Chapter 2
Model Theory

This chapter explains the fundamentals of model theory which can be used for obtaining scaled experiments in a laboratory. In order to obtain complete similarity, all dimensionless products have to be the same in the model experiment and in the technical application. Then results from the model experiment can be transferred immediately to the technical application. If not all dimensionless products can be kept the same in the model experiments compared to the real application incomplete similarity will be obtained. Also here, the results from the model experiments are very useful. This is also shown and discussed in this chapter.

2.1 General Model Theory

The principles of scientific thinking and working involve the observation of the surrounding nature and consequently to develop theories that explain the observed events and predict future behavior. Every theory requires its verification by experiments. The necessary experiments are performed in many applications in physics and the engineering sciences on a basis of scaled models. The spatial extents of the model are often larger or smaller than that of the original case. This is shown in Fig. 2.1.

In many applications, the size of the original object is so large that only scaled models are suitable for experimental investigation. Consider, for example the study of the flow around an aircraft, the flow around skyscrapers or the flow through rivers and lakes. Models with dimensions that are larger than that of the original are displayed if the spatial resolution of the original is too low or the experiment itself would be technically difficult to measure, as for instance in the case of internal cooling of gas turbine blades.

Each model experiment must be preceded by a thorough dimensional analytical consideration of the problem, because the verified physical laws in the experiment depend only on dimensionless quantities. The basics of model theory can be very

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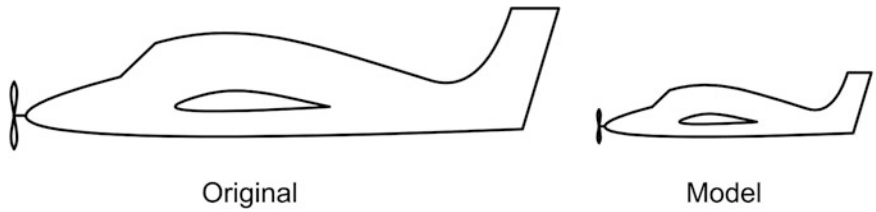

Fig. 2.1 Schematic comparison between geometries for the real application and laboratory testing

easily deduced from the previous considerations. We assume Eq. (1.45) to be solved for one of the dimensionless products (here we follow the derivation of Spurk (1992), since it is very focused):

$$\prod_1 = f\left(\prod_2 \cdots \prod_d\right). \quad (2.1)$$

Obviously $\prod_1$ does not change even when the physical quantities contained in $\prod_i (i = 2\ldots d)$ change, as long as the dimensionless products remain unchanged. Denoting the physical parameters of the original by p_j and those of the model by p_j', the relationship between the two follows to

$$p_j' = M_j p_j, \quad (j = 1\ldots n), \quad (2.2)$$

where M_j denotes the scale factor of the j-th physical quantity. In order to ensure complete physical similarity between model and original, all dimensionless products for model and original must be equal, i.e.

$$\prod_i = \prod_i', \quad (i = 1\ldots d). \quad (2.3)$$

With Eq. (2.2) it follows

$$\prod_{j=1}^{n} p_j^{k_{(i),j}} = \prod_{j=1}^{n} p_j'^{k_{(i),j}} = \prod_{j=1}^{n} p_j^{k_{(i),j}} \prod_{j=1}^{n} M_j^{k_{(i),j}}, \quad (i = 1\ldots d), \quad (2.4)$$

so that the scale factors are obtained by

$$\prod_{j=1}^{n} M_j^{k_{(i),j}} = 1, \quad (i = 1\ldots d). \quad (2.5)$$

By taking the logarithm of this equation it follows

$$\sum_{j=1}^{n} k_{(i),j} \ln\left(M_j\right) = 0, \quad (i = 1\ldots d). \tag{2.6}$$

Equation (2.6) is a system of linear equations with $d = n - r$ equations for the unknown scale factors. According to the considerations that were made in connection with Eq. (1.33), these d equations are linearly independent. One can therefore define $n - d = r$ scale factors arbitrarily, and the remaining scale factors can then be uniquely calculated from Eq. (2.6).

2.2 Complete Similarity

One speaks of “complete similarity” or “full similarity” when all dimensionless products are the same in model and original. As a simple example of complete similarity, we consider the experimental investigation of a wind wheel in the model. The power P of the wind turbine is a function of blade length L, the wind speed U, the rotational speed n and the density of the air ρ, thus

$$P = f\,(L,\ U, n, \rho). \tag{2.7}$$

The corresponding dimension exponents are summarized in the following dimension matrix

	P	L	U	n	ρ
L	2	1	1	0	−3
M	1	0	0	0	1
T	−3	0	−1	−1	0

This yields two dimensionless products so that Eq. (2.7) takes the form

$$\frac{P}{\rho\, U^3\, L^2} = f\left(\frac{U}{n\, L}\right). \tag{2.8}$$

For the model and the full-scale original to show the same physical behavior, the dimensionless numbers must be equal, i.e. it must be insured that

$$\frac{P}{\rho\, U^3\, L^2} = \frac{P'}{\rho'\, U'^3\, L'^2} \quad \text{and} \quad \frac{U}{n\, L} = \frac{U'}{n'\, L'} \tag{2.9}$$

or

$$\frac{P'}{P} = \frac{\rho'}{\rho}\left(\frac{U'}{U}\right)^3\left(\frac{L'}{L}\right)^2 \quad \text{and} \quad \frac{n'}{n} = \frac{U'}{U}\frac{L}{L'}. \tag{2.10}$$

The relations for the scale factors are

$$P' = M_P\, P, \quad L' = M_L\, L, \tag{2.11}$$

$$U' = M_U\, U, \quad n' = M_n\, n, \tag{2.12}$$

and

$$\rho' = M_\rho\, \rho, \tag{2.13}$$

so that one obtains from Eq. (2.9)

$$\frac{M_P}{M_\rho\, M_U^3\, M_L^2} = 1 \quad \text{and} \quad \frac{M_U}{M_n\, M_L} = 1. \tag{2.14}$$

In this example, there are $n = 5$ physical variables, the rank of the dimension matrix is $r = 3$, and the problem is described by $d = 2$ dimensionless products. So we can define $n - d = r = 3$ scale factors at our discretion, and the remaining scale factors can then be uniquely calculated from Eq. (2.14). If both model and original are operated with the same flow medium (e.g. air), it follows that

$$M_\rho = 1. \tag{2.15}$$

Two more scale factors, e.g. M_L and M_U can still be freely chosen, and the two remaining factors are then obtained to be

$$M_P = M_U^3\, M_L^2 \quad \text{and} \quad M_n = \frac{M_U}{M_L}. \tag{2.16}$$

Complete similarity, that is, the equality of all dimensionless products in model and original, cannot be realized in most cases. Almost always, the geometric similarity is violated, for example, because the relative roughness $k_R\,/\,L$ (with k_R as the average roughness and L as the characteristic length) cannot be kept constant in model and original. The same is true in turbomachinery for the ratio of blade gap to blade height. The problems caused by incomplete geometric similarity and the resulting differences between the behavior of the model and the original are denoted as scale effects.

2.3 Incomplete Similarity

If, apart from these scale effects, complete similarity cannot be achieved anyway, and one or more dimensionless products in model and original cannot be kept at the same value, one speaks of "incomplete similarity". As an example, the flow under simultaneous influence of gravity and friction forces or the determination of stresses and deformations in elastic components under the simultaneous influence of single and weight forces are mentioned here. When complete similarity cannot be achieved and only partially similar experiments can be carried out, additional considerations have to be incorporated that cannot be discussed in general terms, but only for specific tasks.

As an example we consider a ship whose flow resistance W has to be determined. The ship is characterized by its length L and moves with the speed U. It is subject to resistance as it moves through a viscous fluid, e.g. water. The viscosity of the liquid shall be η its density is ρ. A part of the necessary energy for locomotion of the ship is transferred to the wave motion of the liquid. The wave motion is affected by gravity, so that the acceleration due to gravity g must also be included as a variable. One thus obtains

$$W = f\,(U,\, L,\, \eta,\, \rho,\, g). \tag{2.17}$$

Now $n = 6$ and $r = 3$ so that the problem is described by $d = 3$ dimensionless products, and one obtains

$$\frac{W}{\rho\, U^2\, L^2} = f\left(\frac{UL}{\nu},\, \frac{U^2}{gL}\right), \tag{2.18}$$

where $\nu = \eta\,/\,\rho$ denotes the kinematic viscosity. For brevity, we introduce the drag coefficient c_w, the Reynolds number Re and Froud number Fr according to

$$c_w = \frac{W}{\rho\, U^2\, L^2}, \quad \mathrm{Re} = \frac{UL}{\nu} \quad \text{and} \quad \mathrm{Fr} = \frac{U^2}{gL}. \tag{2.19}$$

In order to achieve complete similarity, all dimensionless products for the model and for full-scale must be equal, thus

$$\mathrm{Re} = \mathrm{Re}', \quad \mathrm{Fr} = \mathrm{Fr}' \quad \text{and} \quad c_w = c_w'. \tag{2.20}$$

One obtains

$$\frac{UL}{\nu}\frac{M_U\, M_L}{M_\nu} = \frac{U'L'}{\nu'} \quad \text{respectively} \quad \frac{M_U\, M_L}{M_\nu} = 1, \tag{2.21}$$

and

$$\frac{U^2}{g\,L}\frac{M_U^2}{M_g\,M_L} = \frac{U'^2}{g\,L'} \quad \text{respectively} \quad \frac{M_U^2}{M_L} = 1, \tag{2.22}$$

where because of $g = g'$ the scale factor $M_g = 1$ is already set. If full scale ship and model are both operated in water also $M_\nu = 1$ is fixed and one gains

$$M_U\,M_L = 1 \quad \text{and} \quad \frac{M_U^2}{M_L} = 1 \tag{2.23}$$

and thus immediately

$$M_L = 1 \quad \text{and} \quad M_U = 1 \tag{2.24}$$

that is, model and full-scale ship must be of the same size. Thus, when wanting to perform the experiment on a scaled model, complete similarity has to be abandoned. In the present example the similarity in the Reynolds number is disregarded, and it is thus assumed that

$$c_w = c_{wR}(\mathrm{Re}) + c_{wW}(\mathrm{Fr}). \tag{2.25}$$

In this approach it is assumed that both resistance components are decoupled from each other, which indeed represents a good approximation, but is of course not exactly correct. If now

$$\mathrm{Fr} = \mathrm{Fr}', \tag{2.26}$$

and the scale factors M_U and M_L can be freely selected, it follows that

$$c_{wW}(\mathrm{Fr}) = c'_{wW}(\mathrm{Fr}') \quad \text{respectively} \quad c_{wW} = c'_{wW}. \tag{2.27}$$

The resistance coefficient of the model becomes

$$c'_w = c'_{wW}(\mathrm{Fr}') + c'_{wR}(\mathrm{Re}') \tag{2.28}$$

and the full size version applies to

$$c_w = c_{wW}(\mathrm{Fr}) + c_{wR}(\mathrm{Re}). \tag{2.29}$$

From the difference of Eqs. (2.28) and (2.29) one obtains

$$c_w - c'_w = \big(c_{wW}(\mathrm{Fr}) - c'_{wW}(\mathrm{Fr}')\big) + \big(c_{wR}(\mathrm{Re}) - c'_{wR}(\mathrm{Re}')\big) \tag{2.30}$$

and from this, because of (2.27)

$$c_w - c'_w = \left(c_{wR}(\mathrm{Re}) - c'_{wR}(\mathrm{Re}')\right) \tag{2.31}$$

respectively

$$c_w = c'_w + \left(c_{wR}(\mathrm{Re}) - c'_{wR}(\mathrm{Re}')\right). \tag{2.32}$$

To achieve further progress, one must find an approach to include an analytical calculation of $c_{wR} = c_{wR}(\mathrm{Re})$ or $c'_{wR} = c'_{wR}(\mathrm{Re}')$. If the Reynolds numbers of model and full scale are so large that the flow can be assumed to be fully turbulent in both cases, the frictional resistance of a body can be determined by good approximation by

$$c_{wR} = \mathrm{const} \cdot \mathrm{Re}^{-m} \quad \text{with} \quad m \cong 0,2. \tag{2.33}$$

Thus $c_{wR}(\mathrm{Re})$ as well as $c'_{wR}(\mathrm{Re}')$ are available as analytical functions, and the resistance of the full size version can be determined from

$$c_w = c'_w - c'_{wR}(\mathrm{Re}')\left(1 - \frac{c_{wR}(\mathrm{Re})}{c'_{wR}(\mathrm{Re}')}\right) \tag{2.34}$$

respectively

$$c_w = c'_w - c'_{wR}(\mathrm{Re}')\left(1 - \left(\frac{\mathrm{Re}'}{\mathrm{Re}}\right)^m\right). \tag{2.35}$$

The procedure described here for the determination of the resistance at incomplete similarity is called “up-scaling” and Eq. (2.35) represents a simple form of this procedure.

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