

Chapter 2

Models of Threshold Collective Behavior

Consider the following model of an active network structure that includes several interacting *agents*. Each agent chooses between two decisions resulting in one of two admissible *states*, namely, “1” (*active*, the excited state) or “0” (*passive*, the normal or unexcited state). For instance, possible examples are a *social network* [48] or a *mob* [23], where the active state means participation in mass riots.

While making his decision, each agent demonstrates *conformity behavior*, taking into account the so-called *social pressure* [30, 87] as the observed or predicted behavior of the environment: if a definite number (or proportion) of his “neighbors” are active, then this agent chooses activity. The minimum number (or proportion) of neighbors that “excites” a given agent is called his *threshold*. Note that there also exist models of *anti-conformity* behavior and “mixture” of conformity and anti-conformity, see [26].

Numerous models of *threshold collective behavior* [20, 21] extending Granovetter’s basic model [44] define the “equilibrium” state of a mob within collective behavior dynamics via the *distribution function* of agents’ *thresholds*. The framework of the game-theoretic models of threshold behavior [18, 19] also treats thresholds’ distribution as a major characteristic determining the set of Nash equilibria in the agents’ game.

The model adopted below is close to the agent-oriented models such as the bounded-neighborhood model and the spatial proximity model proposed by T. Schelling [82].

If the relationship between the equilibrium state of a system (a social network, a mob) and the threshold distribution function is known, one can pose threshold control problems, e.g., find an appropriate control action modifying agents’ thresholds so that the system reaches a desired equilibrium.

Consider the following *model* of a *mob* as a set $N = \{1, 2, \dots, n\}$ of agents. Agent $i \in N$ is characterized by

- (1) the *influence* $t_{ji} \geq 0$ on agent j (a certain “weight” of his opinion for agent j); for each agent j , we have the normalization conditions $\sum_{i \neq j} t_{ji} = 1, t_{ii} = 0$;

- (2) the decision $x_i \in \{0; 1\}$;
- (3) the *threshold* $\theta_i \in [0; 1]$, defining whether agent i acts under a certain *opponents' action profile* (the vector $\bar{x}_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ comprising the decisions of the other agents except agent i). Formally, define the action x_i of agent i as the best response to the existing opponents' action profile:

$$x_i = BR_i(x_{-i}) = \begin{cases} 1, & \text{if } \sum_{j \neq i} t_{ij} x_j \geq \theta_i \\ 0, & \text{if } \sum_{j \neq i} t_{ij} x_j < \theta_i. \end{cases} \quad (2.1)$$

This book has independent numbering of formulas for each section; while referring to a formula from another section, the double numbering system is used, where the first number indicates the section.

The behavior described by (2.1) is called *threshold behavior*, see surveys in [20, 21]. A *Nash equilibrium* is an agents' action vector $\bar{x}_N$ such that $\bar{x}_N = BR(\bar{x}_N)$, where

$$BR(\bar{x}_N) = (BR_1(\bar{x}_{-1}), \dots, BR_n(\bar{x}_{-n})).$$

Consider the *following discrete-time dynamic model of collective behavior* [23]. At the initial (zero) step, all agents are passive. At each subsequent step, the agents act simultaneously and independently according to the best-response procedure (2.1).

Introduce the notation

$$\begin{aligned} Q_0 &= \emptyset, Q_1 = \{i \in N \mid \theta_i = 0\}, \\ Q_k &= Q_{k-1} \cup \left\{ i \in N \mid \sum_{j \in Q_{k-1}, j \neq i} t_{ij} \geq \theta_i \right\}, k = 1, 2, \dots, n-1. \end{aligned} \quad (2.2)$$

Clearly, $Q_0 \subseteq Q_1 \subseteq \dots \subseteq Q_n \subseteq N$. Let $T = \{t_{ij}\}$ be the influence matrix of the agents and $\theta = (\theta_1, \theta_2, \dots, \theta_n)$ correspond to the vector of their thresholds. Evaluate the following index:

$$q(T, \theta) = \min\{k = \overline{0, n-1} \mid Q_{k+1} = Q_k\}. \quad (2.3)$$

Define the *collective behavior equilibrium* (CBE) [23]

$$x_i^*(T, \theta) = \begin{cases} 1, & \text{if } i \in Q_{q(T, \theta)} \\ 0, & \text{if } i \in N \setminus Q_{q(T, \theta)}, i \in N. \end{cases} \quad (2.4)$$

The value

$$x^* = \frac{\#Q_{q(T,\theta)}}{n} = \frac{1}{n} \sum_{i \in N} x_i^*(T, \theta) \quad (2.5)$$

with $\#$ denoting set power characterizes the proportion of active agents in the CBE.

Further exposition mostly deals with the *anonymous case* where the graph of agents' relations is complete: $t_{ij} = 1/(n-1)$). In the anonymous case, expression (2.1) takes the form

$$x_i = BR_i(\bar{x}_{-i}) = \begin{cases} 1, & \text{if } \frac{1}{n-1} \sum_{j \neq i} x_j \geq \theta_i, \\ 0, & \text{if } \frac{1}{n-1} \sum_{j \neq i} x_j < \theta_i. \end{cases} \quad (2.6)$$

Designate by $F(\cdot): [0, 1] \rightarrow [0, 1]$ the *distribution function of agents' thresholds*, a nondecreasing function defined on the unit segment that is left continuous and possesses right limit at each point of its domain. Let $\{x_t \in [0, 1]\}_{t \geq 0}$ be a discrete sequence of the *proportions of active agents*, where t indicates time step.

Assume that the proportion x_k of active agents at step k is known ($k = 0, 1, \dots$). Then the following recurrent expression describes the dynamics of the proportion of active agents at the subsequent steps [19–27, 44, 56]:

$$x_{l+1} = F(x_l), \quad l = k, k+1, \dots \quad (2.7)$$

(as a matter of fact, in theory of conformity collective behavior, this equation is sometimes called *Granovetter's behavior*).

The equilibria of system (2.7) are defined by the initial point x_0 (as a rule, $x_0 = 0$) and by the intersection points of the distribution function $F(\cdot)$ with the bisecting line of quadrant I, see [19, 23, 44]:

$$F(x) = x. \quad (2.8)$$

Note that 1 forms a trivial equilibrium due to the properties of the distribution function.

Potentially stable equilibria are points where the curve $F(\cdot)$ crosses the bisecting line approaching it “from left and top.”

Denote by $y = \inf\{x : x \in (0, 1], F(x) = x\}$ the least nonzero root of Eq. (2.8). The collective behavior equilibrium (CBE) and, as shown in [23], a Nash equilibrium of the agents' game is the point

$$x^* = \begin{cases} y, & \text{if } \forall z \in [0, y] : F(z) \geq z, \\ 0, & \text{otherwise.} \end{cases} \quad (2.9)$$

According to the properties of the distribution function, for implementing a nonzero CBE a sufficient condition is $F(0) > 0$.

Therefore, given an initial state (the proportion of active agents at step 0), further dynamics of system (2.7) and its equilibrium states depend on the properties of the distribution function of agents' thresholds. Hence, a goal-oriented modification of this function can be treated as *mob control*.

Possible ways of such control that vary the equilibrium states by affecting the parameters of the threshold distribution function will be analyzed in the forthcoming sections.

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