Several Geometries for Movements Generations

Daniel Bennequin and Alain Berthoz

Abstract In previous works we reanalyzed the kinematics of hand movements and locomotion, and suggested that several geometries are used conjointly by the brain for according the shape and the duration along trajectories; this was done in collaboration with Tamar Flash and her collaborators [10, 64, 67], and with Quang-Cuong Pham [79]. The variety of geometries which were implied in this process, were associated to sub-groups of the affine group of a plane: full affine, equi-affine and Euclidean. Other studies have shown how the above geometries constrain the production of the movements [92], or began to use the affine geometry in Robotics [80]. In this article, we propose to use a new variety of geometries which extends the preceding series in another direction, to cover wider contexts and more complex movements, like prehension, initiation of walking, locomotion, navigation, imagined motion. The new spaces adapted to those geometries have no points; they come from topos theory, which is an extension of set theory replacing sets by fields and graphs of dynamics. Any given topos generates a variety of different geometries, which can be mixed as in the preceding studies. Such geometries take into account efforts, forces and dynamics; they do not neglect them aside as does traditional geometry. In this preliminary report we indicate the simplest characteristics of spaces which underly the above examples. The hypothesis is also that these spaces are implemented in different, although overlapping, central nervous system networks in the brain, corresponding to the different action spaces mentioned above. Here, as for the known classical geometries, the most concrete suggestion concerns the timing of movement: we predict that different components of the controlled system are using *different intrinsic time courses*, and that the mapping between these different internal durations is an important part of the dynamic under geometrical control. This reminds us of a well

D. Bennequin (✉)
Institut de Mathématiques de Jussieu, Université Paris VII, Paris, France
e-mail: bennequin@math.univ-paris-diderot.fr

A. Berthoz
Collège-de-France, Paris, France
e-mail: alain.berthoz@college-de-france.fr

J.-P. Laumond et al. (eds.), *Geometric and Numerical Foundations of Movements*,
Springer Tracts in Advanced Robotics 117, DOI 10.1007/978-3-319-51547-2_2

known psychological observation, for instance that time in imagination does not flow as ordinary clocks time, but this also suggests that reaching an object with the hand has its own time, or that equilibrium control in walking works within a specific time, which is different from the walking trajectory displacement time.

1 Introduction. Geometry, from Spaces to Transformations

Geometries as Transformations Groups; Multiple Geometries.

Geometry appeared first as a science of space (or planes) and simple objects (or figures). Accumulation of mathematical knowledge about measuring, constructing, cutting and pasting has begun five thousands of years ago, in Sumer and Babylonia, cf. Eleanor Robson [95], and in Egypt, cf. Annette Imhausen [51]. A formally perfect theory, based on axioms and demonstrations, was invented in Greece, cf. Euclide, Archimedes, Apollonius. But with centuries of practice and reflection, Geometry became gradually a science of *transformations*. A strong emphasis in this direction was proposed by arabian and persian mathematicians of the *IX*-th to *XII*-th centuries, in particular Abd Al-Jalīl al-Sijzī (Al-Sijistani), Abu Sahl Al Quhrī, Ibn Al-Haytham (Alhazeen), cf. [94]. A clear formulation of this evolution was proposed at the end of XIXth century, by F. Klein in his Erlangen program [53], when he told that *the essence of a Geometry is contained in a group and a family of subgroups*, cf. Appendix 1.

For our theory, we shall retain first, the idea of *multiple geometries*, and second the Galois idea of a *spectrum of conjugated subgroups in a group*, giving a new notion of what is a general space in a geometry.

The notion of parallelism was explicitly separated from the notion of distance in geometry by Euler in the XVIIIth century, under the name of affine geometry; then an independent treatment of points, lines and planes was at the core of projective geometry, as it was foreseen by Kepler an Desargues in the first part of the XVIIth century, when they introduced the idea of points at infinity. In addition a theory of curved forms and their qualitative relations, anticipated by Pascal and Leibniz, was offered in the middle of XIXth century by the rising science of Topology, mostly due to Riemann. This was also the time of the discovery of several geometries which violate the axiom of parallels of Euclide, in particular the hyperbolic geometry of Bolyai, Lobatchevski and Gauss, but also the elliptic geometry on the projective space. We therefore have to admit the multiplicity of geometries. These geometries are essentially characterized by the type of change of reference frames allowed by them, which belong to a given group of transformations. Moreover, and crucial to our proposal, the points which constitute a space, are characterized by the sub-groups of transformations that let them unchanged. (Cf. Appendix 1.)

The Origin of Spaces; Indifference Spectrum.

Finally Helmholtz and Poincaré [86–89, 104] invited us to question the nature of space. Their essential conclusion is that the only way for an animal to organize itself for acting in space, is to incorporate what is due to its action and what is due to an external change in the world. For this purpose, the animal (or any living organism) has to perform an active internal comparison between the sensory effects which result of voluntary self motions and the ones resulting from modifications of the world outside. To internalize this difference is a necessity for survival.

A change of apparent visually perceived form of an object can be due either to the movement of the object in space or to my own movements around it, this defines a special set of "ambiguous" transformations (because they deal with this dual potential interpretation); they form a group G. Every group can be interpreted as defining a particular structure of *ambiguity*. However, the form of the thick frontier between the inner and the outer world is not the group itself, it is a certain faithful representation of G by permutations on a set (ensemble). In our case, with Euclidean displacements, Poincaré showed that this set (ensemble) is the collection of certain sub-groups (of the form gHg^{-1}, cf. Appendix 1), that are transformations having no effect on particular end sensors, like the end of fingers or the retinal fovea (cf. [28]); those sub-groups form a structure of *indifference*. All this framework was already present in the seminal work of Galois about the ambiguity on the solutions of an algebraic equation [38]. Forgetting the internal structure of the sub-groups, and considering each one as a "point", induces a set, named a quotient set, on which G acts transitively; by definition, this is a *geometrized space*. (Cf. the up cited Poincaré books, and [8, 28].)

From this point of view, spatial knowledge is equivalent to the organisation of the command of motions, and geometrical rules describe a form of interaction with the world and with agents acting in the world.

From an experimental point of view, D. Philipona, K. O'Reagan, J.-P. Nadal [82] have succeeded to implement the approach of Poincaré on a virtual robot, to recover the dimension 6 of the group of isometries in $3D$ space.

The consideration of variable curvatures induced a revolution in Geometry: starting with plane and space curves and with surfaces in the space (Monge, Gauss), the study of curvature was extended to manifolds of every dimension by Riemann. Here the infinitesimal reference was Euclidean, but after Klein and Sophus Lie, it became evident that all kinds of Klein geometries associated to a differentiable Lie group provide an extended notion of curvature. The complete theory was developed by Elie Cartan in the first part of XXth century, and was named Cartan geometry, cf. [96]. This considerably extended the range of geometrized spaces.

However several new directions appeared in Geometry in the second part of XXth century; for instance, coming from Topology, the geometrical study of the dynamical fields on manifolds and their deformations (Whitney, Thom, Milnor, Smale et al.), cf. [101, 102], or coming from Algebraic Geometry and Arithmetics, the development of categories and topos (Eilenberg, Mac Lane, Grothendieck, Verdier et al.), cf. [42, 62]. As the name "functor" for natural maps between categories is not very appealing for a non-mathematician, we prefer in this text to use the name "field" in its place, which gives a better intuition. This is justified by the following example: the simplest physical field is a scalar depending on the place, for instance the temperature $T(x, y, z)$, where x, y, z denote Cartesian coordinates of the points in the usual space, and it is also a functor from the category $0 \to 1$, with the two objects 0 and 1, and one arrow between them, to the category of sets (mathematical word for ensembles). In the same manner, any vector field $V(x, y, z)$ is such a functor. Note that we can replace the simple oriented graph $0 \to 1$ by any oriented graph Γ, and get in this way a representation of interacting fields. We will see in what follows how this categorical framework permits to enrich our description of geometries adapted to complex movements.

We can summarize the above evolution of geometry as follows:
a geometry is made by a certain set of transformations; in the traditional point of view, this set constitutes a group of transformations of a space; in the new extended point of view, it is a field of natural transformations of a field of spaces into itself.

The Relation with Biology and Neuroscience.

Several biologists and psychologists have suggested that the inner representation of space is associated to movement production; many of them have insisted on the importance of group theory, and geometrical invariance. They explained that groups organize perception and action together. In particular, the experiments and the theories of J.J. and E.J. Gibson [39, 97] deserve to be cited. J. Piaget reported that the psychological evolution of children follows an ordered sequence of different geometries, first topological, then projective, then affine, and finally Euclidean [83, 84].

In the domain of vision, we must mention the works of J. Koenderinck and A. van Dorn [54], about the role of affine geometry in visual motion perception, and F. Wolf and his collaborators, who attributed a decisive role to the group of displacements in the visual plane for the organization of cortical maps of $V1$ [106]. In addition, J. Koenderinck used all kinds of possible groups for planar geometries arising in the perception and the analysis of images [55]. Considerations of Differential geometry and Lie groups theory were also used in the context of visual neuroscience by J. Petitot, P. Chossat and O. Faugeras, D. Barbieri, G. Citti and A. Sarti.

As reminded by R. Llinas in his book, *I of the vortex: From neurons to self* [61], the structure of vertebrates brains appeared in schematic form in the larvae of the ascidian, just before the vertebrates (5.10^8 years ago): the tunicate larva has one eye, one otolith, a chord and several muscles to control movements and to perceive space. In particular, the elements for the Euclidian group were already present. Thus, the origin of our brain's structure and dynamics is motor control, in the wide sense, to orient itself, to move in water, to navigate and decide where it will be the best to stop.

As claimed by A. Pellionisz and R. Llinas [74, 75], the brain is a geometric machine, because there is the need to transform sensory information coded co-variantly in sensor space into the contra-variant space of the effectors. In particular, they had attributed to the cerebellum this task of transformation between covariant and contra-variant coding.

Following their suggestion detailed analysis of the "eigen vectors" or the six vestibular organs, six eye muscles, thirty two neck muscles revealed interesting invariance in their organization. See [18], and the work of Barry Peterson on neck muscles geometry and its correspondence with vestibular neuron geometrical coding, cf. for instance [77, 78].

Hypothesis: Different Geometries for Different Spaces of Actions.

A. Berthoz [14–16] describe many aspects of the fundamental link between brain, movement and decision. In particular he proposed that several geometries are necessary for guiding several networks controlling actions in different spaces. Neuropsychological observations of pathological behaviors following brain lesions have revealed that different neural networks are involved in action in different spaces. (See reviews in [16, 17, 43, 73].) It has been proposed that at least five spaces are subserved by at least four different mechanisms and networks:
(1) ***Body space***, which is reconstructed in a "body schema" in networks located in the temporo-parietal junction, as first shown in epileptic patients by the neurologist Wilder Penfield in Canada, who identified this brain region as responsible for "awareness of body schema and spatial relationships" [76]. It is known that this schema takes into account all the mechanical and dynamic properties of the real physical body, and it has been also proposed that the temporo-parietal junction contains an "internal model" of gravity, cf. [57, 66].
(2) ***Near action and prehension space***, which is equivalent to the space at which we can reach things with the extended hand. In this space the geometries have to include forces and dynamic properties of the objects that one manipulates or obstacles that we may encounter. Simplifying laws of movement are at work to control gestures (see above and [10]). Actions can be made in ego-centric reference frame or in object centered reference frame or, if another person is involved, in hetero-centric reference frame.
(3) ***Far action space***, that is the space that we reach with a short locomotor trajectory (typically a room). In this space it has been shown that optimizing principles induce stereotyped trajectories. Both ego and allocentric reference frames can be used as well as heterocentric ones. Evidence shows that the neural networks involved in this space are not the same as those for near action space (cf. [85, 105]).
(4) ***Environmental navigation space***, that cannot be explored by a short walk. Typically a city or a park that requires an allocentric cartographic coding to be able to navigate and find new paths. Cf. [71]. (5) In addition to this modularity recent studies have identified multiple reference frames and different neural structures for "egocentric" (referred to an observer own body viewpoint), "allocentric" (map like, independent of an observer view point), or even "heterocentric" (taking an other person as a reference) ([6, 12, 24, 37, 58]. This diversity of reference frame has given rise to a number of terminologies (like first or third person perspective etc.).

Our hypothesis is that evolution has applied a principle of modularity and designed different networks for actions and perception in these different spaces because each had different requirements and therefore different "geometries".

In the present paper we show, in addition to the already published combination of Euclidian, affine and equi-affine geometries mentioned above, how a variety of different geometries is useful (and even necessary) to understand various aspects of motor control and sensory-motor interaction with the world. We explain how these geometries intervene for adaptation of neuronal dynamics by virtual systems of “homological nature”, and how the movements durations reflect geometrical invariants and coordinate choices. Moreover, we suggest that new types of generalized geometries without points, are necessary for guiding the neural networks underlying complex actions, movements preparation and execution.

2 Geometries for Motions Timing

2.1 Euclidean and Galilean Brains Structures

It is amazing to see how precisely the geometrical principles of Physics are reflected in the organization and dynamics of the visual and vestibular system for controlling posture, locomotion, active vision and equilibrium in highly dynamic conditions. In particular, the vestibular end sensors of vertebrates, the semi-circular canals and the otoliths which record heads rotations and translations. (See a recent review in [40].) Even at the first level of transduction, in the hair cells, there exists a coherent recording of linear acceleration and rotational velocity, or at higher order, linear jerk and rotational acceleration. We have described recently the remarkable geometrical organisation of the otolithic maculae which allow this transducer through the creation of a “virtual dynamic line” to detect $3D$ acceleration very rapidly and efficiently [27]; and we have shown that a peculiar geometry of the semi-circular canals ampullae optimizes the distribution of forces for the detection or rotational forces [65].

All this is compatible with the natural analysis of a Galilean group. From principles of the Theory of Relativity, linear acceleration and gravitation are a priori non separable; however, after two neuronal relays, in the cerebellum, gravitation and acceleration information are both accessible. (Cf. [3, 107]). With vision (and/or hearing), we get the ten dimensions of the complete *Galilean group* R the rotation (3), V a uniform speed (3), T a spatial translation (3) and τ a time translation (1). Cf. [9, 41].

In addition, vestibular, visual and proprioceptive information flows are able to produce in the hippocampal formation a variety of geometrical neurons for navigation in the Euclidian plane. This is performed by the system of *place cells*, *head direction cells*, *grid cells*, *frame cells*, *boundary cells* etc. Cf. [1]. A variety of frames for navigation can be obtained with this diversity of modes of coding.

Note that this network involves many structures from other regions of the brain, for instance in the Thalamus [52], and it exchanges information with other neocortical areas, for instance prefrontal or parietal cortex [13, 21], and even cerebellum [22, 50].

2.2 Affine Evidence and Multiplicity of Geometries

The brain uses other geometries than Euclidean. For instance Flash and Handzel [31, 32], Pollick and Shapiro [90] have remarked that the $2/3$ law [56] which gives a non-linear relationship between tangential velocity and curvature during a natural movement, can be interpreted in terms of affine geometry. The starting point of the $2/3$ power law was an old observation [19], that when drawing, or writing, the end effector moves slower in the more curved parts of the trajectory; the more precise law tells that the linear velocity $V(t)$ is proportional to $R(t)^{1/3}$, i.e. the radius of curvature of the trajectory at time t elevated to the exponent $1/3$ (which makes $-2/3$ for the angular velocity, and gives its name to the law). Handzel and Flash, Pollick and Shapiro [31, 90] re-expressed this law as follows: the time course along a path corresponds to the unique equi-affine invariant way of parameterizations. The equi-affine group is the subgroup of affine transformations that preserves the area.

For hand movements in space, Maoz et al. [64] have shown that torsion with exponent $1/6$ comes into the play, and this also can be interpreted by the equi-affine invariant parameterization. Concerning the shape of trajectories, Polyakov et al. [91, 92] studying scribbling in monkeys, have shown a dominance of arcs of parabola, which are the curves with the highest dimension of affine symmetries.

A remarkable finding was that the same kind of law also holds for human locomotion, but with an exponent smaller than $1/3$ and depending upon the form of the trajectory. References [47, 103] The idea that similar laws subserve the generation of a trajectory for a similar gesture executed by different effectors was known in Physiology under the name of “the principle of motor equivalence”. In accordance, it was shown recently that in the motor system a large distributed population encode handwriting movements in scale independent manner [44].

Studies on locomotion have also suggested that general optimizing principles probably involving non trivial euclidian geometries subserve the formation of locomotor trajectories [4, 48, 81].

A systematic exploration of kinematics of drawing and walking [10] showed that different geometries (Euclidian, equi-affine, affine) are used together for generating the time course of a trajectory, depending on its local shape. This incorporates the observation of Binet and Courtier that isochrony guides successive productions of point to point motion with the hand. And this is compatible with the necessary interplay between vision and motion, because, for instance, when we have to walk on a circle what we see on the ground is an ellipse. Geometry appears to serve the action/perception coupling. (For the comparison between perception and production with respect to the geometric parameterizations see [60].)

The combination of the different geometries during a movement was modeled by logarithmic combinations of invariant abscissae. This allowed to revisit the time course of velocities from the point of view of shifts between different geometries. As a result, we have compared the duration in drawing and walking, and shown that the main difference between them is a larger impact of Euclidean geometry for walking and a larger impact of pure affine deformation for drawing.

The fact that affine geometry underlies hand movements production was established from a statistical analysis of a large set of scribbling [79]. More precisely, if two paths segments can be transformed accurately one into each other by an affine transformation, then the timing on these segments are accurately matched by the affine transportation, once it is normalized by total time. Also these finding have been recently applied to robotics.[1]

Of course, we do not assert that geometry alone is responsible of movement planing and generation; geometry must conjugate its role with other principles, like min jerk, min variance, min time or min energy (cf. [67]).

3 Geometries for Adaptation

3.1 Homological Spaces and Galois Operations

Below we propose some suggestions to explain how the brain creates several different geometries and why these geometries allow a great flexibility i.e a capacity of adaptation to the variety of conditions in which action has to be made.

Specificity.

According to Poincaré, Euclidean geometry has its origin in the overlap of information between the inner world and the outer world, during movements and explorations. As reminded in the first sections, this overlap produces an ambiguity which possesses a structure, described by a convenient group and a convenient space. Every geometry used by the brain offers a *specific process* to overcome the complexity of the interaction with the external world, it guides the choice of pertinent aspects in an excessively rich set of interaction, and it allows to plan actions in various spaces. This is compatible with the with the concept of modularity and simplexity proposed by Berthoz [16]. Our hypothesis is that during evolution, living organisms have created in the brain new neuronal structures adapted to different action spaces, extending the range and abilities of interactions with the world. These new structures are organized according to specific geometries.

Let it be clear that we do not suppose that geometries are organised as such in the brain, or as described by the abstract concepts of mathematics like those below.

[1] Recently, Q.-C. Pham and Y. Nakamura developed a new trajectory deformation algorithm based on affine transformations. Reference [80]. The idea is to apply a set of predefined affine transformations to a set of trajectory segments, to avoid unexpected obstacles or to achieve a new objective goal. They also conjugate this idea with optimization algorithms for better accuracy, respecting C^1 continuity, keeping fixed final configuration and avoiding joint limits. The method was tested on a virtual planar three-links manipulator, and compared to polynomial interpolations; the main result is a considerable gain in *computation time* for equal accuracy. This can also be efficiently applied for minimizing curvature's changes in $3D$ point to point deformation. The method was applied to rapid motion transfer from humans to robots, with better performance in kinematics than polynomial interpolation methods.

But these geometries correspond to specific processing characteristics of certain cells and networks, defined by particular sets of transformations which occur within these networks and which constrain and modulate the interactions between several brain areas, dedicated to a given type of functions. A simple example is given by Affine geometry, present in movement preparation and execution, as in visual or somatosensory perception; there is apparently no center dedicated to this geometry, to the contrary it seems operating indirectly in some dispersed co-variant states of neuronal networks, premotor and cerebellar areas or in higher visual areas for instance.

Adaptation.

We suggest that adaptation is the main goal of most of the non Euclidean geometries that are used by the brain. In Biology, adaptation is an ubiquitous and essential property of sensory and motor processing, allowing the living systems to sense and anticipate what is changing in the world [14]. An example is the decline in the frequency of firing of a neuron in response to constantly applied environmental conditions, or more generally, any change in the relationship between stimulus and response that is induced by the level of stimulus [59]. For surviving, acting and perceiving efficiently, every living entity must dispose of functional systems that adapt themselves continuously to the changing environment and to internal modifications.

A general hypothesis of *homology for adaptation* [8] can be expressed as follows: for every decision organ, or every function (that are dynamics transforming an input in output, or predicting an output from prior experience), denoted by X, which depends on parameters, or modules, denoted by U (corresponding to the mathematical notion of *unfolding*, that is a deployment), there exists necessarily a secondary organ, or functional system, I, which "builds" the schema (plan) of the functional dynamics in X, and at the same time "shapes" the modules in U (gives them a structure), for guiding the adaptation of X. Up to this point I could be the support of any "internal model" or "supervisor" as it was already suggested in the neurosciences or robotics or artificial intelligence literatures; for instance, a notion of observer was proposed by J.-J. Slotine and W. Lohmiller [99, 100], with interesting applications to dynamical systems that satisfy a property of contraction. However this would not generate a *new geometry*; thus the most important point here is that I comes naturally equipped with a set of *virtual transformations*, forming a group G, which characterizes the Galoisian nature of homology. This group corresponds to a *scheme* of deformations of the modules in U. The definition of a group (or more generally a groupoid, cf. Appendix 2), i.e. operational associativity and reversibility, corresponds to the main requirements for an adaptation scheme. In the simplest examples, where X describes the competitions between minima of a potential function, the space U is made by significant minimal deformations coefficients of the potential, the space I introduces imaginary linear combinations of vanished minima, that do not correspond to realizable situations, and the geometry G on I is an extension of the Galois group of a generic numerical equation. The correspondence between I and U is induced by the map from the roots (real or imaginary) to the coefficients, it is not one to one, thus the induced *real deformations* in U are only the shadows of the geometrical operations of G.

Consequently, the natural homological operations in I create a new world, including the various plans for adaptation. This is a new formulation, leading to a control. In other terms, I is at the same time, able to "understand" when the dynamics of X needs to be adapted, and able to "drive" the required adequate deformation of the parameters of U, by working in an imaginary space. This represents a higher level of control than a simple direct coupling between the dynamics of X and U.

This process involves therefore a true *ternary structure*, with three equals actors, X, U and I, where I expresses the convenient geometry G. This structure can be generalized in the framework of localized categories, to get new geometries without points, corresponding to a field of groups. An interesting example is given by the co-homology of information topos, where X is a category of observables, U a set of probability laws, and I the localized mutual information quantities of Shannon, see [7, 8].

Let us insist on an important point: the geometry G on I introduces a particular structure of ambiguity between the brains states and the external world. As this point is surprising and certainly difficult to accept for lectors who are not familiar with homology or Galois theory, let us take the example of Affine geometry in movement production. In this case, there is no compensation between a transformation in neuronal states induced by a body motion and a transformation in neuronal states induced by a change in the world, but there is a compensation between a transformation in neuronal states induced by an internal dynamics and a virtual coordinates change on the world. For instance, preparing a hand writing "up to dilatation" expresses an internal ambiguity with respect to the achieved motion. A dilatation of the dimensions of the ambient space has no experimental support, it is only a virtual change of the world, that can be reflected by a transformation of the dynamics in a particular structure of the brain and compensated by the dynamics of another structure. (Cf. [44] for an experimental support of this hypothesis in the motor system.)

Properties of Homology.

Thus our hypothesis is that geometries are properties of spaces I which are of homological nature with respect to a neuronal dynamics X, they define the relevant characteristics of X, they structure these characteristics, and define virtual operations on them (see [8, 20]). In the same manner that Poincaré associated a group and a geometrized space to the ambiguity structure of rigid motions, a geometry, associated to a group G and a spectrum of sub-groups (gHg^{-1}), is implemented in I to make standard and flexible the deformations for adaptation in the space U which represents the control of the changes in X (the unfolding of the dynamics). In general, the relationship between I and U is ambiguous, reflecting the ambiguity between I and the real world, but in most known cases, they are related by a locally one-to-one correspondence, in particular they have the same dimension.

We suggest that concretely, X, U and I are probably implemented in interacting neuronal networks, frequently organized in brain areas, or interacting nuclei. For instance, an interesting brain structure for containing homological areas is the thalamus. Even if he used other words and concepts, D. Mumford [68, 69] described the thalamus as a kind of black-board were transiently complex messages are schematized. This is compatible with another important role of the thalamus, which is to transmit information between brains areas, in particular neo-cortical ones [29], and to systematically send traces of this information to the motor system [98].

A paradigmatic example of homological space I, with an affine geometry on it, is for color in LGN (cf. [8]). Of course other regions are possibly of this nature, for instance the Entorhinal cortex EC, in relation to CA3, CA1 and Subiculum in the Hippocampal formation, and several thalamic nuclei (cf. [1, 52]), may play this role for navigation and more generally for memory.

An homological network I is also a dynamical function, thus it can itself be transformed by a higher degree homology for adaptation. This can generate a cascade of homological folds: $H(H(...))...)$. ... In fact, there exist in a developed brain many ternary structures, supported by different but overlapping material structures, that are interacting cellular networks. Several ternary structures can share a given geometry, or several different geometries on different ternary structures can interact coherently. We saw this result in our analysis of duration of curved trajectories [10].

Therefore our general hypothesis is that ***(1) the brain is a creator of ambiguities (on the model of the initial ambiguity between inner and external world, but comparing now internal modulations with external coordinates changes), (2) the associated geometries*** G ***are virtual operations on ideal homology spaces*** I, ***that are engaged in ternary structures*** X, U, I, ***for guiding control and adaptation***.

In reality, during the Evolution, it is likely that the three components and the corresponding geometry evolved together.

3.2 Canonical Times and Moving Frames

A geometry G on a space I can offer a repertoire of virtual movements and plans of actions. In the present text, we focus on trajectories of body systems and their generalizations, and we replace I by a geometrized space E that represents the considered body system. This space E is identified with an homogeneous quotient G/H of G by a sub-group H (cf. Appendix). Elie Cartan (1937) summarized in a beautiful concrete method one century of research about the geometric invariants of curves in E; this is the *moving frames method*. (Cf. [10, 23, 35, 72].) In [10, 32], this method was used to define a series of natural timing on trajectories for drawing and walking, and compare it with experiments.

The Cartan's method is inspired by the Galois theory [38], and consists in establishing a natural bijection from a product of spaces issued from the groups G and H to the manifold of infinitesimal elements of every order of curves in E. On the side of G, this gives a sequence of homogeneous spaces G_n/G_{n+1}, where G_n is a decreasing sequence of groups, starting with $G_0 = G$ and $G_1 = H$, plus numerical values for geometrical invariants until the order n (i.e. quantities which do not depend on the frames and depend only on the derivatives of the trajectory up to the order n), and on the side of curves, this corresponds to the sequence of derivatives (jets) of order n modulo derivatives (jets) of order $n-1$, that define the *infinitesimal curve* up to the order n. The decreasing sequence of sub-groups i.e. $\{e\} = G_N \subset G_{N-1} \subset \cdots \subset G_1 \subset G_0 = G$ ends at the order N, which coincides with the dimension of H.

For the concrete algorithm of the moving frame method, see the above references. [Briefly, for mathematicians readers, it consists to fix a frame F_0 and to choose order after order the frames $g_n F_0$ attached to the given curve in such a manner that the differential equations which describes their displacement along Γ involve the less possible number of free parameters. It follows that a given curve Γ determines step by step, starting with $n = 0$ a canonical sequence of subgroups $H_n = g_n G_n g_n^{-1}$. The element g_n being defined modulo multiplication to the right by a sub-group G_n (ambiguity of order n), the frame at this order is only partially defined. At the order N the frame becomes fixed, then the following orders, larger than N, give new invariant quantities for Γ, that correspond in general to higher order derivatives of a finite set of curvatures.]

Up to a simple ambiguity, like t changes $at + b$, the canonical parametrization emerges as the only one were the moving frames equations keep their form.

Classical examples are the Serret–Frenet frame in Euclidean geometry, parameterized by arc-length, giving rise to usual Euclidean curvature and torsion. Less classical examples are the equi-affine frame, corresponding to the osculating rational helix, in the $1/3$ parametrization, giving the equi-affine curvature, and the more general affine frame, cf. [10]. The order N is given by the dimension of H; in the case of planar geometry, this gives 1 for Euclidean, 3 for equi-affine, 4 for affine, 6 for projective.

The final canonical moving frame equation can be seen as an optimal form of sparseness for transportation description. The decreasing sequence of sub-groups can be intuitively understood as a manner to break progressively the ambiguity on the space surrounding a curve, by choosing canonically a system of coordinates, attached to our trajectory. Thus, we see that in this context, of a trajectory in a geometrized space, information is formulated with groups theory, as in other contexts it is formulated with probability theory.

For preparation of action (or its reverse face, that is perception, as proposed by Rodolfo Llinas, [61]), the above geometrical description represents an economy of planning (e.g. the affine group invariance for seeing or imagining, to prepare walking); then, step by step, from anticipation to execution, a sequence of sub-groups breaks the indetermination.

Timing along trajectories and canonical coordinates on geometrized spaces arise from the group structure of the geometry, by an analysis of the ambiguity about coordinates in space for stabilizing position, velocity, acceleration, jerk, and so on.

However new concepts are needed to address complex organizations in different spaces, like, as mentioned in the introduction: action on *body space*, or reaching in *near action and prehension space*, or *locomotor out of reach space*, or *environmental space*. Composite motions can require more than two geometrized spaces. We would like to consider here what are the neural underlying "spaces" in these cases.

4 Geometrical Spaces for Movements

4.1 Topos and New Geometries Without Points

We have previously, as described above, found evidence of combinations of several geometrical systems for the production and control of point's trajectories, their rhythms and timing of duration. In the case of drawing and writing, the considered effector is the end of a finger, or a pointer, and in the case of walking locomotion, the point which is considered in place of the effector is the projection on the floor of a convenient point in the body (for instance the center of the head), thus we characterised these movements by *precisely defined points* in space or on the ground. However, most of the natural voluntary movements, or gestures, cannot be represented by the motion of only one point.

A first natural suggestion is to work in a large "parameter space" with many dimensions corresponding to the control of many points attached to the body, as it is frequently done in robotics. However, this method can hide a geometry, as for instance was first the case in astronomy by the description of planets motions by composition of circular motions with constant velocity; this method occluded for a long time the Kepler motion on conics. Thus we prefer to look at the possibility of *geometries without points*.

Such geometries and dynamics do exist, they are associated to Topos, (cf. Appendix 2). The idea of topos [42, 93] is to replace sets and points in these sets by networks of arrows with fixed topology, between sets, and we forget about the notion of points. For example a chain $\bullet \to \bullet \to \cdots \to \star$ of fixed length $n+1$, gives rise to all sequences of length $n+1$ of transformations. More generally, an oriented graph Γ gives rise to the articulated structures of several maps between spaces acting coherently as indicated by Γ; we proposed to designate generically these families of maps by the name of "fields". Thus a usual set is replaced by a field. And the usual transformations from a set to a set are now replaced by a collection of maps between the sets that are placed over the same vertex of Γ, which satisfy a condition of compatibility: that going on two different paths from a set to another one, either following the fields or following the transformations of fields, we get the same result. For instance in the above example of the chains, a generalized transformation from a field $A_0 \to A_1 \to \cdots \to A_n$ to a field $B_0 \to B_1 \to \cdots \to B_n$, consists in a collection of transformations $A_0 \to B_0, A_1 \to B_1, \ldots, A_n \to B_n$, such that for any index $k < n$, the two manners of going from A_k to B_{k+1} give the same map.

The theory of topos was applied to intuitionist logic. (William Lawvere et al., cf. [93] and the references inside.) Even the simplest example of the chain of length 2, that is $0 \to 1$, gives a topos (associated in fact to the site which is the dual graph $1 \to 0$, because presheaves are contra-variant functors). This topos is named the *topos of Shadoks* by Alain Prouté, that has a table of truth with more than two values, then violating the contradiction principle. The Boolean calculus of characteristics functions of sets is replaced by a 3-valued logic (Heyting algebras): true, false and

... uncertain. See the exact statement and its proof at the end of the second appendix below.

All the usual objects in mathematics can be reconsidered in this setting. This applies in particular to dynamical systems and their deformation. The construction of homology I can be extended for giving “homological fields”, that are fields $I_0 \to I_1$ in the case of Shadoks. We can look again at the familiar geometries in the plane, affine, equi-affine and Euclidean, where each geometry corresponds to a group G and a subgroup H; in the new setting of Shadoks, G has to be replaced by a morphism of groups $G_0 \to G_1$ and H by has to be replaced by a restricted morphism $H_0 \to H_1$ from a subgroup $H_0 \subset G_0$ to a subgroup $H_1 \subset G_1$.

Then, assume we can develop a generalization of the algorithm of the Cartan’s moving frame in this context, each group of the chain will contain a canonical sequence of sub-groups, order by order; then the usual sequence is replaced by a grid of morphisms.

Continuing this way, we obtain a new notion of moving frames, which is a mapping between sets of frames. But remind that each pair (G, H) determines a parametrization of trajectories that gives a virtual timing in accord with the geometry, thus two pairs (G_0, H_0) and $G_1, H_1)$ gives two notions of timing. As a consequence something very interesting happens with *time*: we discover that there is no reason for a unified timing, or unified rhythm. The canonical parametrization becomes a change between two parametrization times.

Of course the executed motion is realized in a coherent timing, our assertion applies to the working systems which prepare and execute the action; like in elaboration of a movie, each image and scene being worked in their own timings, and then glued together for projection.

Remark this is not in contradiction with the known relation between the timing of actual executions of action and the timing of the mental imagination of this action, cf. [26, 70], because, as said in [25], imagined and executed actions share, to some extent, the same central structures. On the other side, our assertion accords with the adaptive compression of timing in imagination and memory for navigation, cf. [5], because, in this case, there are probably two different topos geometries at work, one for locomotion and another one for navigation.

In fact, the notion of trajectory itself is now problematic: for example, in the chain, like the shadoks one, a real interval I of numbers for parameterizing the time is only needed at the end, over 1, but at the origin, over 0, we can get a manifold Y, not necessarily an interval, equipped with a map $f : Y \to I$. Thus the parametrization of a trajectory by the time is replaced in the case of a shadoks space $\pi : E_0 \to E_1$ by a field $F : Y \to E_0$ underlying the ordinary trajectory $\gamma : I \to\to E_1$, in the sense that $\pi(F(y)) = \gamma(f(y))$ for every y in Y. This leads to the notion of a non-linear time, which can have many dimensions, corresponding to a network in movement preparation and execution.

Thus we obtain a well established psychological fact: that ***inner times are manifolds***.

Applications to Different Action Spaces.

We shall now apply these concepts to the question of the different action spaces: *prehension*, *initiation of locomotion*, *near locomotion*, *environmental imagined navigation*. In these example the action relies on fields of forces of very different nature. It is natural to expect that the timing to keep an object, to stand up, to press the floor with a foot or to imagine and dream of a city are different from the timing of an ordinary clocks. Theses action-times should give an insight of the different geometries at works.

4.2 Examples

We will consider shortly several sorts of natural motions, and sketch a first preliminary essay to model them by topos geometries. Our first approximation will use only two levels, i.e. geometries in *Shad* (the topos of shadoks), which are associated to an homomorphism of groups $G_B \to G_A$, that respects subgroups $H_B \to H_A$ of G_B and G_A respectively. This will correspond to a division in two parts of the motor system, representing the body and a point of interest. In the same spirit, we could have used a longer chain, taking into account the torso, the shoulder, the hand position and finally the end effected move. Of course this should have been more realistic and efficient in applications. But in this section, we want to present the main idea without modelling experiments, and for that purpose, two levels seem sufficient.

A general fact will appear: that G_A is a group, in accord with usual geometry, but G_B is not a group, only a *groupoid* (the notion that extends groups in category theory, cf. appendix), corresponding to the fact that physical articulations of a body's part limit the iteration of its motions. However this makes no profound difference for the following discussion, and we neglect this fact, considering G_B and H_B as if they were ordinary groups.

4.2.1 Prehension

The convenient geometry has to be *polarized*, from the surface of body to the external world containing objects; this polar structure was also underlined by Tamar Flash. We model this geometry by a two levels sequence $G_B \to G_A$, where G_A is a geometry for the classical external space, that could be the Euclidean one or an affine extension, and a two levels sequence of subgroups $H_B \to H_A$. We consider for simplicity that the object of interest must be reached and moved by the hand. The compensation scheme of Poincaré implies that the group G_B describes the configurations frames that the body can attain with respect to the hand position, indifferently on the fact that the movement is voluntary or imposed from another subject. It describes the allowed equilibrium states of the arm. Stabilizing mechanisms of neuro-muscular activity at equilibrium of the arm should play an important role here, for allowing the compensation of transformations of G_B on neural activities. This can be seen as the set of postures of the arm. In this respect, we can cite the results of Tamar Flash on the stiffness field of the hand in multi-joints arm movements [30].

The map from G_B to G_A represents the induced hand movement. The sub-group H_B of G_B must go into H_A, thus it is a set of deformations of the arm with a fixed end. The choice of H_B defines a sort of redundancy. The quotient space $E_B = G_B/H_B$ defines the geometrical degrees of freedom of the arm, or postural schemes. The minimal choice for E_B is the linear space of vectors from the shoulder to the wrist, which is probably not sufficient for most neural control; a more interesting choice considers in addition the three articulations, at the shoulder, at the elbow and at the wrist. In the Euclidean framework, this gives three angles, but in affine geometry the space is more intricate.

If we want to take in account the eyes movements information to the reaching system, a better representation of the geometry should involve two groups, G_{B_1}, G_{B_2}, one for the arm, the other for the eyes, going to the rigid motions G_A, and two sub-groups H_{B_1} in G_{B_1} fixing the hand position, and H_{B_2} in G_{B_2} fixing the direction of the gaze.

What appears on this example is an interesting possibility to revisit the notion of spatial reference frames used in reaching (cf. [2, 46]. For instance the notion of center of frame is replaced by at least two centers, one in the object, with a possible virtual identification with the subject, and one in the articulation points of the arm, and/or the eyes center. The flexibility in defining a reference frame on the aye, the shoulder or even in arbitrary points has been documented in several papers from the groups of John Soechting, Francesco Lacquaniti, Paolo Viviani and Alain Berthoz. In fact, in the spirit of topos, what replaces the notion of a center is the graph which relates arm, eyes and objects, here the graph has two arrows from arm and eye respectively to the object; thus, in some sense, two new geometries are acting together, one for the eye and the object, the other one for the arm.

4.2.2 Initiation of Locomotion

The convenient geometry has also to be polarized, from the support of the body to to the Euclidean objective space, where locomotion is effectuated. Consequently we take again, as for prehension, for G_A and H_A an ordinary space geometry. This geometry can be Euclidean or affine, depending on the level of preparation in the brain. The second level ambiguity group G_B describes the configurations that the body can attain; it corresponds to the *posture* of the whole body. The arrow (morphism) from G_B to G_A is given by the movement of a rigid frame attached to the body, that can be the head, or the torso. We see that probably, several arrows shall be considered, corresponding to several new geometries, working together.

We observe again that G_B is not a complete group, it is only a groupoid, because the body has to keep contact with the floor, and cannot be deformed arbitrarily; in particular the map from G_B to G_A is not surjective. The group H_A is the set of rigid motion fixing the center of gravity of the chosen body frame. The constraint on the sub-group H_B is that it must go to H_A, and that $E_B = B = G_B/H_B$ defines sufficiently simple parameters to be controlled, like particular angles at the articulations.

If we take for H_B the subgroup of all postures giving a fixed body's center of gravity, we obtain for G_B/H_B a part of the ambient space, representing vectors of standing.However this is not the most interesting choice, we could impose more constraints on H_B, at articulations under control, for instance the neck, or head stabilization, this would give a larger space G_B/H_B, defining a more interesting *postural scheme*. This corresponds precisely to a choice of reduction of number of degrees of freedom (cf. [11]).

In addition, as in the case of reaching, we can take into account the visual information, that enlarges greatly the dimension of the adapted frames dynamics. Note also the possibility to enlarge the geometry by coupling to the topos geometry of another moving subject.

4.2.3 Locomotion

Here G_A and H_A respectively correspond to a choice of geometry in the horizontal plane. We must be careful to not confound this plane with the ground floor, which has a role in the second level group G_B, where resistance and stand up are taken into account. Here, the horizontal plane means the plane that is transversal to the vertical gravity direction. In G_A we can suppose that only the horizontal movement is planed.

The group G_B is again a definition of posture, but now far from equilibrium (at least in humans), and H_B determines the posture with respect to the vertical. A possible choice of H_B could be the sub-group which stabilizes a given vector attached to the body, for instance the axis of torso, or the vector joining the contact on the floor with the center of mass.

It would be interesting to enlarge the postural group G_B by including the transformations of the relief and the nature of the ground floor, modulating the feet positions and the control of the center of mass. Also obstacles and objects in the environment could also be integrated in the geometry through a variety of changing reference frames.

We constat that in each situation, prehension, initiation of walking, locomotion, a variety of spaces and topos geometries appear naturally, as we previously saw that several geometries were useful for the analysis of points trajectories in hand drawing and locomotory motion.

Note that in all the above examples, G_B and H_B have their origin in the usual forces fields and dynamical mechanisms that underly the motions of bodies and objects described by G_A and H_A, but they shall not be confounded with these forces and dynamics, they define operational schemes for them, that can be used for instance to overcome environmental changes for maintaining the success of an action, without much changes of its form.

Also, in all these examples, it is tempting to extend the geometry for including an external time into the geometry itself, taking for G_A the Galilean group in place of the Euclidean group. This will accord better with the fundamental role of the visuo-vestibular system for controlling motor coordination, locomotion, navigation, awareness of space, and social interactions.

5 Discussion

In this article, we began by reviewing the evolution of the mathematical point of views on Geometry, underlying the growing role of transformations and group theory. Then we developed the hypothesis that the brain's evolution used a principle of modularity and designed different networks for actions and perception in different spaces implementing different geometries for different requirements, and we added the hypothesis that the necessity of geometries is found in their role for shaping and guiding adaptation. Note that, when Andras pellionisz Rodolfo Llinas claimed that the *the brain is a geometric machine*, they had particularly in mind the fact that neurons with their particular form of dendrites and axon collaterals, and neural systems, in particular in the cerebellum, but also in other structures of the brain, are performing a geometrical work, but they also linked this work with the hidden geometrical functioning, which links actions and perception.

We had previously suggested that several geometries, Euclidean, affine and equi-affine, serve as a guide for humans hand drawing and locomotion. In particular the timing along the trajectories of significant points effectors was reanalyzed and well explained by a mixture of geometrically invariant parameterizations (cf. [10]). More generally, it appeared that preparation and execution of these movements can be better understood by using operations that are organized by those geometries (cf. [31, 32, 34, 64, 67, 79, 80, 92]).

However in more complex situations, the body motion cannot be described conveniently by one point, as an end effector, or even by a finite collection of points. In fact, this remark applies if we want to analyze more precisely the movements of the hand in drawing or the movements of the body in walking. In the present article we have addressed the problem to extend the geometrical approach to such movements. Our new suggestion is to use the extension of the traditional geometries in the context of *topos theory*.

This theory generalizes the theory of sets (i.e. ensembles), by replacing sets by diagrams of fields between sets (cf. the appendices), thus the notion of point disappear but the concept of geometry are maintained and enlarged.

The examples we had in mind were prehension, initiation of walking, locomotion, navigation, imagined motion. In all those cases A. Berthoz had suggested before that several geometries are necessary for guiding several networks in the brain controlling actions in different spaces (cf. [14–16]).

In the present article we have indicated preliminary suggestions of useful topos geometries for prehension, initiation of locomotion and locomotion. In each case, the geometry has a polarized structure, for prehension or initiation of locomotion the polarization goes from the body to the environment, for locomotion the polarization corresponds to the vertical direction opposed to the gravity vector, and in both cases, principles are emerging to define a workspace of postures.

The goal is to describe a field of spaces with scheme for preparing, controlling, executing and adapting movements. For that purpose we have connected the new suggestion to previous developments, in two directions: *homology for adaptation* (cf. [8]) and *time correspondence* (cf. [10, 32]). The first notion is the formation, for each neuronal function, of a secondary functional space that extracts and defines dynamical characteristics and admit ideal operations (Galoisian in nature) forming a geometry, and then applies these operations for guiding the necessary adaptation of the dynamics, at this level invariants are created. The second notion is a kind of Galois correspondence between parameterized manifolds in the geometrized space and frames in the geometry (Cartan moving frame). Then we suggested in the present study, that these two ingredients, homology for adaptation and time correspondence, must be extended in the context of topos geometries.

Topos were invented by Grothendieck and Verdier to unify homological algebra and co-homology theories. Thus, certainly, the co-homological spaces that are adapted to represent the working of chained ternary structures for adaptation of neuronal functions, would come from homological algebras of convenient modules over a ringed topos. We mention that a step in this direction was the definition of Information Homology in [7].

However much has to be done for giving theoretical hypotheses about the geometries underlying the different classes of movements, to be experimentally tested. One direction where we can look for that, would be given by concrete predictions of timing and duration for the body's segments, in relation to stiffness fields and EMG signals. An article in preparation will give more details on the hypothetical brain's networks that are involved in the new geometries.

These new geometries must be compatible with the notion of internal models of the world, as it is well expressed for instance by the Free Energy minimization principle [36], with the optimization of smoothness [33, 49] and with optimal control, including the minimal variance principle [45], but they give a different point of view, for complementing them.

With respect to the more usual applications of geometry and dynamics, as in optimal feedback control, that were used for the understanding of principles guiding voluntary motions in neurosciences or robotics, the main difference of our approach is the research of *invariance principles that are based on a fully developed geometry*.

These invariance principles based on geometry, are starting with physical forces and physical constraints, but in some sense they are integrating them, and become free from them, to get a scheme for preparation, control and execution of movements. For that, planning is elaborated in neuronal networks that are working as if there were ideal geometrized spaces, not as spaces mimicking the real external physical world, because these ideal spaces are structured for allowing the operations of the more convenient geometry for adaptation, and then, paradoxically, only at the end of the process, movements are realized in the physical space through the determination of configurations in time.

6 Appencices

6.1 Groups

Definitions: a *group* is a set G, with a privileged element e, in which a product law is defined, which associates an element gh to every pair of elements g, h of G, satisfying three axioms: (i) associativity $(gh)k = g(hk)$; (ii) neutrality $ge = eg = g$; (iii) the existence of an inverse for each element g, i.e. g^{-1} such that $gg^{-1} = g^{-1}g = e$. The principal example is given by the group $Aut(X)$ of permutations of a set X, that are all the manners to exchange the elements of X between themselves; the product law being the composition of successive permutations; the neutral element e being the identity, when nothing changes; the inverse of a permutation consisting to replace things in the former order. A *subgroup* H of G is a subset containing e which is preserved by products and inversions. Examples are given by subsets of transformations of a set X, containing the identity, and closed by composition and reversibility.

In general, we define an *action* (or *representation*) of G on a set X, as an operation law, $(g, x) \mapsto g.x$, satisfying the two axioms (i) $\forall x$, $e.x = x$ (ii) $\forall g, h, x$, $(gh).x = g.(h.x)$. An action gives a map from G to $Aut(X)$, which respects the neutral elements and the multiplication laws. More generally, such a map between two groups is named an *homomorphism*.

Conjugation by g in the group is the important operation which sends every element g' to the *conjugated element* $gg'g^{-1}$. The set of all conjugations in G is a subgroup $Int(G)$ of $Aut(G)$.

The group $Iso_+(E)$ of rigid isometric displacements of the ordinary Euclidean space E, infinitely extended around us, is a subgroup of the group of all permutations of the points of E. This group $Iso_+(E)$ is made by translations, rotations and twists, i.e. compositions of a rotation and a translation in the direction of the axis of rotation. The group $GA(E)$ of *affine transformations* of E contains $Iso_+(E)$, but its is richer, it is made by all permutations f for which there exists a linear transformation $\varphi = Tf$ of vectors of translations, such that, for any point P and any vector $\overrightarrow{v}$, the image of $P + \overrightarrow{v}$ by f is equal to $f(P) + \varphi(\overrightarrow{v})$; for instance all dilatation, stretching, squeezing are affine transformations. The number of dimensions of $Iso_+(E)$ is 6, and for $GA(E)$ it is 12. They are permutations which send parallel lines to parallel lines. Generally these transformations do not respect the distance, nor the volume. The subgroup which preserves the volume is named *equi-affine*.

Larger than $GA(E)$, we can consider the $3D$ projective group $PGL(\mathbb{P})$, acting on the projective space $\mathbb{P}$, which is obtained from the affine space E by adding a set of *points at infinity*. In fact, these ideal points are in one to one correspondence with the non-oriented directions of parallel lines in E; and a sequence of points x_n in E approaches a point at infinity corresponding to the direction d if it escapes to infinity in the direction of d. Projective transformations are not required to respect the parallelism relation, but they must send any straight line to a straight line, and respect the incidence properties; they depend on 15 dimensions.

Much larger is the *diffeomorphism group*, which preserves continuity and differential regularity of figures; it has an infinite number of dimensions.

More generally, Felix Klein suggested that a Geometry is associated to a group G and a subgroup H, in such a manner that the *points* that G transforms are the subsets of G of the form $xH = \{xh \in G | h \in H\}$ when x describes G; two elements x, y of G are said equivalent modulo H when $xH = yH$, and the set of equivalence classes is denoted by G/H and named a Klein space. However, if we adopt this definition, a special marked point appears in the space, which is H itself. To forget this special point, geometers introduced general *homogeneous spaces*, which are sets X where G acts in such a manner that, each time a point x is chosen in X, the stabilizer G_x of x (i.e. the subgroup of G formed by the elements g such that $g.x = x$), is conjugated to H; therefore X can be identified with a Klein space G/G_x.

However, for most of the useful geometries, it appeared that the set G/H, or the isomorphic homogeneous space X with its action of G, can be identified with the set of the subgroups gHg^{-1} that are conjugated to H; this is due to the fact that in these cases, the application of G onto $Int(G)$ is almost an embedding. Therefore, in such a case, the points of the given geometry identify with the subgroups fixing them.

For all the above examples of groups, there exist natural families of sub-groups that define convenient geometries, which are useful for Mathematics, Physics and Biology. For instance, the traditional Euclidean space, certainly the more useful of all for the ordinary life, is given by the set E of all sub-groups SO_P, where SO_P designates the set of rotations of any angles and any axis containing the point P. It can be easily seen that such a subgroup characterizes a point. Moreover, in this case, the group of translations acts transitively on E: for any pair of points (P, Q) the translation be the vector $\overrightarrow{PQ}$ gives a canonical choice of transformation g in $Iso_+(E)$ that conjugates SO_P to SO_Q, i.e. $SO_P = gSO_Qg^{-1}$. Therefore, once a point of origin is chosen, all other points are described by vectors.

Remark: it is necessary to distinguish a family of conjugated subgroups for characterizing a geometry, the whole group G is not sufficient by itself; this can be seen on a simple example: the group is $G = PSL_2(\mathbb{C})$ is made by the two by two matrices with complex coefficients, where g and λg are identified when λ is a nonzero complex number. If we consider in it, the the family of maximal compact sub-groups conjugated to unitary matrices $H = PU_2$, we get the setting for the hyperbolic space, i.e. the $3D$ simply connected complete Riemannian manifold with curvature -1. The pair G, H also appears to be the setting for the conformal geometry of the Euclidean plane, i.e. the science of circles, the geometry where distances are forgotten but angles are preserved. They are not really two different geometries, one is the natural boundary of the other. We see the pertinence of Klein's formulation. However, G gives also the setting for the complex projective line $\mathbb{P}^1(\mathbb{C})$; every projective transformation in dimension one over complex numbers is an homography. But the conjugated subgroups in this case have to be the stabilizers of lines through 0 in $\mathbb{C}^2$, or equivalently of points in $\mathbb{P}^1(\mathbb{C})$, which are the groups of an affine complex line, whose standard form is made by triangular matrices with one element of the diagonal equals to 1.

6.2 Categories and Topos

A *category* $\mathcal{C}$ is formed by a set of objects $\mathcal{C}_0$ and a family of sets of arrows $\mathcal{C}_1(a, b)$ between pairs of objects (a, b), such that, each time an arrow g goes from an object a to an object c and an arrow f goes from c to b, there exists an associated arrow $f \circ g$ from a to b, satisfying the two following axioms: (i) (associativity) for three consecutive arrows f, g, h we have $(f \circ g) \circ h = f \circ (g \circ h)$, (ii) (neutral elements) for each object c there is an identity arrow 1_c such that for each arrow g coming to c and each arrow f leaving c we have $g = 1_c \circ g$ and $f = f \circ 1_c$. An arrow from a to b in a category is also called a morphism from a to b, and the set of arrows from a to b is also denoted by $Mor(a, b)$ or $Mor_{\mathcal{C}}(a, b)$. A good reference is the book of Mac Lane [62].

A group G can be seen as a category with only one object e, and where any arrow g has an inverse g^{-1}, such that $1_e = gg^{-1} = g^{-1}g$.

Between two categories $\mathcal{C}$, $\mathcal{C}'$ there is also a good notion of morphism; it is a *functor* $T : \mathcal{C} \to \mathcal{C}'$, which associates to each object c an object $T(c)$ and to each arrow $f : a \to b$ an arrow $T(f) : T(a) \to T(b)$, sending the identity morphisms to the identity morphisms and respecting compositions. This kind of functor is named co-variant; there is also a notion of contra-variant functor which reverses the sense of arrows; that is $T^*(f)$ goes from $T^*(b)$ to $T^*(a)$. And between two functors $T : \mathcal{C} \to \mathcal{C}'$ and $S : \mathcal{C}' \to \mathcal{C}''$ of the same variance, there is also a nice notion of morphism: a *natural transformation* α, which associates to each object c of $\mathcal{C}$ of a morphism $\alpha(c) : T(c) \to S(c)$ in $\mathcal{C}'$, in such a way that for each morphism $f : a \to b$ in $\mathcal{C}$ we have the commutativity relation: $\alpha(b) \circ T(a) = S(b) \circ \alpha(a)$. As said by Mac Lane [63], "intuitively, a natural transformation is one which is defined in the same way or by the same formula for every object in the category in question."

The following example makes all these definitions quickly understandable; it corresponds to rectangular matrices of every finite dimensions over a field of numbers K; the set $\mathcal{M}_0$ is the set of natural integers $\mathbb{N}$, including 0, and for any pair (n, m) of natural integers the set $Mor_{\mathcal{M}}(n, m)$ is the set of $m \times n$ matrices with coefficients in K; when $n = m$ we have the identity matrix 1_n, and the composition in the category is given by the multiplication of matrices. An example of a much larger category $\mathcal{V}$ is given by the set of structures of finite dimensional K-vector spaces on a given large set, and by the family of linear applications between these spaces. An example of functor is the embedding ε of $\mathcal{M}$ in $\mathcal{V}$ which sends an integer n to K^n and a matrix M of size $m \times n$ to the operation of multiplication of M with a column vector in K^n. A less canonical functor β in the other direction, from $\mathcal{V}$ to $\mathcal{M}$, is obtained by choosing at the level of objects a basis on every vector space, and at the level of morphisms, the matrix which expresses a linear operator in the chosen basis at the source and the goal. Let us consider the two manner of composing these functors $\varepsilon \circ \beta$ and $\beta \circ \varepsilon$; a natural transformation A from the identity functor $Id_{\mathcal{V}}$ to $\varepsilon \circ \beta$ associates to V the map which sends each vector of V to its coordinates in K^n, and a natural transformation B from the identity functor $Id_{\mathcal{M}}$ to $\beta \circ \varepsilon$ associates to n the element 1_n. For any V, $A(V)$ is a bijection, and for any n, $B(n)$ is a bijection; it is said in such a case that ε and β defines an *equivalence of category*. If we think that the most important things are the properties of morphisms and not the properties of objects, the equivalence of two categories means that they contain essentially the same information; even if one of them contains much less objects, as this is the case for $\mathcal{M}$ with respect to $\mathcal{V}$.

Remark: in category theory it is frequent to encounter collections of objects that are not sets, for instance the collection of all the vector spaces, all the groups, or all the sets; the paradoxes found by Russell and Cantor, for instance, have shown that these examples cannot be sets; but the notion of collection is too vague to justify most aimed theorems, consequently the inventors (Eilenberg, Mac Lane, Grothendieck, ...) managed to constrain the objects and the arrows to be sets $\mathcal{C}_0$, $\mathcal{C}_1$. They tried to find a setting where most of the general constructions that could be expected for "collections" can be done, but also can be controlled. For instance the fact that there cannot exist a set of all sets is overturned by the introduction of a category Set_U, whose objects are the sets belonging to a given set U, which is named the *universe* and which verifies convenient properties. The same thing can be made for groups $Grps_U$ or K-vector spaces $Vect_U^K$. The axioms for a universe are the following ones: (1) if $C \in U$ and $a \in C$ then $a \in U$, (2) if C and D are elements of U, the set with two elements C, D belongs to U, and the set with one element (C, D) (the pair (C, D), which exists in every set theory), and also the product $C \times D$ (made by the pairs (c, d) for $c \in C$ and $d \in D$) belongs to U; (3) the set $\mathcal{P}(C)$ of parts of an element C of U belongs to U, as does the union $\bigcup C$ of all the elements x of the elements a of C; (4) natural integers are included in U; (5) if $C \in U$ and $D \subset U$ (i.e. $d \in D$ implies $d \in U$) and if there exists a surjection f from C onto D, it follows that $D \in U$.

(Again see [62].)

A problem is that the set of integers $\mathbb{N}$ is the only known universe, in fact the existence of other universes was proved to be undecidable in the usual axiomatics of set theory (as Zermelo–Frankel theory for instance). Real numbers are so useful that Grothendieck and Verdier have suggested to add to set theory the following axiom: for every set X there exists a universe U such that $X \in U$. There is no proof that it is safe, i.e. without contradiction, because this is already the case for *ZF* theory.

By definition, a category is a U-category when all the sets $Mor(a, b)$ are elements of U, a category is inside U when $\mathcal{C}_0$ and $\mathcal{C}_1$ are parts of U, a category is said to be U-small when $\mathcal{C}_0$ and $\mathcal{C}_1$ are elements of U.

In general, when $\mathcal{C}_0$ and $\mathcal{C}_1$ are only unspecified "collections", Mac Lane prefers to speak of a meta-category, and he reserves the name of category to the case where $\mathcal{C}_0$ and $\mathcal{C}_1$ are sets. In what follows, we assume that a universe U is chosen, but we do not mention it.

An important sort of category generalizes the notion of group; it is named a *groupoid*. The special axiom specifies that every arrow has an inverse; in particular for every a, the set $Mor(a, a)$ is a group.

When a is an object of a category $\mathcal{C}$, the category $\mathcal{C}|a$ (a fiber over a) has for objects the pairs (b, f) where b is an object of $\mathcal{C}$ and where $f \in Mor(b, a)$, and for set of morphisms between (b, f) and (c, g) the subset of the $h \in Mor(b, c)$ which satisfy $f = g \circ h$. A *refinement* (more frequently named a *sieve*) R of the object a is a sub-category of $\mathcal{C}|a$, that contains $(c, f \circ g)$ each time $g \in Mor(c, b)$ and (b, f) belongs to R.

A *topology* on a category $\mathcal{C}$, in the sense of Grothendieck, is made by the association of a set $J(a)$ of *refinements* to every object a, which satisfies the following axioms: (i) for every $f : b \to a$, if $R \in J(a)$, then $f^*R \in J(b)$, where f^*R is made by the $g : c \to b$ such that $f \circ g$ belongs to R; (ii) for any a and $R \in J(a)$, the refinement R' of a belongs to $J(a)$ if and only if, for any $b \to a$ in R, the category f^*R' belongs to $J(b)$. A *site* is a category equipped with a topology. In the examples below, most of them being in a finite setting, we take the *discrete topology*, where, by definition, for any a, the set $J(a)$ has only one element, which is the category $\mathcal{C}|a$ itself. All the site we will consider are U-small.

An usual topological space X gives a topology in this sense: the objects of the site are the open sets and its morphisms are the inclusions between them, and for every open set U, an element of $J(U)$ is an open covering of U, taken with all the finer coverings.

A *presheaf* on a category $\mathcal{C}$ is a contra-variant functor from $\mathcal{C}$ to the category *Set* of the sets in a given universe U. In the category *Set*, there is a notion of infinite product, and for any presheaf F on $\mathcal{C}$, and any refinement R of a in $\mathcal{C}$, if $F_R : R \to Set$ denotes the restriction, we can consider the projective limit $limF_R$, which is the subset of the product of all the $F(b)$ for $b \in R$, made by the families (x_b); $b \in R$, such that $f : c \to b$, implies $x_c = F(f)(x_b)$. A *sheaf* on a site $\mathcal{C}$ is a presheaf $F : \mathcal{C} \to Set$, such that, for any a and any $R \in J(a)$, the natural map from $F(a)$ to $limF_R$ is a bijection. In the discrete case, every presheaf is a sheaf.

Definition: A *topos* is a category isomorphic to the category of sheaves over a site.

The category *Set* is the simplest topos, associated to the site with one element. An interesting generalization is given by the category of G-sets, where G is a group, a G-set being a set on which G acts to the left, and a G-morphism being an equivariant map.

Most of the constructions that are possible in *Set*, are also possible in every topos. For instance projective and injective limits exist. In particular *Cartesian squares* do exist; they are defined as follows: if $f : a \to c$ and $g : b \to c$, there exists a (non-necessarily unique) object d, equipped with two morphisms $h : d \to a$, $k : d \to b$, satisfying $f \circ h = g \circ k$, such that for any pairs of morphisms $u : x \to a$, $v : x \to c$ satisfying $f \circ u = g \circ v$, there exists a unique morphism $w : x \to d$ which satisfies $u = h \circ w$ and $v = k \circ w$. The object d is denoted by $a \times_c b$.

In every topos, there exists a final object, i.e. an object 1 such that any for other object F the set $Mor(F, 1)$ has one and only one element. This object is interpreted as a singleton.

There exists also an initial object $\emptyset$ which is the empty functor, having a unique morphism to every object in the topos.

Starting with the category having two elements and one arrow between them, a very interesting generalization of set theory occurs, cf. Prouté [93], which corresponds to an intuitionist point of view, where a property can be true, false, or uncertain in various manners. We will come back soon to this example, named the topos of Shadoks.

First we have to generalize the notion of parts of an object in any category, where the notion of point is absent: an arrow $f : b \to a$ is said *monic* (or injective, or a monomorphism) if, for any pairs of arrows $g, h : c \to b$, the equality $f \circ g = f \circ h$ implies $g = h$. Two monics $f : b \to a$ and $f' : b' \to a$ are said equivalent, when there exist arrows (necessarily monics) g and h such that $f = f' \circ g$ and $f' = f \circ h$. By definition and equivalence class of monics going to a is a *subobject* of a in $\mathcal{C}$.

A special Cartesian square occurs when we consider a morphism $f : a \to c$ and an injective morphism $g : b \to c$; the particularity comes from the fact that $k : d \to b$ is determined by g; in this case, $d = a \times_c b$ is named the *pull-back* of the sub-object X defined by g and is written $f^{-1}(X)$; the morphism $h : d \to a$ is a monic. Its universal property is that a morphism $u : x \to a$ can be factorized by a morphism to d (as $u = h \circ w$) if and only if the morphism $f \circ u$ can be factorized by a morphism to b (as $f \circ u = g \circ v$). Therefore in a topos, even if points do not exist, every morphism $f : a \to c$ induces a map f^{-1} from the sub-objects of c to the sub-objects of a.

The relation with intuitionist logic comes from the fact that every set $Sub(X)$, of the sub-objects of an object X, possesses a natural structure of Heyting pre-algebra (cf. [93]). This is the natural origin of *contextual logic*.

An important property of a topos $\mathcal{F}$ is the existence of a classifying object for sub-objects, i.e. an object Ω marked with a special sub-object T, which is given by a morphism from 1 to Ω, such that for any monic $f : G \to F$ there exists a unique morphism $\chi_f : F \to \Omega$ satisfying $T \circ 1_G = \chi_f \circ f$, i.e. G is the pull-back of T by χ_f. Moreover the correspondence between sub-objects f and morphisms χ_f is natural, in the sense that sub-objects of sub-objects go to composition of characteristic morphisms χ, and so on. The sub-objects of Ω give the different values of truth of the logic associated to the topos; for instance T is true, $\emptyset$ is false.

As we said, a simple and surprising example of topos is given by the category *Shad* made by the presheaves over the category with two objects A, B and three arrows, which are the neutral elements 1_A, 1_B and only one more element $\alpha : A \to B$; it is named the topos of Shadoks by Alain Prouté [93]. An object of this topos can be seen as a pair of sets X, Y (X for the shadoks set, Y for their eggs) respectively associated to A and B, and a map f from Y to X, associated to α, like the map which associates to each egg its unique parent. The singleton 1 of *Shad* is given by a point 1 for A and a point 1 for B. But there is a natural embedding of *Set* in *Shad*, made by the shadoks without eggs, which corresponds to the case where Y is empty. In particular, the singleton of *Set* is an intermediary object between the empty element of *Shad* and the singleton of *Shad*; it is denoted by the symbol $1/2$.

A morphism in *Shad* is a pair of maps $G : Y \to Y'$, $F : X \to X'$ making a commutative diagram, i.e. $f' \circ G = F \circ f$. This morphism is monic (resp. epic) if and only if the two maps are injective (resp; surjective). A sub-object of $f : Y \to X$ can be represented by a subset X' of X and a subset Y' of Y (necessarily empty if X'

is empty) such that $f(Y') \subset X'$. If a map $(F, G) : (Z, W) \to (X, Y)$ is given, the pull-back of (X', Y') is simply given by $(F^{-1}(X'), G^{-1}(Y'))$.

Let us consider the sheaf Ω defined by $\Omega(A) = \{0, 1\}$, $\Omega(B) = \{0, 1/2, 1\}$, and $\Omega(\alpha)$ sending 0 to 0 and 1 and 1/2 to 1. We write T for the sub-object of source the shadok singleton 1 (one bird and one egg), which sends the egg to 1 in $\Omega(B)$ and the bird to 1 in $\Omega(A)$. Now, given the sub-object $S = (X', Y')$ of $f : Y \to X$, we define χ_S by sending X' to 1 and $X \backslash X'$ to 0 (no choice here if we want to recover the Boolean characteristic function of set theory), and by sending Y' to 1 (the only possibility if we want that $S = \chi_S^{-1}(T)$), the elements of $Y \backslash Y'$ that are sent in X' by f to 1/2 (the only possibility if we want that χ_S is a morphism), and the rest of the elements of Y to 0 (also the only possibility if we want that χ_S is a morphism). This proves that Ω and T are respectively the classifying space of sub-objects and its universal element.

We also see, by comparison with the case of sets, that the sub-object obtained by sending $1(A)$ to 0 in $\Omega(A)$ and $1(B)$ to 0 in $\Omega(B)$ can be interpreted as describing the failure of S, a strict complementary, but the third possible sub-object of source 1, which is obtained by sending $1(A)$ to 1 in $\Omega(B)$ and $1(B)$ to 1/2 in $\Omega(A)$, gives neither the failure of S neither its success, in some sense, if the egg is considered as the future of the bird, the temporary success of S at the level X becomes a failure at the level of Y, but it could be better to tell that an incertitude is maintained here. This clearly gives a logic with more possibilities than true or false, something like undecidable.

A group object in *Shad* is an homomorphism of ordinary groups $\varphi : G_B \to G_A$, a subgroup is a pair of respective sub-groups H_A, H_B such that $\varphi(H_B) \subset H_A$.

For spaces, behind the usual objective space G_A/H_A or the set of groups conjugated to H_A, several situations are distinguished: the elements of the complementary subset of H_B in G_B that go to H_A, and the elements that do not.

In [7] it is shown that the Shannon information quantities have their origin in the first co-homology of a module associated to probability laws for a canonical sheaf over the site of random variables for observation of a system. Relations with Galois theory was also suggested in this article, and relation with geometry has to be developed. It would be nice to go one step further and connect this information co-homology or a derived more homotopical theory to the structures that are needed in ternary structures for adaptation.

References

1. P. Andersen, R. Morris, D. Amaral, T. Bliss, J. O'Keefe, *The Hippocampus Book* (Oxford University Press, Oxford, 2006)
2. R.A. Andersen, Visual and eye movement function of the posterior parietal cortex. Annu. Rev. Neurosci. **12**, 377–403 (1989)
3. D.E. Angelaki, A.G. Shaikh, A.M. Green, Neurons compute internal models of the physical laws of motion. Nature **430**(6999), 560–564 (2004)

4. G. Arechavaleta, J.-P. Laumond, H. Hicheur, D. Le Bihan, A. Berthoz, The nonholonomic nature of human locomotion: a modeling study, in *Biomedical Robotics and Biomechatronics* (2006), pp. 158–163
5. A.E. Arnold, G. Iaria, A.D. Ekstrom, Mental simulation of routes during navigation involves adaptive temporal compression. Cognition **25**(157), 14–23 (2016)
6. J. Barra, L. Laou, J.-B. Poline, D. Le Bihan, A. Berthoz, Does an oblique/slanted perspective during virtual navigation engage both egocentric and allocentric brain strategies? PLoS One **7**(11), e49537 (2012)
7. P. Baudot, D. Bennequin, The homological nature of entropy. Entropy **17**(5), 3253–3318 (2015)
8. D. Bennequin, Remarks on invariance in the primary visual systems of mammals, in *Neuromathematics of Vision*, ed. by G. Citti, A. Sarti (Springer, Berlin, 2014), pp. 243–333
9. D. Bennequin, A. Berthoz, Non-linear Galilean vestibular receptive fields. Conf. Proc. IEEE Eng. Med. Biol. Soc. **2011**, 2273–2276 (2011)
10. D. Bennequin, R. Fuchs, A. Berthoz, T. Flash, Movement timing and invariance arise from several geometries. PLoS Comput. Biol. **5**(7), e1000426 (2009)
11. N.A. Bernstein, *The Co-ordination and Regulation of Movements* (Pergamon Press, New York, 1967)
12. A. Berthoz, Reference frames for the perception and control of movement, in *Brain and Space*, ed. by J. Paillard (Oxford University Press, Oxford, 1991), pp. 81–111
13. A. Berthoz, Parietal and hippocampal contribution to topokinetic and topographic memory. Philos. Trans. R. Soc. **352**, 1437–1448 (1997)
14. A. Berthoz, *Le sens du mouvement* (Odile Jacob, 1997)
15. A. Berthoz, *La décision* (Odile Jacob, 2003)
16. A. Berthoz, *Simplexity. How to Deal with a Complex World* (Yale University Press, New Haven, 2011)
17. A. Berthoz, *La vicariance* (Odile Jacob, 2013). transl. *Vicariousness* (Harvard University Press, 2016)
18. A. Berthoz, W. Graf, P.P. Vidal, *The Head-Neck Sensory Motor System* (Oxford University Press, Oxford, 1992)
19. A. Binet, J. Courtier, Sur la vitesse des mouvements graphiques. Rev. Philos. **25**, 664–671 (1893)
20. M. Bompard-Porte, D. Bennequin, *Pulsions et politique. Suivi de Le non-être homologique* (Editions L'Harmattan, Paris, 1998)
21. N.E. Burgess, K.J. Jeffery, J.E. O'Keefe, *The Hippocampal and Parietal Foudations of Spatial Cognition* (Oxford University Press, Oxford, 1999)
22. E. Burguiere, A. Arleo, M. reza Hojjati, Y. Elgersma, C.I. De Zeeuw, A. Berthoz, L. Rondi-Reig, Spatial navigation impairment in mice lacking cerebellar LTD: a motor adaptation deficit? Nat. Neurosci. **8**(10), 1292–1294 (2005)
23. E. Cartan, *La théorie des groupes finis et continus et la géométrie différentielle traitées par la méthode du repère mobile* (Gauthier-Villars, 1937)
24. G. Committeri, G. Galati, A.-L. Paradis, L. Pizzamiglio, A. Berthoz, D. Le Bihan, Reference frames for spatial cognition: different brain areas are involved in viewer-, object- and landmark-centered judgments about object location. J. Cogn. Neurosci. **16**(9), 1517–1535 (2004)
25. J. Decety, Do imagined and executed actions share the same neural substrate? Brain Res. Cogn. Brain Res. **3**(2), 87–93 (1996)
26. J. Decety, M. Jeannerod, C. Prablanc, The timing of mentally represented actions. Behav. Brain Res. **34**(1–2), 35–42 (1989)
27. M. Dimiccoli, B. Girard, A. Berthoz, D. Bennequin, Striola magica. A functional explanation of otolith geometry. J. Comput. Neurosci. **35**(2), 125–154 (2013)
28. J. Droulez, D. Bennequin, Perception des symétries et invariances perceptives, in *Symétries, symétries et asymétries du vivant*, ed. by M. Siksou (Lavoisier, 2005), pp. 155–171

29. G.M. Edelman, *The Remembered Present: A Biological Theory of Consciousness* (Basic Books, New York, 1989)
30. T. Flash, The control of hand equilibriul trajectories in multi-joint arm movements. Biol. Cybern. **57**, 257–274 (1987)
31. T. Flash, A. Handzel, Affine differential geometry of human arm trajectories. Abstr. Soc. Neurosci. **22**, 1635 (1996)
32. T. Flash, A. Handzel, Affine differential geometry analysis of human arm movements. Biol. Cybern. **96**(6), 577–601 (2007)
33. T. Flash, N. Hogan, The coordination of arm movements: an experimentally confirmed mathematical model. J. Neurosci. **5**(7), 1688–1703 (1985)
34. T. Flash, Y. Meirovitch, A. Barliya, Models of human movement: trajectory planning and inverse kinematics studies. Robot. Auton. Syst. **61**(4), 330–339 (2013)
35. O. Faugeras, Cartan's moving frame method and its application to the geometry and evolution of curves in the Euclidean, affine and projective planes. Preprint Inria (1994)
36. K. Friston, The free-energy principle: a unified brain theory? Nat. Rev. Neurosci. **11**(2), 127–138 (2010)
37. G. Galati, G. Pelle, A. Berthoz, G. Committeri, Multiple reference frames used by the human brain for spatial perception and memory. Exp. Brain Res. **206**(2), 109–120 (2010)
38. É. Galois, *Œuvres mathématiques* (Gauthier-Villars et fils, 1897)
39. J.J. Gibson, E.J. Gibson, Continuous perspective transformations and the perception of rigid motion. J. Exp. Psychol. **54**(2), 129–138 (1957)
40. J.M. Goldberg, V.J. Wilson, K.E. Cullen, *The Vestibular System: A Sixth Sense* (Oxford University Press, Oxford, 2012)
41. W. Graf, F. Klam, The vestibular system: functional and comparative anatomy, evolution and development. Comptes Rendus Palevol **5**(3–4), 637–655 (2006)
42. A. Grothendieck, J.-L. Verdier, *Théorie des topos (sga 4, exposés i-vi)*, Springer Lecture Notes in Mathematics, pp. 269–270
43. O.J. Grüsser, T. Landis (eds.), *Visual Agnosias and Other Disturbances of Visual Perception and Cognition* (Macmillan, London, 1991)
44. N.K. Harpaz, T. Flash, I. Dinstein, Scale-invariant movement encoding in the human motor system. Neuron **81**, 452–462 (2014)
45. C.M. Harris, D.M. Wolpert, Signal-dependent noise determines motor planning. Nature **394**(6695), 780–784 (1998)
46. B. Heider, A. Karnik, N. Ramalingam, R.M. Siegel, Neural representation during visually guided reaching in macaque posterior parietal cotex. J. Neurophysiol. **104**, 3494–3509 (2010)
47. H. Hicheur, S. Vieilledent, M.J. Richardson, T. Flash, A. Berthoz, Velocity and curvature in human locomotion along complex curved paths: a comparison with hand movements. Exp. Brain Res. **162**(2), 145–154 (2005)
48. H. Hicheur, Q.-C. Pham, G. Arechavaleta, J.-P. Laumond, A. Berthoz, The formation of trajectories during goal-oriented locomotion in humans. I. A stereotyped behaviour. Eur. J. Neurosci. **26**(8), 2376–2390 (2007)
49. N. Hogan, An organizing principle for a class of voluntary movements. J. Neurosci. **4**(11), 2745–2754 (1984)
50. K. Igloi, C.F. Doeller, A. Berthoz, L. Rondi-Reig, N. Burgess, Lateralized human hippocampal activity predicts navigation based on sequence or place memory. Proc. Natl. Acad. Sci. USA **107**(32), 14466–14471 (2010)
51. A. Imhausen, *Mathematics in Ancient Egypt. A Contextual History* (Princeton University Press, Princeton, 2016)
52. M.M. Jankowski, J. Passecker, M.N. Islam, S. Vann, J.T. Erichsen, J.P. Aggleton, S.M. O'Mara, Evidence for spatially-responsive neurons in the rostral thalamus. Front. Behav. Neurosci. **9**, 256 (2015)
53. F. Klein, *Vergleichende Btrachtungen über neurer geometrische Forschungen*. Reedition H. Wussing, Das Erlanger Programm, 1974 (Deichert, 1872)
54. J.J. Koenderink, A.J. van Doorn, Facts on optic flow. Biol. Cybern. **56**(4), 247–254 (1987)

55. J.J. Koenderink, A.J. van Doorn, *Pictorial Space* (MIT Press, Cambridge, 2003)
56. F. Lacquaniti, C. Terzuelo, P. Viviani, The law relating the kinematic and figural aspects of drawing movements. Acta Psychol. **54**(1), 115–130 (1983)
57. F. Lacquaniti, G. Bosco, S. Gravano, I. Indovina, B. La Scaleia, V. Maffei, M. Zago, Gravity in the brain as a reference for space and time perception. Multisens. Res. **28**(5–6), 397–426 (2015)
58. S. Lambrey, C. Doeller, A. Berthoz, N. Burgess, Imagining being somewhere else: neural basis of changing perspective in space. Cereb. Cortex **22**, 166–174 (2011)
59. S.B. Laughlin, The role of sensory adaptation in the retina. J. Exp. Biol. **146**(1), 39–62 (1989)
60. N. Levit-Binnun, E. Schechtman, T. Flash, On the similarities between the perception and production of elliptical trajectories. Exp. Brain Res. **172**, 533–555 (2006)
61. R.R. Llinás, *I of the Vortex: From Neurons to Self* (MIT Press, Cambridge, 2002)
62. S.M. Lane, *Categories for the Working Mathematician*, vol. 5 (Springer, New York, 1998)
63. S.M. Lane, *Homology* (Springer, New York, 2012)
64. U. Maoz, A. Berthoz, T. Flash, Complex unconstrained three-dimensional hand movement and constant equi-affine speed. J. Neurophysiol. **101**(2), 1002–1015 (2009)
65. P. Marianelli, A. Bethoz, D. Bennequin, Crista egregia: a geometrical model of the crista ampullaris, a sensory surface that detects head rotations. Biol. Cybern. **109**(1), 5–32 (2015)
66. J. Mc Intyre, M. Zago, A. Berthoz, F. Lacquaniti, Does the brain model Newton's laws? Nat. Neurosci. **4**, 693–694 (2001)
67. Y. Meirovitch, D. Bennequin, T. Flash, Geometrical invariance and smoothness maximization for task-space movement generation. Preprint (2016)
68. D. Mumford, On the computational architecture of the neocortex. Biol. Cybern. **65**(2), 135–145 (1991)
69. D. Mumford, On the computational architecture of the neocortex. Biol. Cybern. **65**(3), 241–251 (1992)
70. J. Munzert, Temporal accuracy of mentally simulated transport movements. Percept. Mot. Skills **94**(1), 307–318 (2002)
71. F. Nemmi, M. Boccia, L. Piccardi, G. Galati, C. Guariglia, Segregation of neural circuits involved in spatial learning in reaching and navigational space. Neuropsychologia **51**(8), 1561–1570 (2013)
72. P.J. Olver, A survey of moving frames, *Computer Algebra and Geometric Algebra with Applications* (Springer, Berlin, 2005), pp. 105–138
73. J. Paillard (ed.), *Brain and Space* (Oxford University Press, New York, 1991)
74. A. Pellionisz, R. Llinás, Tensorial approach to the geometry of brain function: cerebellar coordination via a metric tensor. Neuroscience **5**(7), 1125–1136 (1980)
75. A. Pellionisz, R. Llinás, Tensor network theory of the metaorganization of functional geometries in the central nervous system. Neuroscience **16**(2), 245–273 (1985)
76. W. Penfield, E. Boldrey, Somatic, motor and sensory represntation in the cerebral cortex of man as studied by electirc stimulation. Bran **60**, 389–443 (1937)
77. B. Peterson, Current approaches and future directions to understanding control of head movement. Prog. Brain Res. **143**, 369–381 (2004)
78. B. Peterson, J. Baker, E. Keshner, Multidimensional analysis of head stabilization-progress and prospects, *The Head-Neck Sensory Motor System* (Oxford University Press, Oxford, 2012)
79. Q.-C. Pham, D. Bennequin, Affine invariance of human hand movements: a direct test (2012). arXiv:1209.1467
80. Q.-C. Pham, Y. Nakamura, A new trajectory deformation algorithm based on affine transformations. IEEE Trans. Robot. **31**(4), 1054–1063 (2015)
81. Q.-C. Pham, H. Hicheur, G. Arechavaleta, J.-P. Laumond, A. Berthoz, The formation of trajectories during goal-oriented locomotion in humans. II. A maximum smoothness model. Eur. J. Neurosci. **26**(8), 2391–2403 (2007)
82. D. Philipona, K. O'Regan, J.-P. Nadal, Is there something out there? Inferring space from sensorimotor dependencies. Neural Comput. **15**(9), 2029–2049 (2003)

83. J. Piaget, B. Inhelder, *La représentation de l'espace chez l'enfant* (Presses Universitaires de France, 1948)
84. J. Piaget, B. Inhelder, A. Szeminska, *La géométrie spontanée de l'enfant* (Presses Universitaires de France, 1948)
85. L. Piccardi, F. Bianchini, R. Nori, A. Marano, F. Iachini, L. Lasala, C. Guariglia, Spatial location and pathway memory compared in the reaching vs. walking domains. Neurosci. Lett. **30**(566), 226-230 (2014)
86. H. Poincaré, *La Science et l'hypothèse* (Flammarion, 1902)
87. H. Poincaré, *La Valeur de la science*. Bibliothèque de philosophie scientifique (Flammarion, 1905)
88. H. Poincaré, *Science et méthode*. Bibliothèque de philosophie scientifique (Flammarion, 1908)
89. H. Poincaré, *Dernières Pensées*. Bibliothèque de philosophie scientifique (Flammarion, 1913)
90. F.E. Pollick, G. Sapiro, Constant affine velocity predicts the $1/3$ power law of planar motion perception and generation. Vis. Res. **37**(3), 347–353 (1997)
91. F. Polyakov, R. Drori, Y. Ben-Shaull, M. Abeles, T. Flash, A compact representation of drawing movements with sequences of parabolic primitives. PLoS Comput. Biol. **5**(7), e1000427 (2009)
92. F. Polyakov, E. Stark, R. Drori, M. Abeles, T. Flash, Parabolic movement primitives and cortical states: merging optimality with geometric invariance. Biol. Cybern. **100**(2), 159–184 (2009)
93. A. Prouté, Introduction à la logique catégorique. Cours à Paris 7. Preprint (2016)
94. R. Rashed, *The Development of Arabic Mathematics: Between Arithmetic and Algebra*, vol. 156 (Springer Science and Business Media, Berlin, 2013)
95. E. Robson, *Mathematics in Ancient Iraq: A Social History* (Princeton University Press, Princeton, 2008)
96. R.W. Sharpe, *Differential Geometry: Cartan's Generalization of Klein's Erlangen Program* (Springer, New York, 1997)
97. R.E. Shaw, M. McIntyre, W.M. Mace, The role of symmetry in event perception, *Perception: Essays in Honor of James J. Gibson* (Cornell University Press, Ithaca, 1974)
98. S.M. Sherman, R.W. Guillery, *Exploring the Thalamus and Its Role in Cortical Function*, 2nd edn. (MIT Press, Cambridge, 2006)
99. J.J.E. Slotine, Modular stability tools for distributed computation and control. Int. J. Adapt. Control Signal Process. **17**(6), 397–416 (2002)
100. J.J. Slotine, W. Lohmiller, Modularity, evolution, and the binding problem: a view from stability theory. Neural Netw. **14**(2), 137–145 (2001)
101. R. Thom, Stabilité structurelle et morphogenèse: essai d'une théorie générale des modèles. Interédition (1977)
102. R. Thom, Esquisse d'une sémiophysique. physique aristotéliciene et théorie des catastrophes (1988)
103. S. Vieilledent, Y. Kerlirzin, S. Dalbera, A. Berthoz, Relationship between velocity and curvature of a human locomotor tralectory. Neurosci. Lett. **305**(1), 65–69 (2001)
104. H. von Helmholtz, The origin and meaning of geometrical axioms. Mind **3**, 301–321 (1876)
105. P.H. Weiss, J.C. Marshall, G. Wunderlich, L. Tellmann, P.W. Halligan, H.-J. Freund, K. Zilles, G.R. Fink, Neural consequences of acting in near versus far space: a physiological basis for clinical dissociations. Brain **123**(12), 2531–2541 (2000)
106. F. Wolf, T. Geisel, Universality in visual cortical pattern formation. J. Physiol.-Paris **97**(2), 253–264 (2003)
107. T.A. Yakusheva, A.G. Shaikh, A.M. Green, P.M. Blazquez, J.D. Dickman, D.E. Angelaki, Purkinje cells in posterior cerebellar vermis encode motion in an inertial reference frame. Neuron **54**(6), 973–985 (2007)

Springer
http://www.springer.com/978-3-319-51546-5

Geometric and Numerical Foundations of Movements
Laumond, J.-P.; Amat, N.; Lasserre, J.-B. (Eds.)
2017, X, 419 p. 106 illus., Hardcover
ISBN: 978-3-319-51546-5