

# Chapter 2

## Initial and Boundary Value Problems of Fractional Order Hadamard-Type Functional Differential Equations and Inclusions

### 2.1 Introduction

Functional and neutral functional differential equations arise in a variety of areas of biological, physical, and engineering applications, see, for example, the books [90, 100] and the references therein. Fractional functional differential equations involving Riemann-Liouville and Caputo type fractional derivatives have been studied by several researchers [1, 3, 4, 45, 46, 68, 78, 106, 175].

In this chapter, we discuss the existence of solutions for initial and boundary value problems of Hadamard-type functional and neutral functional differential equations and inclusions involving retarded as well as advanced arguments.

### 2.2 Functional and Neutral Fractional Differential Equations

This section deals with the existence of solutions for initial value problems (IVP for short) of fractional order functional and neutral functional differential equations. In the first problem, we consider fractional order functional differential equations:

$$D^\alpha y(t) = f(t, y_t), \text{ for each } t \in J = [1, b], \quad 0 < \alpha < 1, \quad b > 1, \quad (2.1)$$

$$y(t) = \phi(t), \quad t \in [1 - r, 1], \quad {}_H J^{1-\alpha} y(t)|_{t=1} = 0, \quad (2.2)$$

where  $D^\alpha$  is the Hadamard fractional derivative,  $f : J \times C([-r, 0], \mathbb{R}) \rightarrow \mathbb{R}$  is a given continuous function and  $\phi \in C([1 - r, 1], \mathbb{R})$  with  $\phi(1) = 0$  and  ${}_H J^{(\cdot)}$  is the

Hadamard fractional integral. For any function  $y$  defined on  $[1 - r, b]$  and any  $t \in J$ , we denote by  $y_t$  the element of  $C([-r, 0], \mathbb{R})$  and define it as

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0].$$

Notice that  $y_t(\cdot)$  represents the history of the state from time  $t - r$  up to the present time  $t$ .

The second problem is concerned with fractional neutral functional differential equations:

$$D^\alpha[y(t) - g(t, y_t)] = f(t, y_t), \quad t \in J, \quad (2.3)$$

$$y(t) = \phi(t), \quad t \in [1 - r, 1], \quad {}_H J^{1-\alpha} y(t)|_{t=1} = 0, \quad (2.4)$$

where  $f$  and  $\phi$  are the same as defined in problem (2.1)–(2.2), and  $g : J \times C([-r, 0], \mathbb{R}) \rightarrow \mathbb{R}$  is a given function such that  $g(1, \phi) = 0$ .

**Theorem 2.1** ([96, p. 213]) *Let  $\alpha > 0$ ,  $n = -[-\alpha]$  and  $0 \leq \gamma < 1$ . Let  $G$  be an open set in  $\mathbb{R}$  and let  $f : (a, b] \times G \rightarrow \mathbb{R}$  be a function such that:  $f(x, y) \in C_{\gamma, \log}[a, b]$  for any  $y \in G$ . Then the following problem*

$$D^\alpha y(t) = f(t, y(t)), \quad \alpha > 0, \quad (2.5)$$

$${}_H J^{\alpha-k} y(a+) = b_k, \quad b_k \in \mathbb{R}, \quad (k = 1, \dots, n, n = -[-\alpha]), \quad (2.6)$$

*satisfies the Volterra integral equation:*

$$y(t) = \sum_{j=1}^n \frac{b_j}{\Gamma(\alpha - j + 1)} \left(\log \frac{t}{a}\right)^{\alpha-j} + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, y(s)) \frac{ds}{s}, \quad t > a > 0, \quad (2.7)$$

*that is,  $y(t) \in C_{n-\alpha, \log}[a, b]$  satisfies the relations (2.5)–(2.6) if and only if it satisfies the Volterra integral equation (2.7).*

*In particular, if  $0 < \alpha \leq 1$ , the problem (2.5)–(2.6) is equivalent to the following equation:*

$$y(t) = \frac{b}{\Gamma(\alpha)} \left(\log \frac{t}{a}\right)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, y(s)) \frac{ds}{s}, \quad s > a > 0. \quad (2.8)$$

Further details can be found in [96].

### 2.2.1 Functional Differential Equations

By  $C(J, \mathbb{R})$ , we denote the Banach space of all continuous functions from  $J$  into  $\mathbb{R}$  with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in J\},$$

where  $|\cdot|$  is a suitable complete norm on  $\mathbb{R}$ . The space  $C([-r, 0], \mathbb{R})$  is endowed with norm  $\|\cdot\|_C$  defined by

$$\|\phi\|_C := \sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}.$$

**Definition 2.1** A function  $y \in \mathcal{C}^1([1-r, b], \mathbb{R})$  is said to be a solution of (2.1)–(2.2) if it satisfies the equation  $D^\alpha y(t) = f(t, y_t)$  on  $J$ , the conditions  $y(t) = \phi(t)$  on  $[1-r, 1]$  and  ${}_H J^{1-\alpha} y(t)|_{t=1} = 0$ .

Our first existence result for the IVP (2.1)–(2.2) is based on the Banach's contraction mapping principle.

**Theorem 2.2** Let  $f : J \times C([-r, 0], \mathbb{R}) \rightarrow \mathbb{R}$ . Assume that:

(2.2.1) there exists  $\ell > 0$  such that

$$|f(t, u) - f(t, v)| \leq \ell \|u - v\|_C, \text{ for } t \in J \text{ and for every } u, v \in C([-r, 0], \mathbb{R}).$$

If  $\frac{\ell(\log b)^\alpha}{\Gamma(\alpha + 1)} < 1$ , then there exists a unique solution for the IVP (2.1)–(2.2) on the interval  $[1-r, b]$ .

*Proof* To transform the problem (2.1)–(2.2) into a fixed point problem, we consider an operator  $N : C([1-r, b], \mathbb{R}) \rightarrow C([1-r, b], \mathbb{R})$  defined by

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [1-r, 1], \\ \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s, y_s)}{s} ds, & \text{if } t \in [1, b]. \end{cases} \quad (2.9)$$

Let  $y, z \in C([1-r, b], \mathbb{R})$ . Then, for  $t \in [1-r, b]$ , we have

$$\begin{aligned} |N(y)(t) - N(z)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |f(s, y_s) - f(s, z_s)| \frac{ds}{s} \\ &\leq \frac{\ell}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \|y_s - z_s\|_C \frac{ds}{s} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\ell}{\Gamma(\alpha)} \|y - z\|_{[1-r, b]} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\
&\leq \frac{\ell(\log t)^\alpha}{\Gamma(\alpha + 1)} \|y - z\|_{[1-r, b]}.
\end{aligned}$$

Consequently,

$$\|N(y) - N(z)\|_{[1-r, b]} \leq \frac{\ell(\log b)^\alpha}{\Gamma(\alpha + 1)} \|y - z\|_{[1-r, b]},$$

which implies that  $N$  is a contraction as  $\frac{\ell(\log b)^\alpha}{\Gamma(\alpha + 1)} < 1$ , and hence the operator  $N$  has a unique fixed point by Banach's contraction mapping principle. Therefore, the problem (2.1)–(2.2) has a unique solution on  $[1 - r, b]$ .  $\square$

We make use of the nonlinear alternative of Leray-Schauder type to obtain our second existence result for the IVP (2.1)–(2.2).

**Theorem 2.3** *Assume that the following hypotheses hold:*

(2.3.1)  $f : J \times C([-r, 0], \mathbb{R}) \rightarrow \mathbb{R}$  is a continuous function;

(2.3.2) there exist a continuous nondecreasing function  $\psi : [0, \infty) \rightarrow (0, \infty)$  and a function  $p \in C([1, b], \mathbb{R}^+)$  such that

$$|f(t, u)| \leq p(t)\psi(\|u\|_C) \text{ for each } (t, u) \in [1, b] \times C([-r, 0], \mathbb{R});$$

(2.3.3) there exists a constant  $M > 0$  such that

$$\frac{M}{\psi(M)\|p\|_\infty \frac{(\log b)^\alpha}{\Gamma(\alpha + 1)}} > 1.$$

Then the IVP (2.1)–(2.2) has at least one solution on  $[1 - r, b]$ .

*Proof* We consider the operator  $N : C([1 - r, b], \mathbb{R}) \rightarrow C([1 - r, b], \mathbb{R})$  defined by (2.9) and show that it is both continuous and completely continuous.

**Step 1:**  $N$  is continuous.

Let  $\{y_n\}$  be a sequence such that  $y_n \rightarrow y$  in  $C([1 - r, b], \mathbb{R})$ . Let  $\eta > 0$  such that  $\|y_n\|_\infty \leq \eta$ . Then

$$\begin{aligned}
|N(y_n)(t) - N(y)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |f(s, y_{ns}) - f(s, y_s)| \frac{ds}{s} \\
&\leq \frac{1}{\Gamma(\alpha)} \int_1^b \left(\log \frac{t}{s}\right)^{\alpha-1} \sup_{s \in [1, b]} |f(s, y_{ns}) - f(s, y_s)| \frac{ds}{s}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\|f(\cdot, y_n) - f(\cdot, y)\|_\infty}{\Gamma(\alpha)} \int_1^b \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\
&\leq \frac{(\log b)^\alpha \|f(\cdot, y_n) - f(\cdot, y)\|_\infty}{\Gamma(\alpha + 1)}.
\end{aligned}$$

Since  $f$  is a continuous function, we have

$$\|N(y_n) - N(y)\|_\infty \leq \frac{(\log b)^\alpha \|f(\cdot, y_n) - f(\cdot, y)\|_\infty}{\Gamma(\alpha + 1)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

**Step 2:**  $N$  maps bounded sets into bounded sets in  $C([1-r, b], \mathbb{R})$ .

Indeed, it is enough to show that for any  $\eta^* > 0$  there exists a positive constant  $\tilde{\ell}$  such that for each  $y \in B_{\eta^*} = \{y \in C([1-r, b], \mathbb{R}) : \|y\|_\infty \leq \eta^*\}$ , we have  $\|N(y)\|_\infty \leq \tilde{\ell}$ . By (2.3.2), for each  $t \in [1, b]$ , we obtain

$$\begin{aligned}
|N(y)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |f(s, y_s)| \frac{ds}{s} \\
&\leq \frac{\psi(\|y\|_{[1-r, b]}) \|p\|_\infty}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\
&\leq \frac{\psi(\|y\|_{[1-r, b]}) \|p\|_\infty}{\Gamma(\alpha + 1)} (\log b)^\alpha.
\end{aligned}$$

Thus

$$\|N(y)\|_\infty \leq \frac{\psi(\eta^*) \|p\|_\infty}{\Gamma(\alpha + 1)} (\log b)^\alpha := \tilde{\ell}.$$

**Step 3:**  $N$  maps bounded sets into equicontinuous sets of  $C([1-r, b], \mathbb{R})$ .

Let  $t_1, t_2 \in (0, b]$ ,  $t_1 < t_2$ ,  $B_{\eta^*}$  be a bounded set of  $C([1-r, b], \mathbb{R})$  as in Step 2, and let  $y \in B_{\eta^*}$ . Then

$$\begin{aligned}
|N(y)(t_2) - N(y)(t_1)| &\leq \frac{1}{\Gamma(\alpha)} \left| \int_1^{t_1} \left[ \left(\log \frac{t_2}{s}\right)^{\alpha-1} - \left(\log \frac{t_1}{s}\right)^{\alpha-1} \right] f(s, y_s) \frac{ds}{s} \right. \\
&\quad \left. + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} f(s, y_s) \frac{ds}{s} \right| \\
&\leq \frac{\psi(\eta^*) \|p\|_\infty}{\Gamma(\alpha + 1)} \left( 2|(\log t_2/t_1)^\alpha| + |(\log t_2)^\alpha - (\log t_1)^\alpha| \right).
\end{aligned}$$

As  $t_1 \rightarrow t_2$ , the right-hand side of the above inequality tends to zero, independent of  $y \in B_{\eta^*}$ . The equicontinuity for the cases  $t_1 < t_2 \leq 0$  and  $t_1 \leq 0 \leq t_2$  is obvious.

In consequence of Steps 1–3, it follows by the Arzelá-Ascoli theorem that  $N : C([1-r, b], \mathbb{R}) \rightarrow C([1-r, b], \mathbb{R})$  is continuous and completely continuous.

**Step 4:** We show that there exists an open set  $U \subseteq C([1-r, b], \mathbb{R})$  with  $y \neq \lambda N(y)$  for  $\lambda \in (0, 1)$  and  $y \in \partial U$ .

Let  $y \in C([1-r, b], \mathbb{R})$  and  $y = \lambda N(y)$  for some  $0 < \lambda < 1$ . Thus, for each  $t \in [1, b]$ ,

$$y(t) = \lambda \left( \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} f(s, y_s) \frac{ds}{s} \right).$$

By the assumption (2.3.2), for each  $t \in J$ , we get

$$\begin{aligned} |y(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} p(s) \psi(\|y_s\|_C) \frac{ds}{s} \\ &\leq \frac{\|p\|_\infty \psi(\|y\|_{[1-r, b]})}{\Gamma(\alpha + 1)} (\log b)^\alpha, \end{aligned}$$

which can be expressed as

$$\frac{\|y\|_{[1-r, b]}}{\psi(\|y\|_{[1-r, b]}) \|p\|_\infty \frac{(\log b)^\alpha}{\Gamma(\alpha + 1)}} \leq 1.$$

In view of (2.3.3), there exists  $M$  such that  $\|y\|_{[1-r, b]} \neq M$ . Let us set

$$U = \{y \in C([1-r, b], \mathbb{R}) : \|y\|_{[1-r, b]} < M\}.$$

Note that the operator  $N : \overline{U} \rightarrow C([1-r, b], \mathbb{R})$  is continuous and completely continuous. From the choice of  $U$ , there is no  $y \in \partial U$  such that  $y = \lambda Ny$  for some  $\lambda \in (0, 1)$ . Consequently, by the nonlinear alternative of Leray-Schauder type (Theorem 1.15), we deduce that  $N$  has a fixed point  $y \in \overline{U}$  which is a solution of the problem (2.1)–(2.2). This completes the proof.  $\square$

## 2.2.2 Neutral Functional Differential Equations

In this subsection, we establish the existence results for the IVP (2.3)–(2.4).

**Definition 2.2** A function  $y \in \mathcal{C}^1([1-r, b], \mathbb{R})$  is said to be a solution of (2.3)–(2.4) if it satisfies the equation  $D^\alpha[y(t) - g(t, y_t)] = f(t, y_t)$  on  $J$ , the conditions  $y(t) = \phi(t)$  on  $[1-r, 1]$  and  ${}_H J^{1-\alpha} y(t)|_{t=1} = 0$ .

**Theorem 2.4 (Uniqueness Result)** Assume that (2.2.1) and the following condition hold:

(2.4.1) there exists a nonnegative constant  $c_1$  such that

$$|g(t, u) - g(t, v)| \leq c_1 \|u - v\|_C, \quad \text{for every } u, v \in C([-r, 0], \mathbb{R}).$$

If

$$c_1 + \frac{\ell(\log b)^\alpha}{\Gamma(\alpha + 1)} < 1, \quad (2.10)$$

then there exists a unique solution for the IVP (2.3)–(2.4) on the interval  $[1 - r, b]$ .

*Proof* Associated with the problem (2.3)–(2.4), we introduce an operator  $N_1 : C([1 - r, b], \mathbb{R}) \rightarrow C([1 - r, b], \mathbb{R})$  defined by

$$N_1(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [1 - r, 1], \\ g(t, y_t) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s, y_s)}{s} ds, & \text{if } t \in [1, b]. \end{cases} \quad (2.11)$$

To show that the operator  $N_1$  is a contraction, let  $y, z \in C([1 - r, b], \mathbb{R})$ . Then, we have

$$\begin{aligned} |N_1(y)(t) - N_1(z)(t)| &\leq |g(t, y_t) - g(t, z_t)| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_1^t |f(s, y_s) - f(s, z_s)| \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\ &\leq c_1 \|y_t - z_t\|_C + \frac{\ell}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \|y_s - z_s\|_C \frac{ds}{s} \\ &\leq c_1 \|y - z\|_{[1-r, b]} + \frac{\ell}{\Gamma(\alpha)} \|y - z\|_{[1-r, b]} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\ &\leq c_1 \|y - z\|_{[1-r, b]} + \frac{\ell(\log t)^\alpha}{\Gamma(\alpha + 1)} \|y - z\|_{[1-r, b]}. \end{aligned}$$

Consequently, we obtain

$$\|N_1(y) - N_1(z)\|_{[1-r, b]} \leq \left[ c_1 + \frac{\ell(\log b)^\alpha}{\Gamma(\alpha + 1)} \right] \|y - z\|_{[1-r, b]},$$

which, in view of (2.10), implies that  $N_1$  is a contraction. Hence  $N_1$  has a unique fixed point by Banach's contraction mapping principle. This, in turn, shows that the problem (2.3)–(2.4) has a unique solution on  $[1 - r, b]$ .  $\square$

**Theorem 2.5** Assume that (2.3.1) and (2.3.2) hold. Further, we suppose that:

(2.5.1) the function  $g$  is continuous and completely continuous, and for any bounded set  $B$  in  $C([1-r, b], \mathbb{R})$ , the set  $\{t \rightarrow g(t, y_t) : y \in B\}$  is equicontinuous in  $C([1, b], \mathbb{R})$ , and there exist constants  $0 \leq d_1 < 1$ ,  $d_2 \geq 0$  such that

$$|g(t, u)| \leq d_1 \|u\|_C + d_2, \quad t \in [1, b], \quad u \in C([-r, 0], \mathbb{R});$$

(2.5.2) there exists a constant  $M > 0$  such that

$$\frac{(1-d_1)M}{d_2 + \frac{\|p\|_\infty \psi(M)}{\Gamma(\alpha+1)} (\log b)^\alpha} > 1.$$

Then the IVP (2.3)–(2.4) has at least one solution on  $[1-r, b]$ .

*Proof* Let us show that the operator  $N_1 : C([1-r, b], \mathbb{R}) \rightarrow C([1-r, b], \mathbb{R})$  defined by (2.11) is continuous and completely continuous.

Using (2.5.1), it suffices to show that the operator  $N_2 : C([1-r, b], \mathbb{R}) \rightarrow C([1-r, b], \mathbb{R})$  defined by

$$N_2(y)(t) = \begin{cases} \phi(t), & t \in [1-r, 1], \\ \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s, y_s)}{s} ds, & t \in [1, b], \end{cases}$$

is continuous and completely continuous. The proof is similar to that of Theorem 2.3, so we omit the details.

We now show that there exists an open set  $U \subseteq C([1-r, b], \mathbb{R})$  with  $y \neq \lambda N_1(y)$  for  $\lambda \in (0, 1)$  and  $y \in \partial U$ .

Let  $y \in C([1-r, b], \mathbb{R})$  and  $y = \lambda N_1(y)$  for some  $0 < \lambda < 1$ . Thus, for each  $t \in [1, b]$ , we have

$$y(t) = \lambda \left( g(t, y_t) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s, y_s)}{s} ds \right).$$

For each  $t \in J$ , it follows by (2.3.1) and (2.3.2) that

$$\begin{aligned} |y(t)| &\leq d_1 \|y_t\|_C + d_2 + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} p(s) \psi(\|y_s\|_C) \frac{ds}{s} \\ &\leq d_1 \|y_t\|_C + d_2 + \frac{\|p\|_\infty \psi(\|y\|_{[1-r, b]})}{\Gamma(\alpha+1)}, \end{aligned}$$



which yields

$$(1 - d_1)\|y\|_{[1-r,b]} \leq d_2 + \frac{\|p\|_\infty \psi(\|y\|_{[1-r,b]})}{\Gamma(\alpha + 1)} (\log b)^\alpha.$$

In consequence, we get

$$\frac{(1 - d_1)\|y\|_{[1-r,b]}}{d_2 + \frac{\|p\|_\infty \psi(\|y\|_{[1-r,b]})}{\Gamma(\alpha + 1)} (\log b)^\alpha} \leq 1.$$

In view of (2.5.2), there exists  $M$  such that  $\|y\|_{[1-r,b]} \neq M$ . Let us set

$$U = \{y \in C([1-r, b], \mathbb{R}) : \|y\|_{[1-r,b]} < M\}.$$

Note that the operator  $N_1 : \overline{U} \rightarrow C([1-r, b], \mathbb{R})$  is continuous and completely continuous. From the choice of  $U$ , there is no  $y \in \partial U$  such that  $y = \lambda N_1 y$  for some  $\lambda \in (0, 1)$ . Thus, by the nonlinear alternative of Leray-Schauder type (Theorem 1.15), we deduce that  $N_1$  has a fixed point  $y \in \overline{U}$  which is a solution of the problem (2.3)–(2.4). This completes the proof.  $\square$

### 2.2.3 An Example

Consider the initial value problem for fractional functional differential equations:

$$D^{1/2}y(t) = \frac{\|y_t\|}{2(1 + \|y_t\|)} + \frac{1}{3}, \quad t \in J := [1, e], \quad (2.12)$$

$$y(t) = \phi(t), \quad t \in [1-r, 1], \quad {}_H J^{1/2}y(t)|_{t=1} = 0. \quad (2.13)$$

Let

$$f(t, x) = \frac{x}{2(1 + x)}, \quad (t, x) \in [1, e] \times [0, \infty).$$

For  $x, y \in [0, \infty)$  and  $t \in J$ , we have

$$|f(t, x) - f(t, y)| = \frac{1}{2} \left| \frac{x}{1+x} - \frac{y}{1+y} \right| = \frac{|x-y|}{2(1+x)(1+y)} \leq \frac{1}{2} |x-y|.$$

Hence the condition (2.2.1) holds with  $\ell = 1/2$ . Since  $\frac{\ell(\log b)^\alpha}{\Gamma(\alpha + 1)} = \frac{1}{\sqrt{\pi}} < 1$ , therefore, by Theorem 2.2, the problem (2.12)–(2.13) has a unique solution on  $[1-r, b]$ .

## 2.3 Functional and Neutral Fractional Differential Inclusions

In this section, we study the existence of solutions for initial value problems of functional and neutral functional Hadamard type fractional differential inclusions given by

$$D^\alpha y(t) \in F(t, y_t), \text{ for each } t \in J := [1, b], \quad 0 < \alpha < 1, \quad (2.14)$$

$$y(t) = \vartheta(t), \quad t \in [1 - r, 1], \quad {}_H J^{1-\alpha} y(t)|_{t=1} = 0, \quad (2.15)$$

and

$$D^\alpha [y(t) - g(t, y_t)] \in F(t, y_t), \quad t \in J, \quad (2.16)$$

$$y(t) = \vartheta(t), \quad t \in [1 - r, 1], \quad {}_H J^{1-\alpha} y(t)|_{t=1} = 0, \quad (2.17)$$

where  $D^\alpha$  is the Hadamard fractional derivative,  $F : J \times C([-r, 0], \mathbb{R}) \rightarrow \mathcal{P}(\mathbb{R})$  ( $\mathcal{P}(\mathbb{R})$  is the family of all nonempty subsets of  $\mathbb{R}$ ) is a given function and  $\vartheta \in C([1 - r, 1], \mathbb{R})$  with  $\vartheta(1) = 0$  and  $g : J \times C([-r, 0], \mathbb{R}) \rightarrow \mathbb{R}$  is a given function such that  $g(1, \vartheta) = 0$ .

### 2.3.1 Functional Differential Inclusions

In this section, we establish the existence criteria for the problem (2.14)–(2.15).

**Definition 2.3** A function  $y \in \mathcal{C}^1([1 - r, b], \mathbb{R})$  is called a solution of problem (2.14)–(2.15) if there exists a function  $v \in L^1(J, \mathbb{R})$  with  $v(t) \in F(t, y_t)$ , a.e. on  $J$  such that  $D^\alpha y(t) = v(t)$  for a.e.  $t \in J$ ,  $y(t) = \vartheta(t)$ ,  $t \in [1 - r, 1]$  and  ${}_H J^{1-\alpha} y(t)|_{t=1} = 0$ .

**Theorem 2.6** Assume that:

- (2.6.1)  $F : J \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$  is  $L^1$ -Carathéodory and has nonempty compact and convex values;
- (2.6.2) there exists a continuous nondecreasing function  $\beta : [0, \infty) \rightarrow (0, \infty)$  and a function  $\zeta \in C(J, \mathbb{R}^+)$  such that

$$\|F(t, y)\|_{\mathcal{P}} := \sup\{|v| : v \in F(t, y)\} \leq \zeta(t)\beta(\|y\|_C),$$

for each  $(t, y) \in J \times C([-r, 0], \mathbb{R})$ ;

(2.6.3) there exists a constant  $\sigma > 0$  such that

$$\frac{\sigma}{\beta(\sigma)\|\zeta\|_{\infty} \frac{(\log b)^{\alpha}}{\Gamma(\alpha+1)}} > 1.$$

Then the initial value problem (2.14) and (2.15) has at least one solution on  $[1-r, b]$ .

*Proof* Define an operator  $\Omega_F : C([1-r, b], \mathbb{R}) \rightarrow \mathcal{P}(C([1-r, b], \mathbb{R}))$  by

$$\Omega_F(y) = \left\{ \begin{array}{l} h \in C([1-r, b], \mathbb{R}) : \\ h(t) = \begin{cases} \vartheta(t), & \text{if } t \in [1-r, 1], \\ \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds, & \text{if } t \in [1, b], \end{cases} \end{array} \right\} \quad (2.18)$$

for  $v \in S_{F,y}$ . It will be shown that the operator  $\Omega_F$  satisfies the assumptions of Theorem 1.15. Firstly, we observe that  $\Omega_F$  is convex for each  $y \in C([1-r, b], \mathbb{R})$  since  $S_{F,y}$  is convex ( $F$  has convex values). Next, we show that  $\Omega_F$  maps bounded sets into bounded sets in  $C([1-r, b], \mathbb{R})$ . For a positive number  $r$ , let  $B_r = \{y \in C([1-r, b], \mathbb{R}) : \|y\|_{[1-r, b]} \leq r\}$  be a bounded ball in  $C([1-r, b], \mathbb{R})$ . Then, for each  $h \in \Omega_F(y)$ ,  $y \in B_r$ , there exists  $v \in S_{F,y}$  such that

$$h(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds.$$

Then, for  $t \in J$ , we have

$$\begin{aligned} |h(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |v(s)| \frac{ds}{s} \\ &\leq \frac{\beta(\|y\|_{[1-r, b]})\|\zeta\|_{\infty}}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\ &\leq \frac{\beta(\|y\|_{[1-r, b]})\|\zeta\|_{\infty}}{\Gamma(\alpha+1)} (\log b)^{\alpha}. \end{aligned}$$

Thus

$$\|h\| \leq \frac{\beta(r)\|\zeta\|_{\infty}}{\Gamma(\alpha+1)} (\log b)^{\alpha} := \tilde{\ell}.$$

Now, we show that  $\Omega_F$  maps bounded sets into equicontinuous sets of  $C([1-r, b], \mathbb{R})$ . Let  $t_1, t_2 \in J$  with  $t_1 < t_2$  and  $y \in B_r$ . For each  $h \in \Omega_F(y)$ , we obtain

$$\begin{aligned}
|h(t_2) - h(t_1)| &\leq \frac{1}{\Gamma(\alpha)} \left| \int_1^{t_1} \left[ \left( \log \frac{t_2}{s} \right)^{\alpha-1} - \left( \log \frac{t_1}{s} \right)^{\alpha-1} \right] f(s, y_s) \frac{ds}{s} \right. \\
&\quad \left. + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left( \log \frac{t_2}{s} \right)^{\alpha-1} f(s, y_s) \frac{ds}{s} \right| \\
&\leq \frac{\psi(\eta^*) \|p\|_\infty}{\Gamma(\alpha + 1)} \left( 2|(\log t_2/t_1)^\alpha| + |(\log t_2)^\alpha - (\log t_1)^\alpha| \right).
\end{aligned}$$

Clearly the right hand side of the above inequality tends to zero independent of  $y \in B_r$  as  $t_2 - t_1 \rightarrow 0$ . As  $\Omega_F$  satisfies the above three assumptions, it follows by the Arzelà-Ascoli Theorem that  $\Omega_F : C([1 - r, b], \mathbb{R}) \rightarrow \mathcal{P}(C([1 - r, b], \mathbb{R}))$  is completely continuous.

In our next step, we show that  $\Omega_F$  is upper semicontinuous. It is known [69, Proposition 1.2] that  $\Omega_F$  will be upper semicontinuous if we establish that it has a closed graph, since  $\Omega_F$  is already shown to be completely continuous. Thus, we will prove that  $\Omega_F$  has a closed graph. Let  $y_n \rightarrow y_*$ ,  $h_n \in \Omega_F(y_n)$  and  $h_n \rightarrow h_*$ . Then, we need to show that  $h_* \in \Omega_F(y_*)$ . Associated with  $h_n \in \Omega_F(y_n)$ , there exists  $v_n \in S_{F, y_n}$  such that for each  $t \in J$ ,

$$h_n(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} v_n(s) \frac{ds}{s}.$$

Thus it suffices to show that there exists  $v_* \in S_{F, y_*}$  such that for each  $t \in J$ ,

$$h_*(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} v_*(s) \frac{ds}{s}.$$

Let us consider the linear operator  $\Theta : L^1(J, \mathbb{R}) \rightarrow C(J, \mathbb{R})$  given by

$$v \mapsto \Theta(v)(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} v(s) \frac{ds}{s}.$$

Notice that

$$\|h_n(t) - h_*(t)\| = \left\| \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} (v_n(s) - v_*(s)) \frac{ds}{s} \right\| \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Thus, it follows by Lemma 1.2 that  $\Theta \circ S_{F, y}$  is a closed graph operator. Further, we have that  $h_n(t) \in \Theta(S_{F, y_n})$ . Since  $y_n \rightarrow y_*$ , we have

$$h_*(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} v_*(s) \frac{ds}{s},$$

for some  $v_* \in S_{F, y_*}$ .

Finally, we show that there exists an open set  $U \subseteq C(J, \mathbb{R})$  with  $y \notin \Omega_F(y)$  for any  $\lambda \in (0, 1)$  and all  $y \in \partial U$ . Let  $\lambda \in (0, 1)$  and  $y \in \lambda \Omega_F(y)$ . Then there exists  $v \in L^1(J, \mathbb{R})$  with  $v \in S_{F,y}$  such that, for  $t \in J$ , we have

$$y(t) = \lambda \left( \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} v(s) \frac{ds}{s} \right).$$

By the assumption (2.6.2), for each  $t \in J$ , we get

$$\begin{aligned} |y(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} \zeta(s) \beta(\|y_s\|) \frac{ds}{s} \\ &\leq \frac{\|\zeta\|_\infty \beta(\|y\|_{[1-r,b]})}{\Gamma(\alpha+1)} (\log b)^\alpha, \end{aligned}$$

which can be expressed as

$$\frac{\|y\|_{[1-r,b]}}{\beta(\|y\|_{[1-r,b]}) \|\zeta\|_\infty \frac{(\log b)^\alpha}{\Gamma(\alpha+1)}} \leq 1.$$

In view of (2.6.3), there exists  $\sigma$  such that  $\|y\|_{[1-r,b]} \neq \sigma$ . Let us set

$$U = \{y \in C([1-r, b], \mathbb{R}) : \|y\|_{[1-r,b]} < \sigma\}.$$

Note that the operator  $\Omega_F : \overline{U} \rightarrow \mathcal{P}(C([1-r, b], \mathbb{R}))$  is upper semicontinuous and completely continuous. From the choice of  $U$ , there is no  $y \in \partial U$  such that  $y \in \lambda \Omega_F(y)$  for some  $\lambda \in (0, 1)$ . Consequently, by the nonlinear alternative of Leray-Schauder type (Theorem 1.15), we deduce that  $\Omega_F$  has a fixed point  $y \in \overline{U}$  which is a solution of the problem (2.14)–(2.15). This completes the proof.  $\square$

Next, we prove the existence of solutions for the problem (2.14)–(2.15) with a nonconvex valued right hand side (Lipschitz case) by applying a fixed point theorem for multivalued maps due to Covitz and Nadler (Theorem 1.18).

**Theorem 2.7** *Assume that:*

- (2.7.1)  $F : J \times \mathbb{R} \rightarrow \mathcal{P}_{cp}(\mathbb{R})$  is such that  $F(\cdot, y) : J \rightarrow \mathcal{P}_{cp}(\mathbb{R})$  is measurable for each  $y \in \mathbb{R}$ ;
- (2.7.2)  $H_d(F(t, y), F(t, \bar{y})) \leq \ell(t) \|y - \bar{y}\|_C$  for almost all  $t \in J$  and  $y, \bar{y} \in C([-r, 0], \mathbb{R})$  with  $\ell \in C(J, \mathbb{R}^+)$  and  $d(0, F(t, 0)) \leq \ell(t)$  for almost all  $t \in J$ .

Then, if  $\frac{(\log b)^\alpha}{\Gamma(\alpha+1)} \|\ell\|_\infty < 1$ , the initial value problem (2.14)–(2.15) has at least one solution on  $[1-r, b]$ .

*Proof* Observe that the set  $S_{F,y}$  is nonempty for each  $y \in C([1-r, b], \mathbb{R})$  by the assumption (2.7.1), so  $F$  has a measurable selection (see Theorem III.6 [57]). Now, we show that the operator  $\Omega_F$ , defined by (2.18), satisfies the hypothesis of Theorem 1.18. To show that  $\Omega_F(y) \in \mathcal{P}_{cl}(C([1-r, b], \mathbb{R}))$  for each  $y \in C([1-r, b], \mathbb{R})$ , let  $\{u_n\}_{n \geq 0} \in \Omega_F(y)$  be such that  $u_n \rightarrow u$  ( $n \rightarrow \infty$ ) in  $C([1-r, b], \mathbb{R})$ . Then  $u \in C([1-r, b], \mathbb{R})$  and there exists  $v_n \in S_{F,y_n}$  such that, for each  $t \in J$ ,

$$u_n(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v_n(s) \frac{ds}{s}.$$

As  $F$  has compact values, we pass onto a subsequence (if necessary) to obtain that  $v_n$  converges to  $v$  in  $L^1(J, \mathbb{R})$ . Thus,  $v \in S_{F,y}$  and for each  $t \in J$ , we have

$$u_n(t) \rightarrow u(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v(s) \frac{ds}{s}.$$

Hence,  $u \in \Omega(y)$ .

Next, we show that there exists  $\delta < 1$  ( $\delta := \frac{(\log b)^\alpha}{\Gamma(\alpha+1)} \|\ell\|_\infty$ ) such that

$$H_d(\Omega_F(y), \Omega_F(\bar{y})) \leq \delta \|y - \bar{y}\|_C \text{ for each } y, \bar{y} \in C([1-r, b], \mathbb{R}).$$

Let  $y, \bar{y} \in C([1-r, b], \mathbb{R})$  and  $h_1 \in \Omega_F(y)$ . Then there exists  $v_1(t) \in F(t, y_t)$  such that, for each  $t \in J$ ,

$$h_1(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v_1(s) \frac{ds}{s}.$$

By (2.7.2), we have

$$H_d(F(t, y), F(t, \bar{y})) \leq \ell(t) \|y - \bar{y}\|_C.$$

So, there exists  $w \in F(t, \bar{y}_t)$  such that

$$|v_1(t) - w| \leq \ell(t) \|y - \bar{y}\|_C, \quad t \in J.$$

Define  $U : J \rightarrow \mathcal{P}(\mathbb{R})$  by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq \ell(t) \|y - \bar{y}\|_C\}.$$

Since the multivalued operator  $U(t) \cap F(t, \bar{y}_t)$  is measurable (Proposition III.4 [57]), there exists a function  $v_2(t)$  which is a measurable selection for  $U$ . So  $v_2(t) \in F(t, \bar{y}_t)$  and for each  $t \in J$ , we have  $|v_1(t) - v_2(t)| \leq \ell(t) \|y - \bar{y}\|_C$ .

For each  $t \in J$ , let us define

$$h_2(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v_2(s) \frac{ds}{s}.$$

Thus,

$$\begin{aligned} |h_1(t) - h_2(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |v_1(s) - v_2(s)| \frac{ds}{s} \\ &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \ell(s) \|y - \bar{y}\|_C \frac{ds}{s} \\ &\leq \frac{1}{\Gamma(\alpha)} \int_1^b \left(\log \frac{t}{s}\right)^{\alpha-1} \ell(s) \|y - \bar{y}\|_{[1-r, b]} \frac{ds}{s} \\ &\leq \frac{(\log b)^\alpha}{\Gamma(\alpha + 1)} \|\ell\|_\infty \|y - \bar{y}\|_{[1-r, b]}. \end{aligned}$$

Hence,

$$\|h_1 - h_2\| \leq \frac{(\log b)^\alpha}{\Gamma(\alpha + 1)} \|\ell\|_\infty \|y - \bar{y}\|_{[1-r, b]}.$$

Analogously, interchanging the roles of  $y$  and  $\bar{y}$ , we obtain

$$H_d(\Omega_F(y), \Omega_F(\bar{y})) \leq \frac{(\log b)^\alpha}{\Gamma(\alpha + 1)} \|\ell\|_\infty \|y - \bar{y}\|_{[1-r, b]}.$$

Since  $\Omega_F$  is a contraction by the given condition, it follows by Theorem 1.18 that  $\Omega_F$  has a fixed point  $y$  which is a solution of (2.14)–(2.15). This completes the proof.  $\square$

### 2.3.2 Neutral Functional Differential Inclusions

This subsection is concerned with the existence of solutions for the problem (2.16)–(2.17).

**Definition 2.4** A function  $y \in \mathcal{C}^1([1-r, b], \mathbb{R})$  is said to be a solution of (2.16)–(2.17) if there exists a function  $v \in L^1([1, b], \mathbb{R})$  with  $v(t) \in F(t, y_t)$ , a.e. on  $[1, b]$  such that  $D^\alpha[y(t) - g(t, y_t)] = v(t)$  on  $J$ ,  $y(t) = \vartheta(t)$  on  $[1-r, 1]$  and  ${}_H J^{1-\alpha} y(t)|_{t=1} = 0$ .

**Theorem 2.8** Suppose that (2.5.1), (2.6.1) and (2.6.2) hold. Further it is assumed that:

(2.8.1) there exists a constant  $M > 0$  such that

$$\frac{(1 - d_1)M}{d_2 + \frac{\|\zeta\|_\infty \beta(M)}{\Gamma(\alpha + 1)} (\log b)^\alpha} > 1.$$

Then the IVP (2.16)–(2.17) has at least one solution on  $[1 - r, b]$ .

*Proof* Define an operator  $Q : C([1 - r, b], \mathbb{R}) \rightarrow \mathcal{P}(C([1 - r, b], \mathbb{R}))$  by

$$Q(y) = \left\{ \begin{array}{ll} h \in C([1 - r, b], \mathbb{R}) : & \\ h(t) = \begin{cases} \vartheta(t), & \text{if } t \in [1 - r, 1], \\ g(t, x_t) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds, & \text{if } t \in [1, b], \end{cases} & \end{array} \right\}$$

for  $v \in S_{F,y}$ .

Using (2.8.1), it suffices to show that the operator  $Q_1 : C([1 - r, b], \mathbb{R}) \rightarrow C([1 - r, b], \mathbb{R})$  defined by

$$Q_1(x) = \left\{ \begin{array}{ll} h \in C([1 - r, b], \mathbb{R}) : & \\ h(t) = \begin{cases} \vartheta(t), & \text{if } t \in [1 - r, 1], \\ \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds, & \text{if } t \in [1, b], \end{cases} & \end{array} \right\}$$

for  $v \in S_{F,y}$ , is continuous and completely continuous. The proof is similar to that of Theorem 2.6, so, we omit the details.

Next, we show that there exists an open set  $U \subseteq C([1 - r, b], \mathbb{R})$  with  $y \neq \lambda Q(y)$  for  $\lambda \in (0, 1)$  and  $y \in \partial U$ .

Let  $y \in C([1 - r, b], \mathbb{R})$  be such that  $y = \lambda Q(y)$  for some  $0 < \lambda < 1$ . Thus, for each  $t \in [1, b]$ , we have

$$y(t) = \lambda \left( g(t, y_t) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds \right).$$

For each  $t \in J$ , it follows by (2.6.2) and (2.5.1) that

$$\begin{aligned} |y(t)| &\leq d_1 \|y_t\|_C + d_2 + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \zeta(s) \beta(\|y_s\|_C) \frac{ds}{s} \\ &\leq d_1 \|y_t\|_C + d_2 + \frac{\|\zeta\|_\infty \beta(\|y\|_{[1-r,b]})}{\Gamma(\alpha + 1)} (\log b)^\alpha, \end{aligned}$$



which yields

$$(1 - d_1)\|y\|_{[1-r, b]} \leq d_2 + \frac{\|\xi\|_\infty \beta(\|y\|_{[1-r, b]})}{\Gamma(\alpha + 1)} (\log b)^\alpha.$$

In consequence, we get

$$\frac{(1 - d_1)\|y\|_{[1-r, b]}}{d_2 + \frac{\|\xi\|_\infty \beta(\|y\|_{[1-r, b]})}{\Gamma(\alpha + 1)} (\log b)^\alpha} \leq 1.$$

In view of (2.8.1), there exists  $M$  such that  $\|y\|_{[1-r, b]} \neq M$ . Let us set

$$U = \{y \in C([1 - r, b], \mathbb{R}) : \|y\|_{[1-r, b]} < M\}.$$

Note that the operator  $Q : \overline{U} \rightarrow C([1 - r, b], \mathbb{R})$  is continuous and completely continuous. From the choice of  $U$ , there is no  $y \in \partial U$  such that  $y = \lambda Qy$  for some  $\lambda \in (0, 1)$ . Thus, by the nonlinear alternative of Leray-Schauder type (Theorem 1.15), we deduce that  $Q$  has a fixed point  $y \in \overline{U}$  which is a solution of the problem (2.16)–(2.17). This completes the proof.  $\square$

**Theorem 2.9** Assume that (2.7.1) and (2.7.2) hold. In addition, we suppose that:

(2.9.1) there exists a constant  $L > 0$  such that

$$|g(t, x) - g(t, y)| \leq L\|x - y\|_C, \text{ for all } t \in [1, b] \text{ and } x, y \in C([-r, 0], \mathbb{R}).$$

Then, if  $L + \frac{(\log b)^\alpha}{\Gamma(\alpha + 1)} \|\ell\|_\infty < 1$ , the IVP (2.16)–(2.17) has at least one solution on  $[1 - r, b]$ .

*Proof* Since the proof is similar to that of Theorem 2.7, it is omitted.  $\square$

### 2.3.3 Examples

*Example 1* For any function  $\vartheta \in C([1 - r, 1], \mathbb{R})$  with  $\vartheta(1) = 0$ , consider the problem

$$D^\alpha y(t) \in F(t, y_t), \text{ for each } t \in J := [1, e], \quad 0 < \alpha < 1, \quad (2.19)$$

$$y(t) = \vartheta(t), \quad t \in [1 - r, 1], \quad {}_H J^{1-\alpha} y(t)|_{t=1} = 0, \quad (2.20)$$

where

$$F(t, y_t) = \left[ \frac{1}{4 + e^{-t}} \left( \frac{|y_t|}{2(1 + |y_t|)} + \frac{1}{4} \right), \frac{1}{16} (1 + e^{-t}) \right].$$

Clearly

$$\|F(t, y_t)\|_{\mathcal{D}} := \sup\{|u| : u \in F(t, y_t)\} \leq \frac{1}{4} \left( \frac{3}{4} \right), \quad y_t \in \mathbb{R}.$$

With  $\zeta(t) = 1/4$ ,  $\beta(\|y_t\|) = 3/4$ , by the condition (2.3.3), we find that

$$M > \frac{3}{16\Gamma(\alpha + 1)}, \quad 0 < \alpha < 1.$$

Hence, by Theorem 2.6, the problem (2.19)–(2.20) has a solution on  $[1 - r, e]$ .

*Example 2* Let us consider the problem (2.19)–(2.20) with

$$F(t, y_t) = \left[ \frac{1}{16}, \frac{1}{\pi\sqrt{t+3}} \tan^{-1}(y_t) + \frac{1}{12} \right]. \quad (2.21)$$

Observe that

$$H_d(F(t, y_t), F(t, \bar{y}_t)) \leq \frac{1}{\pi\sqrt{t+3}} \|y - \bar{y}\|_C.$$

Letting  $\ell(t) = \frac{1}{\pi\sqrt{t+3}}$ , we find that  $d(0, F(t, 0)) \leq \ell(t)$  for almost all  $t \in J$  and  $\frac{(\log b)^\alpha}{\Gamma(\alpha + 1)} \|\ell\|_\infty = \frac{1}{2\pi\Gamma(\alpha + 1)} < 1$ , for  $0 < \alpha < 1$ . Thus all the conditions of Theorem 2.7 are satisfied. Hence, by the conclusion of Theorem 2.7, the problem (2.19)–(2.20) with (2.21) has a solution on  $[1 - r, e]$ .

## 2.4 Boundary Value Problems of Fractional Order Hadamard-Type Functional Differential Equations and Inclusions with Retarded and Advanced Arguments

In this section, we study Hadamard-type fractional functional differential equations and inclusions involving both retarded and advanced arguments with boundary conditions.

### 2.4.1 Fractional Order Hadamard-Type Functional Differential Equations

Here, we investigate a boundary value problem of Hadamard-type fractional functional differential equations involving both retarded and advanced arguments given by

$$D^\alpha x(t) = f(t, x^t), \quad 1 \leq t \leq e, \quad 1 < \alpha < 2, \quad (2.22)$$

$$x(t) = \chi(t), \quad 1 - r \leq t \leq 1, \quad (2.23)$$

$$x(t) = \psi(t), \quad e \leq t \leq e + h, \quad (2.24)$$

where  $D^\alpha$  is the Hadamard fractional derivative,  $f : [1, e] \times C([-r, h], \mathbb{R}) \rightarrow \mathbb{R}$  is a given continuous function,  $\chi \in C([1 - r, 1], \mathbb{R})$  with  $\chi(1) = 0$  and  $\psi \in C([e, e + h], \mathbb{R})$  with  $\psi(e) = 0$ . For any function  $x$  defined on  $[1 - r, e + h]$  and any  $1 \leq t \leq e$ , we denote by  $x^t$  the element of  $C([-r, h], \mathbb{R})$  defined by  $x^t(\theta) = x(t + \theta)$  for  $-r \leq \theta \leq h$ , where  $r, h \geq 0$  are constants.

By  $C := C([-r, h], \mathbb{R})$ , we denote the Banach space of all continuous functions from  $[-r, h]$  into  $\mathbb{R}$  equipped with the norm

$$\|\chi\|_{[-r, h]} = \sup\{|\chi(\theta)| : -r \leq \theta \leq h\}$$

and  $C([1, e], \mathbb{R})$  is the Banach space endowed with norm  $\|x\|_0 = \sup\{|x(t)| : 1 \leq t \leq e\}$ . Also, let  $E = C([1 - r, e + h], \mathbb{R})$ ,  $E_1 = C([1 - r, 1], \mathbb{R})$ , and  $E_2 = C([e, e + h], \mathbb{R})$  be respectively endowed with the norms  $\|x\|_{[1 - r, e + h]} = \sup\{|x(t)| : 1 - r \leq t \leq e + h\}$ ,  $\|x\|_{[1 - r, 1]} = \sup\{|x(t)| : 1 - r \leq t \leq 1\}$ , and  $\|x\|_{[e, e + h]} = \sup\{|x(t)| : e \leq t \leq e + h\}$ .

**Lemma 2.1** *Given  $g \in C([1, e], \mathbb{R})$  and  $1 < \alpha \leq 2$ , the problem*

$$D^\alpha u(t) = g(t), \quad 0 < t < 1, \quad (2.25)$$

$$u(1) = u(e) = 0, \quad (2.26)$$

*is equivalent to the integral equation*

$$u(t) = - \int_1^e G(t, s) \frac{g(s)}{s} ds, \quad (2.27)$$

where

$$G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (\log t)^{\alpha-1} (1 - \log s)^{\alpha-1} - (\log t - \log s)^{\alpha-1}, & 1 \leq s \leq t \leq e, \\ (\log t)^{\alpha-1} (1 - \log s)^{\alpha-1}, & 1 \leq t \leq s \leq e. \end{cases} \quad (2.28)$$

*Proof* As argued in [96], the solution of Hadamard differential equation (2.25) can be written as

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds + c_1 (\log t)^{\alpha-1} + c_2 (\log t)^{\alpha-2}. \quad (2.29)$$

Using the given boundary conditions, we find that  $c_2 = 0$ , and

$$c_1 = -\frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds.$$

Substituting the values of  $c_1$  and  $c_2$  in (2.29), we obtain

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds - (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds \\ &= -\frac{1}{\Gamma(\alpha)} \int_1^t \left[ (\log t)^{\alpha-1} (1 - \log s)^{\alpha-1} - (\log t - \log s)^{\alpha-1} \right] \frac{g(s)}{s} ds \\ &\quad - \int_t^e \frac{1}{\Gamma(\alpha)} (\log t)^{\alpha-1} (1 - \log s)^{\alpha-1} \frac{g(s)}{s} ds \\ &= -\int_1^e G(t, s) \frac{g(s)}{s} ds, \end{aligned}$$

where  $G(t, s)$  is given by (2.28). Converse of the theorem follows by direct computation. This completes the proof.  $\square$

By a solution of (2.22)–(2.24), we mean a function  $x \in \mathcal{C}^2([1-r, e+h], \mathbb{R})$  that satisfies the equation  $D^\alpha x(t) = f(t, x')$  on  $[1, e]$  and the conditions  $x(t) = \chi(t)$ ,  $\chi(1) = 0$  on  $[1-r, 1]$  and  $x(t) = \psi(t)$ ,  $\psi(e) = 0$  on  $[e, e+h]$ .

**Theorem 2.10** *Let  $f : [1, e] \times C([-r, h], \mathbb{R}) \rightarrow \mathbb{R}$  be a continuous function. Assume the following conditions hold:*

(2.10.1) *there exist  $p \in C(J, \mathbb{R})$  and  $\Omega : [0, \infty) \rightarrow (0, \infty)$  continuous and nondecreasing such that*

$$|f(t, u)| \leq p(t)\Omega(\|u\|_{[-r, h]})$$

*for all  $t \in J$  and all  $u \in C([-r, h], \mathbb{R})$ ;*

(2.10.2) *there exists a number  $K_0 > 0$  such that*

$$\frac{K_0}{\frac{2\|p\|_0}{\Gamma(\alpha+1)}\Omega(K_0 + \max\{\|x\|_{[1-r, 1]}, \|x\|_{[e, e+h]}\})} > 1.$$

*Then the boundary value problem (2.22)–(2.24) has at least one solution on the interval  $[1-r, e+h]$ .*

*Proof* To transform the problem (2.22)–(2.24) into a fixed point problem, we consider an operator  $\mathcal{Q} : C([1 - r, e + h], \mathbb{R}) \rightarrow C([1 - r, e + h], \mathbb{R})$  defined by

$$(\mathcal{Q}x)(t) = \begin{cases} \chi(t), & \text{if } t \in [1 - r, 1], \\ \int_1^e G(t, s) \frac{f(s, x^s)}{s} ds, & \text{if } t \in [1, e], \\ \psi(t), & \text{if } t \in [e, e + h]. \end{cases} \quad (2.30)$$

Let  $u : [1 - r, e + h] \rightarrow \mathbb{R}$  be a function defined by

$$u(t) = \begin{cases} \chi(t), & \text{if } t \in [1 - r, 1], \\ 0, & \text{if } t \in [1, e], \\ \psi(t), & \text{if } t \in [e, e + h]. \end{cases}$$

For each  $y \in C([1, e], \mathbb{R})$  with  $y(1) = 0$ , we denote by  $z$  the function defined by

$$z(t) = \begin{cases} 0, & \text{if } t \in [1 - r, 1], \\ y(t), & \text{if } t \in [1, e], \\ 0, & \text{if } t \in [e, e + h]. \end{cases}$$

Let us set  $x(t) = y(t) + u(t)$  such that  $x^t = y^t + u^t$  for every  $1 \leq t \leq e$ , where

$$\begin{aligned} x(t) &= \int_1^e G(t, s) \frac{f(s, x^s)}{s} ds, \\ y(t) &= \int_1^e G(t, s) \frac{f(s, y^s + u^s)}{s} ds. \end{aligned}$$

Next, we define  $B = \{y \in C([1 - r, e + h], \mathbb{R}) : y(1) = 0\}$  and let  $\mathfrak{F} : B \rightarrow B$  be an operator given by

$$(\mathfrak{F}y)(t) = \begin{cases} 0, & 1 - r \leq t \leq 1, \\ \int_1^e G(t, s) \frac{f(s, y^s + u^s)}{s} ds, & 1 \leq t \leq e, \\ 0, & e \leq t \leq e + h. \end{cases} \quad (2.31)$$

Then it is enough to show that the operator  $\mathfrak{F}$  has a fixed point which will guarantee that the operator  $\mathcal{F}$  has a fixed point and in consequence, this fixed point will correspond to a solution of the problem (2.22)–(2.24). In the following three steps, it will be shown that the operator  $\mathfrak{F}$  is continuous and completely continuous.

**Step 1:**  $\mathfrak{F}$  is continuous.

Let  $(y_n)$  be a sequence such that  $y_n \rightarrow y$  in  $B$ . Then, we have

$$\begin{aligned} |(\mathfrak{F}y_n)(t) - (\mathfrak{F}y)(t)| &\leq \int_1^e G(t, s) |f(s, y^{ns} + u^s) - f(s, y^s + u^s)| \frac{ds}{s} \\ &\leq \|f(\cdot, y^{n(\cdot)} + u^{(\cdot)}) - f(\cdot, y^{(\cdot)} + u^{(\cdot)})\|_0 \int_1^e G(t, s) \frac{ds}{s}. \end{aligned}$$

Since the function  $f$  is continuous, we have

$$\|\mathfrak{F}y_n - \mathfrak{F}y\|_{[1-r, e+h]} \leq \|f(\cdot, y^{n(\cdot)} + u^{(\cdot)}) - f(\cdot, y^{(\cdot)} + u^{(\cdot)})\|_0 \int_1^e G(t, s) \frac{ds}{s} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

**Step 2:**  $\mathfrak{F}$  maps bounded sets into bounded sets in  $B$ .

For any  $k > 0$ , it is enough to show that there exists a positive constant  $\hat{L}$  such that, for each  $y \in U_k := \{y \in B : \|y\|_{[1-r, e+h]} \leq k\}$ , we have  $\|\mathfrak{F}y\|_{[1-r, e+h]} \leq \hat{L}$ . For  $y \in B$  and  $s \in J$ , we have

$$\|y^s\|_{[-r, h]} = \max_{\theta \in [-r, h]} |y(s + \theta)| \leq \max_{t \in [1-r, e+h]} |y(t)| = \|y\|_{[1-r, e+h]}$$

and

$$\|y^s + u^s\| \leq \|y^s\|_{[-r, h]} + \|u^s\|_{[-r, h]} \leq \|y\|_{[-r, h]} + \max\{\|x\|_{[1-r, 1]}, \|x\|_{[e, e+h]}\}.$$

Let  $y \in U_k$ . Since  $f$  is continuous, for  $t \in [1, e]$ , we have

$$\begin{aligned} |(\mathfrak{F}y)(t)| &\leq \left| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s, y^s + u^s)}{s} ds \right. \\ &\quad \left. - (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{f(s, y^s + u^s)}{s} ds \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{p(s) \Omega(\|y^s + u^s\|_{[-r, h]})}{s} ds \\ &\quad + (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{p(s) \Omega(\|y^s + u^s\|_{[-r, h]})}{s} ds \\ &\leq \frac{2\|p\|_0 \Omega(k + \max\{\|x\|_{[1-r, 1]}, \|x\|_{[e, e+h]}\})}{\Gamma(\alpha)} \int_1^e \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{1}{s} ds \\ &\leq \frac{2\|p\|_0 \Omega(k + \max\{\|x\|_{[1-r, 1]}, \|x\|_{[e, e+h]}\})}{\Gamma(\alpha + 1)}, \end{aligned}$$

and so

$$\|\mathfrak{F}y\|_{[1-r, e+h]} \leq \frac{2\|p\|_0 \Omega(k + \max\{\|x\|_{[1-r, 1]}, \|x\|_{[e, e+h]}\})}{\Gamma(\alpha + 1)} := \hat{L}.$$

Consequently,  $\mathfrak{F}$  maps bounded sets into bounded sets in  $B$ .

**Step 3:**  $\mathfrak{F}$  maps bounded sets into equicontinuous sets of  $B$ .

Let  $t_1, t_2 \in [1, e]$  with  $t_1 < t_2$  and  $U_k$  be a bounded set of  $B$  as in Step 2. Let  $y \in U_k$ . Then, we have

$$\begin{aligned} & |(\mathfrak{F}y)(t_2) - (\mathfrak{F}y)(t_1)| \\ & \leq \int_1^e |G(t_2, s) - G(t_1, s)| \frac{|f(s, y^s + u^s)|}{s} ds \\ & \leq \|p\|_0 \Omega(k + \max\{\|x\|_{[1-r, 1]}, \|x\|_{[e, e+h]}\}) \int_1^e |G(t_2, s) - G(t_1, s)| \frac{ds}{s}. \end{aligned}$$

As  $t_1 \rightarrow t_2$ , the right-hand side of the last inequality tends to zero, independent of  $y \in U_k$ . The equicontinuity for the cases  $t_1 < t_2 \leq 0$  and  $t_1 \leq 0 \leq t_2$  is obvious.

In view of steps 1 to 3, it follows by the Arzelá-Ascoli Theorem that the operator  $\mathfrak{F}$  is continuous and completely continuous.

**Step 4:** *A priori bounds.*

We will show that there exists an open set  $U \subset B$  with  $y \neq \lambda \mathfrak{F}y$  for  $0 < \lambda < 1$  and  $y \in \partial U$ . Let  $y \in B$  and  $y = \lambda \mathfrak{F}y$  for some  $0 < \lambda < 1$ . Thus, for each  $t \in [1, e]$ , we have

$$y(t) = \lambda \int_1^e G(t, s) f(s, y^s + u^s) \frac{ds}{s}.$$

By our assumptions, for each  $t \in J$ , we get

$$\begin{aligned} |y(t)| & \leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{p(s) \Omega(\|y^s + u^s\|_{[-r, h]})}{s} ds \\ & \quad + (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{p(s) \Omega(\|y^s + u^s\|_{[-r, h]})}{s} ds \\ & \leq \frac{2\|p\|_0 \Omega(\|y\|_{[-r, h]} + \max\{\|x\|_{[1-r, 1]}, \|x\|_{[e, e+h]}\})}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{1}{s} ds \\ & \leq \frac{2\|p\|_0}{\Gamma(\alpha + 1)} \Omega(\|y\|_{[-r, h]} + \max\{\|x\|_{[1-r, 1]}, \|x\|_{[e, e+h]}\}), \end{aligned}$$

which implies that

$$\frac{\|y\|_{[1-r, e+h]}}{\frac{2\|p\|_0}{\Gamma(\alpha+1)}\Omega(\|y\|_{[-r, h]} + \max\{\|x\|_{[1-r, 1]}, \|x\|_{[e, e+h]}\})} \leq 1.$$

By (2.10.1), there exists  $K_0$  such that  $\|y\|_{[1-r, e+h]} \neq K_0$ . Set

$$U = \{y \in B : \|y\|_{[1-r, e+h]} < K_0 + 1\}.$$

By our choice of  $U$ , there is no  $y \in \partial U$  such that  $y = \lambda \mathfrak{F}y$  for some  $0 < \lambda < 1$ . As a consequence of the nonlinear alternative of Leray-Schauder type (Theorem 1.4), we deduce that  $\mathfrak{F}$  has a fixed point  $y \in \bar{U}$  which is a solution to problem (2.22)–(2.24).  $\square$

The next result, concerning the existence of a unique solution of problem (2.22)–(2.24), is based on the Banach's fixed point theorem.

**Theorem 2.11** *Let  $f : [1, e] \times C([-r, h], \mathbb{R}) \rightarrow \mathbb{R}$ . Assume that there exists  $L > 0$  such that*

$$|f(t, u(t)) - f(t, v(t))| \leq L\|u - v\|_{[-r, h]},$$

for  $t \in [1, e]$  and for every  $u, v \in C([-r, h], \mathbb{R})$ .

If

$$\frac{2L}{\Gamma(\alpha+1)} < 1,$$

then the BVP (2.22)–(2.24) has a unique solution on the interval  $[1 - r, e + h]$ .

*Proof* As argued in the proof of the preceding theorem, it will be shown that the operator  $\mathfrak{F} : B \rightarrow B$  defined by (2.31) is a contraction, where  $B = \{y \in C([1 - r, e + h], \mathbb{R}) : y(1) = 0\}$ . For that, let  $y_1, y_2 \in B$ . Then, for  $t \in [1, e]$ , we obtain

$$\begin{aligned} |(\mathfrak{F}y_1)(t) - (\mathfrak{F}y_2)(t)| &\leq \int_1^e G(t, s) |f(s, y_1^s + u^s) - f(s, y_2^s + u^s)| \frac{ds}{s} \\ &\leq L \int_1^e G(t, s) \|y_1^s - y_2^s\|_{[-r, h]} \frac{ds}{s} \\ &\leq \frac{2L}{\Gamma(\alpha)} \|y_1 - y_2\|_{[-r, h]} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{1}{s} ds \\ &\leq \frac{2L}{\Gamma(\alpha+1)} \|y_1 - y_2\|_{[1-r, e+h]}. \end{aligned}$$



Consequently, we get

$$\|\mathfrak{F}y_1 - \mathfrak{F}y_2\|_{[1-r, e+h]} \leq \frac{2L}{\Gamma(\alpha + 1)} \|y_1 - y_2\|_{[1-r, e+h]},$$

which shows that  $\mathfrak{F}$  is a contraction by the given assumption, and hence  $\mathfrak{F}$  has a unique fixed point by means of the Banach's contraction mapping principle. This, in turn, implies that the problem (2.22)–(2.24) has a unique solution on the interval  $[1 - r, e + h]$ .  $\square$

### 2.4.2 Fractional Order Hadamard-Type Functional Differential Inclusions

In this subsection, we extend our study initiated for functional fractional differential equations in the last subsection to the multivalued case:

$$D^\alpha x(t) \in F(t, x^t), \quad 1 \leq t \leq e, \quad 1 < \alpha < 2, \quad (2.32)$$

$$x(t) = \chi(t), \quad 1 - r \leq t \leq 1, \quad (2.33)$$

$$x(t) = \psi(t), \quad e \leq t \leq e + h, \quad (2.34)$$

where  $F : [1, e] \times C([-r, h], \mathbb{R}) \rightarrow \mathcal{P}(\mathbb{R})$  is a multivalued map ( $\mathcal{P}(\mathbb{R})$  is the family of all nonempty subsets of  $\mathbb{R}$ ), while the rest of the quantities are the same as defined in the problem (2.22)–(2.24).

**Theorem 2.12** Assume that (2.10.2) and the following conditions hold:

(2.12.1)  $F : [1, e] \times C([-r, h], \mathbb{R}) \rightarrow \mathcal{P}_{c,cp}(\mathbb{R})$  is an  $L^1$ -Carathéodory multivalued map;

(2.12.2) there exist  $p \in C([1, e], \mathbb{R})$  and a continuous and nondecreasing function  $\Omega : [0, \infty) \rightarrow (0, \infty)$  such that

$$\|F(t, u)\| := \sup\{|v| : v \in F(t, u)\} \leq p(t)\Omega(\|u\|_{[-r, h]}),$$

for almost all  $t \in [1, e]$  and all  $u \in C([-r, h], \mathbb{R})$ .

Then the problem (2.32)–(2.34) has at least one solution on the interval  $[1 - r, e + h]$ .

*Proof* In relation to the problem (2.32)–(2.34), we introduce an operator  $\mathcal{N} : C([1 - r, e + h], \mathbb{R}) \rightarrow \mathcal{P}(C([1 - r, e + h], \mathbb{R}))$  as

$$\mathcal{N}(x) := \left\{ \begin{array}{l} h \in C([1-r, e+h], \mathbb{R}) : \\ h(t) = \begin{cases} \chi(t), & \text{if } t \in [1-r, 1], \\ \int_1^e G(t, s)v(s) \frac{ds}{s}, & \text{if } t \in [1, e], \\ \psi(t), & \text{if } t \in [e, e+h], \end{cases} \end{array} \right\}$$

where

$$v \in S_{F,y} = \{v \in L^1([1, e], \mathbb{R}) : v(t) \in F(t, y^t) \text{ for a.e. } t \in J\}.$$

Observe that the existence of a fixed point of the operator  $\mathcal{N}$  implies the existence of a solution to the problem (2.32)–(2.34).

As in the proof of Theorem 2.10, let  $B = \{y \in C([1-r, e+h], \mathbb{R}) : y(1) = 0\}$  and let  $\mathfrak{T} : B \rightarrow \mathcal{P}(B)$  be defined by

$$\mathfrak{T}(y) := \left\{ \begin{array}{l} h \in C([1-r, E+h], \mathbb{R}) : \\ h(t) = \begin{cases} 0, & \text{if } t \in [1-r, 1], \\ \int_1^e G(t, s)v(s) \frac{ds}{s}, & \text{if } t \in [1, e], \\ 0, & \text{if } t \in [e, e+h]. \end{cases} \end{array} \right\}$$

Now, we show that the operator  $\mathfrak{T}$  has a fixed point which is equivalent to proving that the operator  $\mathcal{N}$  has a fixed point. We do it in several steps.

**Claim 1:**  $\mathfrak{T}(y)$  is convex for each  $y \in C([1-r, e+h], \mathbb{R})$ .

This claim is obvious, since  $F$  has convex values.

**Claim 2:**  $\mathfrak{T}$  maps bounded sets into bounded sets in  $C([1-r, e+h], \mathbb{R})$ .

Let  $y \in U_k = \{y \in B : \|y\|_{[1-r, e+h]} \leq k\}$ . Then, for each  $h \in \mathfrak{T}(y)$ , there exists  $v \in S_{F,y}$  such that

$$h(t) = \int_1^e G(t, s)v(s) \frac{ds}{s}, \quad t \in [1, e],$$

and that

$$\begin{aligned} |h(t)| &\leq \left| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds - (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{p(s)\mathfrak{L}(\|y^s + u^s\|_{[-r, h]})}{s} ds \end{aligned}$$

$$\begin{aligned}
& + (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left( \log \frac{e}{s} \right)^{\alpha-1} \frac{p(s) \Omega(\|y^s + u^s\|_{[-r,h]})}{s} ds \\
& \leq \frac{2\|p\|_0 \Omega(k + \max\{\|x\|_{[1-r,1]}, \|x\|_{[e,e+h]}\})}{\Gamma(\alpha + 1)}.
\end{aligned}$$

Thus

$$\|h\|_{[1-r,e+h]} \leq \frac{2\|p\|_0 \Omega(k + \max\{\|x\|_{[1-r,1]}, \|x\|_{[e,e+h]}\})}{\Gamma(\alpha + 1)} := \hat{L}.$$

This shows that  $\mathfrak{T}$  maps bounded sets into bounded sets in  $B$ .

**Claim 3:**  $\mathfrak{T}$  maps bounded sets in  $C([1-r, e+h], \mathbb{R})$  into equicontinuous sets.

We consider  $B_k$  as in Claim 2 and let  $h \in \mathfrak{T}(y)$  for  $y \in B_k$ ,  $k > 0$ . Now let  $t_1, t_2 \in [1, e]$  with  $t_2 > t_1$ . Then, we have

$$\begin{aligned}
|h(t_2) - h(t_1)| & \leq \int_1^e |G(t_2, s) - G(t_1, s)| |f(s, y^s + u^s)| \frac{ds}{s} \\
& \leq \|p\|_0 \Omega(k + \max\{\|x\|_{[1-r,1]}, \|x\|_{[e,e+h]}\}) \int_1^e |G(t_2, s) - G(t_1, s)| \frac{ds}{s}.
\end{aligned}$$

Clearly the right-hand side of the last inequality tends to zero as  $t_1 \rightarrow t_2$ , independently of  $y \in B_k$ . In view of Claims 2, 3 and the Arzelá-Ascoli Theorem, we conclude that  $\mathfrak{T} : B \rightarrow \mathcal{P}(B)$  is completely continuous.

In our next step, we show that  $\mathfrak{T}$  is upper semicontinuous. We are done if we show that the operator  $\mathfrak{T}$  has a closed graph, since  $\mathfrak{T}$  is already shown to be completely continuous.

**Claim 4:**  $\mathfrak{T}$  has a closed graph.

Let  $x_n \rightarrow x_*$ ,  $h_n \in \mathfrak{T}(x_n)$  and  $h_n \rightarrow h_*$ . Then, we need to show that  $h_* \in \mathfrak{T}(x_*)$ . Associated with  $h_n \in \mathfrak{T}(x_n)$ , there exists  $v_n \in S_{F, x_n}$  such that for each  $t \in [1, e]$ ,

$$h_n(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} \frac{v_n(s)}{s} ds - (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left( \log \frac{e}{s} \right)^{\alpha-1} \frac{v_n(s)}{s} ds.$$

Thus it suffices to show that there exists  $v_* \in S_{F, x_*}$  such that for each  $t \in [1, e]$ ,

$$h_*(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} \frac{v_*(s)}{s} ds - (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left( \log \frac{e}{s} \right)^{\alpha-1} \frac{v_*(s)}{s} ds.$$

Let us consider the linear operator  $\Theta : L^1([1, e], \mathbb{R}) \rightarrow C([1, e], \mathbb{R})$  given by

$$\begin{aligned} f \mapsto \Theta(v)(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds \\ &\quad - (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds. \end{aligned}$$

Clearly

$$\begin{aligned} \|h_n(t) - h_*(t)\| &= \left\| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{(v_n(s) - v_*(s))}{s} ds \right. \\ &\quad \left. - (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{(v_n(s) - v_*(s))}{s} ds \right\| \rightarrow 0, \end{aligned}$$

as  $n \rightarrow \infty$ . Thus, it follows by Lemma 1.2 that  $\Theta \circ S_{F,x}$  is a closed graph operator. Further, we have  $h_n(t) \in \Theta(S_{F,x_n})$ . Since  $x_n \rightarrow x_*$ , we get

$$h_*(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v_*(s)}{s} ds - (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{v_*(s)}{s} ds,$$

for some  $v_* \in S_{F,x_*}$ .

**Claim 5:** We will show that there exists an open set  $U \subset B$  with  $y \neq \lambda \mathfrak{T}y$  for  $0 < \lambda < 1$  and  $y \in \partial U$ .

Let  $y \in B$  be such that  $y \in \lambda \mathfrak{T}(y)$  for some  $0 < \lambda < 1$ . Then there exists  $v \in S_{F,y}$  such that

$$y(t) = \lambda \int_1^e G(t, s) v(s) \frac{ds}{s}, \quad t \in [1, e].$$

By the given assumptions, for each  $t \in [1, e]$ , we have

$$\begin{aligned} |y(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{p(s) \Omega(\|y^s + u^s\|_{[-r,h]})}{s} ds \\ &\quad + (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{p(s) \Omega(\|y^s + u^s\|_{[-r,h]})}{s} ds \\ &\leq \frac{2\|p\|_0 \Omega(\|y\|_{[-r,h]} + \max\{\|x\|_{[1-r,1]}, \|x\|_{[e,e+h]}\})}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{1}{s} ds \\ &\leq \frac{2\|p\|_0}{\Gamma(\alpha+1)} \Omega(\|y\|_{[1-r,e+h]} + \max\{\|x\|_{[1-r,1]}, \|x\|_{[e,e+h]}\}). \end{aligned}$$

Then

$$\frac{\|y\|_{[1-r, e+h]}}{\frac{2\|p\|_0}{\Gamma(\alpha+1)}\Omega(\|y\|_{[1-r, e+h]} + \max\{\|x\|_{[1-r, 1]}, \|x\|_{[e, e+h]}\})} \leq 1.$$

By (2.12.3), there exists  $K_0$  such that  $\|y\|_{[1-r, e+h]} \neq K_0$ . Set

$$U = \{y \in C([1-r, e+h], \mathbb{R}) : \|y\|_{[1-r, e+h]} < K_0 + 1\}.$$

From the choice of  $U$  there is no  $y \in \partial U$  such that  $y \in \lambda \mathfrak{T}(y)$  for  $\lambda \in (0, 1)$ . As a consequence of the Leray-Schauder Alternative for Kakutani maps (Theorem 1.15), we deduce that  $\mathfrak{T}$  has a fixed point. Thus the problem (2.32)–(2.34) has at least one solution.  $\square$

Finally, we present an existence result for the problem (2.32)–(2.34) with nonconvex valued right hand side.

**Theorem 2.13** *Suppose that:*

(2.13.1)  $F : [1, e] \times C([-r, h], \mathbb{R}) \longrightarrow \mathcal{P}_{cp}(\mathbb{R})$  has the property that  $F(\cdot, y) : [1, e] \longmapsto \mathcal{P}_{cp}(\mathbb{R})$  is measurable for each  $y \in C([-r, h], \mathbb{R})$ ;

(2.13.2) there exists  $\ell \in C(J, \mathbb{R})$  such that

$$H_d(F(t, u), F(t, \bar{u})) \leq \ell(t) \|u - \bar{u}\|_{[-r, h]} \quad \text{for every } u, \bar{u} \in C([-r, h], \mathbb{R}),$$

and

$$d(0, F(0, u)) \leq \ell(t), \quad \text{for a.e. } t \in [1, e].$$

If

$$\frac{2}{\Gamma(\alpha+1)} \|\ell\|_0 < 1 \quad (\|\ell\|_0 = \sup_{t \in [1, e]} |\ell(t)|),$$

then there exists at least one solution for the problem (2.32)–(2.34).

*Proof* Transform the problem (2.32)–(2.34) into a fixed point problem by means of the multivalued operator  $\mathfrak{T} : B \rightarrow \mathcal{P}(B)$  introduced in Theorem 2.12. We shall show that  $\mathfrak{T}$  satisfies the assumptions of Theorem 1.18. The proof will be given in two steps.

**Step 1:**  $\mathfrak{T}(y) \in \mathcal{P}_{cl}(B)$  for each  $y \in B$ .

Indeed, let  $(y_n)_{n \geq 0} \in \mathfrak{T}(y)$  such that  $y_n \longrightarrow \tilde{y}$  in  $B$ . Then  $\tilde{y} \in B$  and there exists  $g_n \in S_{F, y}$  such that for each  $t \in [1, e]$ ,

$$y_n(t) = \int_1^e G(t, s) g_n(s) \frac{ds}{s}.$$

Using (2.13.1) together with the fact that  $F$  has compact values, we may pass onto a subsequence to get that  $g_n$  converges weakly to  $g$  in  $L^1([1, e], \mathbb{R})$ . Then,  $g \in S_{F,x}$  and for each  $t \in [1, e]$ , we have

$$y_n(t) \longrightarrow \tilde{y}(t) = \int_1^e G(t, s)g(s) \frac{ds}{s}.$$

So  $\tilde{y} \in \mathfrak{T}(y)$ .

**Step 2:** *There exists  $\gamma < 1$  such that*

$$H_d(\mathfrak{T}(y), \mathfrak{T}(\bar{y})) \leq \gamma \|y - \bar{y}\|_{[1-r, e+h]} \text{ for each } y, \bar{y} \in B.$$

Let  $y, \bar{y} \in B$  and  $h \in \mathfrak{T}(y)$ . Then there exists  $g(t) \in F(t, y^t + u^t)$  such that

$$h(t) = \int_1^e G(t, s)g(s) \frac{ds}{s},$$

for each  $t \in J$ . From (2.13.2), it follows that

$$H_d(F(t, y^t + u^t), F(t, \bar{y}^t + u^t)) \leq \ell(t) \|y - \bar{y}\|_{[-r, h]}, \quad t \in [1, e].$$

Hence there is  $w \in F(t, \bar{y}^t + u^t)$  such that

$$|g(t) - w| \leq \ell(t) \|y - \bar{y}\|_{[-r, h]}, \quad t \in [1, e].$$

Consider  $U : [1, e] \rightarrow \mathcal{P}(E)$ , given by

$$U(t) = \{w \in E : |g(t) - w| \leq \ell(t) \|y - \bar{y}\|_{[-r, h]}\}.$$

Since the multivalued operator  $V(t) = U(t) \cap F(t, \bar{y}^t + u^t)$  is measurable (see Proposition III.4 in [57]), there exists a function  $\bar{g}(t)$ , which is a measurable selection for  $V$ . So,  $\bar{g}(t) \in F(t, \bar{y}^t + u^t)$  and

$$|g(t) - \bar{g}(t)| \leq \ell(t) \|y - \bar{y}\|_{[-r, h]}, \text{ for each } t \in [1, e].$$

Let us define for each  $t \in [1, e]$ ,

$$\bar{h}(t) = \int_1^e G(t, s)\bar{g}(s) \frac{ds}{s}.$$

Then we have

$$\begin{aligned}
 |h(t) - \bar{h}(t)| &\leq \int_1^e |G(t, s)| |g(s) - \bar{g}(s)| \frac{ds}{s} \\
 &\leq \int_1^e |G(t, s)| \ell(s) \|y - \bar{y}\|_{[-r, h]} \frac{ds}{s} \\
 &\leq \frac{2\|\ell\|_0 \|y - \bar{y}\|_{[-r, h]}}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{1}{s} ds \\
 &\leq \frac{2}{\Gamma(\alpha + 1)} \|\ell\|_0 \|y - \bar{y}\|_{[-r, h]}.
 \end{aligned}$$

Thus

$$\|h - \bar{h}\|_{[1-r, e+h]} \leq \frac{2}{\Gamma(\alpha + 1)} \|\ell\|_0 \|y - \bar{y}\|_{[1-r, e+h]}.$$

Analogously, interchanging the roles of  $y$  and  $\bar{y}$ , it follows that

$$H_d(\mathfrak{T}(y), \mathfrak{T}(\bar{y})) \leq \frac{2}{\Gamma(\alpha + 1)} \|\ell\|_0 \|y - \bar{y}\|_{[1-r, e+h]}.$$

So,  $\mathfrak{T}$  is a contraction and hence, by Theorem 1.18,  $\mathfrak{T}$  has a fixed point  $y$ , which is a solution to the problem (2.32)–(2.34).  $\square$

## 2.5 Notes and Remarks

We have established several existence results for initial and boundary value problems of Hadamard type fractional order functional and neutral functional differential equations involving both retarded and advanced arguments. Also, we have discussed the multivalued analog of Hadamard type fractional functional and neutral functional equations. Our results rely on the standard tools of the fixed point theory for single and multivalued maps. Our results are not only new in the given setting but also correspond to some new interesting situations for an appropriate choice of  $r$  and  $h$ . For example, the results for ordinary Hadamard-type fractional differential equations/inclusions follow by taking  $r = h = 0$ . Our results reduce to the retarded and advanced argument cases for  $r > 0; h = 0$  and  $r = 0; h > 0$  respectively. The mixed (both retarded and advanced) case follows by choosing  $r > 0$  and  $h > 0$ . The results of this chapter are adapted from the papers [17, 19] and [13].

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