# 2
# Ill-Posed Problems and Regularization

In this chapter, we consider the equation

$$Ax = y \tag{2.0}$$

where $A$ is a (not necessarily linear) continuous operator acting from a subset $X$ of a Banach space into a subset $Y$ of another Banach space and $x \in X$ is to be found given $y$. We discuss the solvability of this equation when $A^{-1}$ does not exist by outlining basic results of the theory created in the 1960s by Ivanov, John, Lavrent'ev, and Tikhonov. In Section 2.1, we give definitions of well- and ill-posedness, together with important illustrating examples. In Section 2.2, we describe a class of equations (2.0) that can be numerically solved in a stable way. Section 2.3 is devoted to the variational construction of algorithms of solutions by minimizing Tikhonov-stabilizing functionals. In Section 2.4 we show that stability estimates for the equation (2.0) imply convergence rates for numerical algorithms and discuss the relation between the convergence of these algorithms and the existence of a solution to (2.0). The final section, Section 2.5, describes some iterative regularization algorithms.

## 2.1 Well- and Ill-Posed Problems

We say that equation (2.0) represents a well-posed problem in the sense of Hadamard if the operator $A$ has a continuous inverse from $Y$ onto $X$, where $X$ and $Y$ are open subsets of the classical spaces $C^k(\bar{\Omega}), H^k_p(\Omega)$, or of their finite codimensional subspaces. In other words, we require that

(2.1.1) for any $y \in Y$ there is no more than one $x \in X$ satisfying (2.0) (uniqueness of a solution);



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(2.1.2) for any $y \in Y$ there exists a solution $x \in X$ (existence of a solution);

(2.1.3) $\|x - x^{\bullet}\|_X$ goes to 0, when $\|y - y^{\bullet}\|_Y$ goes to 0 (stability of a solution)

The condition that $X$ and $Y$ be subspaces of classical function spaces is due to the fact that those spaces are quite natural for partial differential equations and mathematical physics. They reflect physical reality and serve as a basis for stable computational algorithms.

If one of the conditions (2.1.1)–(2.1.3) is not satisfied, the problem (2.0) is called ill-posed (in the sense of Hadamard).

We observe that these conditions are of quite different degrees of importance. If one cannot guarantee the uniqueness of a solution under any reasonable choice of $X$, then the problem does not make much sense and there is no hope of handling it. The condition (2.1.2) appears not as restrictive, because it shows only that we cannot describe conditions that guarantee existence. In fact, as shown later, even without this condition, one can produce a stable numerical algorithm for finding $x$ given $Ax$. Moreover, in many important inverse problems mentioned in Chapter 1, it is not realistic to describe $\{Ax\}$. It looks as though without condition (2.1.3), problem (2.0) is not physical (as suggested by Hadamard in the 1920s) and is not computable, because, practically, we never know exact data due to errors in computations, measurements, and modeling. However, a reasonable use of a convergence and a change of $X$ can fix this situation.

Now we consider examples of important and still not completely understood ill-posed problems.

Example 2.1.1 (Backward Heat Equation). In the simplest case, the problem is to find a function $u(x,t)$ satisfying the heat equation and the homogeneous lateral boundary conditions

$$\partial_t u - \partial_x^2 u = 0 \,\text{in}\, \Omega \times (0,T), \quad u = 0 \,\text{on}\, \partial\Omega \times (0,T),$$

where $\Omega$ is the unit interval (0, 1), from the final data

$$u(x,T) = u_T(x), \quad x \in (0,1).$$

The functions $u_k(x,t) = e^{-\pi^2 k^2 t} \sin(\pi k x)$ satisfy the heat equation and the boundary conditions. The initial data are $u_k(x,0) = \sin(\pi k x)$. They have $C^0$-norm equal to 1 and $L_2$-norm $(1/2)^{\frac{1}{2}}$. The final data have $C^0$-norm $e^{-\pi^2 k^2 T}$ and $H^m$-norm $e^{-\pi^2 k^2 T}((1 + \cdots + (\pi k)^{2m})/2)^{1/2}$. If we define $Au_0 = u_T$, then the bound $||u_0||_X \leq C||u_T||_Y$ is impossible when $X$, $Y$ are classical function spaces: the norms of $u_{Tk}$ go to zero exponentially when the norms of the $u_{0k}$ are greater than 1/2. Therefore, the problem of finding the initial data from the final data is exponentially unstable in all classical function spaces. This phenomenon is quite typical for many important inverse problems in partial differential equations.

The eigenfunctions $a_k(x) = 2^{1/2} \sin \pi k x$ of the operator $-\partial_x^2$ with eigenvalues $\pi^2 k^2$ form a complete orthonormal basis in the space $L_2(0,1)$, so we can write

$$u(x,t) = \sum u_{0k} e^{-\pi^2 k^2 t} a_k(x),$$

where $u_{0k}$ is the Fourier coefficient of the initial data. In particular, we can see that the operator $A$ is continuous from $L_2(\Omega)$ into $L_2(\Omega)$. It is clear that the existence of a solution with the final data $u_T(x) = \sum u_{Tk} a_k(x)$ is equivalent to the very restrictive condition of the convergence of the series $\sum u_{Tk}^2 e^{\pi^2 k^2 T}$, which cannot be expressed in terms of the classical function spaces defined via a power (but not exponential!) growth of the Fourier coefficients $u_{Tk}$ with respect to $k$. A similar description of the range of the final data operator $A$ for more general parabolic equations is given in Section 3.1. So we have no existence theorem; and the condition (2.1.2) is not satisfied.

In fact, the conditions (2.1.)–(2.1.3) are not independent. For linear closed operators $A$ in Banach spaces, the conditions (2.1.1) and (2.1.2) imply condition (2.1.3) due to the Banach closed graph theorem, which implies that if a continuous linear operator maps a Banach space onto another Banach space and is one-to-one, then the inverse is continuous. Indeed, if $A$ maps an open subset $X$ of a subspace $X_1$ of a Banach space onto an open subset $Y$ of a subspace $Y_1$ (codim $X_1$+ codim $Y_1 < \infty$), then $A$ maps $X_1$ onto $Y_1$, both of which are Banach spaces. So by the Banach theorem, the inverse $A^{-1}$ is continuous from $Y_1$ into $X_1$ with respect to the norms in $X$ and $Y$, and we have (2.1.3).

**Exercise 2.1.2 (A non-hyperbolic Cauchy Problem for the Wave Equation).**
Show that the Cauchy problem

$$\partial_t^2 u - \partial_1^2 u - \partial_2^2 u = 0 \text{ in } (0,T)\times\Omega, \quad u = g_0, \partial_2 u = g_1 \text{ on } (0,T)\times\Gamma,$$

where $\Omega = \{0 < x_1 < 1, 0 < x_2 < H\}$ and $\Gamma$ is the part $\{0 < x_1 < 1, 0 = x_2\}$ of its boundary, is ill-posed in the sense of Hadamard.

$\{$*Hint*: Make use of a separation of variables to construct a sequence of solutions that are bounded (with a finite number of derivatives) on $\Gamma$ while growing exponentially at a distance from $\Gamma$.$\}$

In fact, there is an exponential instability, as in Example 2.1.1.

This problem was analyzed initially by Hadamard in his famous book [H], pp. 26, 33, 254–261, where there is an interesting description of the pairs $\{g_0, g_1\}$ that are the Cauchy data for some solution $u$.

Example 2.1.3 (Integral Equations of the First Kind). Consider equation (2.0) (with $x$ replaced by a function $f$ defined on $\Omega$ and $y$ replaced by a function $F$ defined on $\Omega_1, \Omega, \Omega_1 \subset \mathbb{R}^n$) when

$$\text{(2.1.4)} \qquad Af(x) = \int_{\Omega} K(x,y) f(y)dy,$$

with the kernel $K$ continuous on $\bar{\Omega} \times \bar{\Omega}_1$. The operator $A$ in (2.1.4) is completely continuous from $L_2(\Omega)$ into $L_2(\Omega_1)$.

An important example of such equations is obtained with the Riesz kernels $K(x,y) = |x-y|^{\beta}$ when $\bar{\Omega}$ does not intersect $\bar{\Omega}_1$. When $n = 3$ and $\beta = -1$, we have the inverse problem of gravimetry, which is discussed in Section 4.1, and when $\beta = -2$, we will have the integral equation related to the linearized inverse conductivity problem (see Section 4.5) and to some inverse problems of scattering theory.

**Exercise 2.1.4.** Assume that $\Omega$ is the unit ball $|y| < 1$ in $\mathbb{R}^3$ and $\Omega_1$ is the annular domain $\{2 < |y| < 3\}$. Show that the integral equation $Af = F$ with the Riesz kernel represents an ill-posed problem.

{*Hint*: show that the operator $A$ maps the space $L_2(\Omega)$ into the space of functions that are real analytic in some neighborhood of $\bar{\Omega}_1$ in $\mathbb{R}^3$ (and even in $\mathbb{C}^3$). This space is not a subspace of finite codimension of any of spaces $H_p^k(\Omega_1)$, so (2.1.2) is not satisfied. Actually, $A$ maps distributions supported in $\bar{\Omega}$ into analytic functions.}

Another important example is that of convolution equations. We let $\Omega = \Omega_1 = \mathbb{R}^n$, $X = Y = L_2(\Omega)$, and $K(x,y) = k(x-y)$. Then equation (2.0) takes the form

$$\text{(2.1.5)} \qquad \int_{\mathbb{R}^n} k(x-y) f(y)dy = F(x), x \in \mathbb{R}^n.$$

To study and solve such equations, one can use the Fourier (or Laplace) transform $f \to \hat{f}$, which transforms equation (2.1.5) into its multiplicative form

$$\hat{k}(\xi)\hat{f}(\xi) = \hat{F}(\xi).$$

**Exercise 2.1.5.** Show that this equation is ill-posed if and only if for any natural number $l$ the function $\hat{k}^{-1}(\xi)(1+|\xi|)^{-l}$ is (essentially) unbounded on $\mathbb{R}^n$.

In particular, this equation is ill-posed for $k(x) = \exp(-|x|^2/(2T))$, which reflects the ill-posedness of the backward initial problem for the heat equation in the domain $\mathbb{R}^n \times (0,T)$. Indeed, it is known ([Hö2], sec.7.6) that $\hat{k}(\xi) = (2\pi T)^{\frac{n}{2}} \exp(-T|\xi|^2/2)$. The equation (2.1.5) (or (2.0)) then is equivalent to the well-known representation of the solution at a moment of time $T$ in terms of the initial data $u_0$.

## 2.2 Conditional Correctness: Regularization

The equation (2.0) is called conditionally correct in a *correctness class* $X_M \subset X$ if it does satisfy the following conditions

(2.2.1) A solution $x$ is unique in $X_M$; i.e., $x = x^\bullet$ as soon as $Ax = Ax^\bullet$ and $x$, $x^\bullet \in X_M$ (uniqueness of a solution in $X_M$).

(2.2.2) A solution $x \in X_M$ is stable on $X_M$; i.e., $\|x - x^\bullet\|_X$ goes to zero as soon as $\|Ax - Ax^\bullet\|_Y$ goes to zero and $x^\bullet \in X_M$ (conditional stability).

Sometimes we say also that a solution is unique and stable under a constraint and $X_M$ is called a set of constraints.

We observe that the existence condition is completely eliminated. A reason is that in important applied problems, it is almost never satisfied. Moreover, a stable numerical solution of the problem (2.0) can be obtained only under conditions (2.2.1) and (2.2.2), provided that a solution $x$ to equation (2.0) does exist. Certainly, a choice of the correctness class is crucial: it must not be so narrow as to reflect only some natural a priori information about a solution.

A function $\omega$ such that $||x - x^\bullet||_X \le \omega(||Ax - Ax^\bullet||_Y)$ is called a *stability estimate*. In (2.2.2) this function may depend on $x$. In some cases, it does not depend on $x \in X_M$, and then it is particularly interesting. We give stability estimates in Chapters 3–9 for some sets $X_M$ and for important inverse problems. A stability estimate must satisfy the condition $\omega(\tau) \to 0$ as $\tau$ goes to 0. It can and will be assumed monotone.

We make the simple but important observation that if $X_M$ is compact, then the condition (2.2.1) implies the condition (2.2.2) (the uniqueness guarantees stability). Indeed, $A$ is continuous, by (2.2.1) it is one-to-one, and then the well-known topological lemma gives that $A^{-1}$ is continuous from $A(X_M)$ into $X$ with respect to the norms on $Y$ and $X$. Moreover, there is a stability estimate on $X_M$. This observation, applied to the inverse problem of gravimetry by Tikhonov in 1943, was one of the ideas initiating the contemporary theory of stable solutions of ill-posed problems. Also, it emphasizes the mathematical role of uniqueness.

Let us consider the examples of Section 2.1. A solution of the backward heat equation is unique, so we can expect some stability. As shown in Section 3.1 (Exercise 3.1.2), there is a logarithmic stability estimate $||u_0|| \le \omega(\varepsilon) = -C\varepsilon_1 \ln(\varepsilon_1)$, where $\varepsilon_1 = -1/\ln\varepsilon, \varepsilon = ||u_T||$, and $C$ depends on $M$, provided that

$$||u_0||_2 + ||\partial_x^2 u_0||_2 \le M. \tag{2.2.3}$$

Here we let $X = Y = L_2(0,1)$ and the operator $Au_0 = u_T$.

**Exercise 2.2.1.** Show that the set of functions $u_0$ satisfying condition (2.2.3) is compact in $X$.

We consider another operator $Au_\tau = u_T$ defined on solutions of the heat equations at the moment of time $t = \tau > 0$. Then we have a much better estimate $||u_\tau|| \leq M\epsilon^{\tau/T}$ under the weaker constraint $||u_0|| \leq M$, as shown in Section 3.1.

**Exercise 2.2.2.** Let $\Omega = (0,1)$. Show that the set of functions $u(\tau), \tau > 0$, where $u$ solves the heat equation in $\Omega\times(0,T)$, $||u_0||_0(\Omega) \leq M$ when $0 < t < T$, and $u$ is zero on $\partial\Omega \times (0,T)$, is compact in $X = L_2(0,1)$.

The situation is more complicated if we consider Example 2.1.2. Then a solution $u$ is not unique in the domain $Q = \partial\Omega \times (0,T)$ but only in a subdomain $Q_0$ described in Lemma 3.4.8 or in Exercise 3.4.13. The best-known stability estimate will be in general only of a logarithmic type.

The basic idea in solving (2.0) is to use the regularization, i.e., to replace this equation by a "close" equation involving a small parameter $\alpha$, so that the changed equation can be solved in a stable way and its solution is close to the solution of the original equation (2.0) when $\alpha$ is small.

In the following definition, we need many-valued operators $R$ that map elements $y$ of $Y$ into subsets $\mathfrak{X}$ of $X$. We denote all closed subsets of $X$ by $\mathfrak{A}(X)$. The distance $d$ between two subsets $\mathfrak{X}$ and $\mathfrak{X}^\#$ is defined as $max\{\sup_x \inf_v \|x-v\|_X, \sup_v \inf_x ||x-v||_X\}$ where in the first term we take inf with respect to $v \in \mathfrak{X}^\#$ and then sup with respect to $x \in \mathfrak{X}$ and in the second term interchange $x$ and $v$. A many-valued operator $R$ is continuous at $y$ if $d(Ry^\bullet, Ry)$ goes to zero as $||y^\bullet - y||_Y$ goes to zero.

A family of continuous operators $R_\alpha$ from a neighborhood of $AX_M$ in $Y$ into $\mathfrak{A}(X)$ is called a *regularizer* to the equation (2.0) on $X_M$ when

$$\lim_{\alpha\to 0} R_\alpha Ax = x \quad \text{for any } x \in X_M. \tag{2.2.4}$$

The positive parameter $\alpha$ is called the *regularization parameter*.

Many-valued regularizers are necessary to treat nonlinear equations, while for linear $A$ we can typically build single-valued regularizers that are usual continuous operators from $Y$ into $X$.

We observe that at least for linear $A$ that have no continuous inverse and for one-to-one $R_\alpha$ (examples are in Section 2.3), convergence in (2.2.4) is not uniform with respect to $x$ if $X_M$ contains an open subset of $X$. Indeed, assuming the contrary and using translations and scaling, we can obtain the uniform convergence on $\{x : ||x||_X = 1\}$. In particular, there is $\alpha$ such that $||R_\alpha Ax - x||_X \leq 1/2||x||_X$. Let us consider the equation $x + (R_\alpha Ax - x) = R_\alpha y$. By the Banach contraction theorem, it has a unique solution $x = By$. Moreover, by using the triangle inequality, we obtain

$$||x||_X \leq ||R_\alpha Ax - x||_X + ||R_\alpha y||_X \leq \frac{1}{2}||x||_X + C||y||_Y,$$

because $R_\alpha$ is continuous. Therefore, $||By||_X \leq 2C||y||_Y$. We have $R_\alpha Ax = R_\alpha y$ and consequently $Ax = y$, so $A$ has the continuous inverse $B$, which is a contradiction. This shows that convergence in (2.2.4) is generally only pointwise, i.e., at any fixed $x$.

**Exercise 2.2.3.** Let $x \in X_M$ and $y = Ax$. Let $R_\alpha$ be a single-valued regularizer. Show that for any $\varepsilon > 0$ there are $\delta > 0$ and $\alpha > 0$ such that if $||y^\bullet - y||_Y < \delta$, then $||R_\alpha y^\bullet - x||_X < \varepsilon$.

We observe that in this exercise $\delta$ and $\alpha$ generally depend on $x$.

The result of Exercise 2.2.3 is valid also for many-valued regularizers.

So in principle, given a regularizer, we can solve equation (2.0) in a stable way. We are left with two important questions: how to construct regularizers and how to estimate the convergence rate. In the next section, we will show a variational method for finding $R_\alpha$ for many correctness sets $X_M$, and then we prove that a stability estimate for the initial equation (2.0) implies some convergence rate for regularization algorithms.

Let us consider Example 2.1.3. In a general situation, we cannot expect uniqueness of a solution. For many particular kernels, we obtain important equations, and then it is possible to show uniqueness and (which is usually more difficult) to find a stability estimate. If we consider operators of a convolution, then in terms of the Fourier transforms, uniqueness means that $\hat{k}$ is not zero on any subset of nonzero measure, which is the case when this function is real analytic on $\mathbb{R}^n$.

For equations with the Riesz-type kernels and nonintersecting $\bar{\Omega}, \bar{\Omega}_1$, there is uniqueness in $X = L_2(\Omega)$, provided that $\beta \neq 2k, \beta \neq 2k + 2 - n$ for any $k = 0, 1, \ldots$, where $n$ is the dimension of the space (see the book [Is4], p. 79).

If $n = 3$ and $\beta = -1$, then there is nonuniqueness even in $C_0^\infty(\Omega)$.

**Exercise 2.2.4.** Show that if $f = \Delta\phi$, where $\phi$ is a $C^2$-function, $\phi = 0$ outside $\Omega$, then $Af(x) = 0$ when $x$ is not in $\Omega$, provided that $n = 3$ and $K(x, y) = |x - y|^{-1}$.

Since in this case the problem has very important applications (inverse gravimetry), it is interesting to find $X_M$ where a solution is unique. This is not a simple (and not completely resolved) question. Referring to Section 4.1 and to the book [Is4], Sections 3.1–3.3, we claim that a solution $f$ is unique at least in the following two cases: (1) when $\partial_n f = 0$ on $\Omega$ or (2) $f$ is the characteristic function $\chi(D)$ of a star-shaped (or $x_n$-convex) subdomain of $\Omega$. We will discuss the uniqueness in more detail in Section 4.1. The stability is an even more complicated topic. It is quite well understood for the inverse gravimetric problem, and there are some results for the Riesz-type potentials in the paper of Djatlov [Dj].

A convolution equation (2.1.5) can be studied in terms of the function $k_C$, which is defined as inf $|\hat{k}(\xi)|$ over $|\xi| < C$.

**Exercise 2.2.5.** Assume that $||f||_{(1)} \leq M$. Show that a solution $f$ to the convolution equation (2.1.5) satisfies the following estimate: $||f||_2 \leq ||F||_2/k_C + M/C$. By minimizing the right side with respect to $C$, derive from this estimate the logarithmic-type estimate

$$||f||_2 \leq M(3T/2\ln B)^{-1/2}(3(2\ln B)^{-1}+1) \text{ where } B = 1/3(M^2||F||_2^{-2}T^{-1})^{1/3}$$

for the Gaussian kernel $k(x) = \exp(-\frac{|x|^2}{2T})$.

{*Hint*: Solve the equation for the minimum point, and bound this point from below using the inequality $te^t < e^{2t}$.}

The result of this exercise gives a stability estimate for a solution to the backward heat equation. When $||F||_2$ goes to zero, $B$ goes to $+\infty$, and $||f||_2$ converges to zero at a logarithmic rate.

## 2.3 Construction of Regularizers

We describe a quite general method of a so-called stabilizing functional suggested by Tikhonov [TiA].

We call $\mathcal{M}$ a stabilizing functional for the correctness class $X_M$ if

(2.3.1) $\mathcal{M}$ is a lowersemicontinuous (on $X$) nonnegative functional defined on $X_M$;

(2.3.2) the set $X_{M,\tau} = \{x \in X_M : \mathcal{M}(x) \leq \tau\}$ is bounded in $X$ for any number $\tau$.

We construct a regularizer by using the following minimization problem:

(2.3.3) $$\min(||Av - y||_Y^2 + \alpha\mathcal{M}(v)) \quad \text{over } v \in X_M.$$

**Lemma 2.3.1.** *Under the additional condition that $X_{M,\tau}$ is compact in $X$ for any $\tau$, a solution $R_\alpha(y)$ to the minimization problem (2.3.3) exists, and under another additional condition that $R_\alpha(y)$ contains only one element $R_\alpha$ it is a regularizer.*

PROOF. Let $x_\bullet \in X_M$. We define $\tau = ||Ax_\bullet - y||_Y^2 + \alpha\mathcal{M}(x_\bullet)$. According to the condition (2.3.2), the set $X_\bullet = \{\mathcal{M}(v) \leq \tau/\alpha\} \cap X_M$ is compact, so the lower semicontinuous functional $\Phi(v; y) = ||Av - y||_Y^2 + \alpha\mathcal{M}(v)$ has a minimum point $x_*$ on $X_\bullet$. The value $\Phi(x_*)$ is minimal over $X_M$ because if $\Phi(v) \leq \Phi(x_\bullet)$, then $v \in X_\bullet$. The set of all minimum points is closed in $X$ due to the semicontinuity of $\Phi$. We denote this set by $x(\alpha)$ or by $R_\alpha(y)$.

The next step is a proof of continuity of $R_\alpha$ for any fixed $\alpha$. Let us assume that it is not continuous at $y$. Then there is a sequence $y_k$ converging to $y$ and $\varepsilon > 0$ such that $d(R_\alpha y_k, R_\alpha y) > \varepsilon$. According to the definition of the distance, we have $x_k = R_\alpha y_k$ such that $||x_k - x||_X > \varepsilon$ for any $x = R_\alpha y$.

Let $x_\bullet \in X_M$. We define $\tau$ as sup $\Phi(x_\bullet; y_k)$ with respect to $k$. Since the $y_k$ are convergent and $\Phi$ is continuous with respect to $y$, this sup is finite. As above, we have $x_k \in X_\bullet$, which is a compact set, so by extracting a subsequence, we can assume that the $x_k$ converge to some $x_\infty \in X_M$. Since $\Phi$ is lower semicontinuous with respect to $x_k$ and continuous with respect to $y_k$, we can pass to the limit and obtain the same inequality with $y$ instead of $y_k$ and $x_\infty$ instead of $x$. This means that $x_\infty$ is a minimum point for $\Phi$ on $X_M$, so it is contained in $R_\alpha y$. On the other hand, $||x_\infty - x||_X \geq \varepsilon$ for $x = R_\alpha y$, and we arrived at a contradiction.

Now we will show that the $x(\alpha)$ converge to $x$ when $y = Ax$, provided that $\alpha$ goes to 0. Assuming the opposite, we can find a sequence of points $x_k \in x(\alpha_k), \alpha_k < 1/k$ whose distances to $x$ are greater than some $\varepsilon$. Since $y = Ax$ and $||A(x_k) - y||_Y^2 + \alpha_k \mathcal{M}(x_k) \leq \alpha_k \mathcal{M}(x)$, we conclude that the $x_k$ are contained in the set $X_*$, defined as $\{v : v \in X_M, \mathcal{M}(v) \leq \mathcal{M}(x)\}$. Since $X_*$ is compact, by extracting a subsequence, we can assume that the $x_k$ converge to $x_*$. By continuity of the distance function, we have $x \neq x_*$. On the other hand, by the definition of minimizers, we have

$$||Ax_k - Ax||_Y^2 \leq \alpha_k \mathcal{M}(x) \leq (1/k)\mathcal{M}(x),$$

so using continuity of $A$ and passing to the limit, we obtain $Ax_* = Ax$. By the uniqueness property, we get $x_* = x$, which is a contradiction. Our initial assumption was wrong, and the convergence of $x(\alpha)$ to $x$ is proven. □

The above proof shows that in the general case, a multivalued $R_\alpha(y)$ is semicontinuous in the sense of the (nonsymmetric) semi-metric $d_*(X(1), X(2)) = sup\, inf\, ||x(1) - x(2)||_X$ where $inf$ is over $x(2) \in X(2)$ and $sup$ is over $x(1) \in X(1)$. Moreover, $R_\alpha(y)$ converges to $x$, when $Ax = y$ and $\alpha \to 0$.

We observe that for linear operators $A$, convex sets $X_M$, and strongly convex functionals $\mathcal{M}$, the variational regularizers are single-valued operators, so everything above can be understood in a more traditional sense. Indeed, under these more restrictive assumptions, the functional (2.3.3) to be minimized is convex, so a minimum point is unique. The variational construction is not only a possible way to find regularizers, and there is a very important question about an optimal and natural choice of regularization that agrees with intuition and that allows one to improve convergence by using more information about the problem (in the form of constraints).

For linear operators $A$ and Hilbert spaces $X$ and $Y$, we have a somehow stronger result. For references about convex functionals and weak convergence, we refer to the book of Ekeland and Temam [ET]. For example, we will make use of the fact that a convex lower semicontinuous function in $X$ is lower semicontinuous with respect to the weak convergence in $X$.

**Lemma 2.3.2.** *If $X$, $Y$ are Hilbert spaces, $x^0 \in X$, and $\mathcal{M}(v) = ||v - x^0||_X^2$, then the minimization problem (2.3.3) has a unique solution $R_\alpha(y)$ that is a regularizer of equation (2.0).*

Proof. The functional $\Phi(x) = ||Ax-y||_Y^2 + \alpha\mathcal{M}(x)$ is convex and continuous in $X$.

Let $x_m$ be a minimizing sequence such that $\Phi(x_m) \to \Phi_*$, which is infimum of $\Phi$ over $X$. Then $\alpha||x_m - x^0||_X \le C$, so $\{x_m\}$ is bounded. In any Hilbert space bounded closed sets are weakly compact, so we can assume that $\{x_m\}$ is weakly convergent to $x$. It is known [ET] that $\Phi$ is lower semicontinuous with respect to weak convergence; hence, we have $\Phi_* = \liminf \Phi(x_m) \ge \Phi(x)$. Since $\Phi_*$ is inf $\Phi$ over $X$, we have $\Phi(x) = \Phi_*$.

The uniqueness of $x$ as well as the continuity will follow from the next example, where we will show that $R_\alpha(x)$ is given by the right side of (2.3.4), which is a linear continuous operator in $X$.

The proof that $x(\alpha)$ (strongly) converges to $x$ when $\alpha \to 0$ is the subject of Exercise 2.3.4.

The proof is complete. □

Example 2.3.3. Let $X, Y$ be Hilbert spaces with scalar products $(,)_X, (,)_Y$. Assume that $A$ is a linear compact operator from $X$ to $Y$. Let $\mathcal{M}(x) = ||x - x^0||_X^2$. We let $X = X_M$. In this case, a necessary condition for a minimum point $x(\alpha)$ of the quadratic functional $q(v) = (Av - y, Av - y)_Y + \alpha(v - x^0, v - x^0)_X$ is $d/dt(q(x(\alpha) + tu)) = 0$ at $t = 0$ for any $u \in X$. Calculating this derivative, we obtain

$$2(Au, Ax(\alpha) - y)_Y + 2\alpha(u, x(\alpha) - x^0)_X = 2(u, A^*A + \alpha x(\alpha) - \alpha x^0 - A^*y)$$

by the definition of the adjoint operator $A^*$. Since this derivative must be zero for all $u \in X$, we conclude that $x(\alpha)$ is a solution to the equation

$$(A^*A + \alpha)x(\alpha) = A^*y + \alpha x^0. \tag{2.3.4}$$

Lemma 2.3.2 guarantees the convergence of $x(\alpha)$ to a solution $x$ of equation (2.0) when $\alpha \to 0$, provided that $y$ is convergent to $Ax$. It is easy to observe that the operator $A^*A$ is positive and self-adjoint in $X$, so we have uniqueness of a solution $x(\alpha)$ of equation (2.3.4) and therefore uniqueness of the minimizer obtained in Lemma 2.3.2.

**Exercise 2.3.4.** Show that a solution $x(\alpha)$ to the equation (2.3.4) exists and is unique. Show that $R_\alpha y$ defined as $x(\alpha)$ is a regularizer to equation (2.0) on $X = X_M$, provided that the equation $Ax = 0$ has only the zero solution.

{*Hint*: to show that $R_\alpha$ is a regularizer, first consider compact $A$ and make use of eigenfunctions of $A^*A$. The general case can be studied by using more general results about spectral representation of self-adjoint operators in a Hilbert space.}

The singular value decomposition is a useful theoretical and computational tool in general and in solving ill-posed problems in particular. We recall that if $A$ is a compact linear operator from a Hilbert space $X$ into a Hilbert space $Y$, then the operator $A^*A$ is compact self-adjoint, and therefore it has a complete orthonormal system $a_k$ of eigenvectors (functions) corresponding to (nonnegative) eigenvalues $\lambda_k^2$:

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq \lambda_{k+1} > 0.$$

We assume here that the equation $Ax = 0$ has only the zero solution. We let $b_k = ||Aa_k||_Y^{-1} Aa_k$. From these definitions, we have

$$A^*Aa_k = \lambda_k^2 a_k, AA^*b_k = \lambda_k^2 b_k, A^*b_k = \lambda_k a_k, Aa_k = \lambda_k b_k. \tag{2.3.5}$$

Observe that the system $\{b_k\}$ is orthonormal. Indeed,

$$(b_k, b_m)_Y = ||Aa_k||_Y^{-1} ||Aa_m||_Y^{-1} (Aa_k, Aa_m)_Y,$$

and the last factor is equal to $(A^*Aa_k, a_m)_X = \lambda_k^2 (a_k, a_m)_X = 0$ when $k \neq m$. It is obvious that the $Y$-norm of any $b_k$ is equal to 1.

**Lemma 2.3.5.** *A necessary and sufficient condition of the existence of a solution to the equation* (2.0) *is given by the following Picard test:*

$$\sum_{k=1}^{\infty} \lambda_k^{-2} |(y, b_k)_Y|^2 < \infty. \tag{2.3.6}$$

Proof. Assume that a solution $x$ to (2.0) exists. Then $A^*Ax = A^*y$. Let us calculate the scalar product of both parts of this equality and $a_k$. We obtain

$$(A^*Ax, a_k)_X = (A^*y, a_k)_X = (y, Aa_k)_Y = \lambda_k (y, b_k)_Y$$

from the definitions of $b_k$ and of the adjoint operator. Since $\{a_k\}$ is an orthonormal basis in $X$, we can write $x$ as the sum of the convergent series of $(x, a_k)_X a_k$ and

$$||x||_X^2 = \sum_{k=1}^{\infty} |(x, a_k)_X|^2 = \sum_{k=1}^{\infty} \lambda_k^{-4} |(x, A^*Aa_k)_X|^2 = \sum_{k=1}^{\infty} \lambda_k^{-2} |(Ax, b_k)_Y|^2,$$

where we used (2.3.5) and again the definition of the adjoint operator. Since $Ax = y$, we obtain (2.3.6).

On the other hand, let us assume (2.3.6). Let

$$x = \sum_{k=1}^{\infty} ||Aa_k||_Y^{-1} (y, b_k)_Y a_k.$$

The convergence of the series

$$||Aa_k||_Y^{-2} |(y, b_k)_Y|^2$$

follows from the condition (2.3.6) and the equality

$$||Aa_k||_Y^2 = (Aa_k, Aa_k)_Y = (A^*Aa_k, a_k)_X = \lambda_k^2.$$

Therefore, $x \in X$. By using the definition of $b_k$, it is not difficult to understand that $Ax = y$.

The proof is complete. □

Due to a significant increase of the computational power, a computation of the singular value decomposition for an interesting operator is now a quite realistic task, so the Picard test is currently becoming a tool for a practical solution of inverse problems. Particular numerical methods include range tests which are described in Section 10.5. The main difficulty with this test for strongly ill-posed problems is the very fast (exponential) decay of singular values when $k$ goes to infinity combined with errors in data.

**Exercise 2.3.6.** By using the singular value decomposition, show that for any compact operator $A$, we have

$$||x(\alpha)||_X^2 \leq \frac{1}{2\alpha}||y||_Y^2 + 2\|x^0\|_X^2$$

for any solution $x(\alpha)$ to the regularized equation (2.3.4).

If $A$ itself is positive and self-adjoint, then the equation (2.0) can be regularized by the similar equation

$$(A + \alpha)x(\alpha) = y. \tag{2.3.7}$$

**Exercise 2.3.7.** Prove that a solution $x(\alpha)$ to the equation (2.3.7) exists for any $y \in Y$, is unique, and $||x(\alpha)||_X \leq \alpha^{-1}||y||_X$.

To be more particular, we consider the integral operator from Example 2.1.3. We let $X = L_2(\Omega)$ and $Y = L_2(\Omega_1)$. Then the adjoint operator

$$A^*F(y) = \int_{\Omega_1} K(x,y)F(x)dx$$

and the regularization of the first kind integral equation is the Fredholm integral equation

$$\alpha f(z) + \int_{\Omega_1} K(x,z)\int_{\Omega} K(x,y)f(y)dy\,dx = \int_{\Omega_1} K(x,z)F(x)dx, z \in \Omega,$$

which has a unique solution $f = f(,\alpha)$ for any $\alpha > 0$ according to Exercise 2.3.4.

When $\Omega = \Omega_1$ and $K(x,y) = K(y,x)$, the integral operator is self-adjoint in $X$, and one can use the simpler regularization (2.3.7).

EXAMPLE 2.3.8 (SMOOTHING STABILIZERS). Let $\Omega = \Omega_1 = (a, b)$ be an interval of the real axis $\mathbb{R}$, and $X = Y = L_2(a, b)$. Let us consider

$$\mathcal{M}f = \int_\Omega (f^2 + f'^2).$$

From the well-known results about $L_2$-norms and about compactness in Lebesgue spaces (see, e.g., the book of Yosida [Yo], ch. X., sect 1), we derive the properties (2.3.1) and (2.3.2). This functional justifies the term *stabilizing functional*, because for any $\alpha$ we are looking for solutions of our equation with bounded first-order derivatives, which makes the solution stable by smoothing cusps of its graph.

For a smoothing stabilizer, the regularized equation is more complicated. To derive it, we can repeat the argument from Example 2.3.3, letting $x(\alpha) = f$ and replacing the new stabilizing term $v$ by $f + tu$. Then, instead of the term $(u, f)_X$, we obtain

$$\int_\Omega (fu + f'u') = \int_\Omega (fu - f''u) + \cdots ,$$

where ... denotes boundary integrals containing $f'u$ that are the result of the integration by parts. Using all $u$ that are zero on $\partial\Omega$, we conclude as above that the factor of $u$ in these integrals must be zero or

$$-\alpha f''(z) + \alpha f(z) + \int_\Omega K(x, z) \int_\Omega K(x, y) f(y) dy\, dx$$
$$= \int_\Omega K(x, z) F(x) dx, z \in \Omega,$$

which is an integro-differential equation for $f$.

In the many-dimensional case, one can similarly make use of the regularizer

$$\mathcal{M}f = \int_\Omega (|f|^2 + \nabla f \cdot \nabla f),$$

which is known to improve numerics when a solution to be found is smooth. Similarly, the regularized equation is

$$-\alpha \Delta f(z) + \alpha f(z) + \int_\Omega K(x, z) \int_\Omega K(x, y) f(y) dy\, dx$$
$$= \int_\Omega K(x, z) F(x) dx, z \in \Omega.$$

As we already noted, the simpler regularizer (2.3.4) works quite well for linear problems, so the smoothing stabilizer is more appropriate for nonlinear problems, which we will not discuss here in detail, referring rather to the books of Engl, Hanke, and Neubauer [EnHN] and of Tikhonov and Arsenin [TiA].

Example 2.3.9 (Maximal Entropy Regularizer). Quite useful are regularizers of physical origin. When one is solving the integral equation from Example 2.1.3 in the class of positive functions $f$, a regularized variational method to solve the equation $Af = F$ is to minimize

$$\|Af - y^\delta\|_Y^2 + \alpha E(f), \quad \text{where}\, E(f) = \int_\Omega f \ln(f/m),$$

where $m$ is some positive function reflecting a priori information about $f$. When $n = 1$ and $\Omega = (0, l)$, one often chooses $m(x) = 1/x$. The functional $-E(f)$ is known as the Shannon entropy, which is a measure of the informational content of $f$. While this functional looks different from the $L_2$-norm, it can be transformed into this norm by a (nonlinear) Nemytskii operator $f \to f_*$ defined by the pointwise relation $f \log(f/m)(x) = f_*^2(x)$, so that the above regularization theory can be applied.

Another important example of a regularizing functional is given by the bounded variation described in more detail in Section 10.2.

## *Quasi-solutions*

Let $X_\bullet$ be a compact set in $X$. An element $x_\bullet$ of this set is called a quasi-solution to the equation (2.0) with respect to $X_\bullet$ if $x_\bullet$ is a solution to the minimization problem

$$\min \|Av - y\|_Y \text{over}\, v \in X_\bullet. \tag{2.3.8}$$

Since $X_\bullet$ is a compact set, there is a solution to this problem.

**Exercise 2.3.10.** Let (2.3.8) have a unique minimizer $x(y)$. Prove that $x(y)$ is continuous with respect to $y$.

Now we consider a regularization method based on approximation of $X_M$ by compact subsets and first suggested by V. Ivanov (see the book of Ivanov, Vasin, and Tanana [IvVT]).

Let $X(k)$ be compact subsets of $X_M$. Assume that $X(k) \subset X(k+1)$ and that closure of the union of $X(k)$ over $k$ is $X_M$. A typical example of $X(k)$ is the intersection of $X_M$ and of the ball of radius $k$ of the finite-dimensional space span $\{x(1), \ldots, x(k)\}$ generated by vectors $x(1), \ldots, x(k)$ in $X$.

We will show that the sets $\mathfrak{X}(y_\delta, k)$ that consist of all quasi-solutions with respect to $X(k)$ and for $y = y^\delta$ are convergent to $\{x\}$ when $\|y_\delta - y\|_Y < \delta, \delta \to 0$ and $k \to \infty$. Indeed, let $x_k \in \mathfrak{X}(y_\delta, k)$. Since the $X(k)$ approximate $X_M$ in the above sense, there are $x_k^\bullet \in X(k)$ such that $\|x_k^\bullet - x\|_X = \omega(k, x; X)$ converge to zero as $k \to \infty$. By the minimizing property,

$$\|Ax_k - y_\delta\|_Y \le \|Ax_k^\bullet - y_\delta\|_Y \le \|Ax_k^\bullet - Ax\|_Y + \|Ax - y_\delta\|_Y \le \omega_A(k, x) + \delta,$$

where $\omega_A(k,x) = \|Ax_k^\bullet - Ax\|_Y$ goes to zero as $k \to \infty$ by the continuity of $A$. By the triangle inequality,

$$\|Ax_k - Ax\|_Y \le \|Ax_k - y_\delta\|_Y + \|y_\delta - y\|_Y < 2\delta + \omega_A(k,x).$$

By the property (2.2.2) of $X_M$, we have

$$\|x_k - x\|_X \le \omega(2\delta + \omega_A(k,x)), \tag{2.3.9}$$

where $\omega$ is a stability estimate for (2.0) at the point $x$. Since $x_k$ is an arbitrary point of $\mathfrak{X}(y_\delta, k)$, we have an estimate of the distance between this set and the exact solution $x$.

Summing up, we have a regularization algorithm $R_\alpha y = \mathfrak{X}(y,k)$, where $\alpha$ is $1/k$. Here $\mathfrak{X}(k,y)$ is the set of all solutions to the minimization problem (2.3.8) with $X_\bullet = X(k)$.

This regularization can be used practically for a numerical solution of equation (2.0). Normally, one uses as $\{x(k)\}$ trigonometric or polynomial bases in standard spaces. At each step of the algorithm, one has to solve a complicated (as a rule, non-convex) optimization problem (2.3.8).

**Exercise 2.3.11.** Formulate the minimization problem for the solution of equation (2.0) where $A$ is the operator from Example 2.1.1 (backward heat equation at the initial moment of time) by using the given basis. From the estimate in this example, derive an estimate (2.3.0) for this particular problem with a particular function $\omega$.

## 2.4 Convergence of Regularization Algorithms

Let $\omega$ be a stability estimate for equation (2.0) on the compact set $X_\bullet$, which is defined as $X_M \cap \{\mathcal{M}(x) \le \tau/\alpha\}$, where $\tau = \|Ax_\bullet - y\|_Y^2 + \alpha\mathcal{M}(x_\bullet)$ for certain $x_\bullet \in X_M, \mathcal{M}(x) \le \mathcal{M}(x_\bullet)$. Such a stability estimate exists as soon as we have uniqueness of a solution in $X_M$. We observe that any stability estimate on $X_M$ will be a stability estimate on its subset $X_\bullet$, but for the set $X_\bullet$, we can expect a better stability.

**Lemma 2.4.1.** *If $R_\alpha$ is a variational regularization algorithm from Section 2.3 and $R_\alpha(y)$ contains only one element $x(\alpha)$, then*

$$\|x(\alpha) - x\|_X \le \omega(2\|Ax - y\|_Y + (\alpha\mathcal{M}(x))^{1/2}). \tag{2.4.1}$$

PROOF. We have

$$\|Ax(\alpha) - y\|_Y^2 \le \|Ax(\alpha) - y\|_Y^2 + \alpha\mathcal{M}(x(\alpha)) \le \|Ax - y\|_Y^2 + \alpha\mathcal{M}(x)$$

according to the definition of $x(\alpha)$ by minimization. Therefore,

$$\|Ax(\alpha) - y\|_Y \leq \|Ax - y\|_Y + (\alpha\mathcal{M}(x))^{1/2}.$$

By the triangle inequality,

$$\|Ax(\alpha) - Ax\|_Y \leq \|Ax(\alpha) - y\|_Y + \|Ax - y\|_Y \leq 2\|Ax - y\|_Y + (\alpha\mathcal{M}(x))^{1/2},$$

and the inequality (2.4.1) follows from the definition of the stability estimate $\omega$. $\square$

The bound (2.4.1) suggests the following choice of the regularization parameter. Suppose our data $y_\delta$ are given with the error $\delta : \|Ax - y_\delta\|_Y \leq \delta$ and we have the a priori information

$$\mathcal{M}(x) \leq M. \tag{2.4.2}$$

Then we can choose $\alpha(\delta) = \delta^2/M$ and obtain the bound

$$\|x(\alpha) - x\|_X \leq \omega(3\delta). \tag{2.4.3}$$

Since the stability of the minimization procedure (2.3.3) deteriorates when $\alpha \to 0$, we cannot tell whether this estimate reflects the actual complexity of the problem or whether the choice of the regularization parameter is optimal. Examples of stability estimates from Section 2.2 suggest that in many cases, one can expect logarithmic convergence rates.

What is missing in this discussion is how to solve the minimization problem (2.3.3). A constructive way to do it in the linear case is to solve equation (2.3.4). We discuss this problem in more detail assuming that $A$ is a compact linear operator in a Hilbert space $X$.

First, let $A$ be a self-adjoint, positive, and compact operator. We write the equation (2.3.7) as

$$(\alpha + A)(x + x^\alpha + x^\delta) = y + (y^\delta - y), y = Ax, \|y_\delta - y\|_Y < \delta, \tag{2.4.4}$$

where $x^\delta$ solves the equation $(\alpha + A)x^\delta = y_\delta - y$. It is easy to see that

$$\|x^\delta\|_X \leq \delta/\alpha \tag{2.4.5}$$

and that $(\alpha + A)(x + x^\alpha) = y$, or $(\alpha + A)x^\alpha = -\alpha x$. The self-adjoint, compact operator $A$ has an orthonormal basis of eigenvectors $a_k$ corresponding to eigenvalues $\lambda_k$. We can assume $\lambda_k \geq \lambda_{k+1}$. Moreover, let $X$ be the uniqueness set, so $\lambda_k > 0$. We set $x_k = (x, a_k)_X$. Then the equation for $x^\alpha$ is $(\alpha + \lambda_k)x_k^\alpha = -\alpha x_k$, and we have

$$\|x^\alpha\|_X^2 \leq \sum_{k \leq K} \alpha^2(\alpha + \lambda_k)^{-2}x_k^2 + \sum_{K<k} x_k^2 \leq \alpha^2\lambda_K^{-2}\, \|x\|_X^2 + \sum_{K<k} x_k^2. \tag{2.4.6}$$

This estimate and the estimate (2.4.5) in principle guarantee convergence of our regularization procedure at this particular $x$. Indeed, choose $K$ large enough to make the sum over $k > K$ small; then choose $\alpha$ small to make the first term small, and then choose small $\delta$.

To obtain estimates, we consider the set $X_M$ defined by the inequalities

$$|x_k| \leq m_k \tag{2.4.7}$$

and let $M_K$ be $\left(\sum_{K<k} m_k^2\right)^{1/2}$. We assume that the series is convergent. Then $M_K$ goes to zero as $K$ goes to infinity. From (2.4.6), we have $\|x^\alpha\|_X^2 \leq \alpha^2\lambda_K^{-2}M_1^2 + M_K^2$, so $\|x^\alpha\|_X \leq \alpha\lambda_K^{-1}M_1 + M_K$. Now, from (2.4.5) we have the following estimate:

$$\|x(\alpha) - x\|_X \leq \alpha M_1/\lambda_K + M_K + \delta/\alpha, \tag{2.4.8}$$

and then the optimal choice of $\alpha$ depends on our constraint (2.4.7). Replacing (2.4.2) by the more special constraint (2.4.7) results in more explicit bound.

We consider Example 2.1.1 again. If we let $X = L_2(0,1)$, replace $x$ by $u_0$, and assume $\|u_0\|_X^2 + \|\partial_x u_0\|_X^2 \leq C_1$, and then by using the Fourier expansion with respect to the trigonometric basis, we conclude that $|x_k| \leq Ck^{-1}$, so we let $m_k = Ck^{-1}$.

**Exercise 2.4.2.** Show that for the solution $u(\alpha)$ to the regularized equation $(\alpha + A)u(\alpha) = u_{T\delta}$ to the backward heat equation $Au_0 = u_T, u_0 \in X_M$, we have the error estimate

$$\|u(\alpha) - u_0\|_X \leq \alpha C_1 e^{\pi^2K^2T} + CK^{-1/2} + \delta/\alpha.$$

Show that by an appropriate choice of $K$ and $\alpha$, one can achieve the logarithmic estimate

$$C_2/(-\ln\delta)^{1/2}$$

of the error with respect to $\delta$.

{*Hint*: as in Exercise 2.2.5, minimize with the respect to $K$ and then with respect to $\alpha$.}

A similar scheme can be developed for the regularized equation (2.3.4). Cheng and Yamamoto [CheY1] proposed to replace a minimizer $x(\alpha)$ of the regularized discrepancy functional (2.3.3) by any element $x(\alpha,\varepsilon)$ which is a $\varepsilon$-approximate minimizer, i.e.,

$$\|Ax(\alpha,\varepsilon) - y\|_Y^2 + \alpha\mathcal{M}(x(\alpha,\varepsilon) \leq inf(\|Ax - y\|_Y^2 + \alpha\mathcal{M}(x)) + \varepsilon^2$$

where *inf* is over $x \in X_M$. They showed that a stability estimate on $X_M$ implies a convergence rate similar to the rate of Lemma 2.4.1. Although, theoretically, a compactness assumption is removed, in practical numerical solutions, it does not remove the question about how to find an approximate infimum.

An important issue is the choice of the regularization parameter $\alpha$. There is the well-known discrepancy principle suggested by Morozov: let

$$\left\| Ax(\alpha(\delta)) - y^{\delta} \right\|_Y = \delta. \tag{2.4.9}$$

This is an a posteriori method because it only says that the parameter $\alpha$ we have chosen was consistent with the accuracy $\delta$ of the data $y$.

**Lemma 2.4.3.** *The equation* (2.4.9) *has a unique solution* $\alpha(\delta)$, *and* $x(\alpha(\delta))$ *converges to* $x$ *in* $X$ *when* $\delta \to 0$.

Proof. We will show that the left side of (2.4.9) is increasing with respect to $\alpha$. Observe that using the expansion $y^{\delta} = \sum y_k^{\delta} b_k$ and the relations (2.3.5), we obtain

$$\begin{aligned} Ax(\alpha) &= \sum A(\alpha + A^*A)^{-1} A^*(y_k^{\delta} b_k) = \sum A(\alpha + A^*A)^{-1} y_k^{\delta} \lambda_k a_k \\ &= \sum \lambda_k (\alpha + \lambda_k^2)^{-1} y_k^{\delta} A a_k = \sum \lambda_k^2 (\alpha + \lambda_k^2)^{-1} y_k^{\delta} b_k. \end{aligned}$$

It suffices to prove the monotonicity for the squared norm, which is

$$(Ax(\alpha) - y^{\delta}, Ax(\alpha) - y^{\delta})_Y = \sum_{k=1}^{\infty} \alpha^2 (\alpha + \lambda_k^2)^{-2} y_k^{\delta 2}.$$

It is clear that any term of the last series is increasing with respect to $\alpha$ unless it is zero.

To prove the convergence, we can use the variational formulation (2.3.3) of equation (2.3.4). Then

$$\left\| Ax(\alpha) - y^{\delta} \right\|_Y^2 + \alpha \left\| x(\alpha) \right\|_X^2 \leq \left\| Ax - y^{\delta} \right\|_Y^2 + \alpha \left\| x \right\|_X^2 = \delta^2 + \alpha \left\| x \right\|_X^2.$$

Since due to our choice of $\alpha$ in (2.4.9) the first term is $\delta^2$, we conclude that $\|x(\alpha)\|_X^2 \leq \|x\|_X^2$. Hence,

$$\begin{aligned} \|x(\alpha) - x\|_X^2 &= \|x(\alpha)\|_X^2 - 2(x(\alpha), x)_X + \|x\|_X^2 \\ &\leq 2 \|x\|_X^2 - 2(x(\alpha), x)_X, \end{aligned}$$

and the convergence $x(\alpha) \to x$ in $X$ follows from the convergence of the second term to $2\|x\|_X^2$. We will show this convergence.

According to (2.3.4) and the equality $Ax = y$, we have

$$x(\alpha) - x = (\alpha + A^*A)^{-1} A^*(y^{\delta} - y) - \alpha(\alpha + A^*A)^{-1} x,$$

or in coordinates as above,

$$(x(\alpha) - x)_k = \lambda_k(\alpha + \lambda_k^2)^{-1}(y^\delta - y)_k - \alpha(\alpha + \lambda_k^2)^{-1} x_k.$$

For any fixed $k$, the eigenvalue $\lambda_k > 0$. Besides, $\alpha(\delta) \to 0$ when $\delta \to 0$, so $(x(\alpha) - x)_k \to 0$. It suffices to show the convergence to 0 of

$$(x(\alpha) - x, x)_X = \sum^K (x(\alpha) - x)_k x_k + \sum_{K+1} (x(\alpha) - x)_k x_k. \tag{2.4.10}$$

Let $\varepsilon > 0$. Fix $K$ such that

$$\sum_{K+1} x_k^2 \le \varepsilon^2 (16 \|x\|_X^2)^{-1}.$$

By the Cauchy-Schwarz inequality, the second term on the right side of (2.4.10) is not greater than

$$\|x(\alpha) - x\|_X \left( \sum_{K+1} x_k^2 \right)^{1/2} \le (\|x(\alpha)\|_X + \|x\|_X)\varepsilon(4\|x\|_X)^{-1} \le \varepsilon/2$$

due to the inequality $\|x(\alpha)\|_X \le \|x\|_X$. Since $K$ is fixed, the first term on the right side of (2.4.10) goes to 0 when $\delta$ goes to 0 because of the convergence of coordinates.

The proof is complete. □

One cannot obtain convergence rates without using stability estimates for the original equation (2.0).

Maslov [Mas] first observed that for some regularization algorithms, the convergence is equivalent to the existence of a solution to equation (2.0). Consider the regularizer (2.3.4). Assume that $R(A)$ is dense in $Y$ and ker $A = \{0\}$ (i.e., we have uniqueness of a solution). These assumptions are quite realistic and are satisfied in many applications.

Let $\delta \to 0$ (so the $y_\delta$ converge to $y$) and $\alpha(\delta) \to 0$ and assume that the $x(\alpha)$ converge to $x_\bullet$. Since all operators in (2.3.4) are continuous, we can pass to the limit and obtain $A^*Ax_\bullet = A^*y$. Since $R(A)$ is dense in $Y$, we have ker $A^* = \{0\}$, so $Ax_\bullet = y$, and we have a solution to the equation (2.0). On the other hand, if a solution to (2.0) exists, then Exercises 2.3.3 and 2.2.3 guarantee the convergence of this regularization algorithm.

In the paper of Maslov [Mas], the same result is obtained for the "adjoint" regularization $x^*(\alpha) = A^*(\alpha + AA^*)^{-1} y_\delta$.

**Exercise 2.4.4.** Show that for this regularization, convergence of $x^*(\alpha)$ when $\alpha$ and $\delta$ go to 0 implies that the original equation (2.0) has a solution $x \in X$, provided that the solution is unique and the range of $A$ is dense in $Y$.

## 2.5 Iterative Algorithms

To solve the regularized equations of the previous section in practice, one uses various iterative methods. At any step of such an algorithm, one solves a well-posed problem, and if this algorithm is convergent, one obtains an approximate solution of the original problem, avoiding long and expensive minimization procedures like (2.3.3), which are even more complicated by the non-convexity of functional, which is typical for inverse problems in partial differential equations.

We start with a short description of these algorithms with iterative solutions of the linear regularized equation (2.3.4). The most minimization algorithms make use of the gradient methods, which require that one minimizes the functional $\Phi(v) = \|Av - y\|_X^2 + \alpha\mathcal{M}(v)$ by iterations:

$$x(m+1) = x(m) - 1/2\tau(m)\Phi'(x(m)), \tag{2.5.1}$$

where $\tau(m)$ is a parameter that may not depend on $m$ and $\Phi'(x)$ is a gradient of the functional $\Phi$ at $x$. We will start with the simplest case of a linear operator $A$ in a Hilbert space $X$ and the functional $\mathcal{M}(v) = \left\|v - x^0\right\|_X^2$. Then $\Phi'(x) = 2(A^*Ax - A^*y + \alpha(x - x^0))$, and the relations (2.5.1) are

$$x(m+1) = x(m) - \tau(m)((A^*A + \alpha I)x(m) - A^*y - \alpha x^0). \tag{2.5.2}$$

To analyze the expected convergence, we will assume that $\tau(m) = \tau$ and make use of the singular value decomposition (2.3.5), denoting by $x_k$ the $k$th coordinate of $x$ in the basis $\{a_k\}$ and similarly for $y_k$ in the basis $\{b_k\}$.

**Lemma 2.5.1.** *Let $\left\|y^\delta - Ax\right\|_Y \leq \delta$. Let $x(1), x^0 \in X$. Then under the condition*

$$0 < \tau(\|A\|^2 + \alpha) < 1 \tag{2.5.3}$$

*we have the following estimate:*

$$\begin{aligned} \|x(m+1) - x\|_X^2 \leq\ & 4(1-\tau\lambda_K^2)^{2m}(\|x(1)\|_X^2 + \|x\|_X^2 + \left\|x^0\right\|_X^2) \\ & +\delta^2\alpha^{-1} + 4\alpha^2\lambda_K^{-4}\left\|x - x^0\right\|_X^2 + R(K), \end{aligned} \tag{2.5.4}$$

*where $R(K) = 4\sum_{K<k}(x_k^2(1) + x_k^2 + x_k^{02} + (x - x^0)_k^2)$.*

Proof. From (2.5.2), we have

$$x_k(m+1) = q_k x_k(m) + \tau(A^*y^\delta + \alpha x^0)_k, \quad q_k = 1 - \tau(\lambda_k^2 + \alpha).$$

By induction it is easy to check that

$$\begin{aligned} x_k(m+1) &= q_k^m x_k(1) + (q_k^{m-1} + \cdots + 1)\tau(A^*y^\delta + \alpha x^0)_k \\ &= q_k^m x_k(1) + (1 - q_k^m)/(1 - q_k)\tau(\lambda_k^2 x_k + \lambda_k w_k^\delta + \alpha x_k^0), \end{aligned}$$

because $A^*y^\delta = A^*Ax + A^*w^\delta, w^\delta = y^\delta - y$ and because of the relations (2.3.5). Using the definition of $q_k$, we obtain

$$\begin{aligned} x_k(m+1) - x_k = q_k^m x_k(1) + (1-q_k^m)\lambda_k(\lambda_k^2+\alpha)^{-1}w_k^\delta \\ -q_k^m(\lambda_k^2+\alpha)^{-1}(\lambda_k^2 x_k + \alpha x_k^0) + \alpha(\lambda_k^2+\alpha)^{-1}(x^0-x)_k, \end{aligned} \tag{2.5.5}$$

where $w_k^\delta$ are coefficients with respect to $b_k$, and by using the inequalities

$$\begin{aligned} (a+b+c+d)^2 &\le 4(a^2+b^2+c^2+d^2), \\ (\lambda_k^2+\alpha)^{-2}(\lambda_k^2 x_k + \alpha x_k^0)^2 &\le \max\{x_k^2, x_k^{02}\} \le x_k^2 + x_k^{02} \end{aligned}$$

from (2.5.5), we conclude that

$$\begin{aligned} (x_k(m+1)-x_k)^2 \le 4q_k^{2m}x_k^2(1) + \alpha^{-1}w_k^{\delta 2} \\ +4q_k^{2m}(x_k^2+x_k^{02}) + 4\alpha^2(\lambda_k^2+\alpha)^{-2}(x-x^0)_k^2, \end{aligned}$$

where we have utilized that $\lambda_k(\lambda_k^2+\alpha)^{-1} \le 2^{-1}\alpha^{-1/2}$. Summing over $k$ and using (2.5.3), we obtain

$$\begin{aligned} &\|x(m+1)-x\|_X^2 \\ &\le 4\left(\sum_{k\le K} q_k^{2m}(x_k^2(1)+x_k^2+x_k^{02}) + \sum_{K<k}(x_k^2(1)+x_k^2+x_k^{02})\right) \\ &\quad +\alpha^{-1}\delta^2 + 4\sum_{k\le K}\alpha^2\lambda_k^{-4}(x-x^0)_k^2 + 4\sum_{K<k}(x-x^0)_k^2, \end{aligned}$$

which gives the bound (2.5.4). □

Lemma 2.5.1 implies convergence of the iterations (2.5.2), provided that $\delta \to 0$. Indeed, since the series for the coordinates of $x$, $x(1)$, and $x^0$ are convergent, for any $\varepsilon > 0$, we can find $K$ such that $R(K) < \varepsilon/4$. Fix this $K$ and then fix $\alpha > 0$ such that the third term on the right side of (2.5.4) is less than $\varepsilon/4$. After that, find $\delta$ such that the second term is less than $\varepsilon/4$, and $m$ such that the first term is less than $\varepsilon/4$. In many applied problems $e^{-CK} < \lambda_K < e^{-cK}$, while a smoothness assumption on $x, x(1), x^0$ implies that $R(K) < CK^{-1}$. Then we have to choose $K \sim \varepsilon^{-1}$ and then $\lambda_K \sim e^{-C\varepsilon^{-1}}$, so we must let $\alpha \sim e^{-C\varepsilon^{-1}}$ and $\delta \sim e^{-C\varepsilon^{-1}}$. Since the iterations (2.5.2) constitute a gradient method for minimization of a (coercive for fixed $\alpha$) convex functional, it is natural that for any fixed $\alpha$ they converge.

**Exercise 2.5.2.** Show that for fixed $\alpha > 0$, the iterations $x(m)$ determined by (2.5.2) converge in $X$ to $(\alpha + A^*A)^{-1}(A^*y + \alpha x^0)$.

Sometimes (2.5.2) with $\alpha = 0$ and $x^0 = 0$ is called the Landweber method:

$$x(m+1) = (I - \tau A^*A)x(m) + \tau A^*y.$$

The biggest advantage of iterative algorithms shows up in solving nonlinear equations, because in this case the variational regularization as a rule leads to a non-convex optimization problem with many possible minima and local minima.

A natural analogy of the Landweber method for the nonlinear equation (2.0) is the iterative method

$$x(m+1) = x(m) + \tau A'^*(x(m))(y - A(x(m))),$$

where $A'(x)$ is the Fréchet derivative of the operator $A$ at a point $x$. There are convergence proofs, provided that the initial guess $x(1)$ is sufficiently close to a solution $x$.

Another type of iterative algorithm suitable for ill-posed problems is the conjugate gradients method

$$\begin{aligned} x(m+1) &= x(m) + (w(m), v(m))_X (v(m), v^*(m))_X^{-1} v(m), \\ w(m) &= A^* y - A^* A x(m), v^*(m) = A^* A v(m), \\ v(m+1) &= -w(m) - (w(m), v^*(m))_X (v(m), v^*(m))_X^{-1} v(m), \end{aligned}$$

with some choice of $x(1)$ and $v(1) = y - A^* A x(1)$.

There are proofs of convergence of the conjugate gradient method as well.

These gradient-type methods have the advantage that no inversion of $A'$, $A'^*$ is needed. This is especially important when solving ill-posed problems, because this inversion is not stable and is computationally expensive. However, gradient methods are relatively slow, and sometimes one makes use of Newton-type methods like

$$\begin{aligned} x(m+1) &= x(m) - (\alpha(m) + A'^* A')^{-1} \\ &\quad \times (A'^*(Ax(m) - y) + \alpha(m)(x(m) - x^0)). \end{aligned} \tag{2.5.6}$$

We refer for convergence results and other iterative algorithms to the book of Engl, Hanke, and Neubauer [EnHN]. Besides a need for a good initial guess, the fundamental difficulty with convergence of the Newton-type algorithms (2.5.6) is that the Fréchet derivative $A'$ is not invertible in standard function spaces when the equation (2.0) represents severely ill-posed problem. We call the problem represented by the equation (2.0) mild ill-posed if $A'$ is invertible from one Sobolev space into another. In case of mild ill-posedness, there is a deep and interesting method of Nash-Moser which guarantees local convergence by combining iterations with some smoothing operators [CN], [Ham]. This method is not applicable to severely ill-posed problems. The best result about convergence of the iterations (2.5.6) is due to Hohage [Ho], and we briefly describe his findings.

To guarantee convergence, it was assumed in [EnHN] and [Ho] that the initial guess $x(0)$ satisfies the source-type condition

$$x(0) - x = f(A'^{*}(x)A'(x))v, \tag{2.5.7}$$

where $v \in X$ has "small" norm and $f$ is some continuous function on the spectrum of $A'^{*}(x)A'(x)$. For mildly ill-posed problems $f(\lambda) = \lambda^{\kappa}, 0 < \kappa < 1$ and for typical severely ill-posed inverse problem in partial differential equations, one expects only logarithmic stability, and then it is more natural to choose $f(\lambda) = (-ln\lambda)^{-p}$. Since the operator $A$ (and hence $A'$) is highly smoothing, the source condition (2.5.7) is much stronger that a standard "smallness" condition for convergence of Newton-type methods. Replacing power by logarithm is certainly less restrictive. Another condition in [Ho] for the convergence of (2.5.6) is the assumption that the range of the $A'(x)$ is not changing dramatically with $x$. More precisely, one assumes that for $x_1, x_2$ which are close to $x$, there are continuous linear operators $R(x_1, x_2), Q(x_1, x_2)$ from $Y$ into itself and from $X$ into $Y$ and constants $C(R), C(Q)$ such that

$$A'(x_1) = R(x_1, x_2)A'(x_2) + Q(x_1, x_2) \tag{2.5.8}$$

with

$$\|R(x_1, x_2)\| \leq C(R), \|Q(x_1, x_2)\| \leq C(Q)\|A'(x)(x_1 - x_2)\|_Y.$$

So far the both conditions could not be checked for important severely ill-posed problems, like the inverse gravimetry problem or inverse scattering problem for obstacles which are domains. The condition (2.5.8) seems to be especially delicate since singularities of solutions dictating range of $A'(x)$ depend on shape of unknown domain. It looks that there is a better chance to satisfy conditions (2.5.7), (2.5.8) in problems of determining coefficients of boundary conditions for elliptic problems when the distance to singularities is given.

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