

Random Walks That Find Perfect Objects and the Lovász Local Lemma

Dimitris Achlioptas and Fotis Iliopoulos

Abstract We give an algorithmic local lemma by establishing a sufficient condition for the uniform random walk on a directed graph to reach a sink quickly. Our work is inspired by Moser’s entropic method proof of the Lovász Local Lemma (LLL) for satisfiability, and completely bypasses the Probabilistic Method formulation of the LLL. In particular, our method works when the underlying state space is entirely unstructured. Similarly to Moser’s argument, the key point is that the inevitability of reaching a sink is established by bounding the entropy of the walk as a function of time.

1 Introduction

Let Ω be a (large) set of objects and let F be a collection of subsets of Ω , each subset comprising objects sharing some (negative) feature. We will refer to each subset $f \in F$ as a *flaw* and, following linguistic rather than mathematical convention, say that f is *present in* σ if $f \ni \sigma$. We will say that an object $\sigma \in \Omega$ is *flawless* (perfect) if no flaw is present in σ . For example, given a CNF formula on n variables with clauses $c_1, c_2, \dots, c_m$, we can define a flaw for each clause c_i , comprising the subset of $\Omega = \{0, 1\}^n$ violating c_i .

Given Ω and F we can often establish the *existence* of flawless objects via the Probabilistic Method. To do so, we introduce a probability measure on Ω and consider the collection of (“bad”) events $\mathcal{A}$ corresponding to the flaws (one event per flaw). The existence of flawless objects is thus equivalent to the intersection of the complements of the bad events having strictly positive probability. Clearly, such

D. Achlioptas (✉)
University of California Santa Cruz, Santa Cruz, CA, USA
e-mail: optas@cs.ucsc.edu

F. Iliopoulos
University of California Berkeley, Berkeley, CA, USA
e-mail: fotis.iliopoulos@berkeley.edu

positivity always holds if the events in $\mathcal{A}$ are independent and none of them has measure 1. One of the most powerful tools of the Probabilistic Method is the Lovász Local Lemma (LLL), asserting that such positivity also holds under a condition of limited dependence among the events in $\mathcal{A}$.

General LLL Let $\mathcal{A} = \{A_1, A_2, \dots, A_m\}$ be a set of events and let $D(i) \subseteq [m] \setminus \{i\}$ denote the set of indices of the dependency set of A_i , i.e., A_i is mutually independent of all events in $\mathcal{A} \setminus \{A_i \cup \bigcup_{j \in D(i)} A_j\}$. If there exist positive real numbers $\{\mu_i\}$ such that for all $i \in [m]$,

$$\Pr(A_i) \prod_{j \in \{i\} \cup D(i)} (1 + \mu_j) \leq \mu_i, \quad (1)$$

then the probability that none of the events in $\mathcal{A}$ occurs is at least $\prod_{i=1}^m 1/(1 + \mu_i) > 0$.

In a landmark work [4], Moser and Tardos made the general LLL constructive for *product* measures over explicitly presented variables. Specifically, in the variable setting of [4], each event A_i is determined by a set of variables $\text{vbl}(A_i)$ so that $j \in D(i)$ if and only if $\text{vbl}(A_i) \cap \text{vbl}(A_j) \neq \emptyset$. Moser and Tardos proved that if (1) holds, then repeatedly selecting *any* occurring event A_i (flaw present) and resampling every variable in $\text{vbl}(A_i)$ independently of all others, leads to a flawless object after a linear expected number of resamplings. Beyond the variable setting, Harris and Srinivasan in [2] algorithmized the general LLL for the uniform measure on permutations.

2 A New Framework

Inspired by the breakthrough of Moser [3], we take a more direct approach to finding flawless objects, bypassing the probabilistic formulation of the existence question. Specifically, we replace the measure on Ω by a directed graph D on Ω and we seek flawless objects by taking random walks on D . With this in mind, we refer to the elements of Ω as states. As in Moser’s work [3], each state transformation (step of the walk) $\sigma \rightarrow \tau$ will be taken to *address* a flaw present at σ . Naturally, a step may eradicate other flaws beyond the one addressed but may also introduce new flaws (and, in fact, may fail to eradicate the addressed flaw). By replacing the measure with a directed graph we achieve two main effects:

- (i) both the set of objects Ω and every flaw $f \subseteq \Omega$ can be entirely *amorphous*; that is, Ω does not need to have product form $\Omega = D_1 \times \dots \times D_n$, as in Moser–Tardos [4], or any form of symmetry, as in Harris–Srinivasan [2];
- (ii) the set of transformations for addressing a flaw f can differ *arbitrarily* among the different states $\sigma \in f$, allowing the actions to adapt to the “environment”. This is in sharp contrast with all past algorithmic versions of the LLL, where either no or very minimal adaptivity was possible.

Concretely, for each $\sigma \in \Omega$, let $U(\sigma) = \{f \in F : \sigma \in f\}$, i.e., $U(\sigma)$ is the set of flaws present in σ . For each $\sigma \in \Omega$ and $f \in U(\sigma)$ we require a set $A(f, \sigma) \subseteq \Omega$ that must contain at least one element other than σ , which we refer to as the set of possible *actions* for addressing flaw f in state σ . To address flaw f in state σ we select uniformly at random an element $\tau \in A(f, \sigma)$ and walk to state τ , noting that possibly $\tau = \sigma \in A(f, \sigma)$. Our main point of departure is that now the set of actions for addressing a flaw f in each state σ can depend *arbitrarily* on the state, σ , itself.

We represent the set of all possible state transformations as a multi-digraph D on Ω formed as follows: for each state σ , for each flaw $f \in U(\sigma)$, for each state $\tau \in A(f, \sigma)$ place an arc $\sigma \xrightarrow{f} \tau$ in D , i.e., an arc labeled by the flaw being addressed. Thus, D may contain pairs of states σ, τ with multiple $\sigma \rightarrow \tau$ arcs, each such arc labeled by a different flaw, each such flaw f having the property that moving to τ is one of the actions for addressing f at σ , i.e., $\tau \in A(f, \sigma)$. Since we require that the set $A(f, \sigma)$ contains at least one element other than σ for every flaw in $U(\sigma)$ we see that a vertex of D is a sink if and only if it is flawless. We focus on digraphs satisfying

Atomicity D is *atomic* if for every flaw f and state τ there is *at most* one arc incoming to τ labeled by f .

The purpose of atomicity is to capture “accountability of action”. In particular, note that if D is atomic, then every walk on D can be reconstructed from its final state and the sequence of labels on the arcs traversed, as atomicity allows one to trace the walk backwards unambiguously. To our pleasant surprise, in all applications we have considered so far we have found atomicity to be “a feature not a bug”, serving as a very valuable *aid* in the design of flaws and actions, i.e., of algorithms.

Having defined the multi-digraph D on Ω , we will now define a digraph C on the set of flaws F , reflecting some of the structure of D .

Potential Causality For each arc $\sigma \xrightarrow{f} \tau$ in D and each flaw g present in τ , we say that f *causes* g if $g = f$ or $g \not\in \sigma$. If D contains *any* arc in which f causes g we say that f *potentially causes* g .

Potential Causality Digraph The digraph $C = C(\Omega, F, D)$ of the potential causality relation, i.e., the digraph on F where $f \rightarrow g \Leftrightarrow f$ potentially causes g , is called the *potential causality digraph*. The *neighborhood* of a flaw f is $\Gamma(f) = \{g : f \rightarrow g \text{ exists in } C\}$.

In the interest of brevity, we will call C the causality digraph, instead of the potential causality digraph. It is important to note that C contains an arc $f \rightarrow g$ if there exists *even one* state transition aimed at addressing f that causes g to appear in the new state. In that sense, C is a “pessimistic” estimator of causality (or, alternatively, a lossy compression of D). This pessimism is both the strength and the weakness of our approach. On one hand, it makes it possible to extract results about algorithmic progress without tracking the state. On the other hand, it only gives good results when C remains sparse even in the presence of such stringent

arc inclusion. We feel that this tension is meaningful: maintaining the sparsity of C requires that the actions for addressing each flaw across different states are *coherent* with respect to the flaws they cause.

So far we have not discussed *which* flaw to address in each flawed state, demanding instead a non-empty set of actions $A(f, \sigma)$ for each flaw f present in a state σ . Suffice it to say that we consider algorithms which employ an *arbitrary* ordering π of F and in each flawed state σ address the greatest flaw according to π in a subset of $U(\sigma)$.

Definition 1 If π is any ordering of F , let $I_\pi: 2^F \rightarrow F$ be the function mapping each subset of F to its greatest element according to π , with $I_\pi(\emptyset) = \emptyset$. We will sometimes abuse notation and for a state $\sigma \in \Omega$, write $I_\pi(\sigma)$ for $I_\pi(U(\sigma))$ and also write I for I_π , when π is clear from context.

Definition 2 Let $D_\pi \subseteq D$ be the result of retaining for each state σ only the outgoing arcs with label $I_\pi(\sigma)$.

The next definition reflects that, since actions are selected uniformly, the *number* of actions available to address a flaw, i.e., the breadth of the “repertoire”, is important.

Amenability The *amenability* of a flaw f is

$$A_f = \min_{\sigma \in f} |A(f, \sigma)| . \quad (2)$$

The amenability of a flaw f will be used to bound from below the amount of randomness consumed every time f is addressed. (The minimum in (2) is often inoperative with $|A(f, \sigma)|$ being the same for all $\sigma \in f$.)

3 Statement of Results

Our simplest result, stated below, concerns the case where, after choosing a single fixed permutation π of the flaws, in each flawed state σ the algorithm addresses the greatest flaw present in σ according to π , i.e., the algorithm is the uniform random walk on D_π .

Theorem 3 *If for every flaw $f \in F$,*

$$\sum_{g \in \Gamma(f)} \frac{1}{A_g} < \frac{1}{e} ,$$

then for any ordering π of F and any $\sigma_1 \in \Omega$, the uniform random walk on D_π starting at σ_1 reaches a sink within $(\log_2 |\Omega| + |U(\sigma_1)| + s)/\delta$ steps with probability at least $1 - 2^{-s}$, where $\delta = 1 - \max_{f \in F} \sum_{g \in \Gamma(f)} \frac{e}{A_g}$.

Remark 4 In applications, typically, $\delta = \Theta(1)$.

Theorem 3 has the following three features worth discussing.

Arbitrary initial state: the fact that σ_1 can be arbitrary means that any foothold on Ω suffices to apply the theorem, without needing to be able to sample from Ω according to some measure. While sampling from Ω has generally not been an issue in existing applications of the LLL, this has only been true precisely because the sets and the measures considered have been highly structured.

Arbitrary number of flaws: the running time depends only on the number of flaws present in the initial state, $|U(\sigma_1)|$, not on the total number of flaws $|F|$. This has an implication analogous to the result of Haeupler–Saha–Srinivasan [1] on core events: even when $|F|$ is very large, e.g., super-polynomial in the problem’s encoding length, we can still get an efficient algorithm if we can show that $|U(\sigma_1)|$ is small, e.g., by proving that in every state only polynomially many flaws may be present. This feature provides great flexibility in the design of flaws.

Cutoff phenomenon: the bound on the running-time is sharper than a typical high probability bound, being instead akin to a mixing time cutoff bound, wherein the distance to the stationary distribution drops from near 1 to near 0 in a very small number of steps past a critical point. In our setting, the walk first makes $(\log_2 |\Omega| + |U(\sigma_1)|)/\delta$ steps without any guarantee of progress, but from that point on every single step has constant probability of being the last step. While, pragmatically, a high probability bound would be just as useful, the fact that our bound naturally takes this form suggests a potential deeper connection with the theory of Markov chains.

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