

Chapter I. Ordinary magic squares

The two tenth-century texts construct magic squares with the numbers taken in their natural order beginning with 1. But this is merely a particular case since the same construction could be applied to any arithmetical progression with any chosen initial number. This was indeed known then: when $\mathcal{B}$ defines the magic square, he explicitly mentions the general case and gives examples of such squares ($\mathcal{B}.1$).

The construction of ordinary magic squares, which was to attain a remarkable development from the 11th century on, is almost insignificant in the 10th century; as already said, it is on the one hand seen as difficult for the beginner ($\mathcal{A}.II.2$) and, on the other hand, less esteemed than the construction of bordered squares: whereas the arrangement in bordered squares is called ‘regular’, the ordinary magic arrangement is ‘not regular’ according to $\mathcal{B}$ ($\mathcal{B}.2i$). To construct such a square, as mentioned by $\mathcal{A}$ when stating its difficulty, we are first to construct the *natural square* of the order considered, that is, the square filled with the numbers to be placed in the magic square, but simply taken in their natural order (Fig. 6); since it appears that the sum in the rows, either horizontal or vertical, is in one half less than the magic sum and correspondingly more in the other, exchanges will be made between opposite rows so as to eliminate these differences. This is what, according to $\mathcal{A}$, is difficult for the beginner. The examples given by $\mathcal{B}$, for the orders 3, 4, 5, 6, 8 indeed confirm that: each order has to be treated separately, and the rules applied to one order will not be valid for another. Once again, we are far from the easy methods to be developed in the 11th century, with the natural sequence of numbers being placed directly in the square to be constructed, thus without resorting to the natural square.

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

Fig. 6

§1. Construction of odd-order squares

A. Particular case of order 3

1. UNIQUENESS OF THE SQUARE OF ORDER 3

Supposing that the square is to be filled with the first nine natural numbers, we immediately know that the magic sum will be $M_3 = 15$ (above, p. 3; or $\mathcal{B}.2ii$). Considering that r will occupy the central cell, the sum in each of the four pairs of opposite cells will be $M_3 - r$ (Fig. 7). Thus, the sum on the whole square will be on the one hand $3M_3$, on the other $4(M_3 - r) + r$. Equating these two expressions and putting, as required in our case, $M_3 = 15$, we find $r = 5$. This is indeed the result arrived at in $\mathcal{B}$ ($\mathcal{B}.3i$), but rather by postulating that 5 must be in the middle since such is its place in the natural square.²⁴ Then we are left with pairs of complements adding up to 10 which are to be placed around the centre in opposite cells.

*	†	◦
□	r	□
◦	†	*

Fig. 7

2	9	4
7	5	3
6	1	8

Fig. 8

Filling the remainder of the square is not difficult, for it suffices to consider the place 1 must occupy. It can be only in the middle cell of a lateral row or in a corner, and (to make the sum 15) 9 will face it. If, first, it is in a corner cell, we are to find two pairs of numbers making the complement, thus 14; if, second, it is in a middle cell, only one. Now of the still available numbers only one pair, namely 6 and 8, makes 14. Thus 1 must be in a middle cell with 6 and 8 as its neighbours, say 6 on the left (the other choice just inverts the figure). That enables us to determine the occupants of the other cells (Fig. 8). This is exactly the reasoning found in $\mathcal{B}.3ii$ – iii . Incidentally, this deduction of the construction shows that the magic square of order 3 with the first nine numbers admits of only one form, others being just rotations or inversions of it.

2. CONSTRUCTION OF THE SQUARE OF ORDER 3 ‘BY DISPLACEMENT’

This logical deduction in $\mathcal{B}$ is followed by two ways of obtaining the same magic square but this time *by displacement*, thus by moving the numbers in the natural square of order 3 ($\mathcal{B}.4$).

²⁴ With odd orders n , the median number is always in the central cell for bordered squares, often for ordinary squares.

(a) Consider the natural square of order 3 (Fig. 9). Leaving the occupant of the central cell where it is, we move those of the border. First, we move the numbers in the corner cells to their respective knight's cell by turning in the same direction, namely two cells towards a corner and one cell sideways (Fig. 10). Then we take the elements formerly in those cells and transfer them to the next corner, moving this time in the opposite direction (Fig. 11). The result is the square already seen, but inverted.

1	2	3
4	5	6
7	8	9

Fig. 9

	<u>9</u>	
<u>3</u>	5	<u>7</u>
	<u>1</u>	

Fig. 10

<u>4</u>		<u>2</u>
<u>8</u>		<u>6</u>

Fig. 11

(b) Considering again the natural square, we move each number in the border to the next cell by turning in the same direction (Fig. 12). Since then the sum of the occupants of two corners in any row and the occupant of the middle cell in the opposite row makes 15, we shall just exchange the occupants of opposite middle cells (Fig. 13).

<u>4</u>	<u>1</u>	<u>2</u>
<u>7</u>	5	<u>3</u>
<u>8</u>	<u>9</u>	<u>6</u>

Fig. 12

4	<u>9</u>	2
<u>3</u>	5	<u>7</u>
8	<u>1</u>	6

Fig. 13

3. A CONSTRUCTION 'WITHOUT DISPLACEMENT'

$\mathcal{B}$ adds to these two ways a method which avoids using the natural square ($\mathcal{B}.5$): we are told to place 5 in the central cell, 1 in the middle of a lateral row, 2 in the knight's cell of 1, and 3 in the queen's cell of 1 (the next diagonally), namely, $\mathcal{B}$ adds, in the knight's cell of 2 (this determines the cell). Having placed these four numbers, completing the square is straightforward.

As to $\mathcal{A}$, he also tells us where to place the numbers; but, unlike $\mathcal{B}$, he does it for *all* occupants, without mentioning that the places of some may determine those of others ($\mathcal{A}.II.3$). Such tedious instructions are common with him, and he was blamed for that by a later reader (above, p. 14).

B. Square of order 5

The two subsequent examples are historically quite interesting since (apart from the particular case of order 3) they are the only ones where a displacement method valid for just one odd order is fully described. They have a common feature, made evident by the author ($\mathcal{B}.6i$), which will

recur in all early displacement methods for $n > 3$, odd or even: since the diagonals of the natural square already make the magic sum, they will remain unchanged, thus leaving us with equalizing only the vertical and horizontal rows. (This apparent simplification turned out in fact, as we shall see below, to be an obstacle to further developments, at least for odd-order squares.)

(a) Taking the natural square for order 5 (Fig. 14), we first exchange the elements of the inner square (excepting those of the diagonals) with those of their bishop's cells (that is, next but one diagonally) by turning in the same direction (Fig. 15). Then, considering the pairs of remaining elements (corners excepted) in opposite lateral rows, we exchange their places (Fig. 16). See *B.6ii*.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

Fig. 14

1	2	3	<u>12</u>	5
<u>18</u>	7	<u>20</u>	9	10
11	<u>4</u>	13	<u>22</u>	15
16	17	<u>6</u>	19	<u>8</u>
21	<u>14</u>	23	24	25

Fig. 15

1	<u>23</u>	<u>24</u>	12	5
18	7	20	9	<u>11</u>
<u>10</u>	4	13	22	<u>16</u>
<u>15</u>	17	6	19	8
21	14	<u>2</u>	<u>3</u>	25

Fig. 16

(b) In the natural square, we have, as just seen, dealt with the pairs 12 & 4, 18 & 6, 14 & 22, 8 & 20, with each element in the bishop's cell of the other, and exchanged their places. Here we take 14 & 2, 8 & 20, 12 & 24, 18 & 6 and exchange their places—in other words, we invert the places of each pair adjoining, on either side, a diagonal, here (say) the descending one (Fig. 17). We then proceed as before with the lateral opposite rows (Fig. 18). See *B.6iii*.

1	<u>14</u>	3	4	5
<u>18</u>	7	<u>20</u>	9	10
11	<u>24</u>	13	<u>2</u>	15
16	17	<u>6</u>	19	<u>8</u>
21	22	23	<u>12</u>	25

Fig. 17

1	14	<u>22</u>	<u>23</u>	5
18	7	20	9	<u>11</u>
<u>10</u>	24	13	2	<u>16</u>
<u>15</u>	17	6	19	8
21	<u>3</u>	<u>4</u>	12	25

Fig. 18

These displacements correspond exactly to what $\mathcal{A}$ found fault with in the methods starting from the natural square (*A.II.2*): performing rather complicated exchanges between vertical and horizontal opposite rows (see note 526).

C. Later developments

$\mathcal{B}$ has thus carried out specific moves in the natural square for each of the first two odd orders. But neither his moves for the square of order 3 nor those for the (less particular) order 5 led to an extension to other odd orders. The constant features of his two treatments for the order 5 are, first, to leave the main diagonals as they are and, second, to exchange a pair of numbers between symmetrical, vertical and horizontal, rows. For order 3 there is also one exchange between opposite rows; but in this case the natural diagonals could not be maintained, for —owing to the single possible configuration— we were obliged to replace them by the (natural) middle rows. This latter step, in fact, prefigured the discovery of a general method for odd orders, seen a few decennaries later.

Indeed, the two following properties are verified in each natural square of odd order.

I. The main diagonals, but also the broken diagonals, make the magic sum.

Thus, in our [Fig. 19](#), the sum of 2, 8, 14, 20, 21, or of 11, 7, 3, 24, 20, equals $M_5 = 65$.

This is easily verified generally. Any diagonal of the natural square, either main or broken, contains each unit of the order, from 1 to n , and each multiple of the order, from $0 \cdot n$ to $(n-1) \cdot n$ ([Fig. 20](#)); since the sum of all these elements is

$$1 + 2 + \cdots + n + n \cdot (0 + 1 + \cdots + (n-1)) = \frac{n(n+1)}{2} + n \frac{(n-1)n}{2},$$

we have indeed $\frac{n}{2} [n + 1 + n^2 - n] = M_n$.

II. The two middle rows make the magic sum.

Indeed, the median column contains, as units of the order, $\frac{n+1}{2}$ uniformly, as well as each multiple of the order; its sum will therefore be the same as above. The median line contains each unit and n times the quantity $\frac{n(n-1)}{2}$; its sum will thus again be the same.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

Fig. 19

0,1	0,2	0,3	0,4	0,5
1,1	1,2	1,3	1,4	1,5
2,1	2,2	2,3	2,4	2,5
3,1	3,2	3,3	3,4	3,5
4,1	4,2	4,3	4,4	4,5

Fig. 20

Now it would have sufficed —as had been done by $\mathcal{B}$ for the square of order 3— to exchange the rôles of the median rows and the main diagonals and to have in mind the property of the broken diagonals to reach the easiest general construction, which no longer even requires use of the natural square as a starting-point. It was Ibn al-Haytham (around 965-1041) who deduced it, or at least explained it.²⁵

Let us consider, for instance once again for the order 5, the natural square (Fig. 21), and let us thus put its middle rows as diagonals of the square to be constructed. The main diagonals then meet the magic condition. We next wish to determine the other occupants of the square, for instance, first, that in the top of the last-but-one (right-hand) column. For this, we shall use the property of broken diagonals. Since (Fig. 22) the first line contains 11 and 3, we know that if it also contains 7, 20, 24, which are the other three terms of their broken diagonal in the natural square, it will make the magic sum. On the other hand 8 and 14, in the column considered, belong to the broken diagonal 2, 8, 14, 20, 21. Their common element is 20, which therefore will have its place at the intersection of the line and the column considered. We shall proceed in the same way for the other elements, at first disregarding the median rows where there is only one known element. These median rows will be considered last: in them, we shall simply write the remaining elements, like 7 for the first line after 24 has been placed. (It will be seen that the median rows contain the elements of the diagonals in the natural square.) In this way we shall have constructed the magic square of Fig. 23.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

Fig. 21

11			20	3
	12		8	
		13		
	18		14	
23				15

Fig. 22

11	24	7	20	3
4	12	25	8	16
17	5	13	21	9
10	18	1	14	22
23	6	19	2	15

Fig. 23

In short, the magic property of the diagonals in the constructed square merely reproduces the magic property of the median rows in the natural square, while the magic property of the other rows originates in their reconstituting (succession of the elements excepted), the diagonals, main or broken, of the natural square. Previous attempts involving exchanges

²⁵ See our *Les carrés magiques*, pp. 25–27 (Russian edition, pp. 33–35); or (Arabic text and translation) our *Une compilation arabe*, pp. 163–164, 181–182 (Arabic text also in the Russian edition, pp. 279–280).

between opposite rows in the natural square of odd order may thus be abandoned.

In this very same way we may construct any odd-order square, such as that of Fig. 24, with $n = 9$. Furthermore, and as already hinted at, we can do so without resorting to the natural square, and proceed as follows in an empty square of the order considered:

- put 1 underneath the central cell;
- put the subsequent numbers descending diagonally;
- when the side of the square is reached (as with 4 and then 5 in Fig. 24), move to the opposite side and place the next number in the cell which would have been reached by the diagonal move had the same square been juxtaposed with that of the figure, then resume the diagonal placing;
- when, after placing a sequence of n elements (thus nine in our example) a cell already occupied is reached, go down (whatever the order) two cells in the same column, and then continue diagonally.

37	78	29	70	21	62	13	54	5
6	38	79	30	71	22	63	14	46
47	7	39	80	31	72	23	55	15
16	48	8	40	81	32	64	24	56
57	17	49	9	41	73	33	65	25
26	58	18	50	1	42	74	34	66
67	27	59	10	51	2	43	75	35
36	68	19	60	11	52	3	44	76
77	28	69	20	61	12	53	4	45

Fig. 24

The placement will end with n^2 opposite 1, thus on the other side of the central cell. As a matter of fact, it is a general feature of a square constructed in this way to be *symmetrical*, that is, with two elements adding up to $n^2 + 1$ being placed in opposite cells relative to the centre; accordingly, $\frac{n^2+1}{2}$ must be in the central cell.

Remark. In this construction the only moment of hesitation might occur when the last cell of the descending diagonal is reached: the next cell would be the top cell of the descending diagonal if it were not already occupied; we shall therefore remain in the same column but have to move on to its second upper cell.

It is clear that neither $\mathcal{A}$ nor $\mathcal{B}$ knew of such a method. The first would not have spoken of a *method presenting difficulty for the beginner* ($\mathcal{A}$.II.2) and the second would not have presented two rather inconvenient methods, only valid for one order, particularly since he elsewhere appears to be anxious to *explain a method to improve the student's skill and (also) intended for those who prefer to save themselves the trouble of working out which numbers to arrange in the square* ($\mathcal{B}$.17, for bordered squares).

Remark. Ibn al-Haytham could well mark the turning point between the *method presenting difficulty for the beginner* and the straightforward placing we have just seen. The anonymous writer of the 12th century who relates Ibn al-Haytham's method says that his displacement of elements between *broken diagonals* (and no longer between vertical and horizontal rows) is carried out *under long conditions which it would take time to mention and the realization of which presents difficulties for the beginner*. This indeed singularly recalls $\mathcal{A}$'s criticism of the methods for ordinary magic squares. The difference is that Ibn al-Haytham paved the way for a general and always applicable method even if, for its explanation, he had to justify it with *long conditions*.

§2. Construction of even-order squares

The impossibility of a magic square of order 2 is simply stated by $\mathcal{A}$ ($\mathcal{A}$.II.36) but justified by the other author ($\mathcal{B}$.7): the magic sum, 5, is found in the diagonals of the natural square of order 2, and it should be found, if possible, in the other rows; now putting a number in one corner will impose the *same* quantity, namely its complement to 5, to occupy all three other cells. Thus, in fact, the same sum in all rows will be possible only if all four numbers are equal, which is contrary to the requirement of different numbers ($\mathcal{B}$.1). The first possible even magic square will thus be that of order 4, which is constructed by our authors in various ways. Indeed, as said before (pp. 6, 20), only the square of order 3 has a single possible configuration.

A. Square of order 4

1. CONSTRUCTIONS BY DISPLACEMENT

(a) In $\mathcal{B}$.8 a magic arrangement is obtained in the 4×4 square 'by displacement', using two properties of the natural square (Fig. 25): first, as always, the content of each of the two main diagonals makes the required sum, 34; second, the content of two end cells of one row and that of the two middle cells of the opposite row also make this sum (thus 2, 3 with 13, 16, or 6, 10 with 3, 15). Therefore, we shall keep the two diagonals as

they are, according to the first property, and use the second property to equalize first the lateral rows of the border (Fig. 26), then the inner rows (Fig. 27).

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Fig. 25

1	<u>14</u>	<u>15</u>	4
<u>8</u>	6	7	<u>5</u>
<u>12</u>	10	11	<u>9</u>
13	<u>2</u>	<u>3</u>	16

Fig. 26

1	<u>15</u>	<u>14</u>	4
<u>12</u>	6	7	<u>9</u>
<u>8</u>	10	11	<u>5</u>
13	<u>3</u>	<u>2</u>	16

Fig. 27

(b) $\mathcal{B}$ also constructs this square more rapidly by combining, in a single move, those two properties: he exchanges diagonally the elements in the natural diagonals, thereby inverting them (Fig. 28). This, by a rotation of 180° , gives the same square as before.

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

Fig. 28

Finally, $\mathcal{B}$ observes that the magic sum is also found elsewhere in the magic square obtained ($\mathcal{B}.8v$): (α) in the pairs of middle lateral cells and those opposite (like 12, 8, 9, 5 in Fig. 27); (β) in the pairs of cells adjoining two opposite corners (like 15, 12, 2, 5); (γ) in each 2×2 square in a corner (like 1, 15, 12, 6). Here we may remark that these properties are not all of the same kind: the first two occur in any magic square of order 4 (and also in the natural square) whereas the last is not verified in all constructions.

2. A CONSTRUCTION WITHOUT DISPLACEMENT

4	14	7	9
15	1	12	6
10	8	13	3
5	11	2	16

Fig. 29

1	14	11	8
15	4	5	10
6	9	16	3
12	7	2	13

Fig. 30

$\mathcal{A}$, according to his initial rejection of displacements in the natural square, does not propose a method as above. The only construction he describes is that of Fig. 29 (here inverted). As usual, he merely indicates

each place individually ($\mathcal{A.II.37}$), even though to construct this square there is no need to know more than where to put the first four numbers (see below). $\mathcal{B}$ constructs Fig. 30 (here inverted), which is older than his treatise since he tells us that he has found it in his *predecessors* ($\mathcal{B.9i}$).

These two figures are not fundamentally different since they are just variants of a single pandiagonal structure: they can be deduced from Fig. 31 by shifting the 4×4 frame in one direction or the other. Any square thus found will be magic (and pandiagonal), the only change being that a formerly broken diagonal becomes a main diagonal, and a given number occupies another cell in the resulting square. As already noted (p. 7), this latter property was highly considered at that time, whereas pandiagonality apparently aroused no interest.

Remark. The manuscript of $\mathcal{A}$ has two other figures. One (a_{22} in $\mathcal{A.II.37}$) corresponds to the same pandiagonal structure. The other (a_{23}) is of another type (it displays a different pandiagonal structure, see Fig. 39 below).

13	2	7	12	13	2	7	12	13	2	7	12
3	16	9	6	3	16	9	6	3	16	9	6
10	5	4	15	10	5	4	15	10	5	4	15
8	11	14	1	8	11	14	1	8	11	14	1
13	2	7	12	13	2	7	12	13	2	7	12
3	16	9	6	3	16	9	6	3	16	9	6
10	5	4	15	10	5	4	15	10	5	4	15
8	11	14	1	8	11	14	1	8	11	14	1
13	2	7	12	13	2	7	12	13	2	7	12
3	16	9	6	3	16	9	6	3	16	9	6
10	5	4	15	10	5	4	15	10	5	4	15
8	11	14	1	8	11	14	1	8	11	14	1

Fig. 31

This possibility of choosing the initial cell leads $\mathcal{B}$ to four types of arrangement ($\mathcal{B.9ii}$, note 545): with the first number, here 1, occupying any corner cell; with it in anyone of the central cells; with it in any middle

one of the lateral columns; and with it in any middle one of the lateral lines. In our [figures 32, 35, 38](#), the * show where to place the first four numbers and the † the four subsequent ones, the sequence of placing being simply reversed. The other cases are obtained by rotation.

*			†
	†	*	
†			*
	*	†	

Fig. 32

1	14	11	8
12	7	2	13
6	9	16	3
15	4	5	10

Fig. 33

4	15	10	5
9	6	3	16
7	12	13	2
14	1	8	11

Fig. 34

*			†
	*	†	
†			*
	†	*	

Fig. 35

4	15	10	5
14	1	8	11
7	12	13	2
9	6	3	16

Fig. 36

1	14	11	8
15	4	5	10
6	9	16	3
12	7	2	13

Fig. 37

*			†
	†	*	
*			†
	†	*	

Fig. 38

1	12	13	8
15	6	3	10
4	9	16	5
14	7	2	11

Fig. 39

4	9	16	5
14	7	2	11
1	12	13	8
15	6	3	10

Fig. 40

This arrangement of the first eight numbers gives the explanation for the magic property: in the resulting squares each line contains the same sum, 9, and each one of alternate columns contains the same sum as its conjugate; thus, by writing the complements in the bishop's cells of one of two conjugate rows, we shall complete the required sum in the other conjugate. As to the magic sum of the main diagonals, it is evident since they comprise pairs of complements.

$\mathcal{B}$ observes that any square of four cells in such a figure also contains the magic sum $M_4 = 34$, and *this is what is called full and complete magic arrangement* ($\mathcal{B}.9i$). Indeed, any four-cell square in [Fig. 31](#) makes the sum M_4 . [Fig. 41](#) and [Fig. 42](#) make clear the origin of that for [Fig. 36](#) and [Fig. 39](#), as also the magic property of the broken diagonals (M stands for M_4).

4	$\frac{M}{2} - 2$	$\frac{M}{2} - 7$	5
$\frac{M}{2} - 3$	1	8	$\frac{M}{2} - 6$
7	$\frac{M}{2} - 5$	$\frac{M}{2} - 4$	2
$\frac{M}{2} - 8$	6	3	$\frac{M}{2} - 1$

Fig. 41

1	$\frac{M}{2} - 5$	$\frac{M}{2} - 4$	8
$\frac{M}{2} - 2$	6	3	$\frac{M}{2} - 7$
4	$\frac{M}{2} - 8$	$\frac{M}{2} - 1$	5
$\frac{M}{2} - 3$	7	2	$\frac{M}{2} - 6$

Fig. 42

Remarks.

- (1) Fig. 33 is the one we shall adopt for our commentary, being easier to memorize: starting from a corner with 1, the moves successively used are the knight's, the queen's, the knight's (and the same in reverse for the sequence 5, $\dots$, 8).
- (2) $\mathcal{B}$ twice mentions that a square of order 4 can be made the *foundation* (*aṣl*) for magic arrangement in other even squares ($\mathcal{B}.8i$ & $\mathcal{B}.9i$). Once this is, banally, when he is dealing with the construction of bordered even squares, where the square of order 4, whatever its form, plays the same rôle as the square of order 3 for odd orders: that of a core to be successively surrounded by borders (see below, pp. 43–47). Another time, and less banally, it is for the construction of composite squares, since any evenly even square may be constructed by means of such subsquares (see below, pp. 107–108).

B. Square of order 6

In the construction of the square of order 6 will be seen the limits of $\mathcal{B}$'s method *by displacement*. He operates on the natural square in two steps, moving first the numbers of the inner square of order 4 then those of the border. The way he does that is quite accurately described in $\mathcal{B}.10$.

8	9	10	11
14	15	16	17
20	21	22	23
26	27	28	29

Fig. 43

8	28	9	11	-18
20	15	16	23	0
17	21	22	14	0
26	10	27	29	+18
-3	0	0	+3	

Fig. 44

1	5	33	34	32	6	
30					25	+18
24					13	0
18					19	0
7					12	-18
31	35	4	3	2	36	
	+3	0	0	-3		

Fig. 45

Fig. 43 shows the original occupants in the central part of the natural square and Fig. 44 the result of the displacements ($\mathcal{B}.10iii$). Since the

magic sum in the square of order 6 is $3 \cdot 37 = 111$, the rows of the inner square would accordingly make, if surrounded by a border, $2 \cdot 37 = 74$. But we are dealing here with ordinary magic squares, where that should not be the case. The differences are indicated in the figure (the two diagonals, being kept from the natural square, make the required sum). The subsequent displacements ($\mathcal{B}.10iv$) will fill the outer border and eliminate the differences (Fig. 45).

Remark. These numerical differences are of course intentional since $\mathcal{B}$ wants to construct an ordinary magic square. Indeed, for the inner square of the natural square of order 6 he could have applied the same moves he had used to construct the square of order 4 in methods (a) and (b); but then, having still to add uniformly 37, he would have obtained a bordered square of order 6.

C. Square of order 8

To obtain the square of order 8 ‘by displacements’ ($\mathcal{B}.11$), the previous construction is *repeated for the inner part*, the numbers moved being this time those of the natural square of order 8 ($\mathcal{B}.11i-ii$). As a result, the square of order 6 displays exactly the sum due, and completing the square will just mean adding a border which uniformly increases each row by 65 (Fig. 46). In this sense, the construction is a failure, unlike that in the previous 6×6 square. Even if $\mathcal{B}$ has recognized some equalization principles, his method could not, as we shall see, be extended to an even order larger than 6.

1	58	59	5	4	62	63	8
16	10	14	52	53	51	15	49
24	47	19	45	20	22	42	41
33	39	35	28	29	38	26	32
25	31	30	36	37	27	34	40
48	18	43	21	44	46	23	17
56	50	54	13	12	11	55	9
57	7	6	60	61	3	2	64

Fig. 46

D. Allusion to a generalization

$\mathcal{B}$ seems to have had two general rules; at least, that is what can be inferred from his two examples for small orders (6 and 8) and his assertion that one should proceed like that for higher even orders ($\mathcal{B}.11i$ & iv).

1. EVENLY EVEN ORDERS

Considering the natural square of order $4k$ (Fig. 47, $n = 12$, thus $k = 3$), without touching the corner-cell elements (which belong to the diagonals), we exchange diagonally $2k$ of the $4k$ elements, k in each half (Fig. 48); the four lateral rows will thus make the required sum. Then, we invert the succession of all elements within one horizontal and one vertical row (Fig. 49); with that, the pairs of opposite elements in lines and columns are also equalized. But since pairs of complements now occupy opposite cells, we have *de facto* constructed a border.

1	2	3	4	5	6	7	8	9	10	11	12
13											24
25											36
37											48
49											60
61											72
73											84
85											96
97											108
109											120
121											132
133	134	135	136	137	138	139	140	141	142	143	144

Fig. 47

1	<u>143</u>	<u>142</u>	<u>141</u>	5	6	7	8	<u>136</u>	<u>135</u>	<u>134</u>	12	
<u>132</u>											<u>121</u>	+108
<u>120</u>											<u>109</u>	+84
<u>108</u>											<u>97</u>	+60
49											60	-36
61											72	-12
73											84	+12
85											96	+36
<u>48</u>											<u>37</u>	-60
<u>36</u>											<u>25</u>	-84
<u>24</u>											<u>13</u>	-108
133	<u>11</u>	<u>10</u>	<u>9</u>	137	138	139	140	<u>4</u>	<u>3</u>	<u>2</u>	144	
	+9	+7	+5	-3	-1	+1	+3	-5	-7	-9		

Fig. 48

1	<u>134</u>	<u>135</u>	<u>136</u>	<u>8</u>	<u>7</u>	<u>6</u>	<u>5</u>	<u>141</u>	<u>142</u>	<u>143</u>	12	
<u>24</u>											121	0
<u>36</u>											109	0
<u>48</u>											97	0
<u>85</u>											60	0
<u>73</u>											72	0
<u>61</u>											84	0
<u>49</u>											96	0
<u>108</u>											37	0
<u>120</u>											25	0
<u>132</u>											13	0
133	11	10	9	137	138	139	140	4	3	2	144	
	0	0	0	0	0	0	0	0	0	0		

Fig. 49

Applied to order 8 (Fig. 50–52), these transformations give successively

1	2	3	4	5	6	7	8
9							16
17							24
25							32
33							40
41							48
49							56
57	58	59	60	61	62	63	64

Fig. 50

1	<u>63</u>	<u>62</u>	4	5	<u>59</u>	<u>58</u>	8
<u>56</u>							<u>49</u>
<u>48</u>							<u>41</u>
25							32
33							40
<u>24</u>							17
<u>16</u>							<u>9</u>
57	<u>7</u>	<u>6</u>	60	61	<u>3</u>	<u>2</u>	64

Fig. 51

1	<u>58</u>	<u>59</u>	<u>5</u>	4	<u>62</u>	<u>63</u>	8
<u>16</u>							49
<u>24</u>							41
<u>33</u>							32
<u>25</u>							40
<u>48</u>							17
<u>56</u>							9
57	7	6	60	61	3	2	64

Fig. 52

which is indeed the border previously obtained by $\mathcal{B}$ (Fig. 46).

2. EVENLY ODD ORDERS

If the natural square is of order $n = 4k + 2$ (Fig. 53, $n = 14$, thus $k = 3$), we shall first exchange diagonally $k + 1$ elements between opposite rows (Fig. 54). We then exchange the places of one such element and its opposite (Fig. 55); the lateral rows then make the required sum. We may further invert the succession of the elements within one row, excepting the pair last moved and the symmetrical pair (Fig. 56); this will give the sum $n^2 + 1$ in any two opposite cells, except in the latter two pairs.

1	2	3	4	5	6	7	8	9	10	11	12	13	14
15													28
29													42
43													56
57													70
71													84
85													98
99													112
113													126
127													140
141													154
155													168
169													182
183	184	185	186	187	188	189	190	191	192	193	194	195	196

Fig. 53

1	2	3	<u>193</u>	<u>192</u>	<u>191</u>	<u>190</u>	<u>189</u>	<u>188</u>	<u>187</u>	<u>186</u>	12	13	14
15													28
29													42
<u>154</u>													<u>141</u>
<u>140</u>													<u>127</u>
<u>126</u>													<u>113</u>
<u>112</u>													<u>99</u>
<u>98</u>													<u>85</u>
<u>84</u>													<u>71</u>
<u>70</u>													<u>57</u>
<u>56</u>													<u>43</u>
155													168
169													182
183	184	185	<u>11</u>	<u>10</u>	<u>9</u>	<u>8</u>	<u>7</u>	<u>6</u>	<u>5</u>	<u>4</u>	194	195	196

Fig. 54

1	2	3	<u>11</u>	192	191	190	189	188	187	186	12	13	14
15													28
29													42
154													141
140													127
126													113
112													99
98													85
84													71
70													57
<u>43</u>													<u>56</u>
155													168
169													182
183	184	185	<u>193</u>	10	9	8	7	6	5	4	194	195	196

Fig. 55

1	<u>13</u>	<u>12</u>	11	<u>187</u>	<u>188</u>	<u>189</u>	<u>190</u>	<u>191</u>	<u>192</u>	186	3	2	14
15													<u>182</u>
29													<u>168</u>
154													141
140													<u>57</u>
126													<u>71</u>
112													<u>85</u>
98													<u>99</u>
84													<u>113</u>
70													<u>127</u>
43													56
155													<u>42</u>
169													<u>28</u>
183	184	185	193	10	9	8	7	6	5	4	194	195	196

+98

-98

Fig. 56

Applied to order 6 (Fig. 57–59), this method gives, successively,

1	2	3	4	5	6
7					12
13					18
19					24
25					30
31	32	33	34	35	36

Fig. 57

1	<u>35</u>	<u>34</u>	<u>33</u>	<u>32</u>	6
<u>30</u>					<u>25</u>
<u>24</u>					<u>19</u>
<u>18</u>					<u>13</u>
<u>12</u>					<u>7</u>
31	<u>5</u>	4	3	2	36

Fig. 58

1	<u>5</u>	<u>33</u>	<u>34</u>	32	6
30					25
24					<u>13</u>
18					<u>19</u>
<u>7</u>					<u>12</u>
31	<u>35</u>	4	3	2	36

Fig. 59

the last of which is the border seen in $\mathcal{B}$ (Fig. 45).

As already observed, this method will not work with $n > 6$. The border of order 6 will only make up for the differences in the inner square and so the border of order 8 will necessarily have to be equalized by itself, as for a bordered square; the same will hold for all subsequent borders.

E. Later developments

1. EQUALIZATION RULES

As a matter of fact, $\mathcal{B}$ was not so very far from discovering a general construction method for even orders, or at least evenly even ones. He had noticed two properties of the natural square of order 4: that the main diagonals make the magic sum and that two corner cells and the two middle ones of the opposite row do too. This is in fact the particular case of two general properties of the natural square of even order, which are the following.

I. The sum of the elements in the diagonals, main or broken, equals the magic sum for the order considered.

We have already verified this property for odd orders (p. 23). It was then applied to easily obtain the magic sum in all horizontal and vertical rows by reconstituting the broken diagonals.

II. The sum of half the elements in a row and the other half of the elements in the symmetrically placed row equals the magic sum.

Indeed, the elements of two lines placed symmetrically being (for $i \leq \frac{n}{2}$)

$$(i-1)n+1, (i-1)n+2, (i-1)n+3, \dots, (i-1)n+n$$

$$(n-i)n+1, (n-i)n+2, (n-i)n+3, \dots, (n-i)n+n$$

the sum of the first will be

$$\begin{aligned} (i-1)n^2 + \frac{n(n+1)}{2} &= \frac{n}{2} [2ni - 2n + n + 1] \\ &= \frac{n}{2} [n^2 + 1 - n^2 + 2ni - n] = M_n - \frac{n}{2} [n(n-2i+1)], \end{aligned}$$

and that of the second (replacing i by $n-i+1$)

$$(n-i)n^2 + \frac{n(n+1)}{2} = M_n + \frac{n}{2} [n(n-2i+1)].$$

Now the difference displayed by two of their elements vertically aligned is uniformly $n(n-2i+1)$, which, when multiplied by $\frac{n}{2}$, equals the deficit

in the upper line and the excess opposite. Therefore exchanging between two lines placed symmetrically half of their elements vertically aligned will produce the magic sum in them.

The same reasoning is applicable to the columns. The elements of the j th left-hand column and of its opposite, the right-hand $(n - j + 1)$ th column ($j \leq \frac{n}{2}$), are

$$j, j + n, j + 2n, \dots, j + (n - 1)n$$

$$n - j + 1, 2n - j + 1, 3n - j + 1, \dots, n^2 - j + 1$$

the sums of which are, respectively,

$$\begin{aligned} nj + n \frac{(n - 1)n}{2} &= \frac{n}{2} [2j + n^2 - n] \\ &= \frac{n}{2} [n^2 + 1 + 2j - n - 1] = M_n - \frac{n}{2} [n - 2j + 1], \\ n(n - j + 1) + n \frac{(n - 1)n}{2} &= M_n + \frac{n}{2} [n - 2j + 1]. \end{aligned}$$

Since, here again, the difference between two elements horizontally aligned is $n - 2j + 1$, exchanging between two columns placed symmetrically half of their elements horizontally aligned will eliminate their difference.

We have seen that, for equalizing the rows in the case of odd orders, the property principally used was that of the (main or broken) diagonals of the natural square, that of the middle rows being used only to meet the condition in the diagonals. In the case of even orders, it is the complementarity of opposite rows which will play the main rôle, with half their elements being exchanged. There is here, though, one difficulty: if one starts with the exchanges between lines, opposite columns will no longer display everywhere the same difference between opposite cells, which is a necessary condition for the exchanges. We may remedy that situation as follows.

2. CASE OF EVENLY EVEN ORDERS

Let us consider (Fig. 60) four elements placed symmetrically relative to the median axes, thus horizontally at the same distance from the vertical axis and vertically at the same distance from the horizontal axis. We first carry out the vertical exchange of these pairs (Fig. 61); thus doing, we have reduced the difference between the two opposite lines by its $\frac{4}{n}$ th part—since each exchange of one element with its opposite contributes a reduction in the difference by its $\frac{2}{n}$ th part—while the difference between

the two columns remains unchanged. Then we carry out the horizontal exchange of the elements displaced (Fig. 62); that will not alter the sum in the lines but lead to a reduction by its $\frac{4}{n}$ th part in the difference between the two columns containing the elements considered. Now it then appears that these four exchanges are equivalent to two exchanges, namely of each of the initial elements with the diagonally opposite one. Thus, effecting, between two opposite *lines* of a square of evenly even order n , $\frac{n}{4}$ pairs of such diagonal exchanges will completely eliminate their difference; and that will, at the same time, eliminate the $\frac{4}{n}$ th part for each pair of columns involved. Extending that procedure to the whole square so as to effect exactly $\frac{n}{2}$ exchanges between opposite lines and opposite columns, namely $\frac{n}{4}$ in each line of a quadrant, we shall have a magic square. An *embryonic* stage of this is seen in $\mathcal{B}$; since he restricts himself to one border of the order considered, that is all he equalizes.

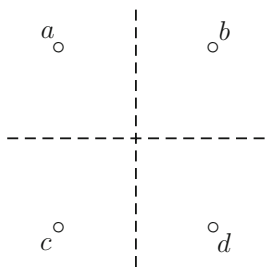


Fig. 60

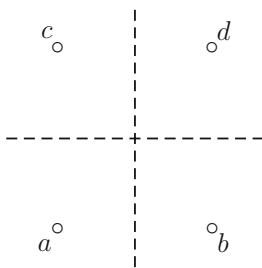


Fig. 61

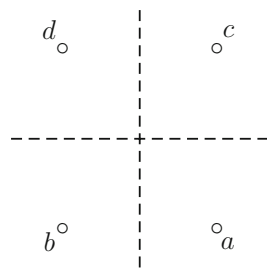


Fig. 62

Remark. It is not necessary for all exchanges to be diagonal. There can be some that are diagonal, or also solely horizontal and vertical ones, provided in all cases that there are in each pair of symmetric rows $\frac{n}{2}$ exchanges. If the exchanges involve elements of the diagonals, they must involve, as above, *two* pairs placed symmetrically in order to maintain in each diagonal the sum $n^2 + 1$ displayed in it by pairs of opposite elements.

The magic square of order 4 obtained ‘by displacement’ in $\mathcal{B}$ is the simplest example of such exchanges (as becomes apparent with the inversion of the natural diagonals —method *b* in the constructions by displacement). With the reasoning to be described now, this square can be constructed directly, without resorting to the natural square. Let us mark with dots in an empty square of order 4 its diagonals (Fig. 63). Then, starting in a corner, say the upper left-hand one, let us count its cells and write in the cells which are marked each number reached; this being done, let us count the cells again, but starting from the opposite corner

and writing each time the number reached in the blank cell. We shall obtain the square of Fig. 64, identical to the square found by $\mathcal{B}$ (Fig. 27). If we count the cells in reverse order—that is, start by filling the cells with dots from the lower right-hand corner—we shall obtain the square found next by $\mathcal{B}$ (Fig. 28), thus the same one rotated by 180° .

•			•
	•	•	
	•	•	
•			•

Fig. 63

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

Fig. 64

This method can easily be extended to any square of order $4k$, as was already done in the 11th century.²⁶ We need merely to mark the diagonals in all subsquares of order 4 and then count the cells as above from two opposite corners (Fig. 65 & Fig. 66). Here too, the dots signify the places unchanged and the blank cells the diagonal exchanges; but, again, reversing their rôles would just produce the same square, rotated by 180° .

•			•	•			•
	•	•			•	•	
	•	•			•	•	
•			•	•			•
•			•	•			•
	•	•			•	•	
	•	•			•	•	
•			•	•			•

Fig. 65

1	63	62	4	5	59	58	8
56	10	11	53	52	14	15	49
48	18	19	45	44	22	23	41
25	39	38	28	29	35	34	32
33	31	30	36	37	27	26	40
24	42	43	21	20	46	47	17
16	50	51	13	12	54	55	9
57	7	6	60	61	3	2	64

Fig. 66

Another less particular way, which enables us to obtain without much thought various configurations of blanks (or dots) indicating diagonal exchanges, is the following. Consider, in the square of order $4k$, its first quadrant, of order $2k$. In it, we shall put dots in such a way that there will be exactly k in each line and in each column. The simplest way to do it is to put k dots in the first line, and then carry on with writing dots along the corresponding broken diagonals of the quadrant; this will

²⁶ *Les carrés magiques*, pp. 44–45 (Russian edition, pp. 53–54).

ensure that there are exactly k dots in each line and in each column of the quadrant. Then we reproduce, in each of the remaining quadrants, the same arrangement mirror-image-wise; we thus again obtain, in the square as a whole, a configuration displaying a central symmetry. Counting the cells as before from two opposite corners will carry out the required exchanges. Here are examples of configurations obtained in this way for order 8 (Fig. 67–70).

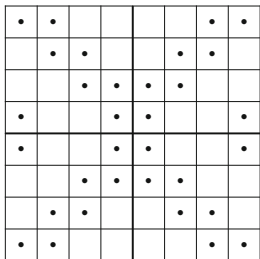


Fig. 67

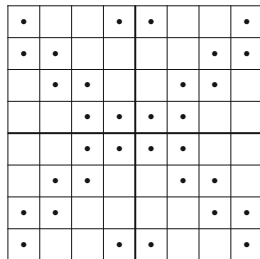


Fig. 68

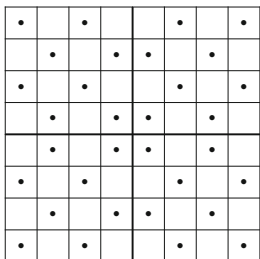


Fig. 69

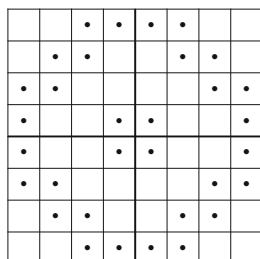


Fig. 70

3. CASE OF EVENLY ODD ORDERS

Such an easily obtained configuration is now no longer possible: since the square is of order $n = 4k + 2$, its quadrant is of order $2k + 1$ and the upper row cannot be half filled with dots. One can, though, resort to a subterfuge: let us put in the first line of the first quadrant $k + 1$ dots, or rather, this time, $\times$ to indicate that they are diagonal exchanges. Next, just as we did earlier, we reproduce these signs first diagonally in the whole quadrant, then in the whole square by reproducing the arrangement (Fig. 71). Now the number of diagonal exchanges, $2k + 2$ by pairs of symmetric rows, would be appropriate for a square of order $4k + 4$; since the order here is $4k + 2$, with just its half, thus $2k + 1$, required horizontal and vertical exchanges, we shall keep $2k$ diagonal exchanges, which will simultaneously effect $2k$ horizontal and $2k$ vertical exchanges, but reduce the two supplementary diagonal exchanges, one to a purely vertical exchange and the other to a purely horizontal exchange, by returning their displaced elements to the neighbouring half; this is the meaning of the

signs — and | in Fig. 72 ($n = 10$, thus $k = 2$), while the remaining $\times$ further play their rôles of diagonal exchanges. One may verify that there are indeed in each line of the *whole* square 5 ($= 2k + 1$) vertical exchanges (signs $\times$ and |) and, in each column, 5 horizontal exchanges ($\times$ and —). The diagonals have been left out of these exchanges.

		$\times$	$\times$	$\times$	$\times$	$\times$	$\times$		
$\times$			$\times$	$\times$	$\times$	$\times$			$\times$
$\times$	$\times$			$\times$	$\times$			$\times$	$\times$
$\times$	$\times$	$\times$					$\times$	$\times$	$\times$
	$\times$	$\times$	$\times$			$\times$	$\times$	$\times$	
	$\times$	$\times$	$\times$			$\times$	$\times$	$\times$	
$\times$	$\times$	$\times$					$\times$	$\times$	$\times$
$\times$	$\times$			$\times$	$\times$			$\times$	$\times$
$\times$			$\times$	$\times$	$\times$	$\times$			$\times$
		$\times$	$\times$	$\times$	$\times$	$\times$	$\times$		

Fig. 71

		$\times$	—		$\times$	—	$\times$		
			$\times$	—	—	$\times$			$\times$
—				$\times$	$\times$			$\times$	—
$\times$	—						$\times$	—	$\times$
	$\times$	—				$\times$	—	$\times$	
	$\times$	$\times$				$\times$	$\times$	$\times$	
$\times$	$\times$						$\times$	$\times$	$\times$
$\times$				$\times$	$\times$			$\times$	$\times$
			$\times$	$\times$	$\times$	$\times$			$\times$
		$\times$	$\times$		$\times$	$\times$	$\times$		

Fig. 72

Applying this to the square of order 10 gives, respectively, Fig. 73 and Fig. 74.

1	2	98	97	96	95	94	93	9	10
90	12	13	87	86	85	84	18	19	81
80	79	23	24	76	75	27	28	72	71
70	69	68	34	35	36	37	63	62	61
41	59	58	57	45	46	54	53	52	50
51	49	48	47	55	56	44	43	42	60
40	39	38	64	65	66	67	33	32	31
30	29	73	74	26	25	77	78	22	21
20	82	83	17	16	15	14	88	89	11
91	92	8	7	6	5	4	3	99	100

Fig. 73

1	2	98	<u>94</u>	<u>6</u>	95	<u>97</u>	93	9	10
<u>20</u>	12	13	87	<u>85</u>	<u>86</u>	84	18	19	81
<u>71</u>	<u>29</u>	23	24	76	75	27	28	72	<u>80</u>
70	<u>62</u>	<u>38</u>	34	35	36	37	63	<u>69</u>	61
41	59	<u>53</u>	<u>47</u>	45	46	54	<u>58</u>	52	50
51	49	48	<u>57</u>	55	56	44	43	42	60
40	39	<u>68</u>	64	65	66	67	33	32	31
30	<u>79</u>	73	74	26	25	77	78	22	21
<u>90</u>	82	83	17	16	15	14	88	89	11
91	92	8	7	<u>96</u>	5	4	3	99	100

Fig. 74

An *embryonic* idea of such exchanges is found in $\mathcal{B}$: he has effected $k + 1$ diagonal exchanges in the lateral rows of each quadrant (Fig. 54), then inverted vertically and horizontally two elements already moved (Fig. 55). But here again — and the final reversal (Fig. 56) shows it — this was carried out for equalizing not only the rows of the outer border but also, as far as possible, opposite cells; and the idea of extending such exchanges to the whole natural square is absent. This will come, at the latest, by the end of the 11th century; the last difficulty posed by the construction

of ordinary magic squares will thus be removed.²⁷

²⁷ *Les carrés magiques*, pp. 84–87 (Russian edition, pp. 95–98).

Magic Squares in the Tenth Century
Two Arabic Treatises by Anṭākī and Būzjānī
Sesiano, J.
2017, VIII, 396 p. 267 illus., Hardcover
ISBN: 978-3-319-52113-8