Electromagnetic Scattering by a Chiral Impedance Screen

C.E Athanasiadis, V. Sevroglou, and K.I. Skourogiannis

Abstract In this paper the solvability of the direct electromagnetic scattering problem by an impedance screen in a chiral environment is presented. Time-harmonic electromagnetic plane waves in a chiral medium are considered as incident fields. These propagating fields are scattered by an obstacle which is a partially coated open surface Γ, well known as the “screen". Uniqueness results are proved using appropriate relations for Beltrami fields, and in addition, existence results are established by using a variational method in suitable functional space setting.

Keywords Chiral media • Beltrami fields • Impedance boundary conditions

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Introduction

In this work the scattering problem of plane time-harmonic electromagnetic waves by a partially coated chiral obstacle embedded in an infinite homogeneous isotropic chiral medium is studied. From the mathematical point of view, chiral media satisfy a set of constitutive relations in which the magnetic and electric fields are coupled. Different expressions exist for the constitutive relations [14]; in this work the well-known Drude-Born-Fedorov (DBF) constitutive relations are used. These constitutive relations are chosen because they are symmetric under time

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reversality and duality transformations. Chiral obstacles are characterized by the so-called chirality (or preferential handedness) and the related electromagnetic fields are composed of left circularly polarized (LCP) and right circularly polarized (RCP) components. These fields have independent directions of propagation and different wave numbers. Chirality is common in a variety of naturally occurring and man-made objects (e.g. DNA in molecular scale, helices) and has also played an important role to the study of optical activity. Properties and scattering problems involving chiral media have been studied by many scientists; for an excellent source we refer to [15, 16] and [17] (and therein references). Solvability results concerning direct scattering problems where the obstacle is a perfect conductor or a dielectric (penetrable scatterer) in chiral media can be found in [5, 6]. In these cases, Bohren decomposition is used and an equivalent boundary integral formulation to the scattering problems is considered. Furthermore, boundary integral equations for electromagnetic scattering by a homogeneous chiral obstacle were studied in [4], by using a generalization of Müller's equations for scattering by a non- chiral obstacle. In [1], existence and uniqueness of the solution to the diffraction problem of a plane electromagnetic field by a chiral curved layer covering a perfectly conducting object have been studied. In particular, approximative impedance conditions are given for thin chiral curved layers and optimal error estimates are obtained (the reader can also see [2]). We end up with the work studied in [3], where the LCP and the RCP Beltrami Herglotz functions were defined by an integral representation over the unit sphere where the corresponding kernels are exactly the Beltrami far-field patterns. These functions will play an important role for the investigation of the inverse electromagnetic problem for a mixed-impedance screen in chiral media. For non-chiral media, mixed boundary value problems which describe model of scattering by obstacles that are covered by a thin layer of material on part of their boundaries are studied in [10]. The direct and inverse scattering problem of a time-harmonic electromagnetic plane wave by a mixed perfectly conducting-impedance screen is studied in [8, 11] and [9]. Further, we mention that problems with mixed-impedance boundary conditions in elasticity have been considered in [7].

Setting Up the Problem

We consider a plane time-harmonic electromagnetic wave $\mathbf{E}^{\mathrm{inc}}$ which is propagated in an infinite homogeneous isotropic chiral medium. This field is disturbed by a very thin partially coated chiral obstacle (the scatterer), known as *screen*, which is an open, bounded, smooth surface $\Gamma \in \mathbb{R}^3$ with two sides coated by impedance material. This surface is also a part of a piecewise smooth surface $\partial \mathrm{D}$ of a bounded domain $\mathrm{D} \subset \mathbb{R}^3$. The domain D as well as the infinite medium is filled up with a homogeneous and isotropic chiral medium of *chirality measure* β. For our case we assume that β is a positive constant. We denote $\hat{\mathrm{n}}$ the unit normal vector to Γ which coincides with the outward normal vector defined almost everywhere on $\partial \mathrm{D}$. The boundary condition on each side of this surface obstacle is described by an impedance boundary condition.

For a vector $\mathbf{u}$ we use the notation $\hat{\mathbf{n}} \times \mathbf{u}^+|_\Gamma, \hat{\mathbf{n}} \cdot \mathbf{u}^+|_\Gamma,\ \gamma_T^+\mathbf{u}|_\Gamma$ for the restriction to Γ of the traces $\hat{\mathbf{n}} \times \mathbf{u}^+|_{\partial D}, \hat{\mathbf{n}} \cdot \mathbf{u}^+|_{\partial D}$ and $\gamma_T^+\mathbf{u}|_{\partial D}$, respectively, from the outside of the ∂D, where $\gamma_T^+\mathbf{u} := \hat{\mathbf{n}} \times (\mathbf{u}^+ \times \hat{\mathbf{n}})$ is the tangential component of $\mathbf{u}^+$. Similar considerations for the traces from the inside of the ∂D which are notated by $\hat{\mathbf{n}} \times \mathbf{u}^-|_\Gamma, \hat{\mathbf{n}} \cdot \mathbf{u}^-|_\Gamma,\ \gamma_T^-\mathbf{u}|_\Gamma$ also hold. We also use the notation $\mathbf{u}^\pm|_\Gamma$ when a relation is hold for both the restrictions of the vector $\mathbf{u}$ on Γ.

The total electric field $\mathbf{E}$ is the superposition of the incident electric field $\mathbf{E}^{\text{inc}}$ and the scattered electric field $\mathbf{E}^{\text{sc}}$, i.e.,

$$\mathbf{E} = \mathbf{E}^{\text{inc}} + \mathbf{E}^{\text{sc}}. \tag{1}$$

The scattering electromagnetic problem by a double impedance screen in chiral media is to determine the total electric field $\mathbf{E}$ that satisfies

$$\nabla \times \nabla \times \mathbf{E} = 2\,\gamma^2\,\beta\,\nabla \times \mathbf{E} + \gamma^2\mathbf{E} \quad \text{in } \mathbb{R}^3 \setminus \overline{\Gamma}, \tag{2}$$

$$\hat{\mathbf{n}} \times \nabla \times \mathbf{E}^- = \frac{i\lambda^-\gamma^2}{k^2}\,\hat{\mathbf{n}} \times \mathbf{E}^- \times \hat{\mathbf{n}} + \gamma^2\,\beta\,\hat{\mathbf{n}} \times \mathbf{E}^- \quad \text{on } \Gamma, \tag{3}$$

$$\hat{\mathbf{n}} \times \nabla \times \mathbf{E}^+ = \frac{i\lambda^+\gamma^2}{k^2}\,\hat{\mathbf{n}} \times \mathbf{E}^+ \times \hat{\mathbf{n}} + \gamma^2\,\beta\,\hat{\mathbf{n}} \times \mathbf{E}^+ \quad \text{on } \Gamma, \tag{4}$$

$$\hat{\mathbf{r}} \times \nabla \times \mathbf{E}^{sc} - \beta\,\gamma^2\,\hat{\mathbf{r}} \times \mathbf{E}^{sc} + \frac{i\,\gamma^2}{k}\,\mathbf{E}^{sc} = o(\frac{1}{r}) \quad r \to \infty, \tag{5}$$

where $\gamma^2 = k^2/(1-k^2\beta^2),\ k = \omega\sqrt{\varepsilon\,\mu}$, with ω the angular frequency, ε, μ been the electric permittivity and magnetic permeability, respectively, and $\lambda^-, \lambda^+ \in L_\infty(\Gamma)$ with $\lambda^-, \lambda^+ \geq \lambda_0 > 0$. The Silver-Müller radiation condition (5) holds uniformly in all directions $\hat{\mathbf{r}} = \mathbf{r}/r$ where $r := |\mathbf{r}|$. We note that the electric field $\mathbf{E}$ is divergence-free, that is $\nabla \cdot \mathbf{E} = 0$. In addition, k is not a wave number and its notation has not any particular physical significance.

In what follows we deal with the uniqueness and existence of the solution of the scattering problem (2)–(5) in an appropriate space setting . Hence, we define the following Sobolev spaces:

$$H(\text{curl}, B_\rho \setminus \overline{\Gamma}) := \{\mathbf{u} \in \left[L^2(B_\rho \setminus \overline{\Gamma})\right]^3 : \text{curl}\mathbf{u} \in \left[L^2(B_\rho \setminus \overline{\Gamma})\right]^3\}, \tag{6}$$

$$L_t^2(\Gamma) := \{\mathbf{u} \in [L^2(\Gamma)]^3 : \mathbf{u} \cdot \hat{\mathbf{n}} = 0, \text{ on } \Gamma\}, \tag{7}$$

$$H_{loc}(\text{curl}, \mathbb{R}^3 \setminus \overline{\Gamma}) := \left\{\mathbf{u} \in H(\text{curl}, B_\rho \setminus \overline{\Gamma}) \text{ for every } B_\rho \text{ such that } D \subset B_\rho\right\}, \tag{8}$$

and

$$X(\text{curl}, \mathbb{R}^3\backslash\overline{\Gamma}) := \left\{\mathbf{u} \in H_{loc}(\text{curl}, \mathbb{R}^3 \setminus \overline{\Gamma}) : \hat{\mathbf{n}} \times \mathbf{u}^-|_\Gamma , \hat{\mathbf{n}} \times \mathbf{u}^+|_\Gamma \in L_t^2(\Gamma)\right\}, \quad (9)$$

where B_ρ is a sphere with radius ρ large enough, containing the bounded domain D. The last space is equipped with the graph norm

$$\begin{aligned}\| \mathbf{u} \|^2_{X(\text{curl}, \mathbb{R}^3\backslash\overline{\Gamma})} := \| \nabla \times \mathbf{u} \|^2_{\left(L^2(B_\rho\backslash\overline{\Gamma})\right)^3} + \| \mathbf{u} \|^2_{\left(L^2(B_\rho\backslash\overline{\Gamma})\right)^3} \\ + \| \hat{\mathbf{n}} \times \mathbf{u}^- \|^2_{L_t^2(\Gamma)} + \| \hat{\mathbf{n}} \times \mathbf{u}^+ \|^2_{L_t^2(\Gamma)} . \qquad (10)\end{aligned}$$

Uniqueness Results

In order to prove uniqueness for the scattering problem (2)–(5) we will be based on the Bohren decomposition of the electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ into the $\mathbf{Q}_L$ (LCP) and $\mathbf{Q}_R$ (RCP) Beltrami fields

$$\mathbf{E} = \mathbf{Q}_L - i\eta\mathbf{Q}_R, \quad \mathbf{H} = \frac{1}{i\eta}\mathbf{Q}_L + \mathbf{Q}_R, \qquad (11)$$

where $\eta = \sqrt{\frac{\mu}{\varepsilon}}$ is the intrinsic impedance of the chiral medium. In view of (11) the Beltrami fields are expressed as

$$\mathbf{Q}_L = \frac{\mathbf{E} + i\eta\,\mathbf{H}}{2}, \quad \mathbf{Q}_R = \frac{i\eta^{-1}\mathbf{E} + \mathbf{H}}{2} . \qquad (12)$$

In addition the Beltrami fields satisfy the following equations:

$$\nabla \times \mathbf{Q}_L = \gamma_L\, \mathbf{Q}_L, \quad \nabla \times \mathbf{Q}_R = -\gamma_R\, \mathbf{Q}_R, \qquad (13)$$

where $\gamma_L = k(1 - k\beta)^{-1}$, $\gamma_R = k(1 + k\beta)^{-1}$ are the wave numbers for the Beltrami fields, $\mathbf{Q}_L, \mathbf{Q}_R$, respectively.

The scattered Beltrami fields $\mathbf{Q}_L^{sc}, \mathbf{Q}_R^{sc}$, satisfy the Silver-Müller type radiation conditions [3, 5]

$$\hat{\mathbf{r}} \times \mathbf{Q}_L^{sc} + i\,\mathbf{Q}_L^{sc} = o(\frac{1}{r}), \quad \hat{\mathbf{r}} \times \mathbf{Q}_R^{sc} - i\,\mathbf{Q}_R^{sc} = o(\frac{1}{r}), \quad \text{as} \quad r \to \infty, \qquad (14)$$

as well as the asymptotic relations

$$\mathbf{Q}_L^{sc} = O(\frac{1}{r}), \quad \mathbf{Q}_R^{sc} = O(\frac{1}{r}), \quad \text{as} \quad r \to \infty. \qquad (15)$$

Relations (14) and (15) are obtained via (12) with the aid of the asymptotic behaviour of **E**, **H** as $r \to \infty$, [12]. In what follows, with the notation $\overline{\mathbf{Q}}_A$, $A = L, R$, the bar "−" will denote the conjugate vector of $\mathbf{Q}_A$ and with $\mathbf{Q}_A^-$, $\mathbf{Q}_A^+$ we denote the limit from inside and outside of the boundary ∂D, respectively. In addition, the notation $\mathbf{Q}_A^{\pm}$ is for both the previous limits. We are now ready to proceed with the following proposition:

Theorem 2.1 *The Beltrami fields* $\mathbf{Q}_A$, $A = L, R$, *with* $\mathbf{Q}_A \in X(curl, \mathbb{R}^3 \setminus \overline{\Gamma})$, *satisfy the following relation:*

$$\int_{S_\rho} \hat{\mathbf{x}} \cdot (\mathbf{Q}_A \times \overline{\mathbf{Q}}_A)\, ds = \int_\Gamma \hat{\mathbf{n}} \cdot (\mathbf{Q}_A^- \times \overline{\mathbf{Q}}_A^-)\, ds - \int_\Gamma \hat{\mathbf{n}} \cdot (\mathbf{Q}_A^+ \times \overline{\mathbf{Q}}_A^+)\, ds, \tag{16}$$

where $S_\rho = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| = \rho\}$ *and* $\hat{\mathbf{x}}$ *is the unit normal vector to the spherical surface* S_ρ.

Proof The reader can be found an analogous proposition in [8], and hence the proof is omitted for brevity. ■

Further we have the following result:

Theorem 2.2 *The Beltrami fields* $\mathbf{Q}_L$, $\mathbf{Q}_R \in X(curl, \mathbb{R}^3 \setminus \overline{\Gamma})$ *satisfy the relation*

$$\begin{aligned} &\Im\left(\frac{1}{\eta}\int_\Gamma \hat{\mathbf{n}} \cdot (\mathbf{Q}_L^{\pm} \times \overline{\mathbf{Q}}_L^{\pm})\, ds - \eta \int_\Gamma \hat{\mathbf{n}} \cdot (\mathbf{Q}_R^{\pm} \times \overline{\mathbf{Q}}_R^{\pm})\, ds\right) \\ &= \frac{k}{\eta}\int_\Gamma \frac{1}{\lambda^{\pm}}(|\mathbf{U}^{\pm}|^2 - |\hat{\mathbf{n}} \cdot \mathbf{U}^{\pm}|^2)\, ds, \end{aligned} \tag{17}$$

where $\mathbf{U}^{\pm} := \mathbf{Q}_L^{\pm} + i\,\eta\,\overline{\mathbf{Q}}_R^{\pm}$.

Proof The boundary conditions (3) and (4) via the relations (11) and (13) and the vector identity $\mathbf{u} = (\hat{\mathbf{n}} \cdot \mathbf{u})\hat{\mathbf{n}} - \hat{\mathbf{n}} \times (\mathbf{u} \times \hat{\mathbf{n}})$ lead to

$$\begin{aligned} \mathbf{Q}_L^{\pm} &= i\eta\mathbf{Q}_R^{\pm} + [\hat{\mathbf{n}} \cdot (\mathbf{Q}_L^{\pm} - i\eta\mathbf{Q}_R^{\pm})]\hat{\mathbf{n}} \\ &\quad + \frac{ik^2}{\lambda^{\pm}}(\beta - \frac{\gamma_L}{\gamma^2})\, \hat{\mathbf{n}} \times \mathbf{Q}_L^{\pm} + \frac{\eta k^2}{\lambda^{\pm}}(\beta + \frac{\gamma_R}{\gamma^2})\, \hat{\mathbf{n}} \times \mathbf{Q}_R^{\pm} \quad \text{on } \Gamma, \end{aligned} \tag{18}$$

but via

$$\beta - \frac{1}{\gamma_R} = -\frac{1}{k} \quad \text{and} \quad \beta + \frac{1}{\gamma_L} = \frac{1}{k} \tag{19}$$

the relation (18) takes the form

$$\begin{aligned} \mathbf{Q}_L^{\pm} &= i\eta\mathbf{Q}_R^{\pm} + [\hat{\mathbf{n}} \cdot (\mathbf{Q}_L^{\pm} - i\eta\mathbf{Q}_R^{\pm})]\hat{\mathbf{n}} \\ &\quad - \frac{ik}{\lambda^{\pm}}\, \hat{\mathbf{n}} \times \mathbf{Q}_L^{\pm} + \frac{\eta}{\lambda^{\pm}}\, \hat{\mathbf{n}} \times \mathbf{Q}_R^{\pm} \quad \text{on } \Gamma. \end{aligned} \tag{20}$$

Multiplying (20) by $\hat{\mathbf{n}}$ and then by $\overline{\mathbf{Q}}_L^{\pm}$ we arrive at

$$\begin{aligned}\hat{\mathbf{n}}\cdot(\mathbf{Q}_L^{\pm}\times\overline{\mathbf{Q}}_L^{\pm}) = {} & i\eta\,\hat{\mathbf{n}}\cdot(\mathbf{Q}_R^{\pm}\times\overline{\mathbf{Q}}_L^{\pm}) \\ & -\frac{ik}{\lambda^{\pm}}(\hat{\mathbf{n}}\cdot\mathbf{Q}_L^{\pm})(\hat{\mathbf{n}}\cdot\overline{\mathbf{Q}}_L^{\pm})+\frac{\eta k}{\lambda^{\pm}}(\hat{\mathbf{n}}\cdot\mathbf{Q}_R^{\pm})(\hat{\mathbf{n}}\cdot\overline{\mathbf{Q}}_L^{\pm}) \\ & +\frac{ik}{\lambda^{\pm}}\mathbf{Q}_L^{\pm}\cdot\overline{\mathbf{Q}}_L^{\pm}-\frac{\eta k}{\lambda^{\pm}}\mathbf{Q}_R^{\pm}\cdot\overline{\mathbf{Q}}_L^{\pm}\end{aligned} \tag{21}$$

as well as

$$\begin{aligned}\hat{\mathbf{n}}\cdot(\mathbf{Q}_R^{\pm}\times\overline{\mathbf{Q}}_R^{\pm}) = {} & -\frac{i}{\eta}\,\hat{\mathbf{n}}\cdot(\mathbf{Q}_L^{\pm}\times\overline{\mathbf{Q}}_R^{\pm}) \\ & +\frac{ik}{\lambda^{\pm}}(\hat{\mathbf{n}}\cdot\mathbf{Q}_R^{\pm})(\hat{\mathbf{n}}\cdot\overline{\mathbf{Q}}_R^{\pm})+\frac{k}{\eta\lambda^{\pm}}(\hat{\mathbf{n}}\cdot\mathbf{Q}_L^{\pm})(\hat{\mathbf{n}}\cdot\overline{\mathbf{Q}}_R^{\pm}) \\ & -\frac{ik}{\lambda^{\pm}}\mathbf{Q}_R^{\pm}\cdot\overline{\mathbf{Q}}_R^{\pm}-\frac{k}{\eta\lambda^{\pm}}\mathbf{Q}_L^{\pm}\cdot\overline{\mathbf{Q}}_R^{\pm}.\end{aligned} \tag{22}$$

For the remaining of the proof of (17), we use (21) and (22) in order to evaluate the quantity

$$\frac{1}{\eta}\hat{\mathbf{n}}\cdot(\mathbf{Q}_L^{\pm}\times\overline{\mathbf{Q}}_L^{\pm})-\eta\hat{\mathbf{n}}\cdot(\mathbf{Q}_R^{\pm}\times\overline{\mathbf{Q}}_R^{\pm}). \tag{23}$$

Taking into account that

$$i\hat{\mathbf{n}}\cdot(\mathbf{Q}_R^{\pm}\times\overline{\mathbf{Q}}_L^{\pm})+i\hat{\mathbf{n}}\cdot(\mathbf{Q}_L^{\pm}\times\overline{\mathbf{Q}}_R^{\pm}) \tag{24}$$

is a real number, after some calculations we can arrive at the relations

$$\frac{ik}{\eta\lambda^{\pm}}\mathbf{Q}_L^{\pm}\cdot\overline{\mathbf{Q}}_L^{\pm}-\frac{k}{\lambda^{\pm}}\mathbf{Q}_R^{\pm}\cdot\overline{\mathbf{Q}}_L^{\pm}+\frac{k}{\lambda^{\pm}}\mathbf{Q}_L^{\pm}\cdot\overline{\mathbf{Q}}_R^{\pm}+\frac{i\eta k}{\lambda^{\pm}}\mathbf{Q}_R^{\pm}\cdot\overline{\mathbf{Q}}_R^{\pm}=\frac{ik}{\eta\lambda^{\pm}}|\mathbf{U}^{\pm}|^2, \tag{25}$$

and

$$\begin{aligned}&-\frac{ik}{\eta\lambda^{\pm}}(\hat{\mathbf{n}}\cdot\mathbf{Q}_L^{\pm})(\hat{\mathbf{n}}\cdot\overline{\mathbf{Q}}_L^{\pm})+\frac{k}{\lambda^{\pm}}(\hat{\mathbf{n}}\cdot\mathbf{Q}_R^{\pm})(\hat{\mathbf{n}}\cdot\overline{\mathbf{Q}}_L^{\pm})-\frac{k}{\lambda^{\pm}}(\hat{\mathbf{n}}\cdot\mathbf{Q}_L^{\pm})(\hat{\mathbf{n}}\cdot\overline{\mathbf{Q}}_R^{\pm}) \\ &-\frac{i\eta k}{\lambda^{\pm}}(\hat{\mathbf{n}}\cdot\mathbf{Q}_R^{\pm})(\hat{\mathbf{n}}\cdot\overline{\mathbf{Q}}_R^{\pm})=-\frac{ik}{\eta\lambda^{\pm}}|\hat{\mathbf{n}}\cdot\mathbf{U}^{\pm}|^2,\end{aligned} \tag{26}$$

and hence, the assertion of the proposition is proved. ■

In the sequel, uniqueness for the boundary value problem (2)–(5) will be established. We will consider the corresponding homogeneous scattering problem of (2)–(5), i.e., incident electric field $\mathbf{E}^{\text{inc}} = \mathbf{0}$. Relations (12), (16) and (23) will be used in order to prove the following uniqueness theorem:

Theorem 2.3 *The electromagnetic scattering problem (2)–(5) in chiral media, for* $\mathbf{E}^{inc} = \mathbf{0}$*, has the trivial solution.*

Proof By radiation conditions (14) we have

$$\lim_{\rho\to\infty}\left(\frac{1}{\eta}\int_{S_\rho}|\hat{\boldsymbol{\rho}}\times\mathbf{Q}_L + i\mathbf{Q}_L|^2 ds + \eta\int_{S_\rho}|\hat{\boldsymbol{\rho}}\times\mathbf{Q}_R - i\mathbf{Q}_R|^2 ds\right) = 0. \tag{27}$$

If $D_{ex} = \mathbb{R}^3\backslash D$, relation (27) with the aid of the divergence theorem in D and $D_{ex}\cap B_\rho$ for the vectors $\mathbf{Q}_A\times\overline{\mathbf{Q}}_A,\ A = L, R$ with $\mathbf{Q}_A\in X(\text{curl}, \mathbb{R}^3\setminus\overline{\Gamma})$, and due to (16), yields to

$$\begin{aligned}&\lim_{\rho\to\infty}\left(\frac{1}{\eta}\int_{S_\rho}|\hat{\boldsymbol{\rho}}\times\mathbf{Q}_L|^2 ds + \frac{1}{\eta}\int_{S_\rho}|\mathbf{Q}_L|^2 ds + \eta\int_{S_\rho}|\hat{\boldsymbol{\rho}}\times\mathbf{Q}_R|^2 ds + \eta\int_{S_\rho}|\mathbf{Q}_R|^2 ds\right)\\&+2\,\Im\left(\frac{1}{\eta}\int_\Gamma(\mathbf{Q}_L^+\times\overline{\mathbf{Q}}_L^+)\cdot\hat{\mathbf{n}}\,ds - \eta\int_\Gamma(\mathbf{Q}_R^+\times\overline{\mathbf{Q}}_R^+)\cdot\hat{\mathbf{n}}\,ds\right)\\&+2\,\Im\left(\frac{1}{\eta}\int_\Gamma(\mathbf{Q}_L^-\times\overline{\mathbf{Q}}_L^-)\cdot\hat{\mathbf{n}}\,ds - \eta\int_\Gamma(\mathbf{Q}_R^-\times\overline{\mathbf{Q}}_R^-)\cdot\hat{\mathbf{n}}\,ds\right) = 0\end{aligned} \tag{28}$$

Taking into account (17) and (28), via Rellich's lemma in chiral media [6], we arrive at $\mathbf{Q}_L = \mathbf{Q}_R = \mathbf{0}$, and from (11) the theorem now easily follows. ■

Existence of the Solution

In this section we will prove the existence of the solution of the scattering problem (2)–(5) using a variational method. Having in mind the Sobolev spaces defined in (6)–(9), we multiply equation (2) by a test function $\mathbf{w}\in X(\text{curl}, \mathbb{R}^3\setminus\overline{\Gamma})$ and we integrate by parts in D and $D_{ex}\cap B_\rho$. If we apply the divergence and the first vector Green's theorem in D and $D_{ex}\cap B_\rho$, in view of the continuity of $\hat{\mathbf{n}}\times\mathbf{E}$ and $\hat{\mathbf{n}}\times\nabla\times\mathbf{E}$ across $\partial D\setminus\overline{\Gamma}$ we can obtain the variational form of the scattering problem

$$\begin{aligned}&\int_D(\nabla\times\mathbf{E})\cdot(\nabla\times\overline{\mathbf{w}})\,du + \int_{D_{ex}\cap B_\rho}(\nabla\times\mathbf{E})\cdot(\nabla\times\overline{\mathbf{w}})\,du\\&-2\gamma^2\beta\int_D\mathbf{E}\cdot(\nabla\times\overline{\mathbf{w}})\,du - 2\gamma^2\beta\int_{D_{ex}\cap B_\rho}\mathbf{E}\cdot(\nabla\times\overline{\mathbf{w}})\,du\\&-\gamma^2\int_D\mathbf{E}\cdot\overline{\mathbf{w}}\,du - \gamma^2\int_{D_{ex}\cap B_\rho}\mathbf{E}\cdot\overline{\mathbf{w}}\,du\end{aligned}$$

$$+\frac{i\gamma^2}{k^2}\int_\Gamma \lambda_+\,\gamma_T^+\mathbf{E}\cdot\gamma_T^+\,\overline{\mathbf{w}}\,ds-\frac{i\gamma^2}{k^2}\int_\Gamma \lambda^-\,\gamma_T^-\mathbf{E}\cdot\gamma_T^-\,\overline{\mathbf{w}}\,ds$$

$$-\gamma^2\beta\int_\Gamma(\hat{\mathbf{n}}\times\mathbf{E})\cdot\gamma_T^+\,\overline{\mathbf{w}}\,ds+\gamma^2\beta\int_\Gamma(\hat{\mathbf{n}}\times\mathbf{E})\cdot\gamma_T^-\,\overline{\mathbf{w}}\,ds$$

$$+\int_{S_\rho}G_{kce}(\hat{\mathbf{x}}\times\mathbf{E})\cdot\gamma_T\overline{\mathbf{w}}ds$$

$$=\ -\frac{i\gamma^{\,2}}{k^2}\int_\Gamma\lambda^+\,\gamma_T^+\,\mathbf{E}^{\mathrm{inc}}\cdot\gamma_T^+\overline{\mathbf{w}}ds+\frac{i\gamma^{\,2}}{k^2}\int_\Gamma\lambda^-\,\gamma_T^-\mathbf{E}^{\mathrm{inc}}\cdot\gamma_T^-\overline{\mathbf{w}}ds$$

$$+\gamma^2\beta\int_\Gamma(\hat{\mathbf{n}}\times\mathbf{E}^{\mathrm{inc}})\cdot\gamma_T^+\overline{\mathbf{w}}\,ds-\gamma^2\beta\int_\Gamma(\hat{\mathbf{n}}\times\mathbf{E}^{\mathrm{inc}})\cdot\gamma_T^-\overline{\mathbf{w}}\,ds$$

$$-\int_{S_\varrho}G_{kce}(\hat{\mathbf{x}}\times\mathbf{E}^{\mathrm{inc}})\cdot\gamma_T\overline{\mathbf{w}}\,ds\ ,\tag{29}$$

where $\mathbf{E}^{\mathrm{inc}}$ is a given field and G_{kce} is a Calderon type operator in chiral media which maps a tangential vector field $\hat{\mathbf{x}}\times\mathbf{E}$ on S_ρ to an also tangential vector field $\hat{\mathbf{x}}\times(\nabla\times\mathbf{E}-2\gamma^2\beta\,\mathbf{E})$ on the same surface space. These operators for non-chiral media have been studied in [13] and [18]. The authors of this article will present Calderon type operators in chiral media, as well as their identities, in a future work.

We are going to prove gradually the existence theorem:

Theorem 3.1 *For any given field $\mathbf{E}^{\mathrm{inc}}\in X(curl,\mathbb{R}^3\setminus\overline{\Gamma})$ the electromagnetic scattering problem in chiral media (29) has a unique solution $\mathbf{E}\in X(curl,\mathbb{R}^3\setminus\overline{\Gamma})$.*

The scattered field $\mathbf{E}$ in (29) also satisfies

$$\nabla\times\nabla\times\mathbf{E}=2\,\gamma^2\beta\,\nabla\times\mathbf{E}+\gamma^2\mathbf{E},\quad \text{in }\mathbb{R}^3\setminus\overline{B}_\rho\tag{30}$$

$$\hat{\mathbf{x}}\times\mathbf{E}=\xi,\quad\text{on }S_\rho\tag{31}$$

$$\hat{\mathbf{r}}\times\nabla\times\mathbf{E}-\beta\,\gamma^2\,\hat{\mathbf{r}}\times\mathbf{E}+\frac{i\gamma^2}{k}\mathbf{E}=o(\frac{1}{r}),\qquad r\to\infty\tag{32}$$

where $\xi\in L_t^2(S_\rho)$. We note that in (29), we have taken into account that for the incident electric field $\mathbf{E}^{\mathrm{inc}}$ the equation

$$\nabla\times\nabla\times\mathbf{E}^{\mathrm{inc}}-2\,\gamma^2\,\beta\,\nabla\times\mathbf{E}^{\mathrm{inc}}-\gamma^2\mathbf{E}^{\mathrm{inc}}=\mathbf{0},\quad\text{in }\mathbb{R}^3\tag{33}$$

holds. We now define the space

$$S:=\left\{p\in H^1(B_\rho\setminus\overline{\Gamma}):\,p^-|_\Gamma=c^-\ \text{and}\ p^+|_\Gamma=c^+\right\},\tag{34}$$

where c^+ and c^- are constant numbers, as well as the space

$$X^0 := \left\{ \mathbf{u} \in X(\mathrm{curl}, \mathbb{R}^3 \setminus \overline{\Gamma}) : < G_{kce}(\hat{\mathbf{x}} \times \mathbf{u}) \, , \nabla_{S_\rho} q > - \gamma^2 (\mathbf{u} \, , \nabla q)_{B_\rho} = 0, \quad \text{for } q \in S \right\}. \quad (35)$$

Then we write (29) in a more compact form

$$A(\mathbf{u}, \mathbf{w}) = B(\mathbf{w}), \tag{36}$$

where

$$\begin{aligned} A(\mathbf{u}, \mathbf{w}) \; &= (\nabla \times \mathbf{u}, \nabla \times \mathbf{w})_D + (\nabla \times \mathbf{u}, \nabla \times \mathbf{w})_{D_{ex} \cap B_\rho} \\ &-2\gamma^2 \beta \left((\mathbf{u}, \nabla \times \mathbf{w})_D + (\mathbf{u}, \nabla \times \mathbf{w})_{D_{ex} \cap B_\rho} \right) \\ &-\gamma^2 ((\mathbf{u}, \mathbf{w})_D + (\mathbf{u}, \mathbf{w})_{D_{ex} \cap B_\rho}) + < G_{kce}(\hat{\mathbf{x}} \times \mathbf{u}), \gamma_T \mathbf{w} >_{S_\rho} \\ &+ \frac{i\gamma^2}{k^2} < \lambda^+ \gamma_T^+ \mathbf{u}, \gamma_T^+ \mathbf{w} >_\Gamma - \frac{i\gamma^2}{k^2} < \lambda^- \gamma_T^- \mathbf{u}, \gamma_T^- \mathbf{w} >_\Gamma \\ &-\gamma^2 \beta < \hat{\mathbf{n}} \times \mathbf{u}, \gamma_T^+ \mathbf{w} >_\Gamma + \gamma^2 \beta < \hat{\mathbf{n}} \times \mathbf{u}, \gamma_T^- \mathbf{w} >_\Gamma , \end{aligned} \tag{37}$$

and the right part of equation (36), due to (29), consists of boundary data

$$\begin{aligned} B(\mathbf{w}) = &\frac{i\gamma^2}{k^2} < \lambda^- \gamma_T^- \mathbf{E}^{\mathrm{inc}}, \gamma_T^- \mathbf{w} >_\Gamma - \frac{i\gamma^2}{k^2} < \lambda^+ \gamma_T^+ \mathbf{E}^{\mathrm{inc}}, \mathbf{w} >_\Gamma \\ &+ \gamma^2 \beta < \hat{\mathbf{n}} \times, \mathbf{E}^{\mathrm{inc}} \gamma_T^+ \mathbf{w} >_\Gamma - \gamma^2 \beta < \hat{\mathbf{n}} \times \mathbf{E}^{\mathrm{inc}}, \gamma_T^- \mathbf{w} >_\Gamma \\ &- < G_{kce}(\hat{\mathbf{x}} \times \mathbf{E}^{\mathrm{inc}}), \gamma_T \mathbf{w} >_{S_\rho} . \end{aligned} \tag{38}$$

The first step is to prove the following:

Lemma 3.2 *The equation $A(\nabla p \, , \nabla q) = B(\nabla q)$ has a unique solution for any $q \in S$.*

Proof We put $\mathbf{u} = \nabla p$ and $\mathbf{w} = \nabla q$ so the equation (36) takes the form

$$-\gamma^2 (\nabla p, \nabla q)_{L^2(B_\rho)} + < G_{kce}(\hat{\mathbf{x}} \times \nabla p), \nabla_{S_\rho} q >_{S_\rho} = B(\nabla q), \tag{39}$$

since $(\nabla p)_T = \hat{\mathbf{n}} \times \nabla p \times \hat{\mathbf{n}} = \gamma_T \nabla p = \mathbf{0}$ for $p \in S$. Then compactness properties of the Calderon type operators allow us to apply the usual procedure of the Fredholm alternative theory to (39) as in [18] in order to complete the proof. ■

We move on with the next step which is the lemma below:

Lemma 3.3 *∇S is a closed subspace of $X(curl, \mathbb{R}^3 \setminus \overline{\Gamma})$ and*

$$X(curl, \mathbb{R}^3 \setminus \overline{\Gamma}) = X^0 \oplus \nabla S. \tag{40}$$

Proof The space ∇S is closed in $X(\text{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})$ since S is closed in $H^1(B_\rho \setminus \overline{\Gamma})$ [11].

Then if $\mathbf{u} \in X(\text{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})$ is a solution of (36), we have $A(\mathbf{u}, \nabla q) = B(\nabla q)$ for any $q \in S$. We consider that $\mathbf{u} = \mathbf{v} + \nabla p_0$, where ∇p_0 is the unique solution of (39), so it holds $A(\mathbf{v}, \nabla q) = 0$ and from the definition (35) we take $\mathbf{v} \in X^0$. Then it is easy to prove that this expression of $\mathbf{u}$ as a sum of elements of ∇S and X^0 is the unique one. ■

Then we are going to deal with the equation $A(\mathbf{u}, \mathbf{v}) = B(\mathbf{v})$, for any $\mathbf{v} \in X^0$, which finally takes the form

$$A(\mathbf{w}, \mathbf{v}) = B(\mathbf{v}) - A(\nabla p_0, \mathbf{v}), \quad \text{for any } \mathbf{v} \in X^0. \tag{41}$$

We continue our proof with the following result, due to [8].

Lemma 3.4 *The space X^0 is compactly imbedded in $L^2(B_\rho)$.*

Proof We consider a sequence $\{\mathbf{u}_n^{ex}\}_{n=1}^{\infty}$ of solutions of the scattering problem

$$\nabla \times \nabla \times \mathbf{u}_n^{ex} = 2\gamma^2 \beta \nabla \times \mathbf{u}_n^{ex} + \gamma^2 \mathbf{u}_n^{ex} \quad \text{in } \mathbb{R}^3 \setminus \overline{B}_\rho, \tag{42}$$

$$\hat{\mathbf{x}} \times \mathbf{u}_n^{ex} = \hat{\mathbf{x}} \times \mathbf{u}_n \quad \text{on } S_\rho, \tag{43}$$

$$\hat{\mathbf{r}} \times \nabla \times \mathbf{u}_n^{ex} - \beta \gamma^2 \hat{\mathbf{r}} \times \mathbf{u}_n^{ex} + \frac{i\gamma^2}{k} \mathbf{u}_n^{ex} = o(\frac{1}{r}) \quad r \to \infty, \tag{44}$$

where $\{\mathbf{u}_n\}_{n=1}^{\infty}$ is a given bounded sequence in X^0. For the solutions $\mathbf{u}_n^{ex}$ we can give series expansions using proper vector wave functions in chiral media analogous to [18]. The boundary condition (43) and the definition (35) lead to the conclusion that the vectors $\mathbf{u}_n^{ex}$ and $\mathbf{u}_n$ have equal normal and tangential components on S_ρ so each element of the sequence $\mathbf{u}_n$ can be extended to a function $\mathbf{u}_n^0 \in H_{loc}(\text{curl}, B_\rho \setminus \overline{\Gamma})$ to all $\mathbb{R}^3$, defined as,

$$\mathbf{u}_n^0 = \begin{cases} \mathbf{u}_n & \text{in } B_\rho, \\ \mathbf{u}_n^{ex} & \text{in } \mathbb{R}^3 \setminus \overline{B}_\rho. \end{cases} \tag{45}$$

Following analogous ideas for chiral media as those in [11], the proof is completed. ■

The above result allows us to define proper compact operators in order to apply again the Fredholm alternative theory to (41), and with the aid of Lemma 3.4 to prove that Eq. (41) has a unique solution. This conclusion completes the proof of Theorem 3.1 for the existence of the solution of (29), and therefore establishes the existence result for the scattering problem (2)–(5).

Conclusions

This paper was concerned with the solvability of the direct electromagnetic scattering problem by a chiral impedance screen in a chiral environment. In particular, the terms $2\gamma^2\,\beta\nabla\times\mathbf{E}^{\mathrm{inc}}$, $\gamma^2\beta\,\hat{\mathbf{n}}\times\mathbf{E}$ in (2)–(4) were the main reason for using the Beltrami fields in order to prove uniqueness for the electromagnetic problem in chiral media. We also make the following remarks:

1. If $\beta = 0$, i.e., non-chiral environment, the approach for existence and uniqueness is similar to the case for the mixed scattering problem in [11], which holds for scattering by a screen in non-chiral media. In addition if $\lambda^+ = 0$ and $\lambda^- = 0$, we can analogous prove that the scattering problem (2)–(5) has a unique solution.
2. In the case where the chirality measure β is not a constant, our method can also be applied, since the modifications that occurred can be handled.

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