A Two-Level Additive Schwarz Domain Decomposition Preconditioner for a Flat-Top Partition of Unity Method

Susanne C. Brenner, Christopher B. Davis, and Li-yeng Sung

Abstract We investigate a two-level additive Schwarz domain decomposition preconditioner for a flat-top partition of unity method. We establish condition number estimates for the biharmonic problem and present numerical results that confirm our analysis.

1 Introduction

Let Ω be a polygonal domain in $\mathbb{R}^2$ and $f \in L_2(\Omega)$. Consider the following model problem: Find $u \in H_0^2(\Omega)$ such that

$$a(u, v) = (f, v) \qquad \text{for all } v \text{ in } H_0^2(\Omega) \tag{1}$$

where

$$a(w, v) = \int_\Omega (D^2 w : D^2 v)dx = \sum_{i,j=1}^{2} \int_\Omega \frac{\partial^2 w}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j}\, dx \tag{2}$$

and $(f, v) = \displaystyle\int_\Omega f v\, dx$.

S.C. Brenner (✉) • L.-Y. Sung
Department of Mathematics, Center for Computation & Technology, Louisiana State University, Baton Rouge, LA 70803, USA
e-mail: brenner@math.lsu.edu; sung@math.lsu.edu

C.B. Davis
Department of Mathematics, Tennessee Technological University, Cookeville, TN 38505, USA
e-mail: cbdavis@tntech.edu

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Let V_h be a finite dimensional subspace of $H_0^2(\Omega)$. The Ritz-Galerkin method for (1) is to find $u_h \in V_h$ such that

$$a(u_h, v) = (f, v) \qquad \forall\, v \in V_h. \tag{3}$$

In this paper we will investigate a two-level additive Schwarz domain preconditioner [8, 30] for the discrete problem (3), where V_h is constructed by a flat-top partition of unity method.

The conditioning of partition of unity methods is an important topic that has recently received some attention. Stable generalized finite element methods whose condition numbers are comparable to standard finite element methods are discussed in [1, 22, 23, 29, 33]. Preconditioners for extended finite element methods have also been investigated. (See [4, 24, 31] and references therein for a non-exhaustive list.) The focus of the aforementioned work is on the ill-conditioning of the discrete problem due to the choice of the enrichment functions. As far as we know, there is only one paper [14] in the literature where an additive Schwarz preconditioner for partition of unity methods is treated, and the preconditioner considered there is a hierarchical multilevel preconditioner.

One of the important features of the partition of unity method is its ability to generate a smooth approximation space with ease, making it a good candidate for higher order problems. While there is a substantial literature on domain decomposition preconditioners for finite element methods for fourth order problems [6, 7, 9, 10, 18, 19, 25, 26, 32], to our knowledge domain decomposition preconditioners for the partition of unity method have not been studied. Our goal is to fill this gap.

The rest of the paper is organized as follows. We present the flat-top partition of unity method and the additive Schwarz preconditioner in Sects. 2 and 3. The condition number estimates are carried out in Sect. 4, followed by numerical results in Sect. 5. The paper ends with some concluding remarks in Sect. 6.

2 A Flat-Top Partition of Unity Method

In this section we describe the construction of V_h using a flat-top partition of unity.

2.1 Partition of Unity

First we recall the definition of a W^2_∞ partition of unity.

Definition 1 Let $\Lambda = \{\Omega_i\}_{i=1}^n$ be an open cover of $\bar{\Omega}$ satisfying a pointwise overlap condition

$$\exists\, M \in \mathbb{N} \quad \text{such that} \quad \operatorname{card}\{i | x \in \Omega_i\} \leq M \quad \forall\, x \in \Omega\,.$$

Let $\{\varphi_i\}_{i=1}^n$ be a family of functions in $W^2_\infty(\mathbb{R}^2)$ satisfying

$$\operatorname{supp} \varphi_i \subset \bar{\Omega}_i \quad 1 \le i \le n,$$

$$\sum_{i=1}^n \varphi_i \equiv 1 \ \text{ on } \Omega,$$

$$|\varphi_i|_{W^m_\infty(\mathbb{R}^2)} \le \frac{C_m}{(\operatorname{diam} \Omega_i)^m} \quad 0 \le m \le 2,\ 1 \le i \le n,$$

where C_m are constants. We will refer to $\{\varphi_i\}_{i=1}^n$ as a W^2_∞ partition of unity subordinate to the cover Λ and the covering sets $\Omega_i \in \Lambda$ as patches.

We will use a variant of the partition of unity in [11–13, 16, 28] and we refer the interested reader to these articles for a more thorough description of the construction. Below we briefly describe our approach for a rectangular domain. Other domains can be treated in a similar fashion.

We begin by choosing two small positive parameters γ_1 and γ_2, and construct the domain Ω_γ by enlarging Ω by a distance of γ_j in the $\pm x_j$ directions for $j = 1$ and 2. We then subdivide Ω_γ into congruent rectangles R_i for $1 \le i \le n$. The lengths of the sides of these rectangles are denoted by h_1 and h_2, which are proportional to γ_1 and γ_2 respectively. The mesh parameter h is the maximum of h_1 and h_2.

The patches Ω_i are formed by enlarging the rectangles R_i by a distance of γ_j in the $\pm x_j$ directions for $j = 1$ and 2. There is a rectangular region in the center of each Ω_i denoted by Ω_i^{flat}. The partition of unity function φ_i is a C^1 piecewise polynomial function such that $\varphi_i = 1$ on Ω_i^{flat} and smoothly decreases to 0 on $\partial\Omega_i$. The construction of the flat-top partition of unity is illustrated in Fig. 1.

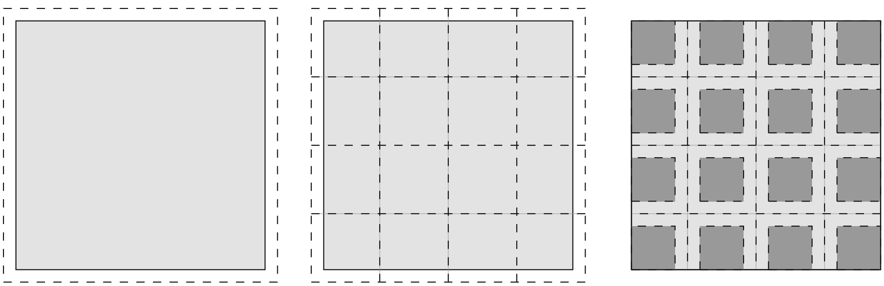

Fig. 1 The construction of the flat-top partition of unity: Ω is expand to Ω_γ (*left*); Ω_γ is subdivided into congruent rectangles (*middle*); Ω_i^{flat} are the dark shaded regions (*right*)

2.2 Generalized Finite Element Space

Let V_i be a subspace of the tensor product Lagrange finite element space $\mathbb{Q}_2$ defined on Ω_i whose members satisfy the homogeneous Dirichlet conditions on $\partial\Omega$. The interpolation nodes for V_i are placed inside the flat-top region Ω_i^{flat}. The generalized finite element space $V_h \subset V$ is given by

$$V_h = \sum_{i=1}^{n} \varphi_i V_i.$$

The interpolation operator $\Pi_h : C(\bar{\Omega}) \to V_h$ is defined by

$$\Pi_h v = \sum_{i=1}^{n} (\Pi_i v)\varphi_i, \tag{4}$$

where Π_i is the local nodal interpolation operator for V_i. The following interpolation error estimate can be established by combining standard interpolation error estimates for the $\mathbb{Q}_2$ finite element and the estimates for the partition of unity functions $\varphi_1, \ldots, \varphi_n$ (cf. [27, 28] for details):

$$\sum_{m=0}^{2} h^m |v - \Pi_h v|_{H^m(\Omega)} \le Ch^s |v|_{H^s(\Omega)} \qquad \forall\, v \in H^s(\Omega) \quad \text{and} \quad 2 \le s \le 3. \tag{5}$$

2.3 Discretization Error Estimate and Conditioning

According to elliptic regularity theory [5, 15, 21] for polygonal domains, we know that $u \in H^{2+\alpha}(\Omega)$, where the index of elliptic regularity α depends only on the angles at the corners of Ω. If Ω is convex, we can take α to be 1, otherwise α belongs to $(1/2, 1)$. It follows from (5) that

$$|u - u_h|_{H^2(\Omega)} = \inf_{v \in V_h} |u - v|_{H^2(\Omega)} \le |u - \Pi_h u|_{H^2(\Omega)} \le Ch^\alpha |u|_{H^{2+\alpha}(\Omega)}. \tag{6}$$

Let V_h' be the dual of V_h, $\langle \cdot, \cdot \rangle$ be the canonical bilinear form on $V_h' \times V_h$, and the linear operator $A_h : V_h \to V_h'$ be defined by

$$\langle A_h w, v \rangle = a(w, v) \quad \forall\, w, v \in V_h. \tag{7}$$

We can then rewrite (3) as

$$A_h x = f_h, \tag{8}$$

where $f_h \in V_h'$ is defined by

$$\langle f_h, v\rangle = \int_\Omega f v \, dx \qquad \forall\, v \in V_h.$$

It can be shown that the condition number of A_h satisfies

$$\kappa(A_h) \approx O(h^{-4}), \tag{9}$$

which is similar to the condition number for standard finite element methods.

3 A Two-Level Additive Schwarz Preconditioner

The two-level additive Schwarz preconditioner was introduced by Dryja and Widlund in [17]. It involves a coarse problem and local problems.

3.1 Coarse Problem

Let $V_0 \subset V$ be the generalized finite element space associated with a coarse mesh with mesh parameter H. We assume there are J coarse patches $\Omega_{j,H}$ $(1 \le j \le J)$ in the construction of V_0.

The coarse space V_0 is connected to V_h by the operator $I_0 : V_0 \longrightarrow V_h$, which is the restriction of the interpolation operator Π_h to V_0. The operator $A_0 : V_0 \longrightarrow V_0'$ is then given by

$$\langle A_0 w, v\rangle = a(I_0 w, I_0 v) \quad \forall w, v \in V_0. \tag{10}$$

3.2 Local Problems

The overlapping subdomains $\tilde{\Omega}_j$ of Ω are obtained by enlarging the coarse patch $\Omega_{j,H}$ $(1 \le j \le J)$ by the amount of $\delta_j (\ge 0)$ in the $\pm x_j$ directions for $j = 1$ and 2. This means that the overlap of the subdomains is given by $\delta = \max\{\delta_1 + \gamma_{1,H}, \delta_2 + \gamma_{2,H}\}$. By adjusting δ_j and $\gamma_{j,H}$, we can align $\partial\tilde{\Omega}_j$ with the boundaries of the patches for V_h and also control the overlap among the subdomains.

The local space $V_i \subset V_h$ is taken to be

$$V_j = \{v \in V_h : v = 0 \text{ on } \Omega \backslash \tilde{\Omega}_j\}$$

and it is connected to V_h by the natural injection $I_j : V_j \longrightarrow V_h$.

The operator $A_j : V_j \longrightarrow V_j'$ is given by

$$\langle A_j w, v\rangle = a(w, v) \quad \forall w, v \in V_j. \tag{11}$$

3.3 The Preconditioner

The two-level additive Schwarz preconditioner is defined by

$$B_{TL} = \sum_{j=0}^{J} I_j A_j^{-1} I_j^T,$$

where the transpose operator $I_j^T : V_h' \longrightarrow V_j'$ is given by

$$\langle I_j^T \mu, v\rangle = \langle \mu, I_j v\rangle \qquad \forall\, \mu \in V_h', v \in V_j.$$

Since $V_h = \sum_{j=0}^{J} I_j V_j$, the operator B_{TL} is symmetric positive definite and we have the following characterizations of the maximum and minimum eigenvalues of $B_{TL}A_h$ (cf. [8, Theorem 7.1.20]).

$$\lambda_{\max}(B_{TL}A_h) = \max_{\substack{v\in V_h\\ v\neq 0}} \frac{\langle A_h v, v\rangle}{\min_{\substack{v=\sum_{j=0}^{J} I_j v_j\\ v_j \in V_j}} \sum_{j=0}^{J} \langle A_j v_j, v_j\rangle} \tag{12}$$

$$\lambda_{\min}(B_{TL}A_h) = \min_{\substack{v\in V_h\\ v\neq 0}} \frac{\langle A_h v, v\rangle}{\min_{\substack{v=\sum_{j=0}^{J} I_j v_j\\ v_j \in V_j}} \sum_{j=0}^{J} \langle A_j v_j, v_j\rangle} \tag{13}$$

4 Condition Number Estimates

To avoid the proliferation of constants, we will use the notation $A \lesssim B$ (or $B \gtrsim A$) to represent the inequality $A \leq (\text{constant})B$, where the positive constant is independent of h, H, δ and J. The notation $A \approx B$ is equivalent to $A \lesssim B$ and $A \gtrsim B$.

4.1 Estimate for $\lambda_{\max}(B_{TL}A_h)$

The following lemma will lead to an upper bound for $\lambda_{\max}(B_{TL}A_h)$.

Lemma 1 *Let* $v_j \in V_j$ *for* $0 \le j \le J$ *and* $v = \sum_{j=0}^{J} I_j v_j$. *Then the following estimate holds*:

$$\langle A_h v, v\rangle \lesssim \sum_{j=0}^{J} \langle A_j v_j, v_j\rangle. \tag{14}$$

Proof Since I_j for $1 \le j \le J$ are natural injections, we derive from (2), (7) and (10)

$$\begin{aligned}\langle A_h v, v\rangle &= \int_\Omega D^2\left(\sum_{j=0}^{J} I_j v_j\right) : D^2\left(\sum_{k=0}^{J} I_k v_k\right) dx \\ &\le 2\int_\Omega |D^2 I_0 v_0|^2\, dx + 2\int_\Omega D^2\left(\sum_{j=1}^{J} I_j v_j\right) : D^2\left(\sum_{k=1}^{J} I_k v_k\right) dx \\ &= 2\langle A_0 v_0, v_0\rangle + 2\sum_{j,k=1}^{J} \int_\Omega D^2 v_j : D^2 v_k\, dx. \end{aligned} \tag{15}$$

Let the constant $c_{j,k}$ $(1 \le j,k \le J)$ be defined by

$$c_{j,k} = \begin{cases} 1 & \text{if } \tilde{\Omega}_j \cap \tilde{\Omega}_k \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $c_{j,k} = c_{k,j}$.

Let N be the maximum number of subdomains that can have nonempty intersection with a subdomain. Then we have

$$\sum_{k=1}^{J} c_{j,k} \le N \qquad \text{for} \quad 1 \le j \le J$$

and hence, in view of (11),

$$\begin{aligned}\sum_{j,k=1}^{J} \int_\Omega (D^2 v_j : D^2 v_k)\, dx &= \sum_{j,k=1}^{J} c_{j,k} \int_\Omega (D^2 v_j : D^2 v_k)\, dx \\ &= \sum_{j,k=1}^{J} c_{j,k} |v_j|_{H^2(\Omega)} |v_k|_{H^2(\Omega)} \\ &\le \left(\sum_{j,k=1}^{J} c_{j,k} |v_j|^2_{H^2(\Omega)}\right)^{\frac{1}{2}} \left(\sum_{j,k=1}^{J} c_{j,k} |v_k|^2_{H^2(\Omega)}\right)^{\frac{1}{2}}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{J} |v_j|^2_{H^2(\Omega)} \sum_{k=1}^{J} c_{j,k} \\
&\leq N \sum_{j=1}^{J} |v_j|^2_{H^2(\Omega)} = N \sum_{j=1}^{J} \langle A_j v_j, v_j \rangle. \qquad (16)
\end{aligned}
$$

The estimate (14) follows by combining (15) and (16). ☐

Combining (12) and (14), we have an upper bound for the eigenvalues of $B_{TL}A_h$:

$$
\lambda_{\max}(B_{TL}A_h) \lesssim 1. \qquad (17)
$$

4.2 Estimate for $\lambda_{\min}(B_{TL}A_h)$

The following lemma will lead to a lower bound for $\lambda_{\min}(B_{TL}A_h)$.

Lemma 2 *Given any $v \in V_h$, there exists a decomposition*

$$
v = \sum_{j=0}^{J} I_j v_j \qquad (18)
$$

where $v_j \in V_j$ for $1 \leq j \leq J$ and

$$
\sum_{j=0}^{J} \langle A_j v_j, v_j \rangle \lesssim \left[1 + \left(\frac{H}{\delta} \right)^3 \right] \langle A_h v, v \rangle. \qquad (19)
$$

Proof It follows from (5) (with $s = 2$) that

$$
\sum_{k=0}^{1} h^k |v - \Pi_h v|_{H^k(\Omega)} + h^2 |\Pi_h v|_{H^2(\Omega)} \lesssim h^2 |v|_{H^2(\Omega)} \qquad \forall\, v \in H^2(\Omega). \qquad (20)
$$

Similarly, we have

$$
\sum_{k=0}^{1} H^k |v - \Pi_H v|_{H^k(\Omega)} + H^2 |\Pi_H v|_{H^2(\Omega)} \lesssim H^2 |v|_{H^2(\Omega)} \qquad \forall\, v \in H^2(\Omega), \qquad (21)
$$

where Π_H is the analog of Π_h for V_H.

Let $v_0 = \Pi_H v \in V_0$, $w = v - I_0 v_0 = v - \Pi_h v_0$ and $v_j = \Pi_h(\theta_j w)$, where $\{\theta_j\}_{j=1}^J$ is a W^2_∞ partition of unity subordinate to the overlapping subdomains $\tilde{\Omega}_j$ such that

$$|\theta_j|_{W^k_\infty(\mathbb{R}^2)} \lesssim \delta^{-k} \qquad \text{for} \quad 0 \le k \le 2. \tag{22}$$

It is easy to check that $v_j \in V_j$ for $0 \le j \le J$ and (18) holds.

In view of (2), (7), (10), (20) and (21), we have

$$\begin{aligned} \langle A_0 v_0, v_0\rangle = |I_0 v_0|^2_{H^2(\Omega)} &= |\Pi_h v_0|^2_{H^2(\Omega)} \\ &\lesssim |v_0|^2_{H^2(\Omega)} \\ &= |\Pi_H v|^2_{H^2(\Omega)} \lesssim |v|^2_{H^2(\Omega)} = \langle A_h v, v\rangle. \end{aligned} \tag{23}$$

Next we consider

$$\langle A_j v_j, v_j\rangle = |v_j|^2_{H^2(\tilde{\Omega})} = |\Pi_h(\theta_j w)|^2_{H^2(\Omega)}$$

for $1 \le j \le J$. In view of (20), we have

$$\begin{aligned} \langle A_j v_j, v_j\rangle &\lesssim |\theta_j w|^2_{H^2(\Omega)} \\ &\lesssim \int_{\tilde{\Omega}_j} (w)^2 |D^2\theta_j|^2\,dx + \int_{\tilde{\Omega}_j} |D\theta_j|^2 |Dw|^2\,dx \\ &\qquad + \int_{\tilde{\Omega}_j} (\theta_j)^2 |D^2 w|^2\,dx \end{aligned} \tag{24}$$

and it only remains to estimate the three terms on the right-hand side of (24).

Observe that (20) and (21) imply

$$\begin{aligned} \|w\|_{L_2(\Omega)} &= \|v - \Pi_h v_0\|_{L_2(\Omega)} \\ &\le \|v - \Pi_h v\|_{L_2(\Omega)} + \|v - \Pi_H v\|_{L_2(\Omega)} \\ &\qquad + \|(v - \Pi_H v) - \Pi_h(v - \Pi_H v)\|_{L_2(\Omega)} \\ &\lesssim h^2|v|_{H^2(\Omega)} + H^2|v|_{H^2(\Omega)} + h^2|v - \Pi_H v|_{H^2(\Omega)} \\ &\lesssim H^2|v|_{H^2(\Omega)}, \end{aligned} \tag{25}$$

and similarly

$$|w|_{H^1(\Omega)} \lesssim H|v|_{H^2(\Omega)}, \tag{26}$$

$$|w|_{H^2(\Omega)} \lesssim |v|_{H^2(\Omega)}. \tag{27}$$

It follows from (22) that

$$\int_{\tilde{\Omega}_j} |D\theta_j|^2 |Dw|^2 \, dx \lesssim \delta^{-2} |w|^2_{H^1(\tilde{\Omega}_j)}, \tag{28}$$

$$\int_{\tilde{\Omega}_j} (\theta_j)^2 |D^2 w|^2 \, dx \lesssim |w|^2_{H^2(\tilde{\Omega}_j)}. \tag{29}$$

Note that $D^2\theta_j$ vanishes except on a strip near $\partial\tilde{\Omega}_j$ with width $\approx \delta$. Therefore it follows from (22) and [6, Lemma 8.1] that

$$\int_{\tilde{\Omega}_j} (w)^2 |D^2\theta_j|^2 \, dx \lesssim \left(\frac{1}{\delta^3 H}\right) \|w\|^2_{L_2(\tilde{\Omega}_j)} + \left(\frac{H}{\delta}\right)^3 |w|^2_{H^2(\tilde{\Omega}_j)}. \tag{30}$$

Combining (2), (7), (24), (25) and (28)–(30), we find

$$\begin{aligned}
\sum_{j=1}^{J} \langle A_j v_j, v_j \rangle &\lesssim \left(\frac{1}{\delta^3 H}\right) \sum_{j=1}^{J} \|w\|^2_{L_2(\tilde{\Omega}_j)} + \left(\frac{1}{\delta^2}\right) \sum_{j=1}^{J} |w|^2_{H^1(\tilde{\Omega}_j)} \\
&\qquad + \left[1 + \left(\frac{H}{\delta}\right)^3\right] \sum_{j=1}^{J} |w|^2_{H^2(\tilde{\Omega}_j)} \\
&\lesssim \left(\frac{1}{\delta^3 H}\right) \|w\|^2_{L_2(\Omega)} + \left(\frac{1}{\delta^2}\right) |w|^2_{H^1(\Omega)} + \left[1 + \left(\frac{H}{\delta}\right)^3\right] |w|^2_{H^2(\Omega)} \\
&\lesssim \left[1 + \left(\frac{H}{\delta}\right)^3\right] |v|^2_{H^2(\Omega)} = \left[1 + \left(\frac{H}{\delta}\right)^3\right] \langle A_h v, v \rangle.
\end{aligned} \tag{31}$$

The estimate (19) follows from (23) and (31) □

Combining (13) and (19), we have a lower bound for the eigenvalues of $B_{TL}A_h$:

$$\lambda_{\min}(B_{TL}A_h) \gtrsim \left[1 + \left(\frac{H}{\delta}\right)^3\right]^{-1}. \tag{32}$$

4.3 Estimate for $\kappa(B_{TL}A_h)$

Putting (17) and (32) together, we have the following result on the condition number of the preconditioned system.

Theorem 1 *There exists a constant C independent of h, H, δ and J such that*

$$\kappa(B_{TL}A_h) = \frac{\lambda_{\max}(B_{TL}A_h)}{\lambda_{\min}(B_{TL}A_h)} \leq C\left[1 + \left(\frac{H}{\delta}\right)^3\right].$$

5 Numerical Results

We have applied the two-level Schwarz preconditioner to the model problem on the unit square and an L-shaped domain. The numerical results presented here were obtained using PETSc [2, 3] and Supermike II, one of the high performance supercomputers at the Louisiana State University.

Throughout these numerical examples we will use the following notation:

- κ, the estimated condition number of the preconditioned system
- its, the number of iterations until the relative residual falls below 10^{-6}
- t_{solve}, amount of (wall) time, in seconds, required to solve the preconditioned system
- H, the maximum width of the coarse mesh
- δ, the amount of overlap among the overlapping subdomains
- h, the maximum width of the fine mesh
- $\|e_h\|_{\text{energy}}$, the error in energy norm given by $|u_h - \pi_h u|_{H^2(\Omega)}$

We run two numerical experiments for each domain to observe the scalability. The first experiment, the case of small overlap, measures strong scalability. This is carried out by keeping the amount of overlap among the subdomains fixed and small, and then refining the coarse mesh. The second experiment, the case of generous overlap, measures weak scalability. This is carried out by keeping the quantity H/δ bounded, and then refining the fine mesh.

The local and coarse problems are solved by using a Cholesky factorization (on their own processors) and the global problem is solved using the preconditioned conjugate gradient method.

5.1 *Results for the Unit Square*

Let Ω be the unit square $(-0.5, 0.5)^2$. We take the exact solution to be

$$u(x) = \frac{35}{2}(x_1^2 - 0.25)^2(x_2^2 - 0.25)^2.$$

The generalized finite element space V_h is constructed through a flat-top partition unity with a background uniform mesh. An example of a fine mesh, a coarse mesh, and typical overlapping subdomains are shown in Fig. 2.

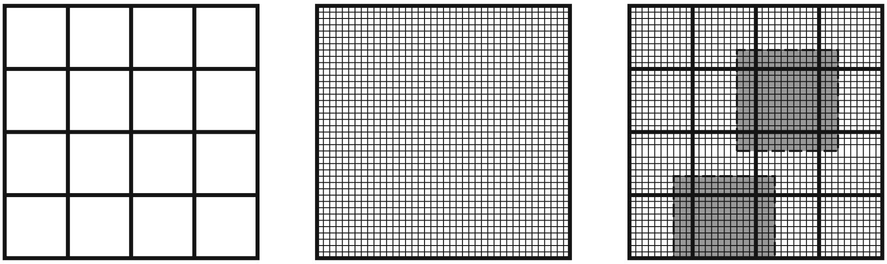

Fig. 2 A coarse mesh (*left*), a fine mesh (*middle*) and overlapping subdomains (*right*) for the unit square

Table 1 Small overlap results for the unit square

n_{sub}	H	κ	Rate	Its	t_{solve}
4	6.0000×10^{-1}	$9.1306\times10^{+6}$	–	1498	$3.8989\times10^{+3}$
16	2.7273×10^{-1}	$6.5953\times10^{+5}$	3.33	1409	$5.5002\times10^{+2}$
64	1.3043×10^{-1}	$7.4148\times10^{+4}$	2.96	400	$4.2519\times10^{+1}$
256	6.3830×10^{-2}	$1.0755\times10^{+4}$	2.70	165	$7.5584\times10^{+0}$
1024	3.1579×10^{-2}	$1.4248\times10^{+3}$	2.87	63	$4.2420\times10^{+0}$

5.1.1 Small Overlap

In the case of small overlap, the number of fine elements is fixed so that $h \approx 3.9113 \times 10^{-3}$. The amount of overlap among the subdomains is also fixed so that $\delta \approx 6.5189 \times 10^{-4}$.

The numerical results are presented in Table 1. We observe that

$$\kappa(B_{TL}A_h) \approx (H/\delta)^{\text{rate}}$$

where the rate is roughly 3, which agrees with Theorem 1. The scalability of the algorithm is also evidenced by the data in the last column.

5.1.2 Generous Overlap

In the case of generous overlap, the total number of subdomains is kept fixed ($n_{\text{sub}} = 64$) so that $H = 1.3043 \times 10^{-1}$. The fine mesh is then refined in such a way that $H/\delta \leq 3$.

The numerical results are presented in Table 2. We observe that $\kappa(B_{TL}A_h)$ is uniformly bounded, which agrees with Theorem 1. We also observe $O(h)$ convergence in the energy error, which agrees with the estimate (6).

Table 2 Generous overlap results for the unit square

h	κ	Its	t_{solve}	$\|e_h\|_{\text{energy}}$	Rate
3.1579×10^{-2}	$3.6247\times10^{+1}$	23	4.6185×10^{-1}	5.6660×10^{-2}	–
1.5707×10^{-2}	$3.1383\times10^{+1}$	23	9.2236×10^{-1}	2.6070×10^{-2}	1.11
7.8329×10^{-3}	$2.7025\times10^{+1}$	21	$5.8779\times10^{+0}$	1.2454×10^{-2}	1.06
3.9113×10^{-3}	$2.5732\times10^{+1}$	21	$7.6505\times10^{+1}$	6.0795×10^{-3}	1.03
1.9544×10^{-3}	$2.4748\times10^{+1}$	21	$1.1662\times10^{+3}$	3.0027×10^{-3}	1.01

5.2 *An L-Shaped Domain*

Let Ω be the L-shaped domain $(-0.5, 0.5)^2 \setminus [0, 0.5]^2$. The exact solution u of the biharmonic problem is given by

$$u = (r^2 \cos^2(\theta) - 0.25)^2 (r^2 \sin^2(\theta) - 0.25)^2 r^{1+\alpha} g(\theta),$$

where the polar coordinate system (r, θ) is centered at the origin so that $\theta = 0$ corresponds to the positive y–axis and $\theta = 3\pi/2$ corresponds to the positive x–axis, and the function g (cf. [20, pp. 107–108]) is given by

$$\begin{aligned} g(\theta) = {} & [\cos((\alpha-1)\omega) - \cos((\alpha+1)\omega)] \\ & \quad \times [(\alpha+1)\sin((\alpha-1)\theta) - (\alpha-1)\sin(\alpha+1)\theta)] \\ & - [\cos((\alpha-1)\theta) - \cos((\alpha+1)\theta)] \\ & \quad \times [(\alpha+1)\sin((\alpha-1)\omega) - (\alpha-1)\sin(\alpha+1)\omega)] \,. \end{aligned}$$

Here $\omega = 3\pi/2$ is the angle of the reentrant corner and

$$\alpha \approx 0.544483736782464 \tag{33}$$

is the index of elliptic regularity.

The generalized finite element space V_h is constructed using a background mesh of quasi-uniform rectangles such that the reentrant corner is inside one of the rectangles. An example of a fine mesh, a coarse mesh, and typical overlapping subdomains are shown in Fig. 3.

5.2.1 Small Overlap

In the case of small overlap, the number of fine elements is fixed so that $h \approx 3.9164 \times 10^{-3}$. The amount of overlap between the subdomains is also fixed so that $\delta \approx 6.5104 \times 10^{-4}$.

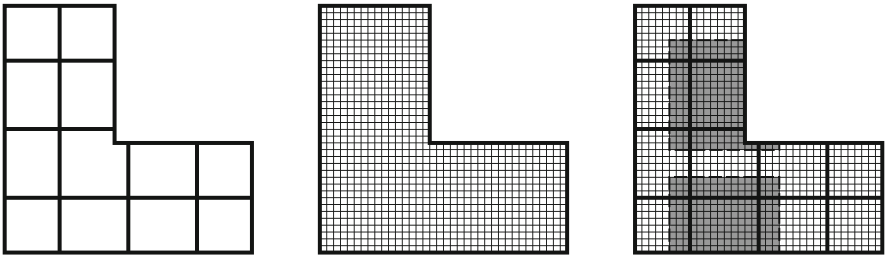

Fig. 3 A coarse mesh (*left*), a fine mesh (*middle*) and overlapping subdomains (*right*) for the *L*-shaped domain

Table 3 Small overlap results for the *L*-shaped domain

n_{sub}	H	κ	Rate	Its	t_{solve}
3	6.6667×10^{-1}	$8.8975\times10^{+6}$	–	1974	$4.5449\times10^{+3}$
12	2.9167×10^{-1}	$5.5286\times10^{+6}$	0.58	4816	$1.5500\times10^{+3}$
48	1.3542×10^{-1}	$1.0507\times10^{+6}$	2.16	1587	$1.3981\times10^{+2}$
192	6.5104×10^{-2}	$1.3655\times10^{+5}$	2.79	529	$1.8855\times10^{+1}$
768	3.1901×10^{-2}	$1.4414\times10^{+4}$	3.15	199	$9.8830\times10^{+0}$

Table 4 Generous overlap results for the *L*-shaped domain

h	κ	Its	t_{solve}	$\Vert e_h\Vert_{\text{energy}}$	Rate
3.1901×10^{-2}	$9.5077\times10^{+1}$	46	6.4991×10^{-1}	3.1239×10^{-3}	–
1.5788×10^{-2}	$1.1376\times10^{+2}$	47	$1.2841\times10^{+0}$	2.0686×10^{-3}	0.59
7.8532×10^{-3}	$1.2383\times10^{+2}$	41	$6.8940\times10^{+0}$	1.3966×10^{-3}	0.56
3.9164×10^{-3}	$1.2052\times10^{+2}$	37	$8.0453\times10^{+1}$	9.5070×10^{-4}	0.55
1.9557×10^{-3}	$1.2187\times10^{+2}$	37	$1.1779\times10^{+3}$	6.4955×10^{-4}	0.55

The numerical results are presented in Table 3. Again we observe that

$$\kappa(B_{TL}A_h) \approx (H/\delta)^3.$$

The scalability of the algorithm is also supported by the data in the last column.

5.2.2 Generous Overlap

In the case of generous overlap, the total number of subdomains is kept fixed ($n_{\text{sub}} = 48$) so that $H = 1.3542 \times 10^{-1}$. The fine mesh is then refined in such a way that $H/\delta \leq 3$.

The numerical results in Table 4 show that $\kappa(B_{TL}A_h)$ is uniformly bounded as predicted by Theorem 1, and that the energy error is $O(h^{0.55})$ as predicted by (6) and (33).

6 Concluding Remarks

We have extended the classical results for two-level additive Schwarz preconditioners to a flat-top partition of unity method for the biharmonic problem.

In the case of nonconvex domains, optimal convergence for the partition of unity method can be restored by including known corner singularities in the local approximation spaces. The preconditioner developed in this paper is also relevant for such methods.

The extension of the results in this paper to partition of unity methods for variational inequalities [11–13] and to partition of unity methods for sixth order problems are ongoing projects.

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