Random Perturbations of a Three-Machine Power System Network

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Abstract This paper develops an asymptotic method based on averaging and large deviations to study the transient stability of a noisy three-machine power system network. We study the dynamics of these nonlinear oscillators (swing equations) as random perturbations of two-dimensional periodically driven Hamiltonian systems. The phase space for periodically driven nonlinear oscillators consists of many resonance zones. It is well known that, as the strengths of periodic excitation and damping go to zero, the measure of the set of initial conditions which lead to *capture in a resonance zone* goes to zero. In this paper we study the effect of weak noise on the escape from a resonance zone and obtain the large-deviation rate function for the escape. The primary goal is to show that the behavior of oscillators in the resonance zone can be adequately described by the (slow) evolution of the Hamiltonian, for which simple analytical results can be obtained, and then apply these results to study the transient stability margin of power system with stochastic loads. The classical swing equations of a power system of three interconnected generators with non-zero damping and small noise is considered as a nontrivial example to derive the "exit time" analytically. This work may play an important role in designing and upgrading existing electrical power system networks.

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1 Introduction

The problem, a three-machine power system network, considered in this paper brings together three interesting topics in dynamical systems. Namely, *resonances* in two-frequency nonlinear systems, where at some moment of time due to nonlinear effects, a linear dependence of frequencies with integer coefficients occurs, giving rise to resonance surfaces; *domains of attraction*, which are formed in the presence of small dissipation when most of the resonant periodic orbits disappear except for a few stable limit cycles with their distinct set of initial conditions that are captured into resonance; and finally *large deviations*, which provide the asymptotic behavior of rare event probabilities, transition pathways, and transition rates in stable systems with small noise. The subtleties of these interactions between noise and nonlinearities are explored in a canonical way by combining the ideas from *dynamical systems*, *homogenization methods* and *large deviations* to develop a general collection of new mathematical techniques. Depending on the time-scale of the rapidly-oscillating periodic dynamics and the strength of the noise, there are two different limits, namely homogenization and large deviations. There is a tug a war between these two scales and we make use of the asymptotic methods that combine homogenization with large deviation [1] to discover a common geometric structure in the phase space and to determine the effects of noisy perturbations on the passage of trajectories through the resonance zones based on the energy barrier heights.

In principal, an infinite number of resonance domains exist, but for two-frequency systems resonance surfaces do not cross each other and the influence of each resonance can be studied separately using a slow angle. At the resonance, a trajectory of the fast systems or the non-resonant angles fills the torus of the lower dimension and these non-resonant angles can be averaged out. In the presence of small dissipation, centers become (stable or unstable) foci, while families of periodic orbits disappear, possibly giving rise to (stable or unstable) limit cycles. Homoclinic and heteroclinic orbits also in general disappear. In light of the above discussion, the intention of this paper is to study the effect of weak noise on the escape from a resonance zone. When the noise is very weak and so large deviations from the corresponding deterministic system occur with very low probability. The phase space for the corresponding deterministic system consists of many resonance zones in which some trajectories of the deterministic system can get "trapped". The rate at which noise facilitates the "escape" from resonance is the subject of this paper. Our goal is to understand a general collection of mathematical techniques which can be applied to and understood through one specific physically-motivated problem. In this paper we deal with swing equations of a 3-machine system to derive the "critical clearing time" analytically.

Problems related to large deviations for stochastic processes have attracted the attention of many physicists and engineers in recent years. For example, in many devices, failure occurs either the first time the response oversteps a particular threshold, as when a vibrating relay contact first touches the frame and shorts out, or due to an accumulation of many small damages inflicted in the duration of the device, as occurs in wear and fatigue. The exit problem is an example of such problems, where most probable transition pathways and the mean transition time τ are useful in determining the direction of failure propagation after the onset of instability.

The transition rate from one stable regime to another along a certain path provides valuable information regarding the time available for mitigating the cascade of failures. The large deviation theory provides the methods to find transition pathways and transition rates in stable systems with small noise. The most probable transition pathways are governed by a first order Hamilton-Jacobi type of equation. In the multidimensional non-gradient vector field case (even in $\mathbb{R}^2$) explicit solutions cannot be obtained in general for the well-known Pontryagin-Witt equation or the HJB equation. However, taking advantage of the fact the Hamiltonian of the unperturbed system evolves slowly (under small perturbations), the Hamiltonian structure is made use of to identify a reduced one-dimensional equation for the evolution of the Hamiltonian $\mathscr{H}$, by averaging the fast dynamics (stochastic averaging). Hence, the escape from the domain of attraction of stable equilibrium points and limit cycles in phase-space can be studied analytically. We present a method based upon an approximation of the Hamiltonian (energy envelope) of the oscillator response as a one-dimensional Markov process, governed either by an appropriate diffusion equation or a one dimensional HJB equation depending on the strength of the noise. For the homogenized nonlinear system, the transition (hopping) rate calculation is based on the energy barrier heights (the maximum load) between local attractors.

The content of this paper is organized as follows. Swing equations of multi-machine system are discussed in Sect. 2. Making use of several assumptions 3-machine equations are modeled as a one degree of freedom periodically driven nonlinear oscillator. The reduction technique detailed in Sect. 2, uses the Hamiltonian structure of the unperturbed system. In Sect. 3 we zoom in to a resonance zone and make a change of variables in order to derive simpler equations that describe the dynamics in the resonance zone. In Sect. 3.1 we consider the deterministic dynamics in the resonance zone and state the well known problem of capture into resonance and identify a variable $\mathscr{H}$ whose value can be used to indicate capture. In Sect. 4 we study the rate of escape from a resonance zone. We achieve this by threading together ideas from averaging and large deviations to derive a large deviation principle for $\mathscr{H}$. It will be shown that the trajectories of the oscillator trickle down close to the bottom of the potential wells. The stochastic dynamics at the bottom of the potential well is not discussed in this paper due to lack of space.

2 Swing Equations with Non-zero Damping and Small Noise

Transient stability in power systems is concerned with the ability of power systems to maintain synchronism in coupled swing dynamics when subject to a severe disturbance. Due to network complexity, power system stability can be divided into smaller areas that include generator rotor angles, frequency and voltage stabilities. We analyze a system of n classical swing equations for a simple power system (related to synchronous generator rotor swing angle) [2]:

$$\dot{\delta}_k = \omega_k - \omega_R,$$
$$\dot{\omega}_k = \left(\frac{\omega_R}{2H_k}\right)\left[-\beta_k(\omega_k - \omega_R) + T_{mk} - G_{kk}E_k^2 - \sum_{i\neq k}^{n} E_k E_i Y_{ki}\cos(\theta_{ki} - \delta_k + \delta_i)\right],$$

where the rotor angle of machine k is δ_k and ω_k denotes the angular velocity of the rotor k. The parameters are constants, ω_R represents system reference frequency, H_k, inertial moment of machine k, β_k, damping coefficient of machine k, T_{mk}, mechanical torque driving machine k, E_k, terminal voltage of machine k, G_{kk}, due to the real power load at machine k. The magnitudes and angles $Y_{ki} = Y_{ik}$ and $\theta_{ki} = \theta_{ik}$ determine the transfer admittance between machines k and i:

$$G_{ki} + jB_{ij} = Y_{ki}\exp^{j\theta_{ki}}.$$

If we assume that resistive loads are located only at the generator buses (i.e. are included in the conductances G_{kk}), then the transfer admittances are purely imaginary and $\theta_{kj} = \frac{\pi}{2}$ (so $G_{kj} = 0$ for $i \neq k$ and $Y_{ki} = B_{ki}$).

Assume that mechanical torque produced equals power absorbed by the loads, so

$$\sum_{k=1}^{n} T_{mk} - G_{kk}E_k^2 = 0.$$

Denote by $M_k := \frac{2H_k}{\omega_R}$, $\bar{P}_k := T_{mk} - G_{kk}E_k^2$, $C_{ki} := E_k E_i Y_{ki}$, and introduce small noise in the power term,

$$P_k := \bar{P}_k + \varepsilon^{\kappa}\tilde{\sigma}_k\eta_k(t), \qquad \text{where} \qquad \bar{P}_K = T_{mk} - G_{kk}E_k^2, \qquad \sum_{k=1}^{n}\bar{P}_k = 0,$$

$0 < \varepsilon \ll 1, \kappa > 0$, and η_ks are modeled as white noise processes. We assume symmetry of transfer admittance between machines, so $C_{kj} = C_{jk}$ for $j \neq k, j, k = 1, \ldots, n$. Finally, we also assume proportional damping, i.e. $\beta_k = \beta M_k$, where damping ratio β is equal for all $k = 1, \ldots, n$. Then, the equations of motion are [2]:

$$\dot{\delta}_k = \omega_k - \omega_R, \tag{1a}$$
$$\dot{\omega}_k = \frac{1}{M_k}\left[-\beta M_k(\omega_k - \omega_R) + \bar{P}_k - \sum_{i\neq k}^{n} C_{ki}\sin(\delta_k - \delta_i) + \varepsilon^{\kappa}\tilde{\sigma}_k\eta_k\right]. \tag{1b}$$

2.1 Coordinate Change

We first describe the dynamics in collective variables that are averages of individual variables, which are well known in power grid stability analysis as the COA (Center-of-Angle) or COI (Center-of-Inertia) variables. Consider new coordinates $(\tilde{\delta}_k, \tilde{\omega}_k)$ that are perturbations of (δ_k, ω_k) from the centers-of-angle and -inertia, $\left(\frac{1}{M}\sum_{k=1}^n M_k\delta_k, \frac{1}{M}\sum_{k=1}^n M_k\omega_k\right)$, where $M := \sum_{k=1}^n M_k$. $\left(\frac{1}{M}\sum_{k=1}^n M_k\delta_k, \frac{1}{M}\sum_{k=1}^n M_k\omega_k\right)$ can be obtained by integrating (1). The perturbation coordinates are

$$\tilde{\omega}_k := \omega_k - \frac{1}{M}e^{-\beta t}\sum_{k=1}^n M_k\omega_k(0) - \frac{\varepsilon^\kappa}{M}\sum_{k=1}^n \tilde{\sigma}_k \int_0^t e^{-\beta(t-s)}\eta_k(s)ds, \tag{2a}$$

$$\begin{aligned}\tilde{\delta}_k := \delta_k - \frac{1}{M}\sum_{k=1}^n M_k\delta_k(0) - \frac{1-e^{-\beta t}}{\beta M}\sum_{k=1}^n M_k\omega_k(0)\\ + \omega_R t - \frac{\varepsilon^\kappa}{M}\sum_{k=1}^n \tilde{\sigma}_k \int_0^t \int_0^s e^{-\beta(s-r)}\eta_k(r)drds.\end{aligned} \tag{2b}$$

The corresponding equations of motion are

$$\dot{\tilde{\delta}}_k = \tilde{\omega}_k, \qquad \dot{\tilde{\omega}}_k = -\beta\tilde{\omega}_k + \frac{\bar{P}_k}{M_k} - \sum_{i\neq k}^n \frac{C_{kj}}{M_k}\sin(\tilde{\delta}_k - \tilde{\delta}_i) + \varepsilon^\kappa \tilde{\sigma}_k \eta_t, \tag{3}$$

where $\eta_t = [\eta_1(t), \eta_2(t), \ldots, \eta_n(t)]^T$, and

$$\sigma_k \in \mathbb{R}^{1\times n}, \quad (\sigma_k)_i = \begin{cases} \tilde{\sigma}_i\left(\frac{1}{M_i} - \frac{1}{M}\right) & \text{if } i = k, \\ -\frac{\tilde{\sigma}_i}{M} & \text{else} \end{cases}.$$

Using (2), we can check that the sum of the angles and angular momenta perturbations are

$$\sum_{k=1}^n M_k\tilde{\omega}_k = \sum_{k=1}^n M_k\tilde{\delta}_k = 0. \tag{4}$$

2.2 3-Machine Setting

Consider the $n = 3$ case. Using the fact that the sum of angles perturbation is zero, (4), we can eliminate one of the $k = 1, 2, 3$ degrees of freedom. Arbitrarily, we eliminate $\tilde{\delta}_2$. We end up with a 2-degree-of-freedom system:

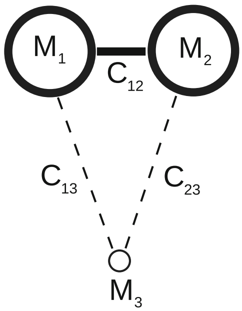

Fig. 1 Heuristic representation of 3-machine scaling

$$
\begin{aligned}
\dot{\tilde{\delta}}_1 &= \tilde{\omega}_1,\\
\dot{\tilde{\omega}}_1 &= -\beta\tilde{\omega}_1 + \frac{\bar{P}_1}{M_1} - \frac{C_{12}}{M_1}\sin\left(\left[1+\frac{M_1}{M_2}\right]\tilde{\delta}_1 + \frac{M_3}{M_2}\tilde{\delta}_3\right) - \frac{C_{13}}{M_1}\sin(\tilde{\delta}_1 - \tilde{\delta}_3) + \varepsilon^{\kappa}\sigma_1\eta_t,\\
\dot{\tilde{\delta}}_3 &= \tilde{\omega}_3,\\
\dot{\tilde{\omega}}_3 &= -\beta\tilde{\omega}_3 + \frac{\bar{P}_3}{M_3} - \frac{C_{13}}{M_3}\sin(\tilde{\delta}_3 - \tilde{\delta}_1) - \frac{C_{23}}{M_3}\sin\left(\frac{M_1}{M_2}\tilde{\delta}_1 + \left[1+\frac{M_3}{M_2}\right]\tilde{\delta}_3\right) + \varepsilon^{\kappa}\sigma_3\eta_t.
\end{aligned}
$$

Consider the $n = 3$ case. As in [3], we assume that (see Fig. 1)

- inertia of machines 1 and 2 are larger than that of machine 3:

$$
M_1 = \frac{\bar{M}_1}{\varepsilon},\ M_2 = \frac{\bar{M}_2}{\varepsilon}, \quad \bar{M}_1, \bar{M}_2, M_3 \sim O(1),
$$

- coupling of machine 1 with 2 is larger than the coupling of machines 1 and 2 with 3:

$$
C_{12} = \frac{\bar{C}_{12}}{\varepsilon}, \quad \bar{C}_{12}, C_{13}, C_{23} \sim O(1), \quad \text{and}
$$

- loads of machines 1 and 2 are larger than that of machine 3

$$
P_1 = \frac{\bar{P}_1}{\varepsilon},\ P_2 = \frac{\bar{P}_2}{\varepsilon}, \quad \bar{P}_1, \bar{P}_2, P_3 \sim O(1).
$$

Using the above scaling and assuming small damping, $\beta = \varepsilon\beta$, we have

$$
\begin{aligned}
\dot{\tilde{\delta}}_1 &= \tilde{\omega}_1,\\
\dot{\tilde{\omega}}_1 &= -\varepsilon\beta\tilde{\omega}_1 + \alpha_1 - c_{12}\sin\left([1+\mu_1]\tilde{\delta}_1 + \varepsilon\mu_3\tilde{\delta}_3\right) - \varepsilon c_{13}\sin\left(\tilde{\delta}_1 - \tilde{\delta}_3\right) + \varepsilon^{\kappa}\sigma_1\eta_t,
\end{aligned}
\tag{5}
$$

$$\dot{\tilde{\delta}}_3 = \tilde{\omega}_3,$$
$$\dot{\tilde{\omega}}_3 = -\varepsilon\beta\tilde{\omega}_3 + \alpha_3 - c_{31}\sin\left(\tilde{\delta}_3 - \tilde{\delta}_1\right) - c_{32}\sin\left(\mu_1\tilde{\delta}_1 + [1+\varepsilon\mu_3]\tilde{\delta}_3\right) + \varepsilon^\kappa\sigma_3\eta_t, \tag{6}$$

along with $\tilde{\delta}_2 = -\mu_1\tilde{\delta}_2 - \varepsilon\mu_3\tilde{\delta}_3$, where

$$\mu_1 := \frac{\bar{M}_1}{\bar{M}_2}, \quad \mu_3 := \frac{M_3}{\bar{M}_2}, \quad \alpha_1 := \frac{\bar{P}_1}{M_1}, \quad \alpha_3 := \frac{\bar{P}_3}{M_3},$$
$$c_{12} := \frac{\bar{C}_{12}}{M_1}, \quad c_{13} := \frac{C_{13}}{M_1}, \quad c_{31} := \frac{C_{13}}{M_3}, \quad c_{32} := \frac{C_{23}}{M_3}.$$

If $\varepsilon \equiv 0$, then $(\tilde{\delta}_1, \tilde{\omega}_1)$ given by (5) is independent of $(\tilde{\delta}_3, \tilde{\omega}_3)$, with equilibrium $(\tilde{\delta}_1^*, \tilde{\omega}_1^*)$ given by

$$(\tilde{\delta}_1^*, \tilde{\omega}_1^*) = \left(\frac{1}{[1+\mu_1]}\sin^{-1}\left(\frac{\alpha_1}{c_{12}}\right), 0\right).$$

We will assume that $\tilde{\delta}_1$ is a small perturbation from $\tilde{\delta}_1^*$, i.e.

$$\tilde{\delta}_1(t) = \tilde{\delta}_1^* + \varepsilon\hat{\delta}_1(t),$$

with initial conditions $\tilde{\delta}_1(0) = \tilde{\delta}_1^* + \varepsilon\tilde{\delta}_{10}$, and $\dot{\tilde{\delta}}_1(0) = \varepsilon\dot{\tilde{\delta}}_{10}$. The behavior of $\hat{\delta}_1$ can be studied by considering small perturbations of $\tilde{\delta}_1$ in (5) about zero, with initial conditions $\tilde{\delta}_{10}$ and $\dot{\tilde{\delta}}_{10}$ (in other words, shift the initial condition for $\tilde{\delta}_1$ by $-\tilde{\delta}_1^*$ and study small perturbations about zero: For $\varepsilon \ll 1$, we replace $\tilde{\delta}_1(t)$ with $\varepsilon\hat{\delta}_1(t)$ in (5), with initial conditions $\tilde{\delta}_1(0) = \varepsilon\hat{\delta}_1(0) = \varepsilon\tilde{\delta}_{10}$ and $\dot{\tilde{\delta}}_1(0) = \varepsilon\dot{\hat{\delta}}_1(0) = \varepsilon\dot{\tilde{\delta}}_{10}$).

Taylor expanding the $\tilde{\delta}_1$ terms in (5) about zero and keeping only the leading order terms, we have the following second order ODE:

$$\varepsilon\ddot{\hat{\delta}}_1 + \varepsilon^2\beta\dot{\hat{\delta}}_1 + \varepsilon c_{12}[1+\mu_1]\hat{\delta}_1 - \frac{1}{2}\varepsilon^3 c_{12}[1+\mu_1]\mu_3^2\tilde{\delta}_3^2\hat{\delta}_1 + \cdots + \varepsilon^2 c_{13}\cos(\tilde{\delta}_3)\hat{\delta}_1 = 0.$$

We asymptotically expand $\hat{\delta}_1$ as

$$\hat{\delta}_1(t) = u(t) + \varepsilon\psi(t) + \varepsilon^2 R(t) + \mathscr{O}(\varepsilon^3),$$

with $(u(0), \dot{u}(0)) = (\tilde{\delta}_{10}, \dot{\tilde{\delta}}_{10})$, $(\psi(0), \dot{\psi}(0)) = (R(0), \dot{R}(0)) = (0,0)$ and substitute this expansion into the preceding ODE. Collecting terms by orders of ε, we have a set of ODEs of orders $\varepsilon, \varepsilon^2, \varepsilon^3, \ldots$ At order ε, u is an undamped, unforced oscillator. Assuming $c_{12}, \mu_1 > 0$, the squared natural frequency $c_{12}[1+\mu_1]$ is > 0, so

$$u(t) = \sin(\nu t + \varphi), \quad \text{where} \quad \nu := \sqrt{c_{12}[1+\mu_1]}, \quad \varphi := \tan^{-1}\left(\frac{\tilde{\delta}_{10}\nu}{\dot{\tilde{\delta}}_{10}}\right). \tag{7}$$

Therefore,

$$\tilde{\delta}_1(t) = \tilde{\delta}_1^* + \varepsilon \sin(\nu t + \varphi) + \varepsilon^2 \psi(t) + \varepsilon^3 R(t) + \mathscr{O}(\varepsilon^4), \tag{8}$$

where ψ and R satisfy the higher order equations. Substituting (8) into (6), we have a single-degree-of-freedom system in $(\tilde{\delta}_3, \tilde{\omega}_3)$ along with the higher order equations for ψ and R. Discarding the higher order deterministic terms and rewriting $(\tilde{\delta}_3, \tilde{\omega}_3)$ as $(\delta_t^\varepsilon, \omega_t^\varepsilon)$, (6) becomes

$$\begin{aligned}
d\delta_t^\varepsilon &= \omega_t^\varepsilon dt \\
d\omega_t^\varepsilon &= \left[-c \sin(\delta_t^\varepsilon - r) + \alpha_3\right] dt - \varepsilon \left[\beta \omega_t^\varepsilon + c_{32}\, \mu_3 \delta_t^\varepsilon \cos(\delta_t^\varepsilon + \mu_1 \delta_1^*)\right] dt \\
&\quad + \varepsilon \left[c_{13} \sin(\nu t + \varphi) \cos(\delta_t^\varepsilon - \delta_1^*) - c_{32} \mu_1 \sin(\nu t + \varphi) \cos(\delta_t^\varepsilon + \mu_1 \delta_1^*)\right] dt \\
&\quad + \mathscr{O}(\varepsilon^2) dt + \varepsilon^\kappa \sigma dW_t,
\end{aligned}$$

where W is a Wiener process, and c and r are such that

$$c \cos r = c_{13} \cos \tilde{\delta}_1^* + c_{32} \cos(\mu_1 \tilde{\delta}_1^*), \quad c \sin r = c_{13} \sin \tilde{\delta}_1^* - c_{32} \sin(\mu_1 \tilde{\delta}_1^*),$$

The Hamiltonian associated with the unperturbed system is

$$H(\delta, \omega) := \frac{1}{2}\omega^2 - \alpha_3 \delta - c \cos(\delta - r) = \frac{1}{2}\omega^2 + U(\delta). \tag{9}$$

3 Dynamics Close to a Resonance Zone: Capture into Resonance

Let (I, φ) be action angle variables and assume

$$I = I(\delta, \omega), \qquad \varphi = \varphi(\delta, \omega),$$

$$\delta = \delta(I, \varphi), \qquad \omega = \omega(I, \varphi)$$

can be written. The system (9) with $\varepsilon = 0$ can be written as

$$\dot{I} = 0, \qquad \dot{\varphi} = \Omega(I). \tag{10}$$

Suppose we want to study the dynamics of the system (9) close to m:n resonance. We then consider dynamics in the region where I is close to the resonant value I_r defined by

$$m\Omega(I_r) = n\nu.$$

Here r is short for resonance m:n. For notational convenience we use $\Omega_r = \Omega(I_r) \neq 0$ and $\Omega'_r = \frac{\partial \Omega}{\partial I}\big|_{I=I_r}$ etc. Without loss of generality we have taken $\alpha_3 = \varepsilon\alpha$ and $c = 1$. We also rename a number of variables $\delta = \delta^{old} - r, r + \mu_1\delta_1^* = \tau_1, r - \delta_1^* = \tau_2, c_{32} = c_2, c_{13} = c_1$ and define

$$
\begin{aligned}
g_2(\delta, \omega, \theta) &\stackrel{\text{def}}{=} -\left[\beta\omega - \alpha + c_2\,\mu_3(\delta + r)\cos(\delta + \tau_1)\right] \\
&\quad + \left[c_1\cos(\delta + \tau_2) - c_2\mu_1\cos(\delta + \tau_1)\right]\sin(\nu t + \varphi),
\end{aligned}
$$

$$
\mathfrak{F}(I, \varphi, \theta) \stackrel{\text{def}}{=} \frac{\partial I(\delta, \omega)}{\partial \omega} g_2(\delta, \omega, \theta)\bigg|_{\delta(I,\varphi),\omega(I,\varphi)}, \mathfrak{G}(I, \varphi, \theta) \stackrel{\text{def}}{=} \frac{\partial \varphi(\delta, \omega)}{\partial \omega} g_2(\delta, \omega, \theta)\bigg|_{\delta(I,\varphi),\omega(I,\varphi)}.
$$

Let $I_t^\varepsilon = I(\delta_t^\varepsilon, \omega_t^\varepsilon)$, $\varphi_t^\varepsilon = \varphi(\delta_t^\varepsilon, \omega_t^\varepsilon)$ and define the slow angle and resonant frequency

$$
\psi_t^\varepsilon := \varphi_t^\varepsilon - \frac{n}{m}\theta_t, \quad \Omega_r := \frac{n}{m}\nu.
$$

where θ_t evolves according to $d\theta_t = \nu dt$. Then, using Ito formula we get

$$
\begin{cases}
dI_t^\varepsilon = \varepsilon\mathfrak{F}(I_t^\varepsilon, \psi_t^\varepsilon + \frac{n}{m}\theta_t, \theta_t)dt + \varepsilon^\kappa\sigma\frac{\partial I}{\partial q_2}\Big|_{(I_t^\varepsilon, \psi_t^\varepsilon + \frac{n}{m}\theta_t)} dW_t + \frac{1}{2}\varepsilon^{2\kappa}\sigma^2\frac{\partial^2 I}{\partial^2 q_2}\Big|_{(I_t^\varepsilon, \psi_t^\varepsilon + \frac{n}{m}\theta_t)} dt, \\
d\psi_t^\varepsilon = (\Omega(I_t^\varepsilon) - \Omega_r)dt + \varepsilon\mathfrak{G}(I_t^\varepsilon, \psi_t^\varepsilon + \frac{n}{m}\theta_t, \theta_t)dt \\
\qquad\qquad + \varepsilon^\kappa\sigma\frac{\partial \psi}{\partial q_2}\Big|_{(I_t^\varepsilon, \psi_t^\varepsilon + \frac{n}{m}\theta_t)} dW_t + \frac{1}{2}\varepsilon^{2\kappa}\sigma^2\frac{\partial^2 \psi}{\partial^2 q_2}\Big|_{(I_t^\varepsilon, \psi_t^\varepsilon + \frac{n}{m}\theta_t)} dt, \\
d\theta_t = \nu dt
\end{cases}
\tag{11}
$$

Since we are interested in the dynamics close to the resonance $I = I_r$ and (I, ψ) are slow variables, we make a change of variables in order to derive simpler equations that describe the dynamics in the resonance zone. Substituting the following standard [4] space and time scaling

$$
h_t^\varepsilon \stackrel{\text{def}}{=} \frac{1}{\sqrt{\varepsilon}}(I^\varepsilon_{t/\sqrt{\varepsilon}} - I_r), \qquad \hat{\psi}_t^\varepsilon \stackrel{\text{def}}{=} \psi^\varepsilon_{t/\sqrt{\varepsilon}}, \qquad \theta_t^\varepsilon \stackrel{\text{def}}{=} \theta_{t/\sqrt{\varepsilon}}, \tag{12}
$$

into the above equations and Taylor-expanding in powers of $\sqrt{\varepsilon}$ about I_r, we get, with higher order terms subsumed in $\mathfrak{R}$

$$
dh_t^\varepsilon = \mathfrak{F}dt + \sqrt{\varepsilon}\,\mathfrak{F}' h_t^\varepsilon dt + \varepsilon^{\kappa - \frac{3}{4}}\sigma\frac{\partial I}{\partial q_2}dW_t + \mathfrak{R}^\varepsilon_{1,t}dt + \hat{\mathfrak{R}}^\varepsilon_{1,t}dW_t, \tag{13}
$$

$$d\widehat{\psi}_t^\varepsilon = \Omega_r' h_t^\varepsilon dt + \sqrt{\varepsilon}\left(\Omega_r''\frac{1}{2}(h_t^\varepsilon)^2 + \mathfrak{G}\right)dt + \mathfrak{R}_{2,t}^\varepsilon dt + \hat{\mathfrak{R}}_{2,t}^\varepsilon dW_t, \tag{14}$$

$$d\theta_t^\varepsilon = \frac{1}{\sqrt{\varepsilon}}\nu dt, \tag{15}$$

where $'$ indicates differentiation w.r.t I and all terms (except $\mathfrak{R}$) are evaluated at $(I_r, \widehat{\psi}_t^\varepsilon + \frac{n}{m}\theta_t^\varepsilon, \theta_t^\varepsilon)$. When $\kappa \geq 1$, the higher order terms are $\mathfrak{R}_i^\varepsilon \sim O(\varepsilon)$ and $\hat{\mathfrak{R}}_i^\varepsilon \sim O(\varepsilon^{\kappa-\frac{1}{4}})$, for $i = 1, 2$.

3.1 Capture into Resonance

From (13)–(15) it is clear that θ_t^ε and φ_t^ε evolve at a faster rate than h_t^ε and $\widehat{\psi}_t^\varepsilon$. Hence we average out the fast variable θ. For this purpose define an averaging operator $\langle\rangle$ as follows: for a function f periodic in θ with period $2m\pi$ we define $\langle f\rangle = \frac{1}{2m\pi}\int_0^{2m\pi} f(\theta)d\varphi$. Note that the functions $\theta \mapsto \mathfrak{F}(I_r, \psi + \frac{n}{m}\theta, \theta)$ and $\theta \mapsto \mathfrak{G}(I_r, \psi + \frac{n}{m}\theta, \theta)$ are periodic in θ with period $2m\pi$. To clearly indicate the dependence of the corresponding averaged function on ψ, we denote the averaged functions by $\langle\mathfrak{F}(\psi)\rangle$ and $\langle\mathfrak{G}(\psi)\rangle$.

For the analysis in this section, we neglect the stochastic term. To this end, in (13)–(15) lets set $\sigma = 0$, ignore higher order terms $\mathfrak{R}$ and perform averaging w.r.t θ. Then we get

$$\begin{pmatrix} dh \\ d\psi \end{pmatrix} = \begin{pmatrix} \langle\mathfrak{F}(\psi)\rangle + \sqrt{\varepsilon}\langle\mathfrak{F}'(\psi)\rangle h \\ \Omega_r' h + \sqrt{\varepsilon}(\frac{1}{2}\Omega_r'' h^2 + \langle\mathfrak{G}(\psi)\rangle) \end{pmatrix} dt, \tag{16}$$

General structure of the averaged terms are, for $\frac{m}{n} \in 2Z^+$

$$\langle\mathfrak{F}(\psi)\rangle = -\beta I_r + J_c\cos(m\psi/n), \quad \langle\mathfrak{F}'(\psi)\rangle = -\beta + J_c'\cos(m\psi/n), \tag{17}$$

$$\langle\mathfrak{G}(\psi)\rangle = -\frac{n}{m}J_c'\sin(m\psi/n), \tag{18}$$

For $\frac{m}{n} \in 2Z^+ + 1$

$$\langle\mathfrak{F}(\psi)\rangle = -\beta I_r + J_c\cos(m\psi/n) + J_s\sin(m\psi/n) \tag{19}$$

$$\langle\mathfrak{F}'(\psi)\rangle = -\beta + J_c'\cos(m\psi/n) + J_s'\sin(m\psi/n) \tag{20}$$

$$\langle\mathfrak{G}(\psi)\rangle = -\frac{n}{m}J_c'\sin(m\psi/n) + \frac{n}{m}J_s'\cos(m\psi/n) \tag{21}$$

where the method to obtain the above (17)–(21) and the quantities J_s, and J_c is discussed in [1, 5] and in the appendix. We can restrict ourselves to the case of $\frac{m}{n} \in 2Z^+$ as the structure of the equations for $\frac{m}{n} \in 2Z^+ + 1$ is qualitatively equivalent.

We can study (16) as a perturbation of a Hamiltonian system

$$\begin{pmatrix} dh \\ d\psi \end{pmatrix} = \begin{pmatrix} \langle \mathfrak{F}(\psi) \rangle \\ \Omega_r' h \end{pmatrix} dt, \tag{22}$$

with the Hamiltonian

$$\mathscr{H}(\psi, h) = \frac{1}{2}\Omega' h^2 - \int_0^{\psi} \langle \mathfrak{F}(\psi) \rangle d\psi. \tag{23}$$

Such Hamiltonians typically occur in resonant problems and (23) represents a "pendulum" under the action of an external torque [4, 6]. Note that (22) has fixed point only if

$$\beta I_r \leq |J_c|. \tag{24}$$

The fixed points are given by

$$\cos(m\psi/n) \approx \frac{\beta I_r}{J_c}, \qquad h = 0.$$

There are many ψ which satisfy the above equation. Typical phase portrait (with $\Omega_r' > 0$) for (16) is shown in the Fig. 2. The saddle (sd) and center (sk) fixed point pairs (i.e. the homoclinic orbit of the saddle encloses the center) for (22) can be easily obtained. All the fixed points have $h = 0$. Recall the definitions (25). Note that $h = 0$ means $I = I_r$, i.e. the system is exactly at resonance. The Fig. 2 shows a finite region around $h = 0$. In terms of I coordinates this region is a neighborhood of I_r of a width of order $\sqrt{\varepsilon}$. This is called a *resonance zone*.

A trajectory which starts at the top of the Fig. 2 ($h > 0$) but not in the narrow neck region would reach the bottom of the figure ($h < 0$), i.e. the trajectory 'passes' through the resonance zone. A trajectory which starts at the top of the Fig. 2 ($h > 0$) in the narrow neck region enters the region A and is trapped there. Lets call the region A as 'trap zone'.

For (16) the region marked A (in Fig. 2) is a trap—the trajectories originating in A cannot exit from it at all. However, when $\sigma \neq 0$, the noise facilitates the escape. We want to study how the noise facilitates the escape from the trap zone.

We denote by $\mathscr{H}|_{sd}$ the value of $\mathscr{H}$ evaluated at one saddle fixed point of (22) and denote by $\mathscr{H}|_{sk}$ the value of $\mathscr{H}$ evaluated at the corresponding center fixed point of (22).

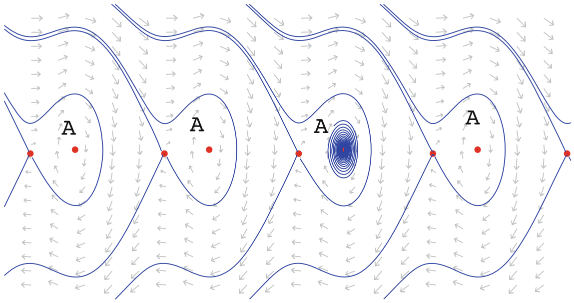

Fig. 2 Typical phase portrait for (16) with $\Omega'_r > 0$. Abscissa is ψ and ordinate is h. The system cannot leave the region A in the absence of noise. The measure of the set of initial conditions that lead to trap in A is small

4 Stochastic Dynamics Close to a Resonance Zone: Case $\kappa > 1$

To see the fluctuations of $\mathscr{H}(\hat{\psi}^\varepsilon_t, h^\varepsilon_t)$, we need to look on an even longer $O(1/\sqrt{\varepsilon})$ time scale. Hence we redefine the $h, \hat{\psi}, \varphi$ process using the following space and time scaling

$$h^\varepsilon_t = \frac{1}{\sqrt{\varepsilon}}(I^\varepsilon_{t/\varepsilon} - I_r), \qquad \hat{\psi}^\varepsilon_t = \psi^\varepsilon_{t/\varepsilon}, \qquad \theta^\varepsilon_t = \theta_{t/\varepsilon}. \tag{25}$$

After doing a Taylor-expansion about I_r, we get, with higher order terms subsumed in $\mathfrak{R}$

$$dh^\varepsilon_t = \frac{1}{\sqrt{\varepsilon}}\mathfrak{F}dt + \mathfrak{F}' h^\varepsilon_t dt + \varepsilon^{\kappa-1}\sigma\frac{\partial I}{\partial q_2}dW_t + \mathfrak{R}^\varepsilon_{1,t}dt + \hat{\mathfrak{R}}^\varepsilon_{1,t}dW_t, \tag{26}$$

$$d\hat{\psi}^\varepsilon_t = \frac{1}{\sqrt{\varepsilon}}\Omega'_r h^\varepsilon_t dt + \left(\Omega''_r\frac{1}{2}(h^\varepsilon_t)^2 + \mathfrak{G}\right)dt + \mathfrak{R}^\varepsilon_{2,t}dt + \hat{\mathfrak{R}}^\varepsilon_{2,t}dW_t, \tag{27}$$

$$d\theta^\varepsilon_t = \frac{1}{\varepsilon}\nu dt, \tag{28}$$

where $'$ indicates differentiation w.r.t I and all terms (except $\mathfrak{R}$) are evaluated at $(I_r, \hat{\psi}^\varepsilon_t + \frac{n}{m}\theta^\varepsilon_t, \theta^\varepsilon_t)$. When $\kappa \geq 1$, the higher order terms are $\mathfrak{R}^\varepsilon_i \sim O(\sqrt{\varepsilon})$ and $\hat{\mathfrak{R}}^\varepsilon_i \sim O(\varepsilon^{\kappa-1/2})$, for $i = 1, 2$.

Since the system (26)–(28) (after averaging φ) can be seen as a perturbation of the Hamiltonian system (22); to the system (26)–(28) we adjoin $\mathscr{H}^\varepsilon_t := \mathscr{H}(\hat{\psi}^\varepsilon_t, h^\varepsilon_t)$, where $\mathscr{H}$ is defined in (23). The evolution of $\mathscr{H}^\varepsilon_t$ can be obtained by applying Ito formula as

$$d\mathscr{H}^\varepsilon_t = \frac{1}{\sqrt{\varepsilon}}\Omega'_r h^\varepsilon_t(\mathfrak{F} - \langle\mathfrak{F}\rangle)dt + \left((\Omega'_r\mathfrak{F}' - \langle\mathfrak{F}\rangle\frac{1}{2}\Omega''_r)(h^\varepsilon_t)^2 - \langle\mathfrak{F}\rangle\mathfrak{G}\right)dt \tag{29}$$
$$+\varepsilon^{\kappa-1}\sigma\Omega'_r h^\varepsilon_t\frac{\partial I}{\partial q_2}dW_t + \mathfrak{R}^\varepsilon_{3,t}dt + \hat{\mathfrak{R}}^\varepsilon_{3,t}dW_t,$$

where arguments for $\mathfrak{F}$, $\langle\mathfrak{F}\rangle$, $\mathfrak{G}$, $\frac{\partial I}{\partial q_2}$ are suppressed; and $\mathfrak{R}$ are higher order terms. Since $\langle\mathfrak{F} - \langle\mathfrak{F}\rangle\rangle = 0$, $\mathscr{H}_t^\varepsilon$ evolves even slowly compared to $(\widehat{\psi}_t^\varepsilon, h_t^\varepsilon)$.

Since our goal is to study the escape from the region marked A we set the initial conditions to (26)–(27) in this region. In terms of $\mathscr{H}_t^\varepsilon$ this amounts to specifying that $\mathscr{H}_0^\varepsilon$ lies in between[1] $\mathscr{H}|_{sk}$ and $\mathscr{H}|_{sd}$. When $\sigma = 0$ the behaviour of $\mathscr{H}_t^\varepsilon$ is to reach $\mathscr{H}|_{sk}$. When $\sigma \neq 0$ the noise facilitates the escape. A good indicator of whether escape occurred is $\mathscr{H}_t^\varepsilon \geq \mathscr{H}|_{sd}$ in the case[2] $\Omega_r' > 0$. Further, $\mathscr{H}_t^\varepsilon$ could be a bit greater than $\mathscr{H}|_{sd}$ and still be in the small neck region which still leads to capture. Let $\mathscr{H}_*$ be the value for which we can be sure that escape occured if $\mathscr{H}_t^\varepsilon \geq \mathscr{H}_*$. Then $\mathscr{H}|_{sd}$ differs from $\mathscr{H}_*$ by a very small amount that goes to zero as $\varepsilon \to 0$. Keeping these caveats in mind, we still study the probability with which $\mathscr{H}_t^\varepsilon$ exceeds $\mathscr{H}|_{sd}$ in presence of noise. However such transition is extremely unlikely because of the smallness of the noise. Hence, our intention is to obtain a large deviation principle for the $\mathscr{H}_t^\varepsilon$ process.

4.1 *Large Deviations Principle (LDP) for $\mathscr{H}^\varepsilon$*

We employ the technique described in [7, 8] to obtain the rate function governing the probability of rare events of $\mathscr{H}_t^\varepsilon$. Averaging would be of help in this regard: because $\mathscr{H}_t^\varepsilon$ evolves slowly compared to $(\widehat{\psi}_t^\varepsilon, h_t^\varepsilon)$ we can average out the fast $(\widehat{\psi}_t^\varepsilon, h_t^\varepsilon)$ dynamics. For this purpose define an averaging operator $\mathbb{A}$ as follows:

Definition 1 For a function f of $(\widehat{\psi}, h)$, the averaged function $\mathbb{A}[f]$ is given by

$$\mathbb{A}[f](\mathfrak{h}) = \frac{1}{\mathfrak{T}(\mathfrak{h})}\int_0^{\mathfrak{T}(\mathfrak{h})} f(\widehat{\psi}(t), h(t))dt$$

where $(\widehat{\psi}(t), h(t))$ is the solution of the Hamiltonian system $\dot{\widehat{\psi}} = \frac{\partial\mathscr{H}}{\partial h}, \dot{h} = -\frac{\partial\mathscr{H}}{\partial\widehat{\psi}}$ with $\mathscr{H}(\widehat{\psi}, h) = \mathfrak{h}$ and $\mathfrak{T}(\mathfrak{h})$ is the time-period of the solution. The $\mathfrak{h}$ is restricted to be in between $\mathscr{H}|_{sk}$ and $\mathscr{H}|_{sd}$; outside these values the orbit of the Hamiltonian system is not closed and the time-period is not defined.

Since $\mathfrak{h}$ is restricted to be in between $\mathscr{H}|_{sk}$ and $\mathscr{H}|_{sd}$ we define a stopping time

$$\mathfrak{e}^\varepsilon := \inf\{t > 0 \; : \; \mathscr{H}_t^\varepsilon \text{ is not in between } \mathscr{H}|_{sk} \text{ and } \mathscr{H}|_{sd}\}. \tag{30}$$

More precisely, if $\Omega_r' > 0$ then $\mathfrak{e}^\varepsilon := \inf\{t > 0 \; : \; \mathscr{H}_t^\varepsilon \geq \mathscr{H}|_{sd}\}$ and if $\Omega_r' < 0$ then $\mathfrak{e}^\varepsilon := \inf\{t > 0 \; : \; \mathscr{H}_t^\varepsilon \leq \mathscr{H}|_{sd}\}$.

[1] $\mathscr{H}|_{sd} > \mathscr{H}|_{sk}$ if $\Omega_r' > 0$ and $\mathscr{H}|_{sd} < \mathscr{H}|_{sk}$ if $\Omega_r' < 0$.

[2] If $\Omega_r' < 0$ then a good indicator is $\mathscr{H}_t^\varepsilon \leq \mathscr{H}|_{sd}$.

Following the standard techniques, first we derive the LDP for the random variable $\mathscr{H}^{\varepsilon}_{T\wedge \mathfrak{e}^{\varepsilon}}$ where $\mathscr{H}$ is governed by (29) with the initial condition $\mathscr{H}_0$ at $t = 0$. Define

$$g^{\varepsilon}_{T,\mathscr{H}_0}(p) \stackrel{\text{def}}{=} \varepsilon^{2(\kappa-1)} \log \mathbb{E}_{\mathscr{H}_0} \exp\left(\frac{1}{\varepsilon^{2(k-1)}} p\mathscr{H}^{\varepsilon}_{T\wedge \mathfrak{e}^{\varepsilon}}\right), \tag{31}$$

where the expectation $\mathbb{E}_{\mathscr{H}_0}$ indicates that the process $\mathscr{H}^{\varepsilon}$ starts at $\mathscr{H}_0$. Let

$$g_{T,\mathscr{H}_0}(p) \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0} g^{\varepsilon}_{T,\mathscr{H}_0}(p). \tag{32}$$

Then $\mathscr{H}^{\varepsilon}_{T\wedge \mathfrak{e}^{\varepsilon}}$ has LDP with rate function

$$\mathscr{V}_{T,\mathscr{H}_0}(\mathfrak{h}) \stackrel{\text{def}}{=} \sup_{p\in\mathbb{R}} \left(p\mathfrak{h} - g_{T,\mathscr{H}_0}(p)\right), \tag{33}$$

for $\mathfrak{h}$ in between $\mathscr{H}|_{sk}$ and $\mathscr{H}|_{sd}$. So, now we evaluate $g_{T,\mathscr{H}_0}(p)$. The idea of using averaging for obtaining large-deviation principle is implemented in, for example, [1, 7, 8].

Theorem 1 (Lingala et al. [1]) *Let $\mathscr{A}_1(\widehat{\psi},\theta)$ be defined by*

$$\nu\mathscr{A}_1(\widehat{\psi},\theta) = \int_0^{\theta} \left(\mathfrak{F}(\widehat{\psi} + \frac{n}{m}\tilde{\theta},\tilde{\theta}) - \langle \mathfrak{F}(\widehat{\psi})\rangle\right) d\tilde{\theta}.$$

Define

$$\mathfrak{B}(\mathfrak{h}) = \mathfrak{B}_1(\mathfrak{h}) + \mathfrak{B}_2(\mathfrak{h}), \qquad \varXi = \sigma^2(\varOmega_r')^2 \mathbb{A}\left[\left\langle \left(h^{\varepsilon}_t \frac{\partial I}{\partial q_2}\right)^2 \right\rangle\right],$$

where[3]

$$\mathfrak{B}_1 = -\varOmega_r' \mathbb{A}\left[\left\langle \mathscr{A}_1\mathfrak{F} + \varOmega_r' h^2 \frac{\partial \mathscr{A}_1}{\partial \widehat{\psi}} \right\rangle\right],$$
$$\mathfrak{B}_2 = \mathbb{A}\left[\left\langle (\varOmega_r'\mathfrak{F}' - \langle\mathfrak{F}\rangle \frac{1}{2}\varOmega_r'')(h^{\varepsilon}_t)^2 - \langle\mathfrak{F}\rangle\mathfrak{G} \right\rangle\right].$$

Then

$$g_{T,\mathscr{H}_0}(p) = p\mathscr{H}_0 + p\int_0^{T\wedge\mathfrak{e}} \mathfrak{B}(\hat{\mathfrak{h}}_t)dt + \frac{1}{2}p^2 \int_0^{T\wedge\mathfrak{e}} \varXi(\hat{\mathfrak{h}}_t)dt \tag{34}$$

[3] In $\mathfrak{B}_1$ the term $\langle \mathscr{A}_1\mathfrak{F}\rangle$ should be interpreted as the average w.r.t θ of the function $\theta \mapsto \mathscr{A}_1(\widehat{\psi},\theta)\mathfrak{F}(\widehat{\psi} + \frac{n}{m}\theta,\theta)$.

where $\hat{\mathfrak{h}}_t$ *is simulated according to*

$$d\hat{\mathfrak{h}}_t = \left(\mathfrak{B}(\hat{\mathfrak{h}}_t) + p\Xi(\hat{\mathfrak{h}}_t)\right)dt, \qquad \hat{\mathfrak{h}}_0 = \mathscr{H}_0, \tag{35}$$

and $\mathfrak{e}$ *is defined by*

$$\mathfrak{e} := \inf\{t > 0 \;:\; \hat{\mathfrak{h}}_{\mathfrak{e}} \textit{ is not in between } \mathscr{H}|_{sk} \textit{ and } \mathscr{H}|_{sd}\}.$$

Proof See [1]. □

Theorem 2 *The rate functional on the path space is*

$$S_{0T}(x) = \frac{1}{2}\int_0^T \frac{(\dot{x}_t - \mathfrak{B}(x_t))^2}{\Xi(x_t)}dt.$$

for $x \in C([0,T], \mathbb{R}_{\mathscr{H}})$ *absolutely continuous where* $\mathbb{R}_{\mathscr{H}}$ *is the set of real numbers lying in between* $\mathscr{H}|_{sk}$ *and* $\mathscr{H}|_{sd}$.

Proof See [1]. □

4.2 Evaluation of $\mathfrak{B}$ and Ξ in Theorem 1

Using $\frac{\partial I}{\partial q_2} = \frac{\partial I}{\partial H}\frac{\partial H}{\partial q_2} = \frac{1}{\Omega}q_2$ and that at the resonance $\langle q_2^2\rangle = I_r\Omega_r$ we have that

$$\mathfrak{B}_1 \equiv 0, \quad \mathfrak{B}_2 = -\beta\Omega_r'\mathbb{A}[h^2], \quad \text{and} \quad \Xi = \frac{\sigma^2(\Omega_r')^2 I_r}{\Omega_r}\mathbb{A}[h^2]. \tag{36}$$

4.3 Escape from the Trap Zone

Since we are interested in the escape from the trap zone (region A in the Fig. 2), we need to consider the probabilities $\mathbb{P}_{\mathfrak{h}^0}[\mathfrak{e}^\varepsilon \le t]$ where $\mathfrak{e}^\varepsilon$ is defined in (30) and $\mathfrak{h}^0$ indicates that the initial condition is such that $\mathscr{H}_0^\varepsilon = \mathfrak{h}^0$. We restrict to the case that $\mathfrak{h}^0$ lies between $\mathscr{H}|_{sk}$ and $\mathscr{H}|_{sd}$.

Define

$$\mathscr{V}(t, \mathfrak{h}^0, \mathfrak{h}) := \inf\{S_{0t}(x) \;:\; x \in C([0,t], \mathbb{R}_{\mathscr{H}}),\, x(0) = \mathfrak{h}^0,\, x(t) = \mathfrak{h}\}, \tag{37}$$

for $\mathfrak{h}^0, \mathfrak{h}$ lying in between $\mathscr{H}|_{sk}$ and $\mathscr{H}|_{sd}$. Applying[4] Theorem 4.1.2 and remarks following it in [9], we have

$$\lim_{\varepsilon\to 0} \varepsilon^{2(\kappa-1)} \log \mathbb{P}_{\mathfrak{h}^0}[\mathfrak{e}^\varepsilon \le t] = -\min_{0\le s\le t} \mathscr{V}(t, \mathfrak{h}^0, \mathscr{H}|_{sd}).$$

The function $\mathscr{V}$ satisfies the Hamilton-Jacobi equation (see Eq. 4.1.11 in [9]):

$$\begin{cases} \frac{\partial \mathscr{V}(t,\mathfrak{h}^0,\mathfrak{h})}{\partial t} + \mathfrak{B}(\mathfrak{h}) \frac{\partial \mathscr{V}(t,\mathfrak{h}^0,\mathfrak{h})}{\partial \mathfrak{h}} + \frac{1}{2}\varXi(\mathfrak{h}) \left(\frac{\partial \mathscr{V}(t,\mathfrak{h}^0,\mathfrak{h})}{\partial \mathfrak{h}}\right)^2 = 0, \\ \mathscr{V}(t, \mathfrak{h}^0, \mathfrak{h}^0) = 0. \end{cases}$$

Solution could not be found explicitly. However, it can be solved by numerical methods.

Define the quasipotential

$$\mathscr{V}(\mathfrak{h}) := \inf\{S_{T_1T_2}(x) \;:\; x \in C([T_1, T_2], \mathbb{R}_{\mathscr{H}}),\, T_1 \le T_2,\, x(T_1) = \mathscr{H}|_{sk},\, x(T_2) = \mathfrak{h}\}. \quad (38)$$

Then, Theorem 4.4.1 of [9] shows that the mean exit time satisfies

$$\lim_{\varepsilon\to 0} \varepsilon^{2(\kappa-1)} \log \mathbb{E}_{\mathfrak{h}^0}[\mathfrak{e}^\varepsilon] = -\mathscr{V}(\mathscr{H}|_{sd}),$$

for any $\mathfrak{h}^0$ between $\mathscr{H}|_{sd}$ and $\mathscr{H}|_{sk}$. The function $\mathscr{V}(\mathfrak{h})$ satisfies

$$\mathfrak{B}(\mathfrak{h}) \frac{\partial \mathscr{V}(\mathfrak{h})}{\partial \mathfrak{h}} + \frac{1}{2}\varXi(\mathfrak{h}) \left(\frac{\partial \mathscr{V}(\mathfrak{h})}{\partial \mathfrak{h}}\right)^2 = 0, \qquad \mathscr{V}(\mathscr{H}|_{sk}) = 0,$$

which can be easily solved to give

$$\mathscr{V}(\mathfrak{h}) = -\int_{\mathscr{H}|_{sk}}^{\mathfrak{h}} \frac{2\mathfrak{B}(y)}{\varXi(y)} dy = \frac{2\beta\varOmega_r}{\sigma^2 \varOmega_r' I_r}(\mathfrak{h} - \mathscr{H}|_{sk}). \quad (39)$$

In particular the following gives a measure of difficulty of escape from the trap zone:

$$\mathscr{V}(\mathscr{H}|_{sd}) = \frac{2\beta\varOmega_r}{\sigma^2 \varOmega_r' I_r}(\mathscr{H}|_{sd} - \mathscr{H}|_{sk}). \quad (40)$$

[4]This application should be taken in a heuristic sense. In the problem considered in Theorem 4.1.2 of [9] the vector field does not vary with ε. However, in the problem considered in this paper we are averaging an oscillating vector field to get simple equation for $\mathscr{H}$ only in the limit as $\varepsilon \to 0$.

The above can be evaluated to be

$$\mathcal{V}(\mathcal{H}|_{sd}) = \frac{2\Omega_r(n/m)}{\sigma^2|\Omega_r'|}\beta^2\left(-2\cos^{-1}|\chi| + \pi + 2\frac{\sqrt{1-|\chi|^2}}{|\chi|}\right), \quad \chi := \frac{\beta I_r}{J_c}.$$

Since the function in the brackets is monotonically decreasing in $|\chi|$, it can be deduced that for a fixed β, $\mathcal{V}(\mathcal{H}|_{sd})$ is monotonically increasing in $|J_c|$, i.e. the higher the strength of periodic excitations the more difficult the escape from the trap. For a fixed J_r, $\mathcal{V}(\mathcal{H}|_{sd})$ has a unique maximum as a function of β. As β increases to $\frac{|J_r|}{I_r}$, $\mathcal{V}(\mathcal{H}|_{sd})$ decreases to 0, because the area of the trap zone decreases to zero. As β decreases to 0, $\mathcal{V}(\mathcal{H}|_{sd})$ also decreases to zero—this behaviour is not intuitive. Hence, for a fixed strength of periodic excitations, both high and low damping makes the escape easier—intermediate values of damping makes the escape difficult.

4.4 Post Escape from the Trap

Immediately outside the trap region A, the deterministic dynamics alone is enough to take the system out of the resonance zone (see Fig. 2). Since the noise is small, getting re-trapped is a rare event, i.e. the system moves out of the resonance zone quickly. Once outside the resonance zone, full-averaging i.e. averaging w.r.t (φ, θ) can be done. The full-averaged system shows that damping results in a decrease of I with time. However as I decreases the system might enter a different resonance zone—from results of [6] we know that the measure of the set of initial conditions which get trapped is small. Those that get trapped, escape at a rate governed by the large-deviations principle obtained above. In such fashion the system evolves until it reaches close to $(\delta, \omega) = (\pm n\pi + r, 0)$, i.e. the bottom of the wells in the potential U of (9).

Note that we have not analysed the behaviour near the homoclinic orbit. So, the description in the above paragraph is valid for those trajectories which start within the region bounded by the homoclinic orbit of the original unperturbed hamiltonian. However, the analysis in previous sections is valid also for the resonance zones that lie outside the region bound by the homoclinic orbit.

If the action at the bottom of the well $I_b := I|_{\delta=\pm n\pi+r,\omega=0}$ is such that $\Omega(I_b)$ is in resonance with ν, then interesting dynamics occurs. Such a situation is discussed in [10] in an attempt to explain phase-flip of electrons in external fields. Due to page limits the dynamics when $\nu \approx 2\Omega(I_b)$ are not presented here.

5 Conclusion

The full United States power grid presents a high dimensional complex network for which any attempt at analytical analysis is near impossible. However there are many important examples of lower dimensional systems governed by key system dynamics that present a rich dynamic behavior that can be studied in order to provide insight into the phenomena that occur on much larger scales.

The model presented in [3] is an example of a fundamental unit that is often studied in power system theory, that is, three interconnected synchronous machines. This paper offers an analytical method to characterize the stability of a resonant equilibrium mode of operation that such a network may find itself in dependent on initial conditions.

Understanding the effect on stability that random fluctuations on the grid have—caused both by load (consumers) and generation (renewable energy inputs)—is a difficult problem and one that has garnered interest in recent years due to the increased penetration of renewable sources on the grid. The first section of this paper presented a formulation that enabled a three machine system with load fluctuations to be reduced to the study of a one-dimensional, two degree of freedom problem with small periodic fluctuations. An explicit analytical method that allows us to understand the relationship between the stability of the system and these random fluctuations by quantifying the dependence of minimum action to escape, damping, and periodic excitation is presented. It is seen that there are a number of modes of operation that will lead to optimal (higher) escape times and thus increased stability of the resonant fixed point.

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Appendix: Calculation of J_s and J_c in (19)–(21)

The reduced order system with $\varepsilon = 0$ with $c = 1$ are the equations for a non-linear pendulum. The pendulum has two modes of motion dependent on total system energy. When $H \in (-1, 1)$ the system is described by oscillatory solutions. Denoting k as the elliptic modulus we have [5]

$$H = 2k^2 - 1$$

With $K = K(k)$ and $E = E(k)$ being complete elliptic integrals of first and second kind respectively,

$$I(k) = [\frac{8}{\pi}[E - (k^2 - 1)K], \quad \Omega = \pi 2K, \quad \dot{\varphi} = \Omega \tag{41}$$

The oscillating displacement and velocity in terms of the angle variable φ are

$$\delta(\varphi) = 2\arcsin(kSn(\frac{2K\varphi}{\pi})), \quad \omega(\varphi) = 2kCn(\frac{2K\varphi}{\pi}) \cdot \frac{2K}{\pi}$$

We have that

$$\mathfrak{F}(I,\varphi,\frac{m}{n}(\psi-\varphi)) = (\alpha - \beta\omega - \mu_3 c_2(q_1(\varphi)+r)\cos(q_1(\varphi)+\tau_1) +$$
$$c_1\sin(\frac{m}{n}(\psi-\varphi))\cos(q_1(\varphi)+\tau_2) - \mu_1 c_2\sin(\frac{m}{n}(\psi-\varphi))\cos(q_1(\varphi)+\tau_1))q_2(\varphi).$$

Noting that $\frac{1}{2m\pi}\int_0^{2m\pi}\mathfrak{F}(I,\psi+\frac{n}{m}\theta,\theta)d\theta = \frac{1}{2n\pi}\int_0^{2n\pi}\mathfrak{F}(I,\varphi,\frac{m}{n}(\psi-\varphi))d\varphi$ due to the resonance condition. Even though it is natural to choose θ_t as the fast variable for multi-phase averaging, in order to simplify the averaging of certain elliptic functions in the expressions $\mathfrak{F}$ and $\mathfrak{G}$ φ is used as the fast angle for multi-phase averaging. We can evaluate the more tractable form $\frac{1}{2n\pi}\int_0^{2n\pi}\mathfrak{F}(I,\varphi,\frac{m}{n}(\psi-\varphi))d\varphi$ which gives

$$\begin{aligned}
\left\langle \mathfrak{F}(I_r,\varphi,\frac{m}{n}(\psi-\varphi))\right\rangle = \beta I_r &+ A_1\left(\frac{\pi^3}{K^3k}\right)\frac{q^{\frac{m}{2n}}}{1+q^{\frac{m}{n}}}\sin\left(\frac{m}{n}\psi\right)\mathbf{1}_{\left\{\frac{m}{n}\in 2Z^+ +1\right\}} \\
&- A_2\left(\frac{\pi^3}{K^3k}\right)\left(\frac{m}{n}\right)^2\frac{q^{\frac{m}{n}}}{1-q^{\frac{2m}{n}}}\cos\left(\frac{m}{n}\psi\right)\mathbf{1}_{\left\{\frac{m}{n}\in Z^+\right\}} \\
&- \tilde{A}_1\left(\frac{\pi^3}{K^3k}\right)\frac{q^{\frac{m}{2n}}}{1+q^{\frac{m}{n}}}\sin\left(\frac{m}{n}\psi\right)\mathbf{1}_{\left\{\frac{m}{n}\in 2Z^+ +1\right\}} \\
&+ \tilde{A}_2\left(\frac{\pi^3}{K^3k}\right)\left(\frac{m}{n}\right)^2\frac{q^{\frac{m}{n}}}{1-q^{\frac{2m}{n}}}\cos\left(\frac{m}{n}\psi\right)\mathbf{1}_{\left\{\frac{m}{n}\in Z^+\right\}} + \mathscr{C},
\end{aligned}$$

where $I_r \stackrel{\text{def}}{=} \frac{8}{\pi n}\left((k^2-1)K+E\right)$ is the resonant value of the action, $\mathscr{C}$ represents the contribution due to the term $\langle q_1(\varphi)\cos(q_1(\varphi)+\tau_1)q_2(\varphi)\rangle$, which can be argued to be negligible, and

$$\begin{aligned}
q &= \exp\left(-\tfrac{\pi K'}{K}\right), \quad A_1 = \frac{4kC_1}{2n\pi^2}\cos\tau_2, \quad A_2 = \frac{4kC_1}{2n\pi^2}\sin\tau_2, \\
\tilde{A}_1 &= \tfrac{4k\mu_1 C_1}{2n\pi^2}\cos\tau_1, \quad \tilde{A}_2 = \frac{4k\mu_1 C_2}{2n\pi^2}\sin\tau_2.
\end{aligned}$$

$\mathbf{J_c}$ and $\mathbf{J_s}$ are the coefficients of the $\cos(\frac{m}{n}\psi)$ and $\sin(\frac{m}{n}\psi)$ terms. Neglecting the $\sin(\frac{m}{n}\psi)$ terms means m:n is even, this can be done without loss of generality.

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