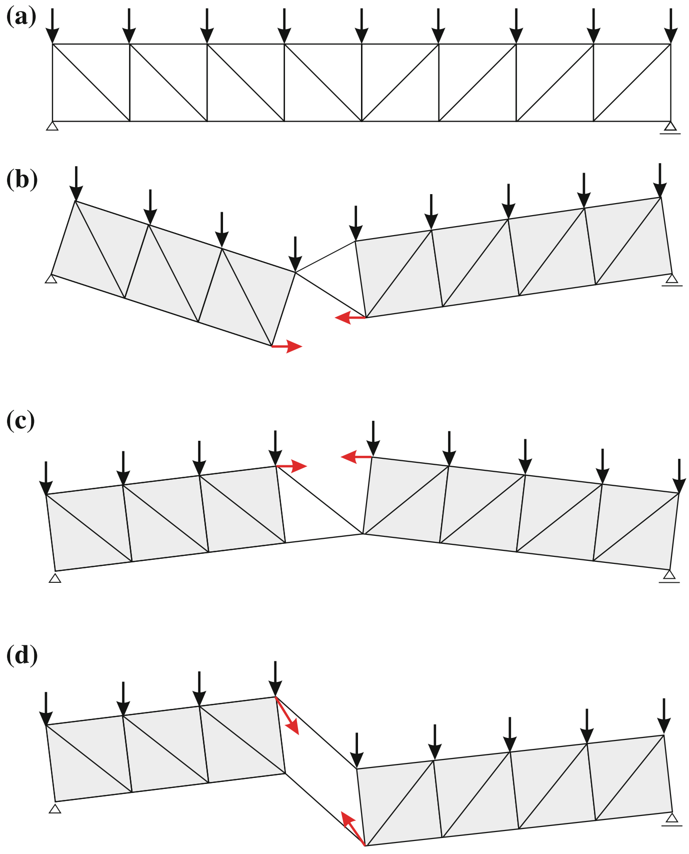
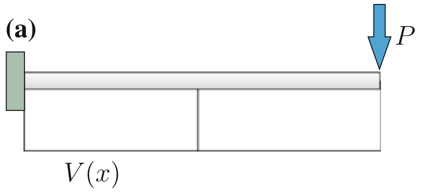
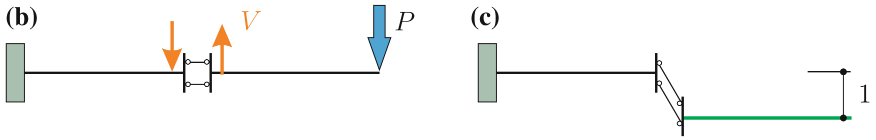
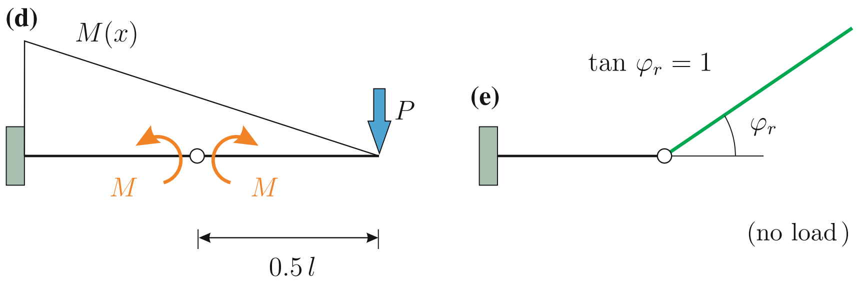


Fig. 2.14 Truss and influence functions for lower and upper chord and a diagonal chord



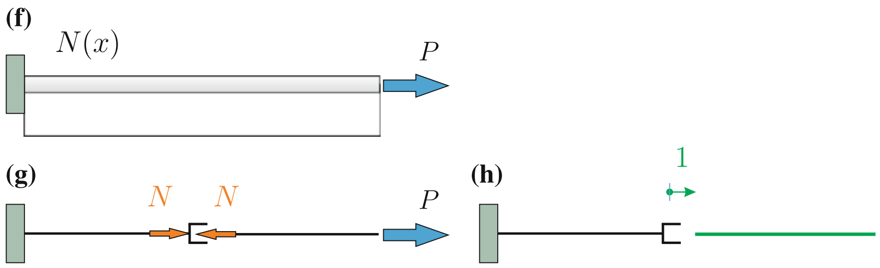


$$W_{1,2} = -V \cdot 1 + P \cdot 1 = W_{2,1} = 0 \qquad \text{(no load)}$$



$$W_{1,2} = -M \cdot \tan\varphi_r - P \cdot \tan\varphi_r \cdot 0.5\,l = W_{2,1} = 0$$

$$M = -P \cdot 0.5\,l$$



$$W_{1,2} = -N \cdot 1 + P \cdot 1 = W_{2,1} = 0 \qquad \text{(no load)}$$

Fig. 2.15 Each internal action can be made 'visible' by an appropriate hinge

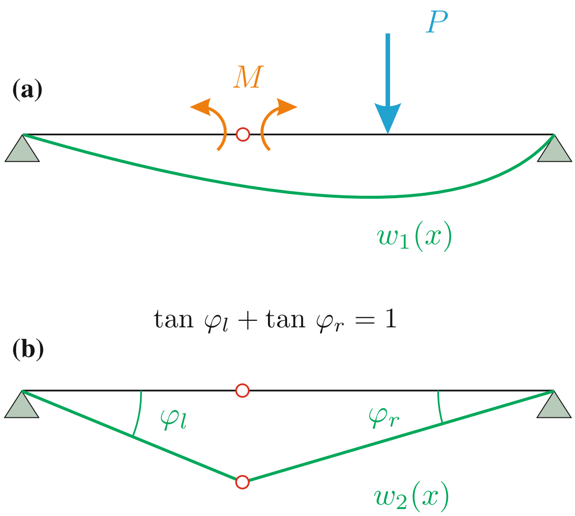


Fig. 2.16 *Betti's theorem*—influence function for a moment, **a** beam and loading and **b** zero loads but a unit spread $\tan\varphi_l + \tan\varphi_r = 1$ of the hinge

How these functions are obtained may be illustrated by detailing the single steps which lead to the influence function for the bending moment of the beam in Fig. 2.16. In the first step, we installed a hinge. This hinge interrupted the flow of the bending moments so that the action of the previous interior moment $M(x)$ had to be compensated by an equivalent pair $\pm M(x)$ of exterior moments. In a second picture, we duplicated the configuration and removed the load so that the structure was load free, and we spread the hinge by one unit, $\tan\varphi_l + \tan\varphi_r = 1$. No forces were required to perform this spread. It was a 'noiseless' operation.

According to *Betti's theorem*, we should then have

$$\mathscr{B}(w_1, w_2) = W_{1,2} - W_{2,1} = 0\,, \tag{2.45}$$

and so we started to count. The work $W_{2,1} = 0$ was zero because the zero forces on beam #2 did not contribute any work on acting through the deflection $w_1(x)$ of beam #1, and consequently, also $W_{1,2}$ had to be zero

$$W_{1,2} = -M_L\,\tan\,\varphi_l + M_R\,\tan\,\varphi_r + P\,w_2(x) = -M \cdot 1 + P\,w_2(x) = 0 \tag{2.46}$$

or

$$1 \cdot M = P\,w_2(x)\,, \tag{2.47}$$

and so $w_2(x)$ had to be the influence function for $M(x)$.

Remark 2.2 The internal forces in a statically determinate structure do not change if the stiffness changes. An 'academic' proof of this well-known fact is the following: If

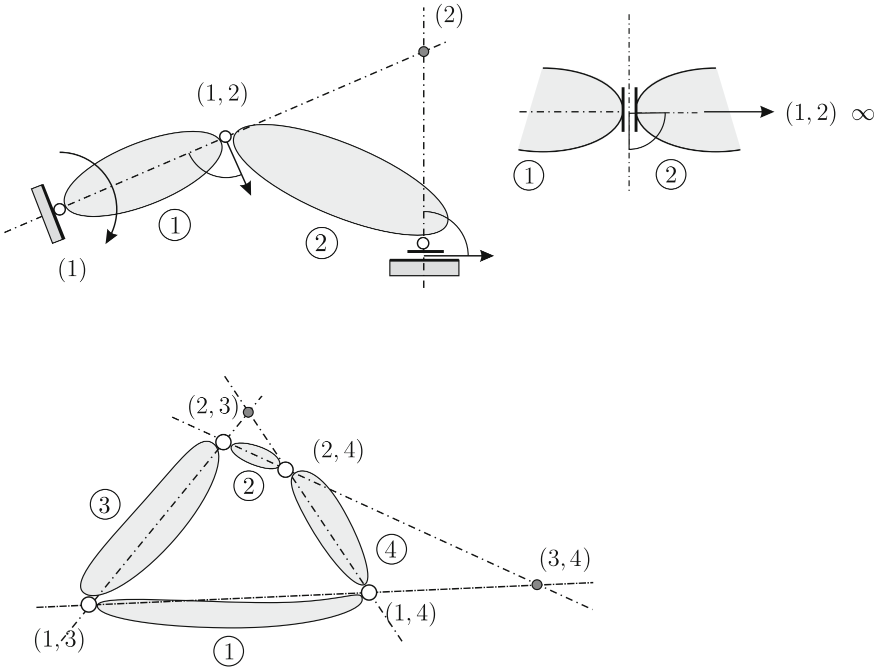


Fig. 2.17 Rules for pole-plans

EI or *EA* changes in a beam, then the associated forces f^+ balance (see Chap. 5), and because equilibrium forces are orthogonal to all rigid body modes, that is, kinematic chains (the influence functions), $N, M,$ *and* V do not change.

2.4.1 Pole-Plans

To draw the displaced shape of a mechanism requires the knowledge of the *instant centers of rotation* or poles of the single segments. We call the plan of all these poles *pole-plan*.

The following rules apply to the construction of pole-plans (see Fig. 2.17):

1. Each fixed hinge is the main pole of the attached segment.
2. Each hinge forms the secondary pole of all the segments connected with the segment.
3. The line orthogonal to a roller support forms the location of the main pole of the segment attached to the support.

4. The secondary pole of two segments which are connected by a sliding hinge (normal force hinge or shear force hinge) lies on the line orthogonal to the sliding movement at infinity.
5. The main poles of two segments and their common secondary pole lie on a straight line: $(i) - (i, j) - (j)$, e.g., $(1) - (1, 2) - (2)$.
6. The secondary poles (i, j), (j, k), *and* (i, k) of three segments I, J, *and* K lie on one line: $(i, j) - (j, k) - (i, k)$, e.g., $(1, 3) - (1, 4) - (3, 4)$.

2.4.2 Construction of Pole-Plans and the Shape of the Displaced Figure

It is best to start at hinged supports because these are (see rule #1) the main pole of the segment attached to the hinge. Hinges in between segments are the secondary poles of the segments connected at the hinge (see rule #2).

All the other poles can be found with the help of the so-called *trace lines*. A trace line or locus is the straight line on which the pole must lie according to rules #3 to #6.

- The intersection of two trace lines of the same pole marks the location of the pole.
- If two trace lines for the same pole run parallel, then they intersect at infinity.
- If the main pole of a segment lies at infinity, then the segment can only move parallel to itself; its rotation is zero.
- If the secondary pole of two segments lies at infinity, then the two segments rotate about their main poles by equal amounts. If two segments are parallel to each other, then they remain parallel to each other after the movement started.

These rules allow to determine the effects when an M-, V-, or N-hinge is spread by one unit. The spread is done in such a way that the movement is opposite to the direction of the forces acting at the hinge. That part of the deformed structure which has the same direction as the traveling load is the influence function. We also say the influence function is the 'projection' of the displaced shape of the structure into the direction of the traveling load.

2.4.3 How to Determine the Magnitude of Rotations

Eventually, we need the rotation angles of the single segments of a mechanism and the relation between these angles. This information is easily obtained when you understand the connection between the rotation of two segments (i) and (k), which are connected via a secondary pole (i, k). Figure 2.18 illustrates how we proceed.

The rotation φ_i of the segment i is given, and we want to find the rotation of the segment k as a function of φ_i.

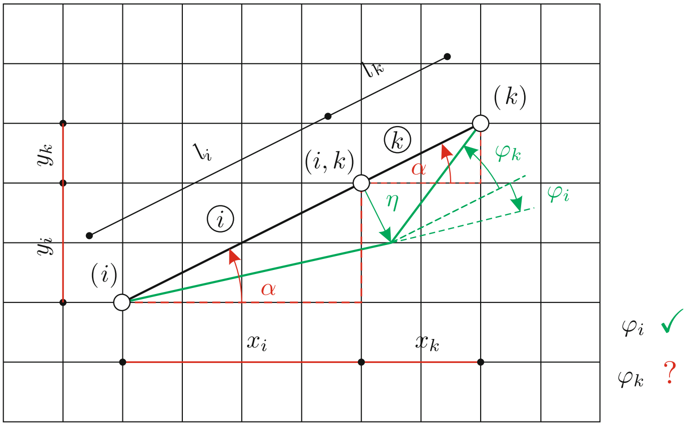


Fig. 2.18 The relative rotations between two segments

Let x_i denote the horizontal distance of the main pole (i) from the secondary pole (i, k) and x_k the horizontal distance of the main pole (k) from the secondary pole (i, k). Correspondingly, do y_i and y_k denote the vertical distances, and l_i and l_k are the distances of the main poles (i) and (k) from the secondary pole (i, k).

So we have

$$\tan\ \varphi_i = \frac{\eta}{l_i} \qquad \tan\ \varphi_k = \frac{\eta}{l_k} \tag{2.48}$$

and also

$$\boxed{l_i \cdot \tan\ \varphi_i = l_k \cdot \tan\ \varphi_k} \tag{2.49}$$

The main poles of the two segments and their secondary poles lie—as always—on a straight line which, here, is inclined by an angle α, and hence, it follows

$$\sin\ \alpha = \frac{y_i}{l_i} = \frac{y_k}{l_k} \tag{2.50}$$

or solved for the lengths

$$l_i = \frac{y_i}{\sin\ \alpha} \qquad l_k = \frac{y_k}{\sin\ \alpha} \tag{2.51}$$

and with (2.49) follows as well

$$\boxed{y_i \cdot \tan\ \varphi_i = y_k \cdot \tan\ \varphi_k} \tag{2.52}$$

By the same procedure does

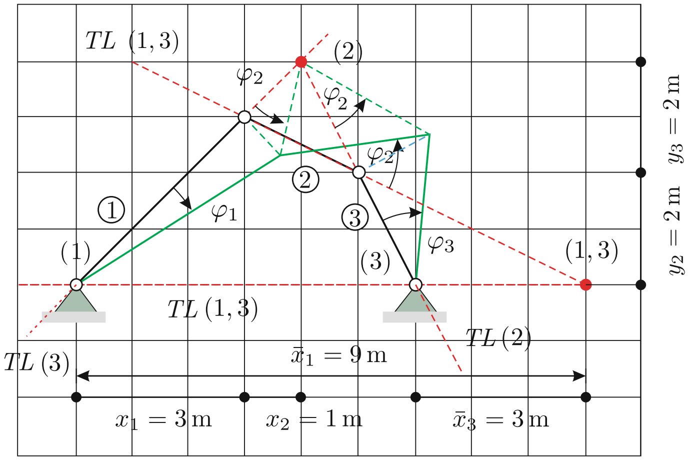


Fig. 2.19 Computing of the displacements of segments 2 and 3, respectively, when the rotation of segment 1 is given, *TL* = trace line

$$\cos\alpha = \frac{x_i}{l_i} = \frac{x_k}{l_k} \tag{2.53}$$

imply that

$$\boxed{x_i \cdot \tan\varphi_i = x_k \cdot \tan\varphi_k} \tag{2.54}$$

A system composed of three segments (see Fig. 2.19) may serve to illustrate the technique. We are given the angle φ_1 having the value $\tan\varphi_1 = 1/3$. We want to find the other two angles φ_2 and φ_3. Because of (2.54), the rotation of segment #2 is

$$\tan\varphi_2 = \frac{x_1 \cdot \tan\varphi_1}{x_2} = \frac{3 \cdot 1/3}{1} = 1\,. \tag{2.55}$$

The rotation of segment #3 follows from (2.52)

$$\tan\varphi_3 = \frac{y_2 \cdot \tan\varphi_2}{y_3} = \frac{2 \cdot 1}{2} = 1 \tag{2.56}$$

or with (2.54) by considering the rotation of segment #1

$$\tan\varphi_3 = \frac{\bar{x}_1 \cdot \tan\varphi_1}{\bar{x}_3} = \frac{9 \cdot 1/3}{3} = 1\,. \tag{2.57}$$

So the rotations of the single segments are

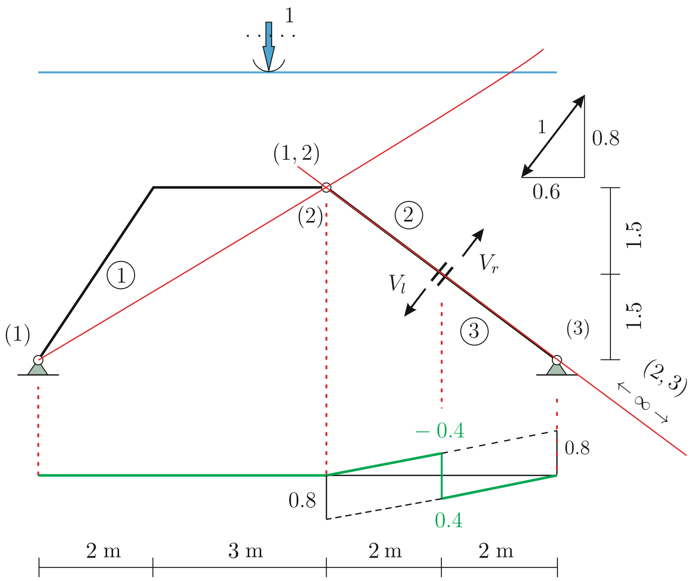


Fig. 2.20 Traveling load and influence function for a shear force, (1), (2), and (3) are the main poles of the segments 1, 2, and 3 and (1, 2), (2, 3) are the secondary poles of segments 1 and 2 and segments 2 and 3, respectively

$$\begin{bmatrix} \tan\varphi_1 \\ \tan\varphi_2 \\ \tan\varphi_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} \cdot \tan\varphi_1\,. \tag{2.58}$$

2.4.4 Influence Function for a Shear Force (Fig. 2.20)

In Fig. 2.20, we determine the influence function for the shear force in the slanted beam. The secondary pole (2,3) lies on any trace line which is orthogonal to the sliding shear hinge and therefore also on the trace line that passes through the main pole (3). This straight line is at the same time the trace line of the main pole (2), as well as the connecting line of (1) and (1,2). The main pole (2) lies at the intersection of these two lines.

We start by spreading the shear hinge by one unit. In vertical direction which is also the direction of the traveling load, this means a relative displacement of the segments at the hinge and also above the main poles of 0.8 m.

The relative displacement atop the main poles can be used to trace the movement of the segments in the projection. Since the main poles are fixed, the displacement of segment #2 atop the main pole (3) amounts to 0.8 m and vice versa amounts the displacement of segment #3 atop the main pole (2) to the same value 0.8 m.

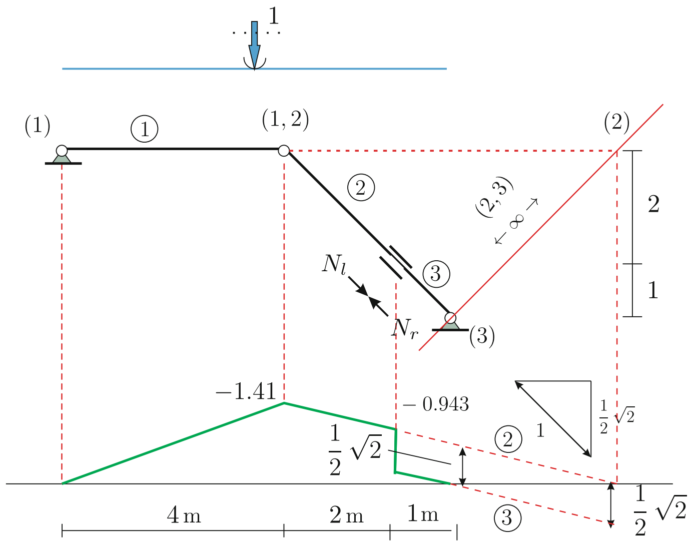


Fig. 2.21 Traveling load and influence function for a normal force

The vertical movements determine the influence function.

2.4.5 *Influence Function for a Normal Force (Fig. 2.21)*

In Fig. 2.21, we construct the influence function for the normal force in the slanted beam. We know that the trace line of the secondary pole (2,3) is found at infinity when we follow any straight line which is orthogonal to the gliding movement of the N-hinge. Such a line passes also through the main pole (3), and therefore, this line is also the trace line for the main pole (2). The line that passes through the main pole (1) and the secondary pole (1,2) is also a trace line for the main pole (2). The intersection of both lines is therefore the main pole (2).

To construct the influence function, we spread the N-hinge by one unit. In vertical direction, in the direction of the traveling load, this corresponds to a relative displacement of the segments at the hinge and atop the main poles of $0.5 \cdot \sqrt{2}$ m. The relative displacement atop the main poles can be used to construct the displacements in the projection. Since the main poles are fixed, the displacement of segment #2 atop the main pole (3) is $0.5 \cdot \sqrt{2}$ m and vice versa the displacement of segment 3 atop the main pole (2) is $0.5 \cdot \sqrt{2}$ m.

The vertical displacements form the influence function.

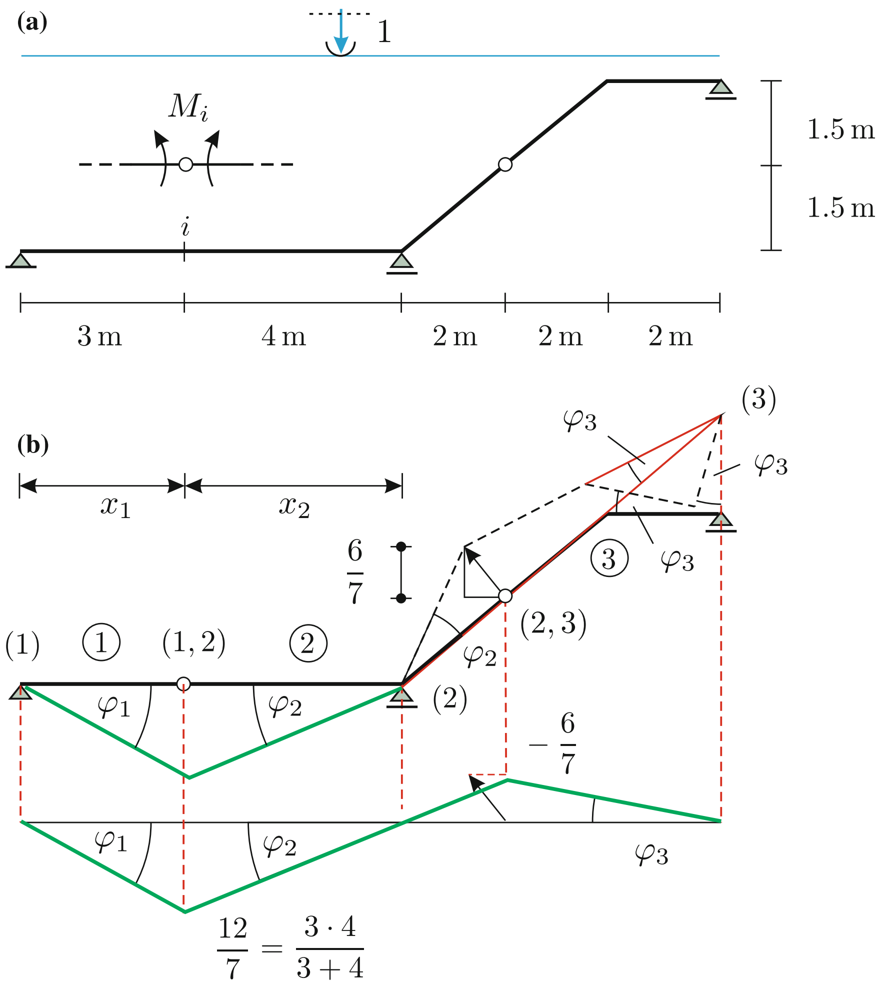


Fig. 2.22 Traveling load and influence function for the moment at point i

2.4.6 Influence Function for a Moment (Fig. 2.22)

In Fig. 2.22, we construct the influence function for the bending moment at the point i. After we have installed a hinge at this point, the previously statically determinate structure can now rotate freely about the hinge. We want a rotation where $\tan \varphi_r + \tan \varphi_l = 1$, and this requires that the vertical displacement at the source point i has the value

$$\eta = \frac{x_1 \cdot x_2}{x_1 + x_2} = \frac{3 \cdot 4}{3+4} = \frac{12}{7} \,. \tag{2.59}$$

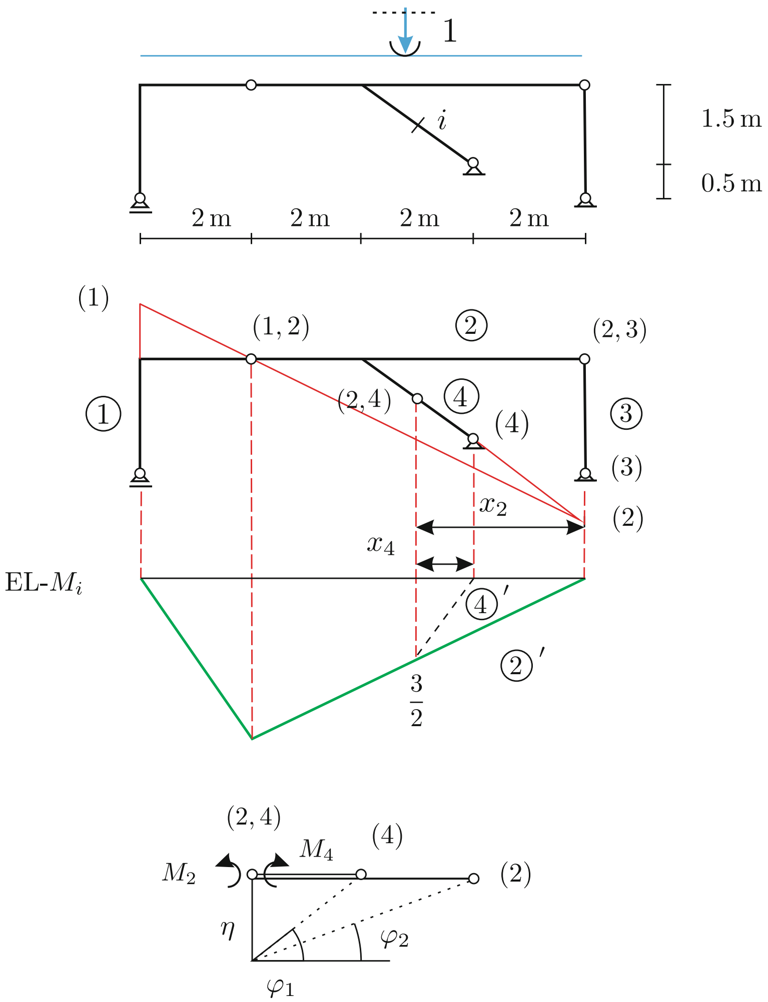


Fig. 2.23 Traveling load and influence function for the moment at point *i*

In the final step, the displaced structure must be projected into the direction of the traveling load. The rotations of the three segments in the direction of the projections agree with the rotations of the structure. At the main poles, the displacement is zero and therewith also in the projection. With these hints, it is easy to draw the displacements which the traveling load experiences that is the influence function.

2.4.7 Influence Function for a Moment (Fig. 2.23)

In Fig. 2.23, we construct the influence function for the bending moment at a point i, and to this end, we install a hinge at this point whereby the structure becomes a mechanism. Unlike the previous example in Fig. 2.22, both main poles of the two segments lie on the right side of the hinge, of this secondary pole (2,4).

We want $\tan\,\varphi_4 + \tan\,\varphi_2 = 1$. This is guaranteed if the relative rotation between the two segments #2 and #4 equals 1, and this is the case if the vertical displacement η at the source point i has the value

$$\eta = \frac{x_2 \cdot x_4}{x_2 - x_4} = \frac{3 \cdot 1}{3 - 1} = \frac{3}{2}\,. \tag{2.60}$$

2.4.8 Influence Function for a Shear Force (Fig. 2.24)

In Fig. 2.24, we construct the influence function for the shear force at a point i and we start by installing a shear hinge at this point. As in the previous example in Fig. 2.23, both main poles of the two segments which are joined at this hinge lie to the right of the source point i.

To construct the influence function, we spread the shear hinge by one unit apart. The work is negative if segment #2 and segment #3 rotate anticlockwise about the main poles. These rotations are also seen in the projection.

In vertical direction, the direction of the traveling load, the relative displacement of the segments atop the main poles is 1 m. The relative displacement atop the main poles can be used to construct the displacements in the projection, because due to the fact that the main poles are fixed, the displacement of segment #2 atop main pole (3) is 1 m and vice versa. The parts of the load-carrying floor in the projection belong to the influence function.

2.4.9 Influence Function for Two Support Reactions (Fig. 2.25)

In Fig. 2.25, we construct the influence functions for two support reactions, R_A and R_B. Because the support reactions follow the direction of the projection, we only need to push the points A and B, respectively, by one unit in negative direction, that is, downward.

The influence function for support reaction R_A is easily found because the displaced support lies on segment #1 which is also the path of the traveling load. In the case of the support reaction R_B, the displaced point belongs to segments #3 and #4

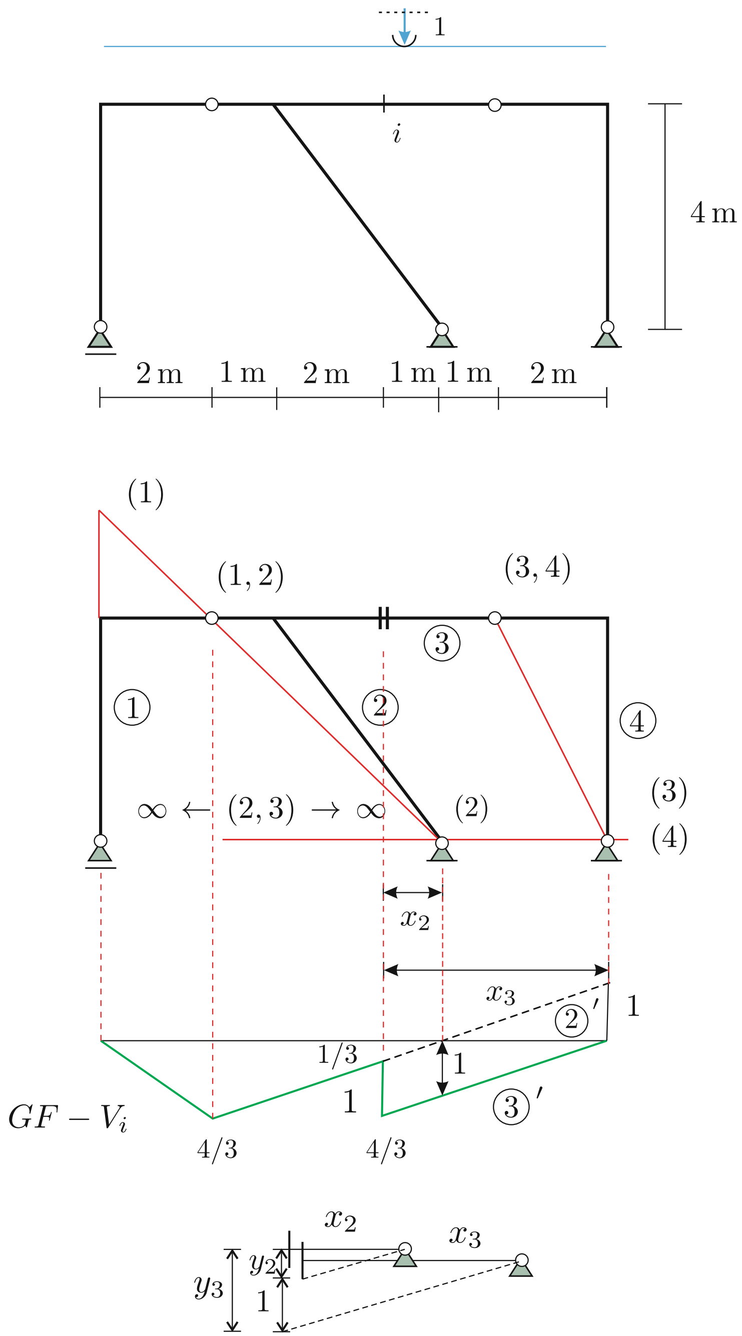


Fig. 2.24 Traveling load and influence function for the shear force at point *i*

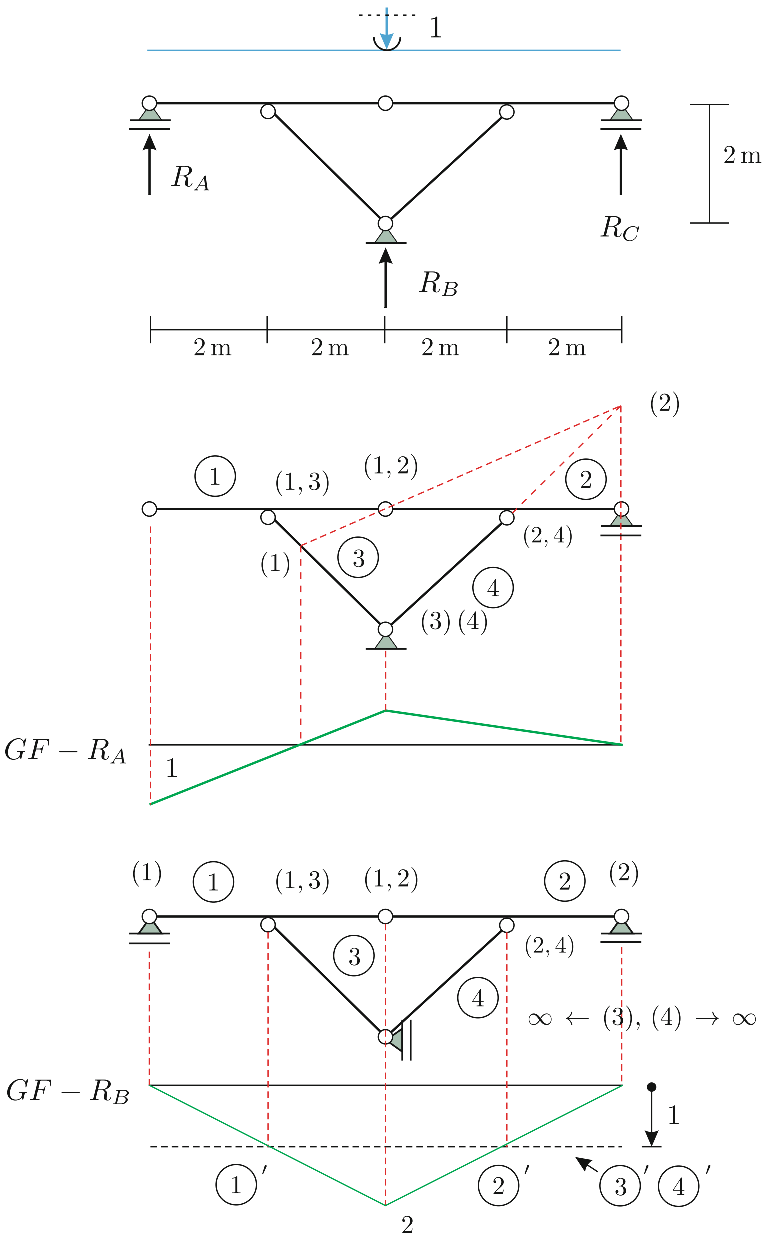


Fig. 2.25 Two influence functions for support reactions

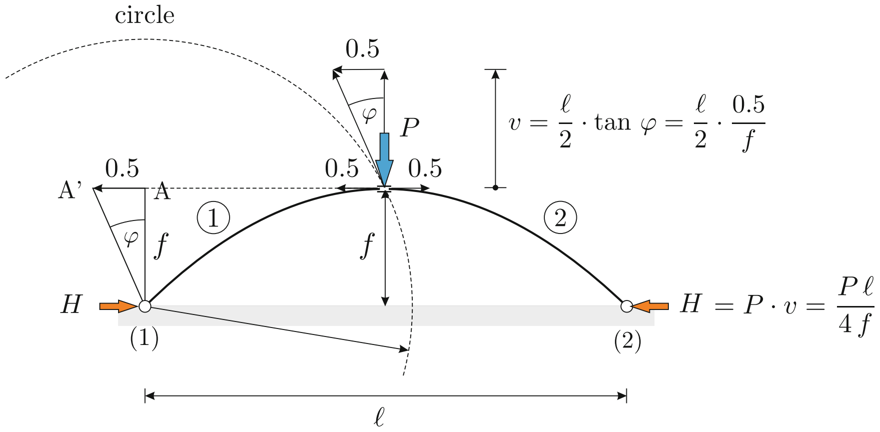


Fig. 2.26 Spread of the top of the circular arch by 1 m. The kinematics of the arch played a decisive role in the history of structures [1]

and the main poles of these two segments lie at infinity. Therefore, these segments move in parallel one unit down.

On these two segments, we find the secondary poles (1,3) and (2,4) of the segments #1 and #2. The connection of (1) with (1,3) and of (2) with (2,4) in the projection determines the segments #1 and #2 and therewith the influence function.

2.4.10 Abutment Reaction (Fig. 2.26)

The horizontal support reaction H of an arch (see Fig. 2.26)

$$H = \frac{M}{f} \tag{2.61}$$

is equal to the bending moment M in a beam of length l which carries the same load, divided by the height f of the arch.

In principle, a two-hinged arch is statically indeterminate, but if the load is symmetric, then $V = 0$ is zero on the symmetry axis and so we can install a shear hinge at the top without disturbing the equilibrium of the arch. With this additional shear hinge, the arch becomes statically determinate and we can apply our technique to derive Eq. (2.61).

In Fig. 2.26, the normal force $N = H$ at the top of the arch gets exposed by installing an N-hinge. When the hinge is spread by one unit, the left side of the arch rotates about the main pole #1 and all points which have the same height f above the main pole swing to the left by an amount of 0.5 m, as point A in Fig. 2.26. This allows to compute tan $\varphi = 0.5/f$ and all other points which have the horizontal distance

$\ell/2$ from the main pole swing by $v = \ell/2 \cdot \tan\,\varphi$ upward, and hence, it follows that $H = P \cdot v = P \cdot l/(4 \cdot f) = M/f$. In the case of a uniform load p, we have

$$H = 2 \cdot p \int_0^{\ell/2} x \cdot \tan\,\varphi\, dx = \frac{p \cdot \ell^2}{8 \cdot f} = \frac{M}{f}\,. \tag{2.62}$$

2.5 Statically Indeterminate Structures

In a statically indeterminate structure, some effort is needed to spread a hinge and this means that the system #2, the structure with the unit spread of the hinge, is not load free, but we will see that even then the work $W_{2,1} = 0$ is zero and *Betti's theorem* reduces as before to the equation

$$\mathscr{B}(w_1, w_2) = W_{1,2} = 0\,. \tag{2.63}$$

Let us exemplify this with the beam in Fig. 2.27. To generate the influence function for the bending moment at midspan, we install a hinge at this point and we spread the hinge by one unit apart

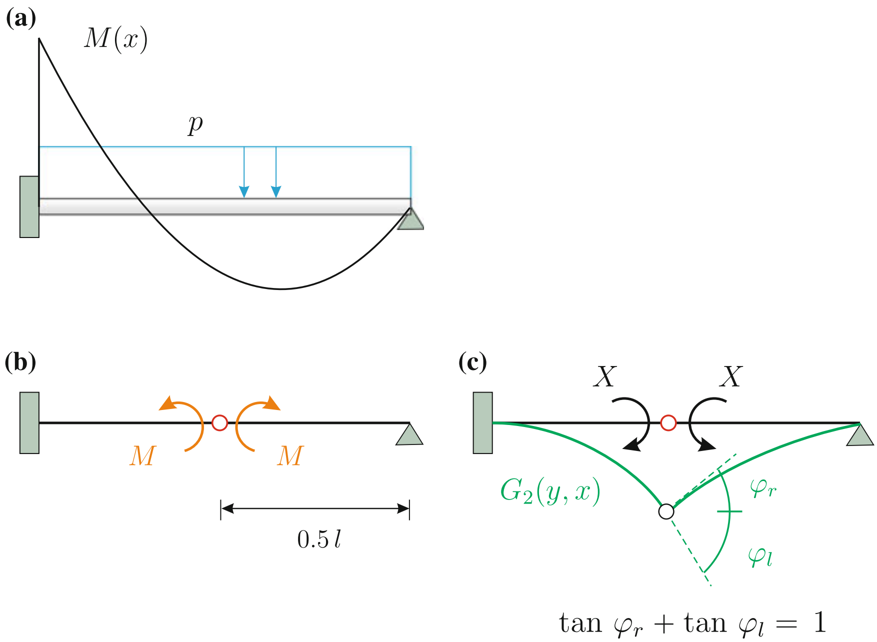


Fig. 2.27 Hinges reveal the internal forces

$$\tan\,\varphi_l + \tan\,\varphi_r = 1\,. \tag{2.64}$$

Then, we integrate from 0 to x, step over the point, and continue up to the end

$$\mathscr{B}(G_2, w) = \mathscr{B}(G_2, w)_{(0,x)} + \mathscr{B}(G_2, w)_{(x,l)}\,. \tag{2.65}$$

The work done at the beam ends is zero, and at the transition point x, only two terms are left over

$$\underbrace{-M(x)\,w'(x_-)}_{\text{on the left}} + \underbrace{M(x)\,w'(x_+)}_{\text{on the right}} = -M(x)\cdot(\tan\,\varphi_l + \tan\,\varphi_r)\,, \tag{2.66}$$

and hence, we have

$$\mathscr{B}(G_2, w) = -M(x)\cdot\underbrace{(\tan\,\varphi_l + \tan\,\varphi_r)}_{=1} + \int_0^l G_2(y, x)\,p(y)\,dy = 0 \tag{2.67}$$

or

$$M(x) = \int_0^l G_2(y, x)\,p(y)\,dy\,. \tag{2.68}$$

It remains to find the shape of $G_2(y, x)$. The spread of the hinge requires the action of two moments $\pm X$ which we determine as follows: First, we apply two moments $\pm X = \pm 1$, and then, we scale these moments so that the resulting spread is exactly one.

The work

$$W_{1,2} = \int_0^l G_2(y, x)\,p(y)\,dy - M\cdot(\tan\,\varphi_l + \tan\,\varphi_r)\,, \tag{2.69}$$

done by the forces of system #2 on acting through the deflections of system #1 is zero

$$W_{2,1} = -X\cdot w'(x) + X\cdot w'(x) = (-X + X)\,w'(x) = 0\,. \tag{2.70}$$

This is no accident, and it is always zero. The two moments have opposite signs but the same magnitude, and because the rotation $w'(x)$ is continuous at the source point, the total work is zero.

In the case of influence functions for force terms *Betti's theorem* reduces to the equation

$$W_{1,2} = 0 \qquad (2.71)$$

because $W_{2,1} = 0$ is always zero.

2.6 Influence Functions for Support Reactions

A support reaction becomes an external force when we install a corresponding hinge, that is, when we remove the constraint in the direction of the support. The influence function is then the result of a corresponding movement of the support. If the foundation is rigid, then only the structure can move, and so the base of the structure (set free by the hinge) must travel the full distance of -1 units (see Fig. 2.28) (Fig. 2.29).

If the foundation is elastic, then a local analysis must clarify how much the soil contributes to the movement (-1) and how much the structure.

The force X required to spread the support by 1 m is

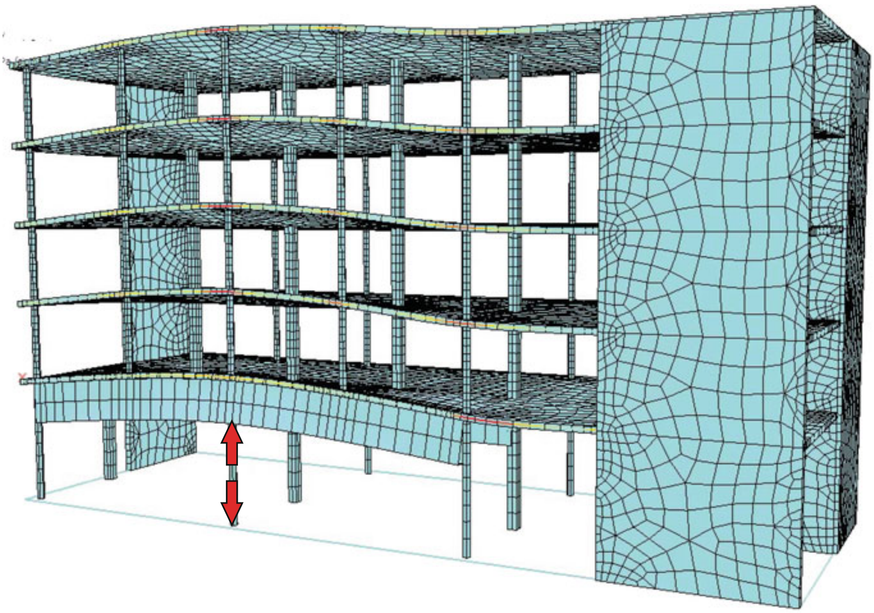


Fig. 2.28 Influence function for the normal force in a pier

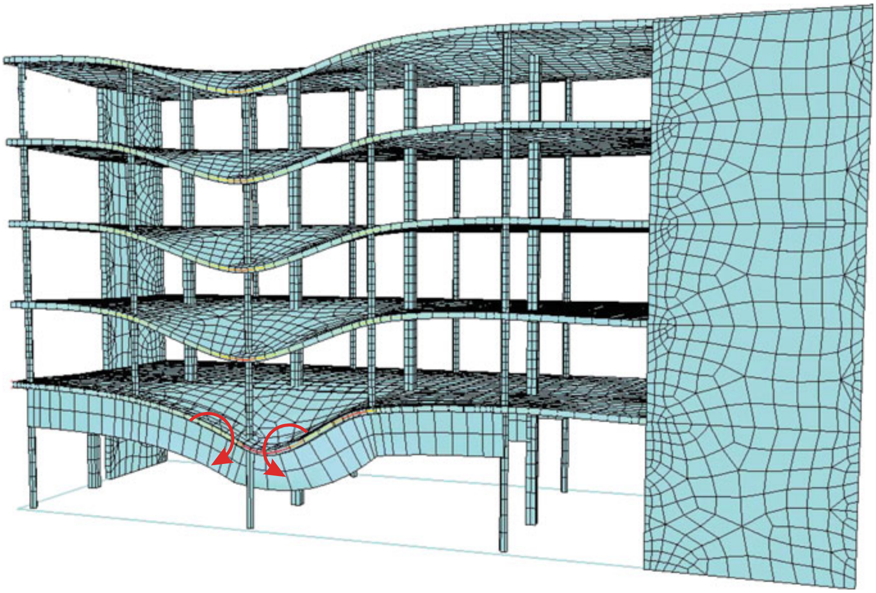


Fig. 2.29 Influence function for a moment in a girder

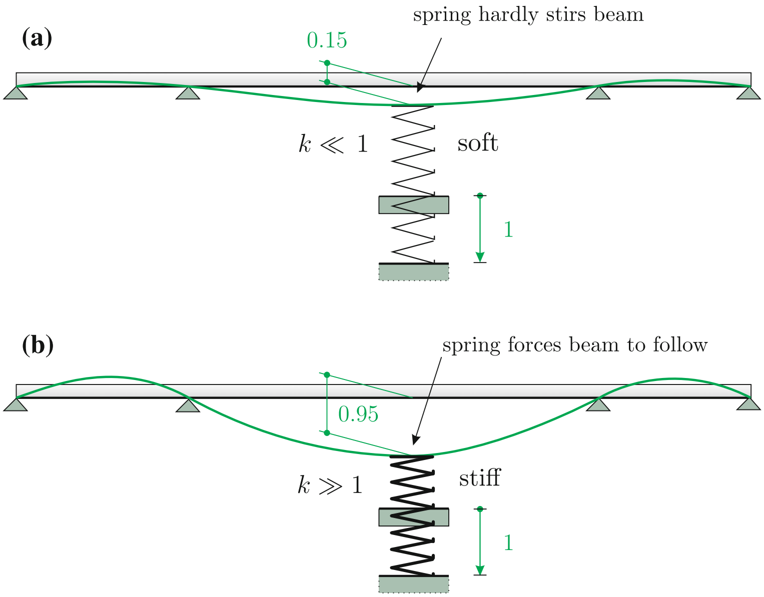


Fig. 2.30 A soft spring absorbs much of the movement of the soil, while a stiff spring transfers most of it to the beam

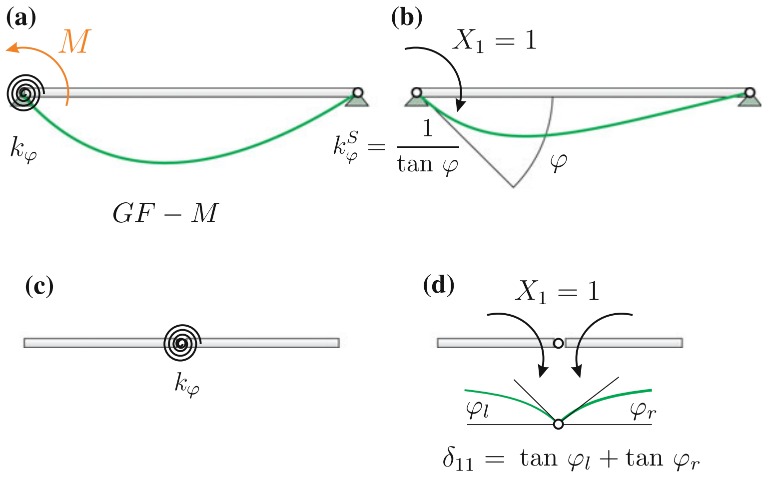


Fig. 2.31 Rotational springs coupled with beam ends

$$X = \frac{k_S\, k_B}{k_S + k_B} \qquad k_B = k\text{-soil} \qquad k_S = k\text{-structure}\,. \tag{2.72}$$

To find the stiffness k_S of the structure, we dismantle the connection between the structure and the foundation and we watch by how much a force $X = 1$ pulls the structure down. The reciprocal of the displacement is k_S.

Piers basically act like springs (see Fig. 2.30). If the pier is soft, then a unit spread -1 traveled by the base of the pier is consumed to a large part by the pier and the beam hardly notices the spread at the base of the pier, which means that the influence function has a very low profile in the beam. Vice versa if the pier is quite stiff, then nearly the full signal will reach the beam, and consequently, the pier will carry a large part of the moving load.

In rotational springs as in Fig. 2.31, the moment is coupled to the rotation φ of the spring via the rotational stiffness k_φ

$$M = k_\varphi\, \tan\,\varphi\,. \tag{2.73}$$

To determine the joined rotational stiffness k_Φ of the spring and the beam end(s), we let two moments $X = \pm 1$ that act on the two sides of the hinge determine the rotational stiffness of the structure (S)

$$k_\varphi^S = \frac{1}{\tan\,\varphi_l + \tan\,\varphi_r}\,, \tag{2.74}$$

and so the stiffness k_Φ is

$$\frac{1}{k_\Phi} = \frac{1}{k_\varphi} + \frac{1}{k_S}\,. \tag{2.75}$$

2.7 The Zeros of the Shear Force

A side effect of the equation $EI\,w^{IV} = -V'(x) = p$ is that in a continuous beam, the distance between the zeros of the shear force is related to the magnitude of the support reaction (see Fig. 2.32).

Let the zeros be the two points x_a and x_b and let the support be located in between at a point x_s. Integrating from x_a up to the support, we find

$$\int_{x_a}^{x_s} V'(x)\,dx = V_L(x_s) - V(x_a) = V_L(x_s) - 0 = V_L(x_s) \tag{2.76}$$

and on the right side of the support, we find as well

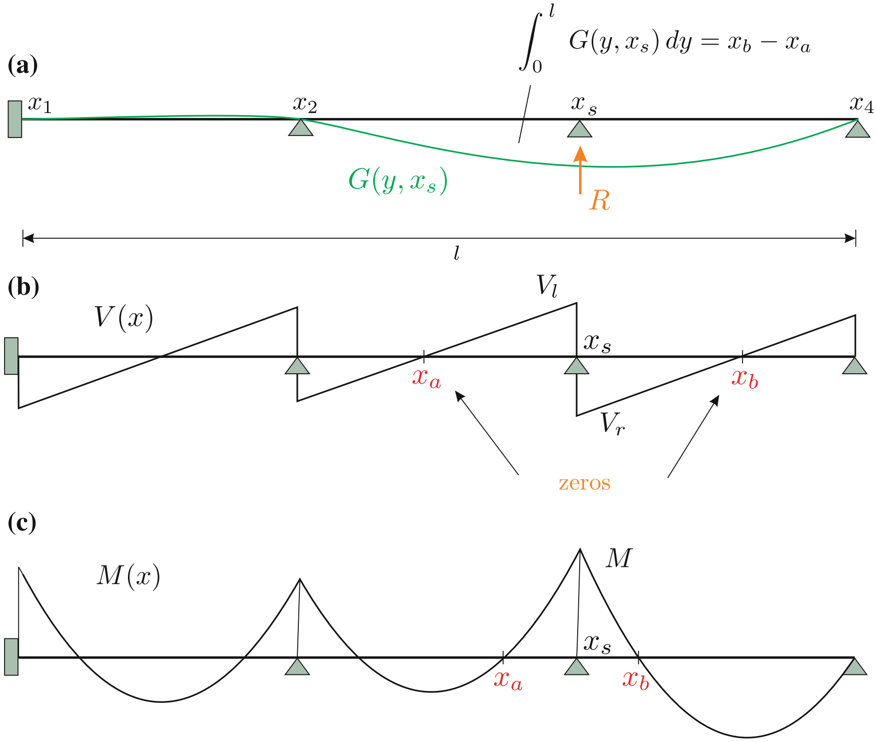


Fig. 2.32 Continuous beam and uniform load, **a** the influence function for the support reaction, **b** the zeros of the shear force, and **c** moment distribution

$$\int_{x_s}^{x_b} V'(x)\,dx = V(x_b) - V_R(x_s) = 0 - V_R(x_s) = V_R(x_s) \tag{2.77}$$

so that the support reaction is

$$R = V_R(x_s) - V_L(x_s) = \int_{x_s}^{x_b} p(x)\,dx + \int_{x_a}^{x_s} p(x)\,dx = \int_{x_a}^{x_b} p(x)\,dx\,. \tag{2.78}$$

In all load cases with a constant load p, the position and the distance between the zeros are the same and the distance is equal to the area under the influence function

$$R = \int_0^l G(y, x_s) \cdot p\,dy = (x_b - x_a) \cdot p\,. \tag{2.79}$$

2.8 Dirac Deltas

Up to now, our approach was rather technical. We reached our goal by splitting the span $(0, l)$ into two parts, formulated on each part separately *Betti's theorem* and extracted from the zero sum $0 + 0 = 0$ the influence function.

The Dirac delta in contrast allows a much more direct approach. It has been invented just for this purpose. The Dirac delta is a load which, like a real point force, is zero at all points y not identical with the source point x

$$\delta_0(y - x) = 0 \qquad y \neq x\,, \tag{2.80}$$

and which contributes the work $w(x) \cdot 1$ on acting through a function $w(x)$

$$\int_0^l \delta_0(y - x)\,w(y)\,dy = w(x) \qquad x \in (0, l)\,. \tag{2.81}$$

The deflection curve generated by such a point force is the solution of the differential equation

$$EI \frac{d^4}{dy^4} G_0(y, x) = \delta_0(y - x) \tag{2.82}$$

and so prepared the influence function for $w(x)$ is obtained nearly automatically

$$\begin{aligned} \mathscr{B}(G_0, w) &= \int_0^l \delta_0(y - x)\,w(y)\,dy - \int_0^l G_0(y, x)\,p(y)\,dy \\ &= w(x) - \int_0^l G_0(y, x)\,p(y)\,dy = 0\,. \end{aligned} \tag{2.83}$$

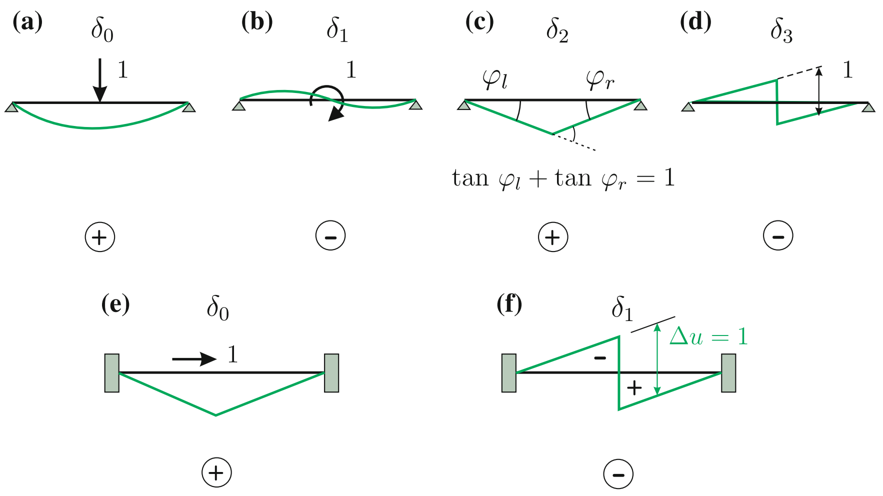


Fig. 2.33 *Upper row* the four influence functions of a beam for **a** w, **b** w', **c** M, and **d** V, each at midspan. The *second row* shows the influence functions for a bar, **e** u, **f** N. The influence functions integrate, $+$, or differentiate, $-$, the load

To extend this approach to the slope $w'(x)$ and to $M(x)$ and $V(x)$, we introduce additional Dirac deltas

$$\begin{array}{ll} \delta_0(y-x) & \text{force } P = 1 \\ \delta_1(y-x) & \text{moment } M = 1 \\ \delta_2(y-x) & \text{bend } \varDelta w' = 1 \\ \delta_3(y-x) & \text{dislocation } \varDelta w = 1 \end{array}$$

with corresponding properties (see Fig. 2.33),

$$\int_0^l \delta_0(y-x)\, w(y)\, dy = w(x) \tag{2.84a}$$

$$\int_0^l \delta_1(y-x)\, w(y)\, dy = w'(x) \tag{2.84b}$$

$$\int_0^l \delta_2(y-x)\, w(y)\, dy = M(x) \tag{2.84c}$$

$$\int_0^l \delta_3(y-x)\, w(y)\, dy = V(x)\,. \tag{2.84d}$$

Operating with Dirac deltas is a very elegant approach because you circumvent all the cumbersome details necessitated by splitting the domain of integration into two parts, etc., but only this analytical approach

$$\mathscr{B}(G_0, w) = \mathscr{B}(G_0^L, w)_{(0,x)} + \mathscr{B}(G_0^R, w)_{(x,l)} = 0 + 0 \tag{2.85}$$

will at the end provide the results which you need to take the shortcut.

One additional remark: The point values such as $1 \cdot w(x)$ do not come from the domain integral (p, δ), as the Dirac delta wants us to make believe, but they come from the jump of the internal forces at the interface between the two halves of the beam

$$\underbrace{(V_0^L(x) - V_0^R(x))}_{=1}\, w(x) = 1 \cdot w(x)\,. \tag{2.86}$$

The same holds true in 2-D and 3-D problems where the point values are the limits of certain boundary integrals along the perimeter of the shrinking circular holes which surround the source point.

The most important property of Dirac deltas is that we can integrate them or rather that there are definitive rules for how to interpret an integral such as

$$\int_0^l \delta(y - x)\, \varphi_i(y)\, dy \tag{2.87}$$

because this allows us to associate with any Dirac delta equivalent nodal forces

$$f_i = \int_0^l \delta_0(y - x)\, \varphi_i(y)\, dy = \varphi_i(x) \tag{2.88a}$$

$$f_i = \int_0^l \delta_1(y - x)\, \varphi_i(y)\, dy = \varphi_i'(x) \tag{2.88b}$$

$$f_i = \int_0^l \delta_2(y - x)\, \varphi_i(y)\, dy = M(\varphi_i)(x) \tag{2.88c}$$

$$f_i = \int_0^l \delta_3(y - x)\, \varphi_i(y)\, dy = V(\varphi_i)(x)\,. \tag{2.88d}$$

The expression $M(\varphi_i)(x)$ is the moment of φ_i at the source point x and analogously is $V(\varphi_i)(x)$ the shear force of φ_i at the source point x.

2.9 Dirac Energy

When we pull the rope of a pulley, the work done by our hand and the load is the same (see Fig. 2.34).

Influence functions express the same balance, an energy balance. The work done by a single force $P = 1$ on acting through $w(x)$

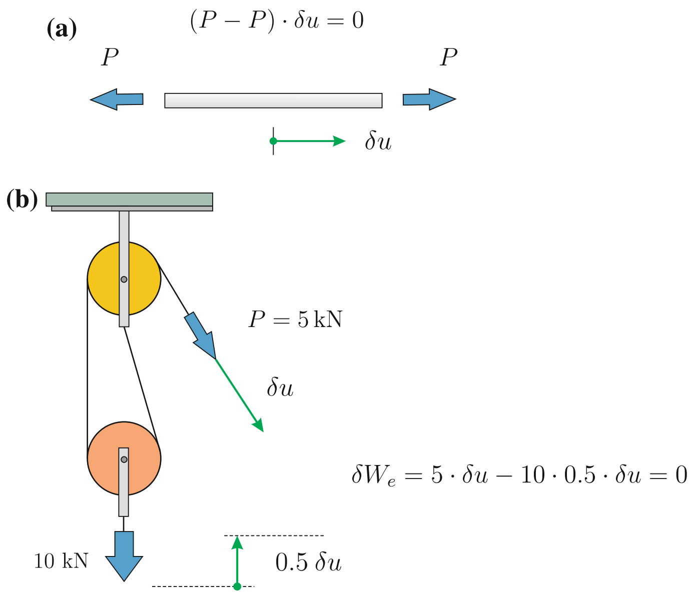


Fig. 2.34 **a** Equilibrium of forces and **b** balance of exterior work at a pulley

$$1 \cdot w(x) = \int_0^l G_0(y, x)\, p(y)\, dy\,, \tag{2.89}$$

is the same as the work done by the distributed load p on acting through $G_0(y, x)$, the deflection of the beam due to the single force.

The factor 1 is essential because otherwise the dimensions would not agree

$$\begin{aligned} \mathit{force} \cdot \mathit{disp.} = 1 \cdot u(x) &= \int_0^l G_0(y, x)\, p(y)\, dy \\ &= \mathit{disp.} \cdot \mathit{force}/\mathit{length} \cdot \mathit{length}\,. \end{aligned} \tag{2.90}$$

This means that the output of an influence function is an *energy*. We call this energy quantum the *Dirac energy*.

The Dirac energy is the work done by the load on acting through the influence function.

The simplest example of this interpretation is provided by a seesaw (see Fig. 2.35). The work done by the two weights at each turn φ of the seesaw is zero

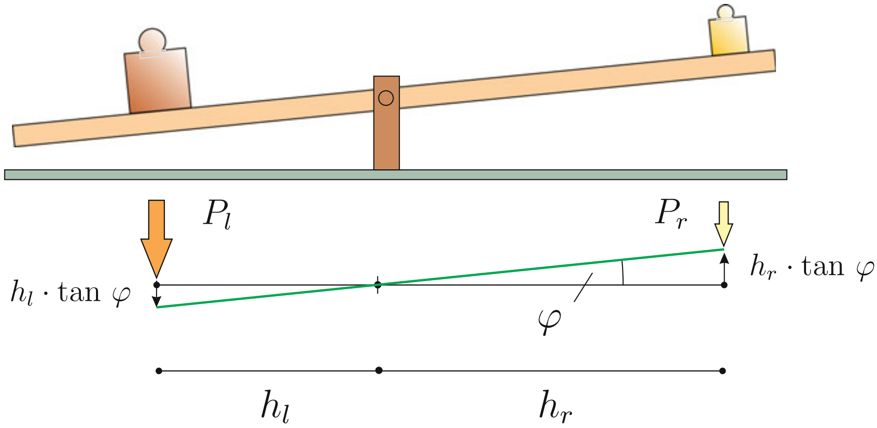


Fig. 2.35 Seesaw

$$P_l\, u_l - P_r\, u_r = P_l\, \tan\varphi\, h_l - P_r\, \tan\varphi\, h_r = (P_l\, h_l - P_r\, h_r)\, \tan\varphi = 0\,, \tag{2.91}$$

because the two forces balance, $P_l\, h_l = P_r\, h_r$.

In this sense, each influence function can be considered a seesaw. To compute the shear force $V(x)$ of a beam at a point x as in Fig. 2.6, p. 84, we install at x a shear hinge and we spread the hinge by one unit apart so that the two shear forces perform the work $-V(x) \cdot 1$

$$-\, V(x)\, w(x_-) - V(x)\, w(x_+) = -V(x)\,(w(x_-) + w(x_+)) = -V(x) \cdot 1 \tag{2.92}$$

and the work done by the point load P on acting through w, the response of the beam to this spread, must exactly be the opposite

$$\underbrace{-V(x) \cdot 1 + P\, w}_{W_{1,2}} = 0\,. \tag{2.93}$$

This is *Betti's theorem*, $W_{1,2} = W_{2,1} = 0$. (The work $W_{2,1}$ is zero (see (2.71)).)

So to each internal action $V(x), N(x), M(x)$, etc., belongs a certain mechanism, a kind of a seesaw (see Fig. 2.36), and if we unlock the hinge and compute the work done by the load on acting through the resulting displacement, then we learn how large the internal action must be to balance the exterior load (Fig. 2.37).

The problem with finite elements is that we hinder the displacements of a structure we hobble so to speak the structure because the *shape functions* $\varphi_i(x)$ are not flexible enough to allow any possible shape and so the hinge gets the wrong signal. The displacement at the base of the point load P is w_h

$$-\, V_h(x) \cdot 1 + P\, w_h = 0 \tag{2.94}$$

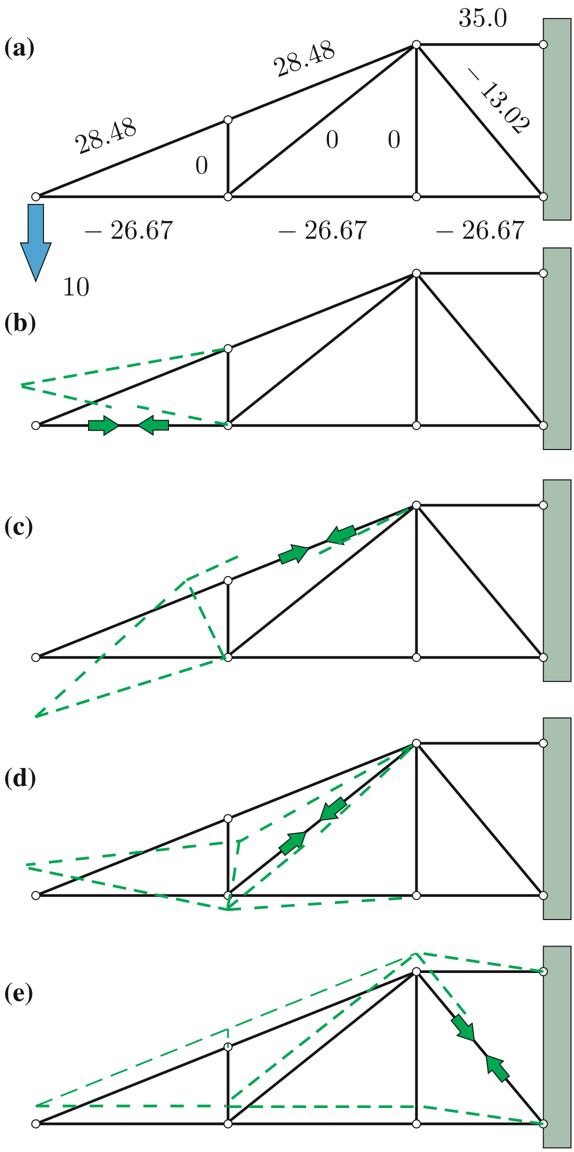


Fig. 2.36 The kinematics determines the distribution of the forces in a structure

and not the exact value w

$$-\,V(x)\cdot 1 + P\,w = 0\,, \tag{2.95}$$

and this is why $V_h(x) \neq V(x)$. *An FE-program gets the Dirac energies wrong* (Fig. 2.38).

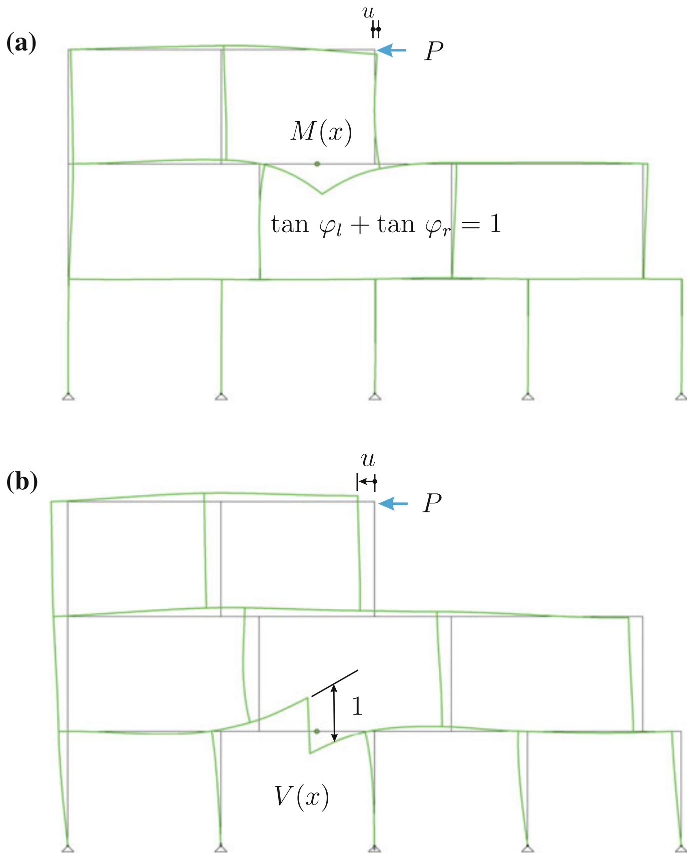


Fig. 2.37 Influence function for a moment and a shear force in a frame

This establishes that the kinematics of an FE-mesh determines the accuracy of an FE-solution.

Mesh = kinematics = precision of influence functions = quality of results

We now can also resolve the question what distinguishes a good design from a mediocre design. The logic of the seesaw

$$V(x) = \frac{P \cdot w}{1} = P \cdot \frac{w}{1} \qquad \leftarrow \qquad \text{make } w \text{ small!} \tag{2.96}$$

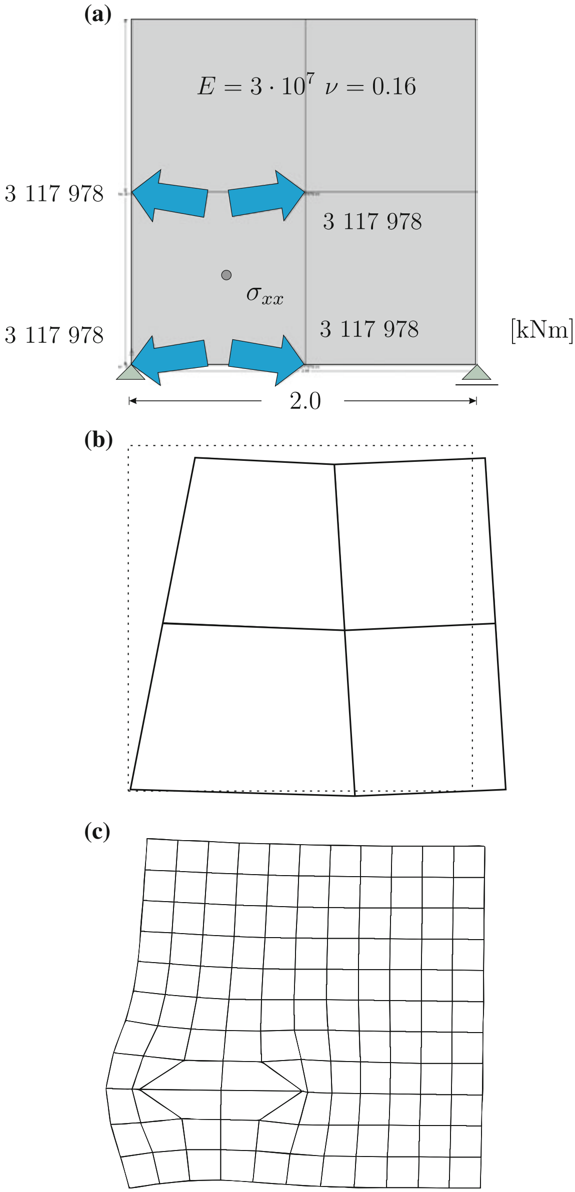


Fig. 2.38 Influence function for σ_{xx} (bilinear elements), **a** nodal forces $\boldsymbol{f}$, **b** approximate influence function, and **c** refined mesh

signals, that in a good design the unit spread of the shear hinge, the 1 in the denominator, causes a displacement w as small as possible at the foot of the point load P.

Throw a stone into the pond and watch the ripples! The smaller the waves that reach the load, the better the design.

Fig. 2.39 The effect of a point load on the top of the Eiffel tower on a truss element depends on how sensitive the top is to a unit spread of the truss element

Archimedes lever is (willingly) the exact opposite of a good design. A unit displacement of the short lever on the left leads to a very large displacement at the opposite end.

This seesaw logic is universal. When a force $\boldsymbol{P} = \{P_x, P_y, P_z\}^T$ acts at the top of the Eiffel tower (see Fig. 2.39), its effect on the normal force N in a truss element—way down below—is determined by how large the displacements g_x, g_y, *and* g_z at the top of the tower are when the truss element is spread one unit apart

$$1 \cdot N = P_x \cdot g_x + P_y \cdot g_y + P_z \cdot g_z\,. \tag{2.97}$$

Remark 2.3 Actually, in frame analysis, FE-programs get the Dirac energy right because the unit spread of a shear hinge propagates correctly over a frame (disregarding the frame element with the shear hinge itself) and so $w_h(x) = w(x)$. This is different when the beams are shaped or other deviations from the beam equation $EI\,w^{IV}(x) = p$ occur.

But in 2-D and 3-D problems, it is neither possible to generate the required unit spread nor would the effect of the (pseudo) unit spread propagate correctly over the mesh. In standard 1-D problems[1], the unit spread too cannot be generated by the shape functions, but outside the element with the source point, the effects of the 'pseudo-spread' propagate as if they were the result of an exact unit spread and this suffices to obtain correct results.

[1]Such as $EI\,w^{IV} = p_z$ or $-EA\,u'' = p_x$ but not $EI\,w^{IV} + c\,w = p_z$.

2.10 Point Values in 2-D and 3-D

To formulate an influence function for the deflection $w(x)$ of a beam is simple, you only have to split the beam into two parts and the discontinuity of the shear force at the interface

$$[V\,w - M\,w']_0^x + [V\,w - M\,w']_x^l = (V(x_-) - V(x_+))\,w(x) = -1 \cdot w(x) \quad (2.98)$$

makes the single term $w(x)$ jump out

$$1 \cdot w(x) = \int_0^l G_0(y, x)\,p(y)\,dy\,. \quad (2.99)$$

In 2-D and 3-D problems, the situation is different. Green's second identity of the Laplace operator, for example,

$$\begin{aligned}\mathscr{B}(w, \hat{w}) = \int_\Omega -\Delta\,w\,\hat{w}\,d\Omega + \int_\Gamma \frac{\partial w}{\partial n}\,\hat{w}\,ds \\ - \int_\Gamma w\,\frac{\partial \hat{w}}{\partial n}\,ds - \int_\Omega w\,(-\Delta\hat{w})\,d\Omega = 0\,, \quad (2.100)\end{aligned}$$

contains no point value $w(\boldsymbol{x})$ which you could isolate on the left side to formulate an influence function.

The maneuver succeeds because the deflection surface $G_0(\boldsymbol{y}, \boldsymbol{x})$, which belongs to a point load $P = 1$, has the property that the integral of the shear force $\partial G_0/\partial n$ over circles Γ_{N_ε} of shrinking radius ε has the value 1 in the limit

$$\lim_{\varepsilon\to 0} \int_{\Gamma_{N_\varepsilon}} \frac{\partial G_0}{\partial n}\,ds = 1\,. \quad (2.101)$$

So we start the formulation of *Betti's theorem* on a punctured domain $\Omega_\varepsilon = \Omega - N_\varepsilon$, that is, we spare a circular neighborhood N_ε of the source point from the domain, and we let the radius $\varepsilon \to 0$ tend to zero. This provides in the limit the influence function for $w(\boldsymbol{x})$ at the source point $\boldsymbol{x}$

$$\lim_{\varepsilon\to 0} \mathscr{B}(G_0, w)_{\Omega_\varepsilon} = \lim_{\varepsilon\to 0} \Big[\int_{\Gamma_{N_\varepsilon}} \frac{\partial G_0}{\partial n}\,w(\boldsymbol{y})\,ds_{\boldsymbol{y}} - \int_{\Omega_\varepsilon} G_0(\boldsymbol{y}, \boldsymbol{x})\,p(\boldsymbol{y})\,d\Omega_{\boldsymbol{y}}\Big]\,, \quad (2.102)$$

that is

$$\lim_{\varepsilon\to 0} \mathscr{B}(G_0, w)_{\Omega_\varepsilon} = 1 \cdot w(\boldsymbol{x}) - \int_\Omega G_0(\boldsymbol{y}, \boldsymbol{x})\,p(\boldsymbol{y})\,d\Omega_{\boldsymbol{y}} = 0\,. \quad (2.103)$$

Remark 2.4 $\Omega_\varepsilon = \Omega - N_\varepsilon$ is the domain Ω minus the area N_ε of the punch-out and Γ_{N_ε} is the edge of the punch-out.

This derivation of influence functions of 2-D and 3-D problems is very technical and not always simple (see [2]). Fortunately, with finite elements, things are much simpler (see Chap. 3).

2.11 Duality

Duality is the theme behind *Betti's theorem*. The easiest introduction provides a stiffness matrix $\boldsymbol{K}_{(n\times n)}$. If you multiply the matrix with a vector $\boldsymbol{u}$ and then this vector $\boldsymbol{K}\,\boldsymbol{u}$ with a second vector $\boldsymbol{\delta u}$, the result is a scalar, a number $\boldsymbol{\delta u}^T\,\boldsymbol{K}\,\boldsymbol{u}$.

Because a real number like π does not change if it is transposed, $\pi^T = \pi$, it holds

$$\boldsymbol{\delta u}^T\,\boldsymbol{K}\,\boldsymbol{u} = \boldsymbol{u}^T\,\boldsymbol{K}\,\boldsymbol{\delta u} \tag{2.104}$$

or

$$\mathscr{B}(\boldsymbol{u}, \boldsymbol{\delta u}) = \boldsymbol{\delta u}^T\,\boldsymbol{K}\,\boldsymbol{u} - \boldsymbol{u}^T\,\boldsymbol{K}\,\boldsymbol{\delta u} = 0\,. \tag{2.105}$$

This is *Betti's theorem* for symmetric, that is, self-adjoint matrices. Symmetry of matrices is the same as self-adjointness in differential equations.

Imagine that the vector $\boldsymbol{f}$ in the system

$$\boldsymbol{K}\,\boldsymbol{u} = \boldsymbol{f} \tag{2.106}$$

represents the nodal forces f_i acting on a truss, and we want to know which value the first component u_1 of the displacement vector $\boldsymbol{u}$ of the truss has.

To this end, we solve an auxiliary problem

$$\boldsymbol{K}\,\boldsymbol{g}_1 = \boldsymbol{e}_1\,, \tag{2.107}$$

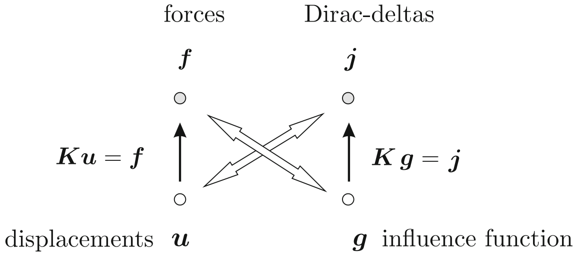
where $\boldsymbol{e}_1$ is the first unit vector $\boldsymbol{e}_1^T = \{1, 0, 0, \ldots, 0\}$, and with the solution $\boldsymbol{g}_1$ and the vector $\boldsymbol{u}$, we formulate then the identity (2.105)

$$\mathscr{B}(\boldsymbol{g}_1, \boldsymbol{u}) = \boldsymbol{g}_1^T\boldsymbol{f} - \boldsymbol{u}^T\,\boldsymbol{e}_1 = \boldsymbol{g}_1^T\boldsymbol{f} - u_1 = 0 \tag{2.108}$$

which promptly provides the answer

$$u_1 = \boldsymbol{g}_1^T\boldsymbol{f}\,. \tag{2.109}$$

We can do this with each component u_i because to each u_i belongs such a vector $\boldsymbol{g}_i$



$$\boxed{J(\boldsymbol{u}) = \boldsymbol{j}^T\,\boldsymbol{u} = \boldsymbol{g}^T\,\boldsymbol{f}}$$

Fig. 2.40 Duality = 'crosswise'

$$\boldsymbol{K}\,\boldsymbol{g}_i = \boldsymbol{e}_i\,, \tag{2.110}$$

which determines u_i (see Fig. 2.40),

$$u_i = \boldsymbol{g}_i^T\boldsymbol{f}\,. \tag{2.111}$$

Stated differently, by projecting the vector $\boldsymbol{f}$ onto the n vectors $\boldsymbol{g}_i$, we can construct the solution $\boldsymbol{u} = u_1\,\boldsymbol{e}_1 + \ldots + u_n\,\boldsymbol{e}_n$, since the projection of the vector $\boldsymbol{f}$ onto the vectors $\boldsymbol{g}_i$ is the same as the projection of $\boldsymbol{u}$ onto the unit vectors $\boldsymbol{e}_i$

$$u_i = \boldsymbol{g}_i^T\boldsymbol{f} = \boldsymbol{u}^T\,\boldsymbol{e}_i\,. \tag{2.112}$$

This is exactly what is happening (with all u_i simultaneously), when we multiply the vector $\boldsymbol{f}$ with the inverse matrix $\boldsymbol{K}^{-1}$

$$\boldsymbol{u} = \boldsymbol{K}^{-1}\boldsymbol{f}\,, \tag{2.113}$$

because the rows (and columns) of the symmetric matrix $\boldsymbol{K}^{-1}$ are the vectors $\boldsymbol{g}_i$, and so it follows that

$$\boldsymbol{u} = (\boldsymbol{g}_1^T\boldsymbol{f})\,\boldsymbol{e}_1 + (\boldsymbol{g}_2^T\boldsymbol{f})\,\boldsymbol{e}_2 + \ldots + (\boldsymbol{g}_n^T\boldsymbol{f})\,\boldsymbol{e}_n\,. \tag{2.114}$$

When we replace the vector by a function, say the solution of the boundary value problem

$$-\,EA\,u''(x) = p(x) \qquad u(0) = u(l) = 0\,, \tag{2.115}$$

the matrix $\boldsymbol{K}$ has infinitely many columns and the unit vectors turn into Dirac deltas

$$-\,\frac{d^2}{dy^2}\,G(y,x) = \delta(y-x)\,, \tag{2.116}$$

but the formalism is the same. By projecting the right-hand side p onto the solutions $G(y, x)$, that is, by forming the scalar product of the two functions, we are able to find the value of the solution at any point x

$$u(x) = \underbrace{\int_0^l G(y, x)\, p(y)\, dy}_{\boldsymbol{g}_i^T \boldsymbol{f}} = \underbrace{\int_0^l \delta(y - x)\, u(y)\, dy}_{\boldsymbol{u}^T\, \boldsymbol{e}_i} . \tag{2.117}$$

2.12 Monopoles and Dipoles

The influence function for the rotation w' of a beam is generated by a single moment $M = 1$

$$M = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\, \Delta x = 1\,, \tag{2.118}$$

which we can identify with the action of two opposite forces, $P = \pm 1/\Delta x$, which become infinitely large when their distance Δx tends to zero. Such a pair of forces is called a *dipole*.

A single force in contrast constitutes a *monopole*. The influence function for a deflection $w(x)$ is generated by a monopole.

Influence functions which are generated by monopoles sum, they resemble dents or sinks, see Figs. 2.41 and 2.44a. Everything that falls into the sink increases the deflection of the slab.

Dipoles instead generate shear deformations which react to imbalances, and they differentiate (see Figs. 2.41 and 2.44b).

Monopoles integrate and dipoles differentiate.

Each of the four influence functions in Fig. 2.41 corresponds to one of these two actions:

- G.F. for deflections and moments *sum*.
- G.F. for rotations, stresses, and shear forces *differentiate*

The influence function for the shear force V is generated by a dipole, while the influence function for the bending moment M is generated by two moments

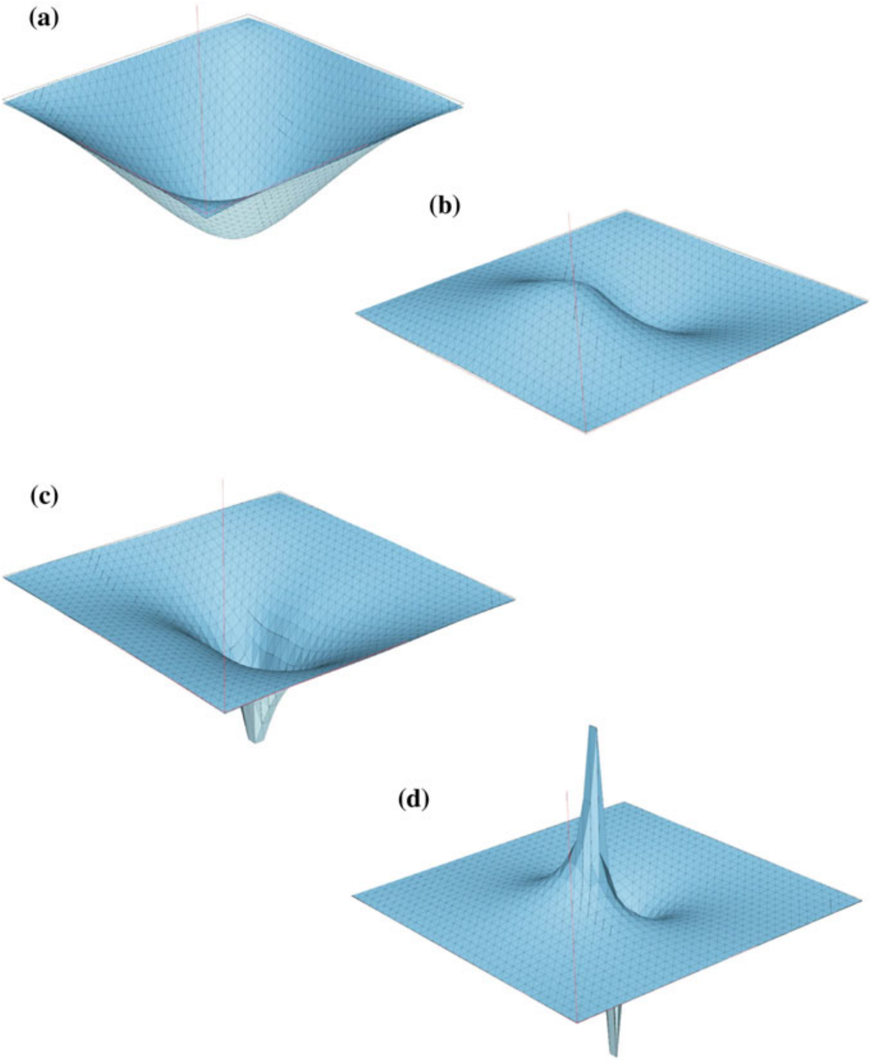


Fig. 2.41 Influence functions are generated by monopoles (*left side*) or dipoles (*right side*). **a** Influence function for deflection, **b** rotation $w_{,x}$, **c** moment m_{xx}, and **d** shear force q_x

$M = \pm 1/\Delta x$ which rotate the beam inward and so generate a symmetric deflection curve with a sharp bend at the source point.[2]

The maximum value is obtained if the load and the influence function are of the same type (*symmetric–symmetric* or *antisymmetric–antisymmetric*) and the minimal

[2]To be precise, the correct sequence is monopole–dipole–quadrupole–octopole, corresponding to the finite differences of $w, w', M,$ *and* V (see Fig. 7.5 p. 335), but for our purposes, the simple division, monopole–dipole or sum–differentiate, may suffice.

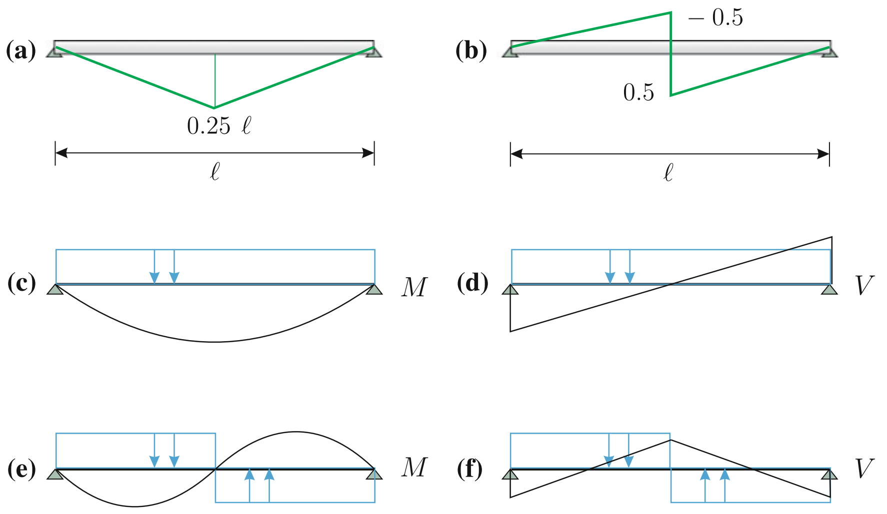


Fig. 2.42 Topmost row influence functions for **a** the bending moment and **b** the shear force at midspan, **c** and **d** moments and shear forces under symmetric load and antisymmetric load, (**e**) and (**f**)

effect if they are of opposite type (see Fig. 2.42).

> The difference between monopoles and dipoles is the reason why it is easier to approximate displacements and moments than stresses and shear forces. It is the difference between numerical integration and numerical differentiation (see Fig. 2.43).

Remark 2.5 All influence functions for support reactions integrate though the support reactions are normal forces (stresses) or shear forces, and we therefore would expect that the influence functions differentiate. But at a rigid support, the movement is hindered by the foundation, and so the other part must go all the way—alone—to effect a unit dislocation, $[[u]] = 1$, and the influence function becomes a one-sided integral.

Remark 2.6 Not all influence functions tend to zero. If parts of the mechanism (after the installation of an N-, V-, or M-hinge) can perform rigid body motions, it can happen that the influence functions blow up (see Fig. 2.45b).

Remark 2.7 The speed with which influence functions decay depends on the order of the derivative 0, 1, 2, *and* 3 of the target value, as in a beam

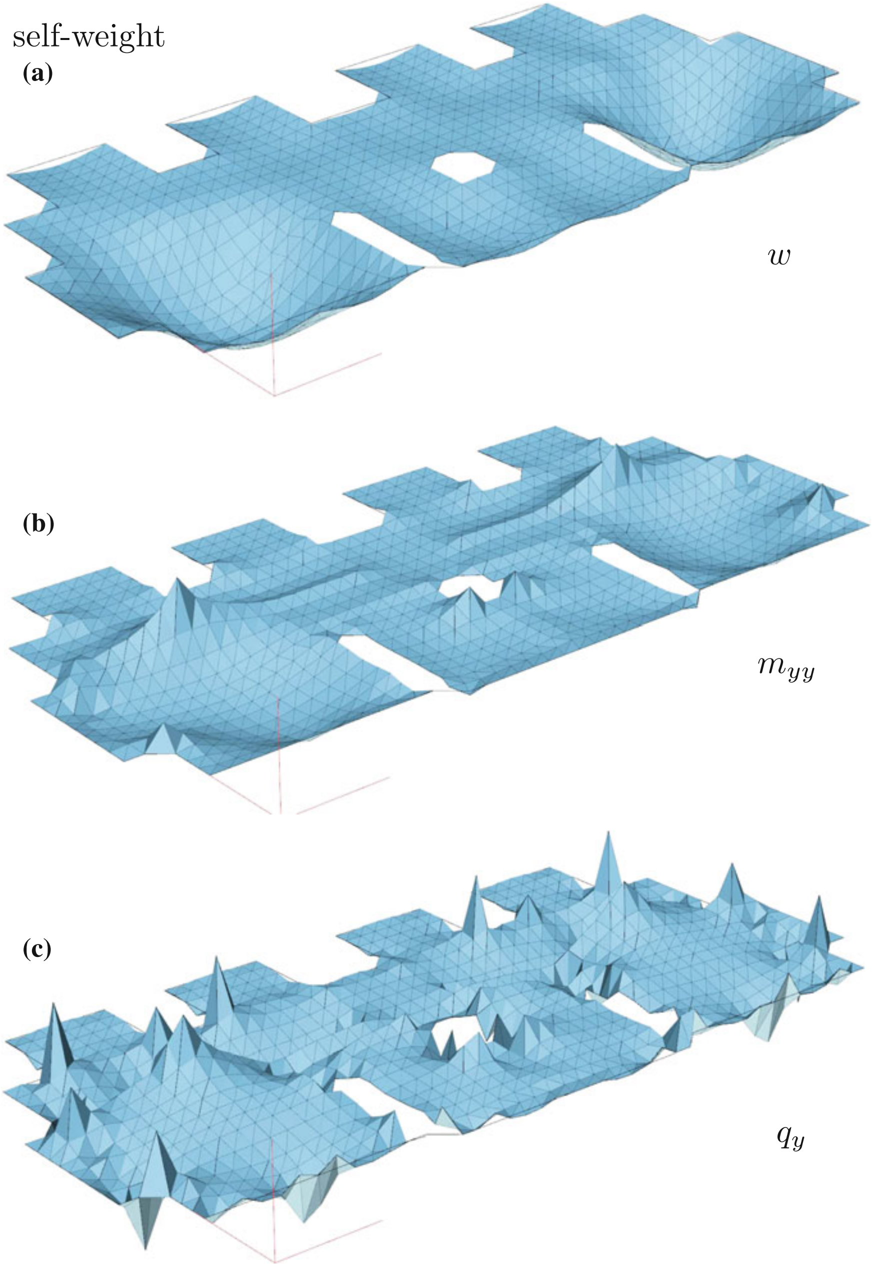


Fig. 2.43 Increasing complexity, **a** deflection w, **b** moments m_{yy}, and **c** shear forces q_y

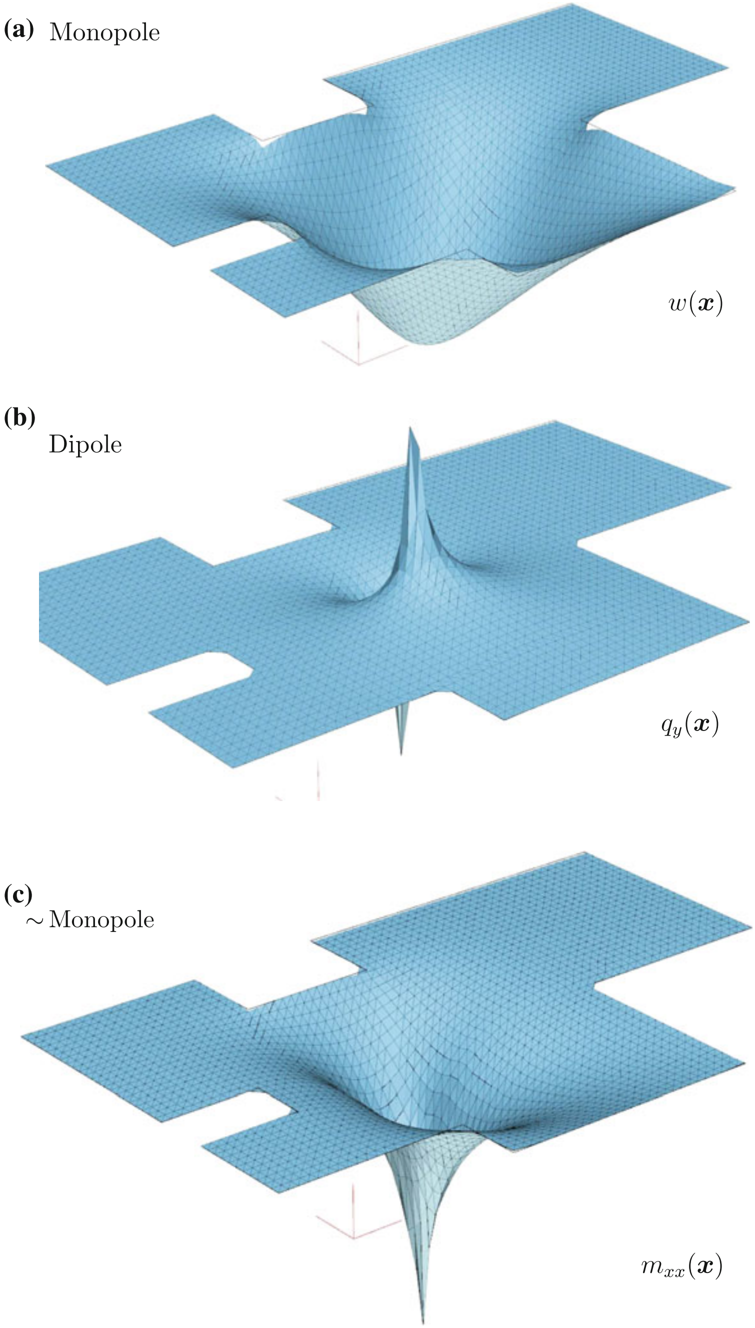


Fig. 2.44 Slab and three influence functions **a** for a deflection, **b** for a shear force, and **c** for a moment

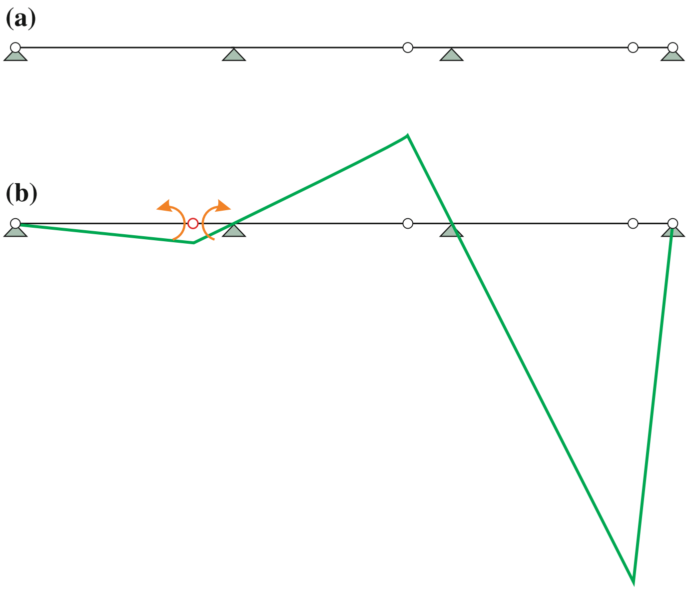


Fig. 2.45 **a** Gerber beam and **b** influence function for a moment M. Not all influence functions decay!

$$w(x), \quad w'(x), \quad M(x) = -EI\, w''(x), \quad V(x) = -EI\, w'''(x)\,. \tag{2.119}$$

The lower the order, the more the influence function spreads out and the more slowly it declines as the influence function for the deflection $w(\boldsymbol{x})$ of a slab demonstrates (see Fig. 2.44a). The influence function for the shear force q_x instead is a tightly concentrated dipole (see Fig. 2.44b), with two infinitely large opposite peaks but from which extreme values the influence function decays rapidly (Fig. 2.46).

Of course, the particular behavior also depends on the support conditions (see Figs. 2.46 and 2.47), because structures with large overhanging parts (cantilevers) play a special role in this regard. Such parts can swing widely and they can easily blow up any influence function.

Remark 2.8 The influence functions in Figs. 2.46 and 2.47 closely follow the beam solutions. Only in the immediate neighborhood of the source point do they exhibit the typical singular behavior. This makes you wonder whether we can associate with any source point a radius outside of which the solution is 'standard' but once we transgress this imaginary boundary the influence function takes off. In problematic situations this radius can be quite large as the example of the influence function for the shear force q_x near a pier in Fig. 3.59 p. 217 demonstrates or even infinite as in the case of the cantilever plate in Fig. 6.13 p. 300 where there is simply no hope that the influence function for the stress σ_{xx} in the extreme fiber at the fixed edge returns to a more ordinary path.

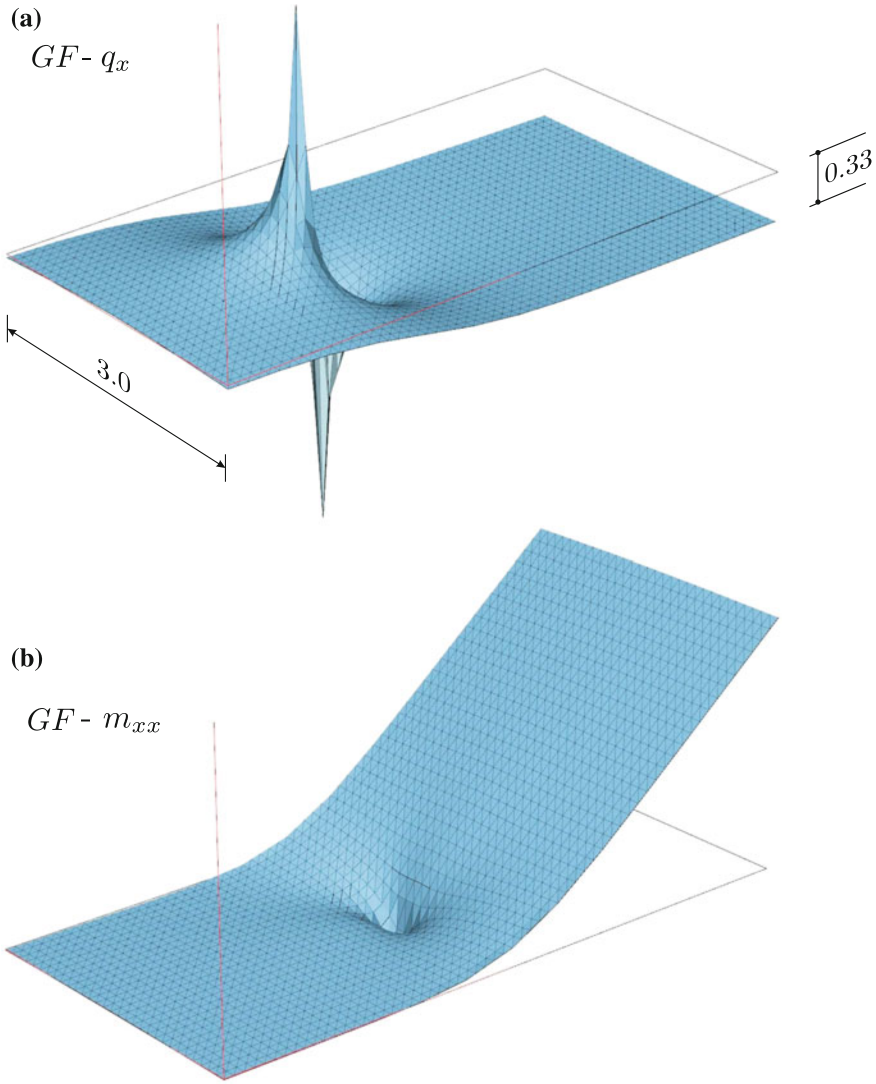


Fig. 2.46 Cantilever plate, **a** influence function for the shear force q_x and **b** for the moment m_{xx}; it is amazing how numerical Dirac deltas (BE-solutions) manage to generate these nearly perfect surfaces

2.13 Influence Functions for Integral Values

By integrating stresses over a short length, we can easily circumvent the sometimes erratic behavior of point values (see Fig. 2.48). That this improves the results is easily understood when we study the associated influence functions. The influence function

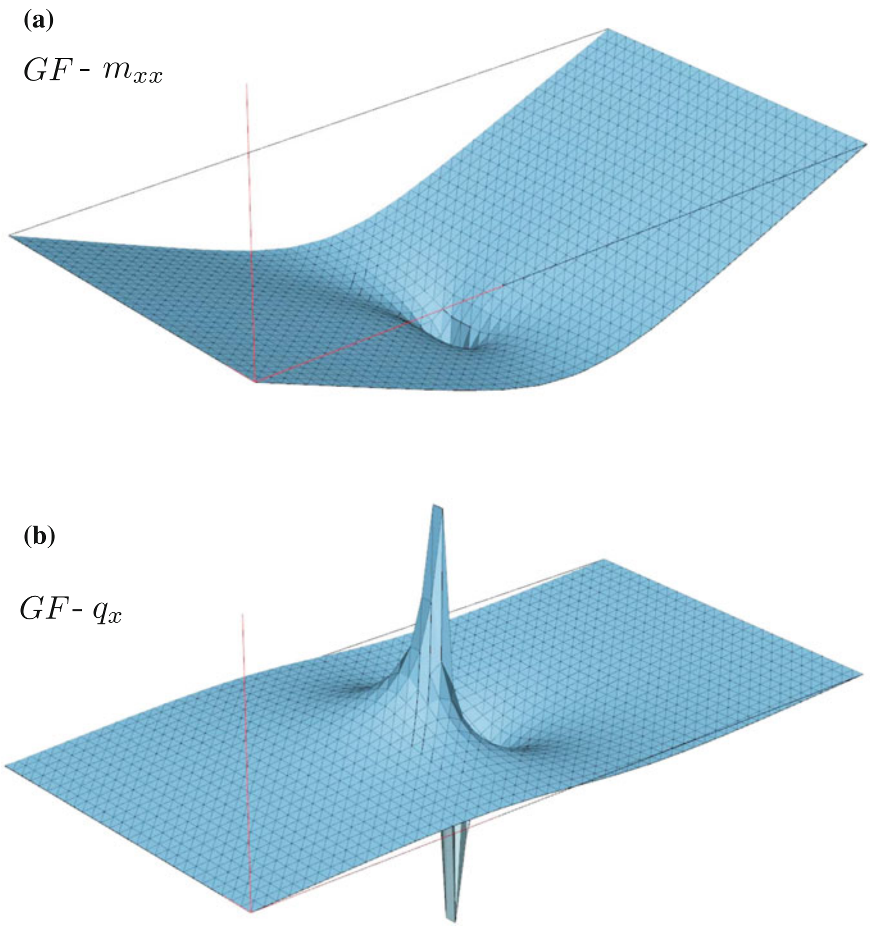


Fig. 2.47 Bridge deck, **a** influence function for the moment m_{xx} and **b** for the shear force q_x at the center (BE-solutions); the influence function for the integral of q_x along the cross section should be identical with the beam solution

for the stress σ_{yy} at a point is a dislocation of the source point in vertical direction (see Fig. 2.48b). When we integrate over a short length ℓ instead

$$N_{yy} = \int_0^\ell \sigma_{yy}\, ds\,, \tag{2.120}$$

the influence function resembles a dislocation of the stretch $(0, \ell)$ (see Fig. 2.48c), and it is far easier to approximate such a movement with finite elements than a dislocation at one point. This is the reason why average values are more tempered than point values.

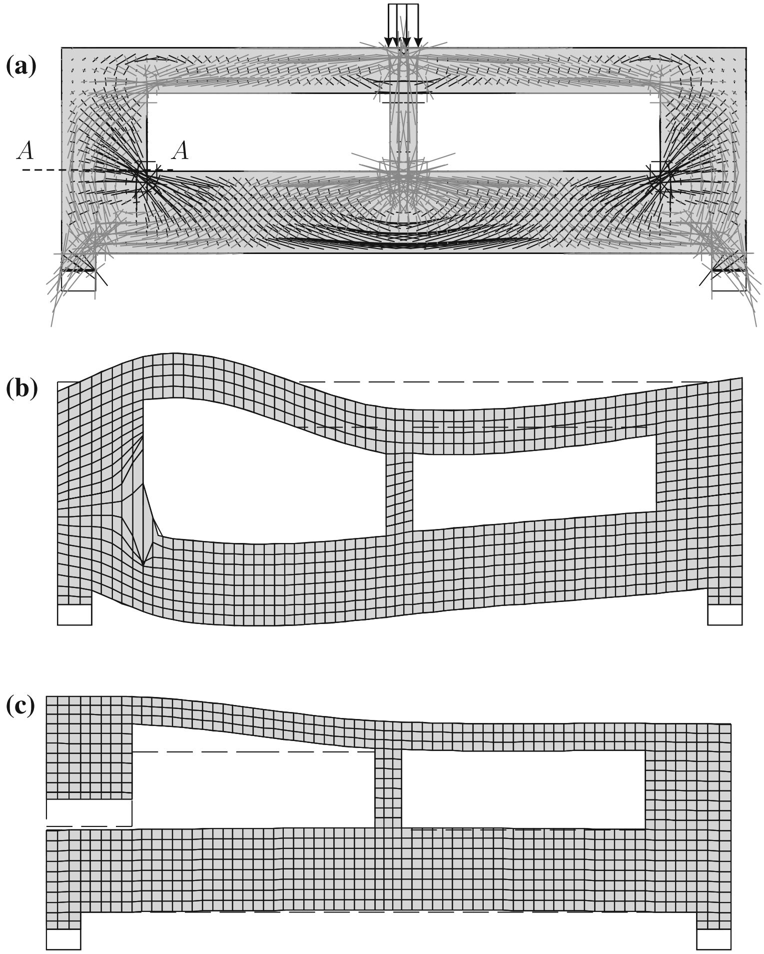


Fig. 2.48 Plate, **a** stress distribution, **b** influence function for σ_{yy}, and **c** influence function for N_{yy} in cross section $A-A$

2.14 Influence Functions Integrate

When we differentiate, we make a step forward, and when we integrate, we make a step back just as influence functions do. They begin at the end of the chain, $-EA\,u'' = p$ and $EI\,w^{IV} = p$, respectively, and do steps back, to find the derivatives of lower order

$$u,\, N = EA\,u' \qquad w,\; w',\; M = -EI\,w'',\; V = -EI\,w'''\,. \tag{2.121}$$

The influence function $G_1(y, x)$ for the normal force in a bar integrates the load once

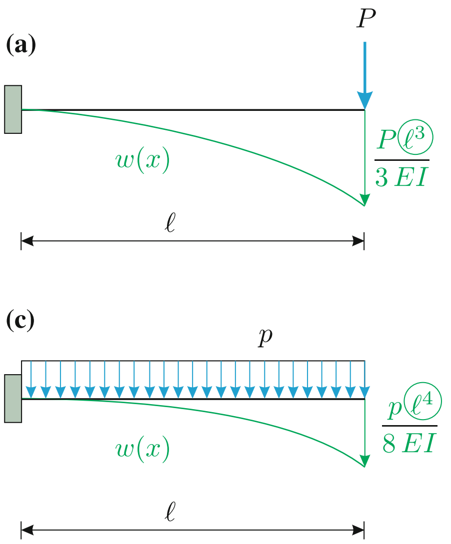


Fig. 2.49 Deflection at the beam end due to **a** a single force, **b** a moment, and **c** a distributed load

$$N(x) = \int_0^l G_1(y, x)\, p(y)\, dy \qquad ('') \rightarrow (') \tag{2.122}$$

and the influence function $G_0(y, x)$ for the longitudinal displacement $u(x)$ integrates twice

$$u(x) = \int_0^l G_0(y, x)\, p(y)\, dy \qquad ('') \rightarrow (\,)\,. \tag{2.123}$$

These steps easily show through in the cantilever beam in Fig. 2.49. The deflection w is the triple indefinite integral of the shear force $V = -EI\, w'''$

$$w = \int\!\!\int\!\!\int V\, dx\, dx\, dx = \int\!\!\int\!\!\int P\, dx\, dx\, dx \tag{2.124}$$

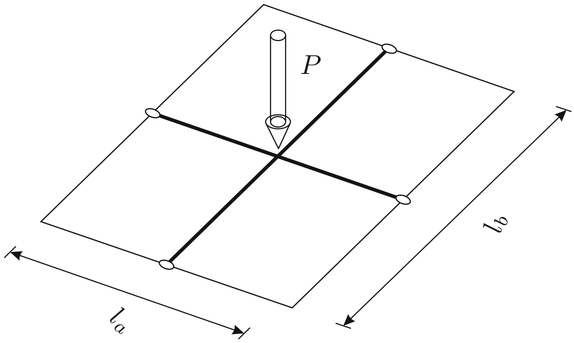
and so we find an ℓ^3 in the end deflection produced by a point load P

$$w(\ell) = \frac{P\,\ell^3}{3\,EI}\,, \tag{2.125}$$

and if the beam carries a single moment $M = -EI\, w''$, then we see an ℓ^2

$$w(\ell) = \frac{M\,\ell^2}{2\,EI}\,. \tag{2.126}$$

Fig. 2.50 Two-way grid



In the case of a distributed load p, the end deflection is

$$w(\ell) = \frac{p\,\ell^4}{8\,EI} \tag{2.127}$$

where the ℓ^4 matches the fourth order of the differential equation $EI\,w^{IV} = p$.

We find the l^3 of (2.125) also in the equation

$$\frac{P_a}{P_b} = \frac{l_b^3}{l_a^3}\,. \tag{2.128}$$

which splits a point load $P = P_a + P_b$ on a two-way grid structure (see Fig. 2.50) into two parts.

A stiffness matrix, $\boldsymbol{K}\,\boldsymbol{u} = \boldsymbol{f}$, instead differentiates, and so we find the 'inverse factor' EI/l^3 up front in a beam matrix and EA/l in a bar matrix.

2.15 Second-Order Beam Theory

The governing equation of second-order beam theory is

$$EI\,w^{IV}(x) + P\,w''(x) = p(x) \tag{2.129}$$

where P is the compressive force in the beam and $p(x)$ the lateral load (see Fig. 2.51a). This is a linear self-adjoint fourth-order differential equation with constant coefficients, but the problem in this case is that the coefficient P is load case dependent and so also the shape of the influence functions depends on P.

The more P nears the critical buckling load P_{crit}, the more the influence function for the end rotation of the beam (Fig. 2.51b), or the end deflection of the column (Fig. 2.51c) buckles.

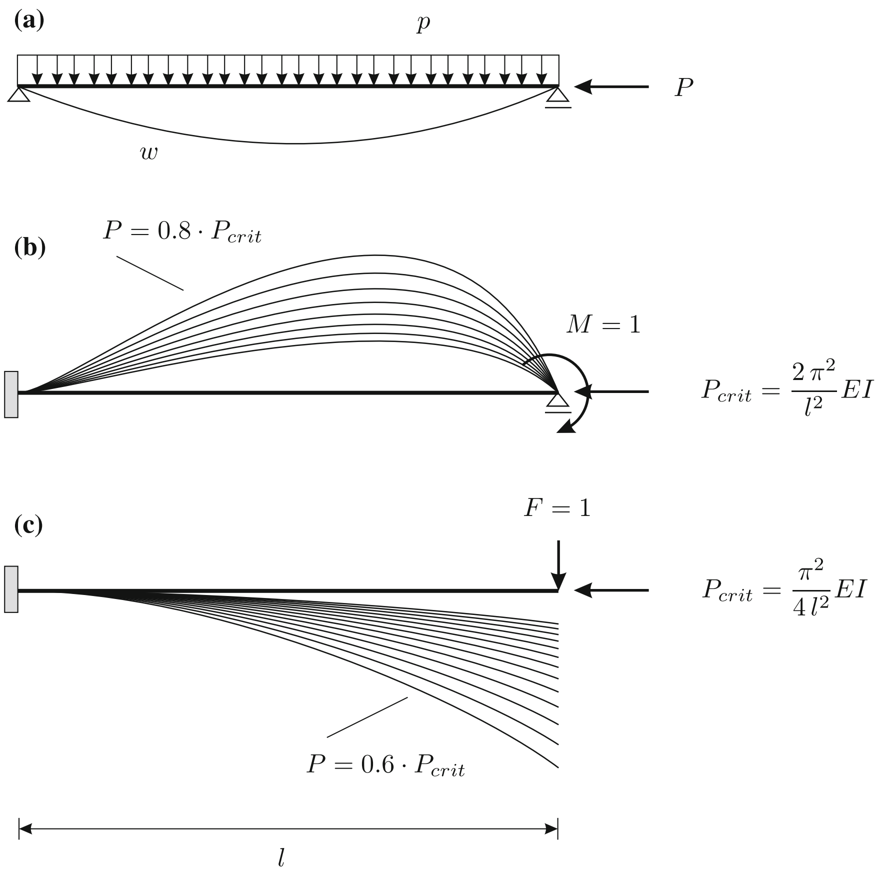


Fig. 2.51 Second-order beam theory **a** compressive force P, **b** influence function for $w'(l)$, and **c** end deflection $w(l)$

This dependence on the normal forces N in the single frame elements is also the reason why it is not possible to derive influence functions for such frames. The equilibrium position of the frame itself first must be found by an iterative analysis which begins with a first-order solution and then iteratively corrects this solution (by applying second-order theory) until equilibrium is achieved.

In principle is second-order beam theory a nonlinear problem where the longitudinal displacement $u(x)$ and the lateral deflection $w(x)$ satisfy the system

$$-EA\left(u' + \frac{1}{2}\,(w')^2\right)' = p_x \tag{2.130a}$$

$$EI\,w^{IV} - \left(EA(u' + \frac{1}{2}\,(w')^2)\,w'\right)' = p_z\,. \tag{2.130b}$$

Only if the normal force

$$N = EA(u' + \frac{1}{2}\,(w')^2) \tag{2.131}$$

is given and constant, does this system simplify to the single equation (2.129). Note that a negative N is a positive P in (2.129).

References

1. Kurrer K-E (2012) The History of the Theory of Structures: From Arch Analysis to Computational Mechanics. Ernst & Sohn, Berlin
2. Hartmann F (1989) Introduction to Boundary Elements. Springer, Berlin

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