Chapter 2
Theory of Integral Invariants

1 Various Properties of the Equations of Dynamics

Let F be a function of a double series of variables:

$$x_1, x_2, \ldots x_n, y_1, y_2, \ldots y_n$$

and of time t.

Suppose that we have differential equations:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\mathrm{d}F}{\mathrm{d}y_i}, \quad \frac{\mathrm{d}y_i}{\mathrm{d}t} = -\frac{\mathrm{d}F}{\mathrm{d}x_i}. \tag{1}$$

Consider two infinitesimally close solutions of these equations:

$$\begin{aligned}
&x_1, x_2, \ldots x_n, y_1, y_2, \ldots y_n, \\
&x_1 + \xi_1, x_2 + \xi_2, \ldots x_n + \xi_n, y_1 + \eta_1, y_2 + \eta_2, \ldots y_n + \eta_n,
\end{aligned}$$

where the ξ and the η are small enough that their squares can be neglected.

The ξ and the η will then satisfy the linear differential equations:

$$\begin{aligned}
\frac{\mathrm{d}\xi_i}{\mathrm{d}t} &= \sum_k \frac{\mathrm{d}^2F}{\mathrm{d}y_i \mathrm{d}x_k}\xi_k + \sum_k \frac{\mathrm{d}^2F}{\mathrm{d}y_i \mathrm{d}y_k}\eta_k, \\
\frac{\mathrm{d}\eta_i}{\mathrm{d}t} &= -\sum_k \frac{\mathrm{d}^2F}{\mathrm{d}x_i \mathrm{d}x_k}\xi_k - \sum_k \frac{\mathrm{d}^2F}{\mathrm{d}x_i \mathrm{d}y_k}\eta_k,
\end{aligned} \tag{2}$$

which are the perturbation equations of equations (1) (first-order Taylor series expansions).

H. Poincaré, *The Three-Body Problem and the Equations of Dynamics*,
Astrophysics and Space Science Library 443,
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Let ξ'_i, η'_i be another solution of these linear equations such that:

$$\begin{aligned}\frac{\mathrm{d}\xi'_i}{\mathrm{d}t} &= \sum_k \frac{\mathrm{d}^2F}{\mathrm{d}y_i\mathrm{d}x_k}\xi'_k + \sum_k \frac{\mathrm{d}^2F}{\mathrm{d}y_i\mathrm{d}y_k}\eta'_k,\\ \frac{\mathrm{d}\eta'_i}{\mathrm{d}t} &= -\sum_k \frac{\mathrm{d}^2F}{\mathrm{d}x_i\mathrm{d}x_k}\xi'_k - \sum_k \frac{\mathrm{d}^2F}{\mathrm{d}x_i\mathrm{d}y_k}\eta'_k.\end{aligned} \tag{2'}$$

Multiply Eqs. (2) and (2′), respectively, by $\eta'_i, -\xi'_i, -\eta_i, \xi_i$ and add up all these equations, the result is:

$$\begin{aligned}&\sum_i\left(\eta'_i\frac{\mathrm{d}\xi_i}{\mathrm{d}t} - \xi'_i\frac{\mathrm{d}\eta_i}{\mathrm{d}t} - \eta_i\frac{\mathrm{d}\xi'_i}{\mathrm{d}t} + \xi_i\frac{\mathrm{d}\eta'_i}{\mathrm{d}t}\right)\\ &\quad= \sum_i\sum_k\left(\xi_k\eta'_i\frac{\mathrm{d}^2F}{\mathrm{d}y_i\mathrm{d}x_k} + \eta_k\eta'_i\frac{\mathrm{d}^2F}{\mathrm{d}y_i\mathrm{d}y_k} + \xi_k\xi'_i\frac{\mathrm{d}^2F}{\mathrm{d}x_i\mathrm{d}x_k} + \eta_k\xi'_i\frac{\mathrm{d}^2F}{\mathrm{d}x_i\mathrm{d}y_k}\right)\\ &\quad - \sum_i\sum_k\left(\eta_i\xi'_k\frac{\mathrm{d}^2F}{\mathrm{d}y_i\mathrm{d}x_k} + \eta_i\eta'_k\frac{\mathrm{d}^2F}{\mathrm{d}y_i\mathrm{d}y_k} + \xi_i\xi'_k\frac{\mathrm{d}^2F}{\mathrm{d}x_i\mathrm{d}x_k} + \xi_i\eta'_k\frac{\mathrm{d}^2F}{\mathrm{d}x_i\mathrm{d}y_k}\right)\end{aligned}$$

or

$$\sum_i \frac{\mathrm{d}}{\mathrm{d}t}\left[\eta'_i\xi_i - \xi'_i\eta_i\right] = 0$$

or finally

$$\eta'_1\xi_1 - \xi'_1\eta_1 + \eta'_2\xi_2 - \xi'_2\eta_2 + \ldots \eta'_n\xi_n - \xi'_n\eta_n = \text{const.} \tag{3}$$

This is a relation which connects the two arbitrary solutions of the linear equations (2) to each other.

It is easy to find other analogous relations.

Consider for solutions of Eq. (2)

$$\begin{gathered}\xi_i, \xi'_i, \xi''_i, \xi'''_i\\ \eta_i, \eta'_i, \eta''_i, \eta'''_i.\end{gathered}$$

Then consider the sum of the determinants:

$$\sum_i\sum_k\begin{vmatrix}\xi_i & \xi'_i & \xi''_i & \xi'''_i\\ \eta_i & \eta'_i & \eta''_i & \eta'''_i\\ \xi_k & \xi'_k & \xi''_k & \xi'''_k\\ \eta_k & \eta'_k & \eta''_k & \eta'''_k\end{vmatrix},$$

where the indices i and k vary from 1 to n. It can be verified without difficulty that this sum is again a constant.

More generally, if the sum of determinants is formed using $2p$ solutions of Eq. (2):

$$\sum_{\alpha_1,\alpha_2,\ldots\alpha_p} \left|\xi_{\alpha_1}\eta_{\alpha_1}\xi_{\alpha_2}\eta_{\alpha_2}\ldots\xi_{\alpha_p}\eta_{\alpha_p}\right|, (\alpha_1,\alpha_2,\ldots\alpha_p = 1,2,\ldots n)$$

this sum will be a constant.

In particular, the determinant formed by the values of the $2n$ quantities ξ and η in $2n$ solutions of Eq. (2) will be a constant.

Using these considerations it is possible to find a solution of Eq. (2) when an integral of them is known and vice versa.

Suppose in fact that

$$\xi_i = \alpha_i, \quad \eta_i = \beta_i$$

is a specific solution of Eq. (2) and designate an arbitrary solution of the same equations by ξ_i and η_i. We will then have:

$$\sum \xi_i\beta_i - \eta_i\alpha_i = \text{const.}$$

which will be an integral of Eq. (2).

And the other-way-around, let

$$\sum A_i\xi_i + \sum B_i\eta_i = \text{const.}$$

be an integral of Eq. (2), we will then have:

$$\sum_i \frac{\mathrm{d}A_i}{\mathrm{d}t}\xi_i + \sum_i \frac{\mathrm{d}B_i}{\mathrm{d}t}\eta_i + \sum_i A_i\left[\sum_k \frac{\mathrm{d}^2F}{\mathrm{d}y_i\mathrm{d}x_k}\xi_k + \sum_k \frac{\mathrm{d}^2F}{\mathrm{d}y_i\mathrm{d}y_k}\eta_k\right]$$
$$-\sum_i B_i\left[\sum_k \frac{\mathrm{d}^2F}{\mathrm{d}x_i\mathrm{d}x_k}\xi_k + \sum_k \frac{\mathrm{d}^2F}{\mathrm{d}x_i\mathrm{d}y_k}\eta_k\right] = 0,$$

hence by aligning terms:

$$\frac{\mathrm{d}A_i}{\mathrm{d}t} = \sum_k \frac{\mathrm{d}^2F}{\mathrm{d}y_i\mathrm{d}x_k}A_k + \sum_k \frac{\mathrm{d}^2F}{\mathrm{d}y_i\mathrm{d}y_k}B_k,$$
$$\frac{\mathrm{d}B_i}{\mathrm{d}t} = -\sum_k \frac{\mathrm{d}^2F}{\mathrm{d}x_i\mathrm{d}x_k}A_k - \sum_k \frac{\mathrm{d}^2F}{\mathrm{d}y_i\mathrm{d}y_k}B_k,$$

which shows that:

$$\xi_i = B_i, \quad \eta_i = -A_i$$

is a specific solution of Eq. (2).

If now:

$$\Phi(x_i, y_i, t) = \text{const.}$$

is an integral of Eq. (1), then

$$\sum \frac{d\Phi}{dx_i}\xi_i + \sum \frac{d\Phi}{dy_i}\eta_i = \text{const.}$$

will be an integral of Eq. (2) and consequently:

$$\xi_i = \frac{d\Phi}{dy_i}, \quad \eta_i = -\frac{d\Phi}{dx_i}$$

will be a specific solution of these equations.

If $\Phi = \text{const.}$ and $\Phi_1 = \text{const.}$ are two integrals of Eq. (1), then we will have

$$\sum\left(\frac{d\Phi}{dx_i}\frac{d\Phi_1}{dy_i} - \frac{d\Phi}{dy_i}\frac{d\Phi_1}{dx_i}\right) = \text{const.}$$

This is Poisson's theorem.

Consider the specific case where the x designate rectangular coordinates of n spatial points; we will designate them using double index notation:

$$x_{1i}, \quad x_{2i}, \quad x_{3i}\,,$$

where the first index refers to the three rectangular coordinates and the second index to the n material points. Let m_i be the mass of material point i. We will then have:

$$m_i\frac{d^2x_{ki}}{dt^2} = \frac{dV}{dx_{ki}},$$

where V is the potential energy.

We will then have the equation for the conservation of energy:

$$F = \sum\frac{m_i}{2}\left(\frac{dx_{ki}}{dt}\right)^2 - V = \text{const.}$$

Next set:

$$y_{ki} = m_i\frac{dx_{ki}}{dt}$$

hence:

$$F = \sum \frac{y_{ki}^2}{2m_i} - V = \text{const.} \tag{4}$$

and

$$\frac{\mathrm{d}x_{ki}}{\mathrm{d}t} = \frac{\mathrm{d}F}{\mathrm{d}y_{ki}}, \quad \frac{\mathrm{d}y_{ki}}{\mathrm{d}t} = -\frac{\mathrm{d}F}{\mathrm{d}x_{ki}}. \tag{1'}$$

Let:

$$x_{ki} = \varphi_{ki}(t), \quad y_{ki} = m_i \varphi'_{ki}(t) \tag{5}$$

be a solution of this Eq. (1′) and another solution be:

$$x_{ki} = \varphi_{ki}(t+h), \quad y_{ki} = m_i \varphi'_{ki}(t+h),$$

where h is an arbitrary constant.

By thinking of h as infinitesimal, a solution of Eq. (2′) can be obtained which correspond to (1′) as Eq. (2) correspond to (1):

$$\xi_{ki} = h\varphi'_{ki}(t) = h\frac{y_{ki}}{m_i}, \quad \eta_{ki} = hm_i\varphi''_{ki}(t) = h\frac{\mathrm{d}V}{\mathrm{d}x_{ki}},$$

where h designates a very small constant factor which can be dropped when only linear Eq. (2′) are considered.

Knowing a solution:

$$\xi = \frac{y}{m}, \quad \eta = \frac{\mathrm{d}V}{\mathrm{d}x}$$

of these equations, an integral can be deduced:

$$\sum \frac{y\eta}{m} - \sum \frac{\mathrm{d}V}{\mathrm{d}x}\xi = \text{const.}$$

But this same integral can be obtained very easily by differentiating the energy conservation Eq. (4).

If the material points are free of any outside action, another solution can be deduced from solution (5):

$$\begin{aligned} x_{1i} &= \varphi_{1i}(t) + h + kt, & y_{1i} &= m_i\varphi'_{1i}(t) + m_i k, \\ x_{2i} &= \varphi_{2i}(t), & y_{2i} &= m_i\varphi'_{2i}(t), \\ x_{3i} &= \varphi_{3i}(t), & y_{3i} &= m_i\varphi'_{3i}(t), \end{aligned}$$

where h and k are arbitrary constants. By thinking of these constants as infinitesimally small, we get two solutions of Eq. (2′)

$$\xi_{1i} = 1, \xi_{2i} = \xi_{3i} = \eta_{1i} = \eta_{2i} = \eta_{3i} = 0,$$
$$\xi_{1i} = t, \xi_{2i} = \xi_{3i} = \eta_{2i} = \eta_{3i} = 0, \eta_{1i} = m_i.$$

Thus two integrals of (2′) can be obtained:

$$\sum_i \eta_{1i} = \text{const.},$$

$$\sum_i \eta_{1i} t - \sum_i m_i \xi_{1i} = \text{const.}$$

These integrals can also be obtained by differentiating the equations of motion of the center of gravity:

$$\sum_i m_i x_{1i} = t \sum_i y_{1i} + \text{const.},$$

$$\sum_i y_{1i} = \text{const.}$$

By rotating the solution (5) through an angle ω around the z-axis, another solution is obtained:

$$\begin{aligned}
x_{1i} &= \varphi_{1i} \cos\omega - \varphi_{2i} \sin\omega, & \frac{y_{1i}}{m_i} &= \varphi'_{1i} \cos\omega - \varphi'_{2i} \sin\omega,\\
x_{2i} &= \varphi_{1i} \sin\omega + \varphi_{2i} \cos\omega, & \frac{y_{2i}}{m_i} &= \varphi'_{1i} \sin\omega + \varphi'_{2i} \cos\omega,\\
x_{3i} &= \varphi_{3i}, & \frac{y_{3i}}{m_i} &= \varphi'_{3i}.
\end{aligned}$$

By regarding ω as infinitesimally small, we find a solution of (2′)

$$\begin{aligned}
\xi_{1i} &= -x_{2i}, & \eta_{1i} &= -y_{2i},\\
\xi_{2i} &= x_{1i}, & \eta_{2i} &= y_{1i},\\
\xi_{3i} &= 0, & \eta_{3i} &= 0,
\end{aligned}$$

and hence the integral for (2′)

$$\sum_i (x_{1i}\eta_{2i} - y_{1i}\xi_{2i} - x_{2i}\eta_{1i} + y_{2i}\xi_{1i}) = \text{const.}$$

that can also be obtained by differentiating the integral of the areas from (1′)

$$\sum_i (x_{1i}y_{2i} - x_{2i}y_{1i}) = \text{const.}$$

Now suppose that the function V is homogeneous and of degree -1 in x which is the case in nature.

Equation (1′) does not change when t is multiplied by λ^3, the x by λ^2, and the y by λ^{-1}, where λ is an arbitrary constant. From the solution (4), the following solution can be deduced:

$$x_{ki} = \lambda^2 \varphi_{ki}\left(\frac{t}{\lambda^3}\right) \quad y_{ki} = \lambda^{-1} m_i \varphi'_{ki}\left(\frac{t}{\lambda^3}\right).$$

If λ is thought of as very close to unity, we will get the following results for the solutions of Eq. (2′)

$$\xi_{ki} = 2\varphi_{ki} - 3t\varphi'_{ki}, \quad \eta_{ki} = -m_i\varphi'_{ki} - 3m_i t\varphi''_{ki},$$

or

$$\xi_{ki} = 2x_{ki} - 3t\frac{y_{ki}}{m_i}, \quad \eta_{ki} = -y_{ki} - 3t\frac{\mathrm{d}V}{\mathrm{d}x_{ki}}, \tag{6}$$

and hence the following integral for Eq. (2′), which, unlike those which we have considered up to here, cannot be obtained by differentiating a known integral of Eq. (1′):

$$\sum (2x_{ki}\eta_{ki} + y_{ki}\xi_{ki}) = 3t\left[\sum\left(\frac{y_{ki}\eta_{ki}}{m_i} - \frac{\mathrm{d}V}{\mathrm{d}x_{ki}}\xi_{ki}\right)\right] + \text{const.}$$

2 Definitions of Integral Invariants

Consider a system of differential equations:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = X_i,$$

where X_i is a given function of $x_1, x_2, \ldots x_n$. If we have:

$$F(x_1, x_2, \ldots x_n) = \text{const.},$$

then this relationship is called an integral of the given equations. The left-hand side of this relationship can be called an invariant because it is not altered when the x_i are increased by infinitesimal increases $\mathrm{d}x_i$ compatible with the differential equations.

Now let

$$x_1', x_2', \dots x_n'$$

be another solution of the same differential equations, such that we have:

$$\frac{\mathrm{d}x_i'}{\mathrm{d}t} = X_i'$$

where X_i' is a function formed with $x_1', x_2', \dots x_n'$ as X_i was formed with $x_1, x_2, \dots x_n$.

It is possible that there could be a relationship of the following form between the $2n$ quantities x and x':

$$F_1\left(x_1, x_2, \dots x_n, x_1', x_2', \dots x_n'\right) = \text{const.}$$

The left-hand side, F_1, could also be called an invariant of our differential equations, because instead of depending on a single solution of these equations, it will depend on two solutions.

It can be assumed that $x_1, x_2, \dots x_n$ represents the coordinates of a point in n dimensional space and that the given differential equations define the laws of motion of this point. If we think about the two solutions of these equations, there are two different moving points, moving under a single law defined by our differential equations. The invariant F_1 will then be a function of the coordinates of these two points and the invariant will retain its initial value during the motion of these two points.

Similarly, instead of two moving points, three or even a large number of moving points could obviously be considered.

Now assume that infinitely many moving points are being considered and that the initial positions of these points form a specific arc of curve C in the n dimensional space.

When we are given the initial position of a moving point and the differential equations which define its laws of motion, the position of the point at an arbitrary moment is then completely determined.

If we therefore know that our moving points, infinitely many, form an arc C at the origin of time, we will know their positions at an arbitrary time t and we will see that the moving points at the moment t form a new arc C' in the n dimensional space. We therefore have an arc of curve which moves while changing shape because its various points move according to the laws defined by the given differential equations.

Now assume that during this motion and this deformation, the following integral:

$$\int \left(Y_1 \mathrm{d}x_1 + Y_2 \mathrm{d}x_2 + \dots Y_n \mathrm{d}x_n\right) = \int \sum Y_i \mathrm{d}x_i$$

(where the Y are given functions of the x and which extends the entire length of the curve) does not change value. This integral will again be an invariant for our differential equations, no longer depending on one, two or three points, but on infinitely many moving points. To indicate what its shape is, I will call it an integral invariant.

Similarly it can be imagined that an integral of the following form:

$$\int \sqrt{\sum Y_{ik} \mathrm{d}x_i \mathrm{d}x_k},$$

over the entire arc of the curve could remain invariant; this again would be an integral invariant.

Integral invariants can also be imagined which are defined by double or multiple integrals.

Imagine that we are considering a fluid in continuous motion such that the three components X, Y, Z of the speed of an arbitrary molecule are given functions of the three coordinates x, y, z of this molecule. Then it would be possible to state that the laws of motion of an arbitrary fluid molecule are defined by the differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = Y, \quad \frac{\mathrm{d}z}{\mathrm{d}t} = Z.$$

It is known that the partial differential equation

$$\frac{\mathrm{d}X}{\mathrm{d}x} + \frac{\mathrm{d}Y}{\mathrm{d}y} + \frac{\mathrm{d}Z}{\mathrm{d}z} = 0$$

expresses that the fluid is incompressible. Therefore assume that the functions X, Y, Z satisfy this equation and consider an ensemble of molecules occupying a specific volume at the origin of time. The molecules will move, but because the fluid is incompressible the volume that they occupy will remain unchanged. In other words the volume, meaning the triple integral:

$$\iiint \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

will be an integral invariant. More generally, if we consider the equations:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = X_i \quad (i = 1, 2, \ldots n)$$

and we have the relationship:

$$\sum_{i=1}^{n} \frac{\mathrm{d}X_i}{\mathrm{d}x_i} = 0,$$

the nth order integral

$$\int \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n$$

which I will continue to call the volume, will be an integral invariant.

This is what will happen in particular for the general equations of dynamics; because on consideration of these equations:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\mathrm{d}F}{\mathrm{d}y_i}, \frac{\mathrm{d}y_i}{\mathrm{d}t} = -\frac{\mathrm{d}F}{\mathrm{d}x_i},$$

it is easy to see that

$$\sum \frac{\mathrm{d}\left(\frac{\mathrm{d}F}{\mathrm{d}y_i}\right)}{\mathrm{d}x_i} + \sum \frac{\mathrm{d}\left(-\frac{\mathrm{d}F}{\mathrm{d}x_i}\right)}{\mathrm{d}y_i} = 0$$

But as it relates to the general equations of dynamics, there is in addition to the volume, another integral invariant that will be even more useful to us. We have in fact seen that:

$$\sum \left(\xi_i \eta_i' - \xi_i' \eta_i\right) = \text{const.}$$

Which translated into our new language means that the double integral

$$\iint \sum_i \mathrm{d}x_i \mathrm{d}y_i$$

is an integral invariant, as I will prove below.

To express this result in another way, take the case of the n-body problem.

We will represent the state of the system of n bodies by the position of $3n$ points in a plane. The abscissa of the first point will be the x of the first body and the ordinate the projection on the x-axis of the momentum of this body; the abscissa of the second point will be the y of the same body and the ordinate the projection on the y-axis of its momentum and so on.

Imagine a double infinity of initial states of the system. A position of our $3n$ points corresponds to each of them and if all of these states are considered, it will be seen that the $3n$ points fill $3n$ plane areas.

If the system now moves according to the law of gravitational attraction the $3n$ points which represent its state are also going to move; the plane areas that I just defined are going to deform, but *their sum will remain constant.*

The theorem on the conservation of volume is just one consequence of the preceding.

In the case of the n-body problem there is another integral invariant to which I want to draw attention.

Consider a single infinity of initial positions of the system which forms an arc of curve in the $6n$ dimensional space. Let C_0 and C_1 be the values of the constant of total energy at two ends of this arc. I will demonstrate later that the expression

$$\int \sum (2x_i \mathrm{d}y_i + y_i \mathrm{d}x_i) + 3(C_1 - C_0)t$$

(where the integral is along the arc of the entire curve and where the time does not enter if $C_0 = C_1$) is again an integral invariant; it is furthermore possible to easily deduce the other integral invariants which were covered above.

We will state that an integral invariant is of first-order, second-order, …, or of nth order according to whether it is a single, double, …, or n times integral.

Among the integral invariants we will distinguish the *positive invariants* that we will define as follows.

The nth order integral invariant:

$$\int M \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n$$

will be a positive invariant in some domain, if M is a function of $x_1, x_2, \ldots x_n$ which remains positive, finite and one-to-one in this domain.

I still need to prove the various results which I just stated; this proof can be done by a very simple calculation.

Let:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = X_1, \quad \frac{\mathrm{d}x_2}{\mathrm{d}t} = X_2, \quad \ldots \quad \frac{\mathrm{d}x_n}{\mathrm{d}t} = X_n \tag{1}$$

be a system of differential equations where $X_1, X_2, \ldots X_n$ are functions of $x_1, x_2, \ldots x_n$ such that:

$$\frac{\mathrm{d}X_1}{\mathrm{d}x_1} + \frac{\mathrm{d}X_2}{\mathrm{d}x_2} + \cdots \frac{\mathrm{d}X_n}{\mathrm{d}x_n} = 0. \tag{2}$$

Let there be a solution to this system of equations which depends on n arbitrary constants:

$$\alpha_1, \alpha_2, \ldots \alpha_n.$$

This solution will be written:

$$\begin{aligned} x_1 &= \varphi_1(t, \alpha_1, \alpha_2, \ldots \alpha_n), \\ x_2 &= \varphi_2(t, \alpha_1, \alpha_2, \ldots \alpha_n), \\ &\vdots \\ x_n &= \varphi_n(t, \alpha_1, \alpha_2, \ldots \alpha_n). \end{aligned}$$

It is a matter of demonstrating that the integral

$$J = \int \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n = \int \Delta \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \ldots \mathrm{d}\alpha_n$$

where

$$\Delta = \begin{vmatrix} \frac{\mathrm{d}x_1}{\mathrm{d}\alpha_1} & \frac{\mathrm{d}x_2}{\mathrm{d}\alpha_1} & \cdots & \frac{\mathrm{d}x_n}{\mathrm{d}\alpha_1} \\ \frac{dx_1}{\mathrm{d}\alpha_2} & \frac{\mathrm{d}x_2}{\mathrm{d}\alpha_2} & \cdots & \frac{\mathrm{d}x_n}{\mathrm{d}\alpha_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\mathrm{d}x_1}{\mathrm{d}\alpha_n} & \frac{\mathrm{d}x_2}{\mathrm{d}\alpha_n} & \cdots & \frac{\mathrm{d}x_n}{\mathrm{d}\alpha_n} \end{vmatrix}$$

is a constant.

In fact we have:

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \int \frac{\mathrm{d}\Delta}{\mathrm{d}t} \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \ldots \mathrm{d}\alpha_n$$

and

$$\frac{\mathrm{d}\Delta}{\mathrm{d}t} = \Delta_1 + \Delta_2 + \ldots \Delta_n,$$

where Δ_k is the determinant Δ in which the kth column

$$\begin{matrix} \frac{\mathrm{d}x_k}{\mathrm{d}\alpha_1} \\ \frac{\mathrm{d}x_k}{\mathrm{d}\alpha_2} \\ \vdots \\ \frac{\mathrm{d}x_k}{\mathrm{d}\alpha_n} \end{matrix} \quad \text{is replaced by} \quad \begin{matrix} \frac{\mathrm{d}^2 x_k}{\mathrm{d}\alpha_1 \mathrm{d}t} \\ \frac{\mathrm{d}^2 x_k}{\mathrm{d}\alpha_2 \mathrm{d}t} \\ \vdots \\ \frac{\mathrm{d}^2 x_k}{\mathrm{d}\alpha_n \mathrm{d}t} \end{matrix}.$$

But we have

$$\frac{\mathrm{d}x_k}{\mathrm{d}t} = X_k,$$

hence

$$\frac{\mathrm{d}^2 x_k}{\mathrm{d}\alpha_i \mathrm{d}t} = \frac{\mathrm{d}X_k}{\mathrm{d}x_1} \frac{\mathrm{d}x_1}{\mathrm{d}\alpha_i} + \frac{\mathrm{d}X_k}{\mathrm{d}x_2} \frac{\mathrm{d}x_2}{\mathrm{d}\alpha_i} + \ldots \frac{\mathrm{d}X_k}{\mathrm{d}x_n} \frac{\mathrm{d}x_n}{\mathrm{d}\alpha_i}.$$

We deduce from that:

$$\Delta_k = \Delta \frac{\mathrm{d}X_k}{\mathrm{d}x_k},$$

hence

$$\begin{aligned}\frac{\mathrm{d}J}{\mathrm{d}t} &= \int (\Delta_1 + \Delta_2 + \cdots \Delta_n)\mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \ldots \mathrm{d}\alpha_n \\ &= \int \left(\frac{\mathrm{d}X_1}{\mathrm{d}x_1} + \frac{\mathrm{d}X_2}{\mathrm{d}x_2} + \cdots \frac{\mathrm{d}X_n}{\mathrm{d}x_n}\right)\Delta \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \ldots \mathrm{d}\alpha_n = 0.\end{aligned}$$

Which was to be proved.

Now suppose that instead of the relation (2) we had:

$$\frac{\mathrm{d}MX_1}{\mathrm{d}x_1} + \frac{\mathrm{d}MX_2}{\mathrm{d}x_2} + \cdots \frac{\mathrm{d}MX_n}{\mathrm{d}x_n} = 0 \tag{2'}$$

where M is an arbitrary function of $x_1, x_2, \ldots x_n$.

I state that:

$$J = \int M \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n = \int M\Delta \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \ldots \mathrm{d}\alpha_n$$

is a constant.

In fact we have:

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \int \left(\Delta \frac{\mathrm{d}M}{\mathrm{d}t} + M\frac{\mathrm{d}\Delta}{\mathrm{d}t}\right)\mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \ldots \mathrm{d}\alpha_n.$$

It must be shown that:

$$\Delta \frac{\mathrm{d}M}{\mathrm{d}t} + M\frac{\mathrm{d}\Delta}{\mathrm{d}t} = 0.$$

In fact we have [because of Eq. (1)]

$$\frac{\mathrm{d}M}{\mathrm{d}t} = X_1 \frac{\mathrm{d}M}{\mathrm{d}x_1} + X_2 \frac{\mathrm{d}M}{\mathrm{d}x_2} + \cdots X_n \frac{\mathrm{d}M}{\mathrm{d}x_n}$$

and (according to what we just saw):

$$\frac{\mathrm{d}\Delta}{\mathrm{d}t} = \Delta\left(\frac{\mathrm{d}X_1}{\mathrm{d}x_1} + \frac{\mathrm{d}X_2}{\mathrm{d}x_2} + \cdots \frac{\mathrm{d}X_n}{\mathrm{d}x_n}\right).$$

It therefore follows that:

$$\Delta \frac{\mathrm{d}M}{\mathrm{d}t} + M\frac{\mathrm{d}\Delta}{\mathrm{d}t} = \Delta\left(\frac{\mathrm{d}MX_1}{\mathrm{d}x_1} + \frac{\mathrm{d}MX_2}{\mathrm{d}x_2} + \cdots \frac{\mathrm{d}MX_n}{\mathrm{d}x_n}\right) = 0.$$

Which was to be proved.

Now move on to the equations of dynamics.

Let the equations be:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\mathrm{d}F}{\mathrm{d}y_i}, \quad \frac{\mathrm{d}y_i}{\mathrm{d}t} = -\frac{\mathrm{d}F}{\mathrm{d}x_i}. \quad (i = 1, 2, \ldots n) \tag{1'}$$

Let there be a solution containing two arbitrary constants α and β and written:

$$x_i = \varphi_i(t, \alpha, \beta)$$
$$y_i = \psi_i(t, \alpha, \beta).$$

I state that:

$$J = \int (\mathrm{d}x_1\mathrm{d}y_1 + \mathrm{d}x_2\mathrm{d}y_2 + \cdots \mathrm{d}x_n\mathrm{d}y_n) = \int \sum_{i=1}^{n} \left(\frac{\mathrm{d}x_i}{\mathrm{d}\alpha}\frac{\mathrm{d}y_i}{\mathrm{d}\beta} - \frac{\mathrm{d}x_i}{\mathrm{d}\beta}\frac{\mathrm{d}y_i}{\mathrm{d}\alpha}\right)\mathrm{d}\alpha\mathrm{d}\beta$$

is a constant.

It follows in fact that:

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \int \sum \left(\frac{\mathrm{d}^2x_i}{\mathrm{d}t\mathrm{d}\alpha}\frac{\mathrm{d}y_i}{\mathrm{d}\beta} + \frac{\mathrm{d}^2y_i}{\mathrm{d}t\mathrm{d}\beta}\frac{\mathrm{d}x_i}{\mathrm{d}\alpha} - \frac{\mathrm{d}^2x_i}{\mathrm{d}t\mathrm{d}\beta}\frac{\mathrm{d}y_i}{\mathrm{d}\alpha} - \frac{\mathrm{d}^2y_i}{\mathrm{d}t\mathrm{d}\alpha}\frac{\mathrm{d}x_i}{\mathrm{d}\beta}\right)\mathrm{d}\alpha\mathrm{d}\beta.$$

It then follows:

$$\frac{\mathrm{d}^2x_i}{\mathrm{d}t\mathrm{d}\alpha} = \sum_k \frac{\mathrm{d}^2F}{\mathrm{d}y_i\mathrm{d}x_k}\frac{\mathrm{d}x_k}{\mathrm{d}\alpha} + \sum_k \frac{\mathrm{d}^2F}{\mathrm{d}y_i\mathrm{d}y_k}\frac{\mathrm{d}y_k}{\mathrm{d}\alpha},$$
$$\frac{\mathrm{d}^2x_i}{\mathrm{d}t\mathrm{d}\beta} = \sum_k \frac{\mathrm{d}^2F}{\mathrm{d}y_i\mathrm{d}x_k}\frac{\mathrm{d}x_k}{\mathrm{d}\beta} + \sum_k \frac{\mathrm{d}^2F}{\mathrm{d}y_i\mathrm{d}y_k}\frac{\mathrm{d}y_k}{\mathrm{d}\beta},$$
$$\frac{\mathrm{d}^2y_i}{\mathrm{d}t\mathrm{d}\alpha} = -\sum_k \frac{\mathrm{d}^2F}{\mathrm{d}x_i\mathrm{d}x_k}\frac{\mathrm{d}x_k}{\mathrm{d}\alpha} - \sum_k \frac{\mathrm{d}^2F}{\mathrm{d}x_i\mathrm{d}y_k}\frac{\mathrm{d}y_k}{\mathrm{d}\alpha},$$
$$\frac{\mathrm{d}^2y_i}{\mathrm{d}t\mathrm{d}\beta} = -\sum_k \frac{\mathrm{d}^2F}{\mathrm{d}x_i\mathrm{d}x_k}\frac{\mathrm{d}x_k}{\mathrm{d}\beta} - \sum_k \frac{\mathrm{d}^2F}{\mathrm{d}x_i\mathrm{d}y_k}\frac{\mathrm{d}y_k}{\mathrm{d}\beta}.$$

From that we conclude that:

$$\sum\left(\frac{\mathrm{d}^2x_i}{\mathrm{d}t\mathrm{d}\alpha}\frac{\mathrm{d}y_i}{\mathrm{d}\beta}-\frac{\mathrm{d}^2y_i}{\mathrm{d}t\mathrm{d}\alpha}\frac{\mathrm{d}x_i}{\mathrm{d}\beta}\right)$$
$$=\sum\sum\left(\frac{\mathrm{d}^2F}{\mathrm{d}y_i\mathrm{d}x_k}\frac{\mathrm{d}x_k}{\mathrm{d}\alpha}\frac{\mathrm{d}y_i}{\mathrm{d}\beta}+\frac{\mathrm{d}^2F}{\mathrm{d}y_i\mathrm{d}y_k}\frac{\mathrm{d}y_k}{\mathrm{d}\alpha}\frac{\mathrm{d}y_i}{\mathrm{d}\beta}+\frac{\mathrm{d}^2F}{\mathrm{d}x_i\mathrm{d}x_k}\frac{\mathrm{d}x_k}{\mathrm{d}\alpha}\frac{\mathrm{d}x_i}{\mathrm{d}\beta}+\frac{\mathrm{d}^2F}{\mathrm{d}x_i\mathrm{d}y_k}\frac{\mathrm{d}x_i}{\mathrm{d}\beta}\frac{\mathrm{d}y_k}{\mathrm{d}\alpha}\right).$$

The right-hand side of the equation does not change on permuting α and β, and therefore we have:

$$\sum\left(\frac{\mathrm{d}^2x_i}{\mathrm{d}t\mathrm{d}\alpha}\frac{\mathrm{d}y_i}{\mathrm{d}\beta}-\frac{\mathrm{d}^2y_i}{\mathrm{d}t\mathrm{d}\alpha}\frac{\mathrm{d}x_i}{\mathrm{d}\beta}\right)=\sum\left(\frac{\mathrm{d}^2x_i}{\mathrm{d}t\mathrm{d}\beta}\frac{\mathrm{d}y_i}{\mathrm{d}\alpha}-\frac{\mathrm{d}^2y_i}{\mathrm{d}t\mathrm{d}\beta}\frac{\mathrm{d}x_i}{\mathrm{d}\alpha}\right).$$

This equality expresses that the quantity under the integral sign in the expression for $\mathrm{d}J/dt$ is zero and consequently that

$$\frac{\mathrm{d}J}{\mathrm{d}t}=0.$$

Which was to be proved.

It remains to consider the last of the integral invariants which comes up in the case of the n-body problem.

Return to the equations of dynamics, but by setting:

$$F=T+U,$$

where T depends only on y and U only on x. Additionally, T is homogeneous and second-degree and U homogeneous and -1 degree.

Take a solution

$$x_i=\varphi_i(t,\alpha),\quad y_i=\psi_i(t,\alpha)$$

which depends solely on a single arbitrary constant, α.

Consider the single integral:

$$J=\int\sum\left(2x_i\frac{\mathrm{d}y_i}{\mathrm{d}\alpha}+y_i\frac{\mathrm{d}x_i}{\mathrm{d}\alpha}\right)\mathrm{d}\alpha+3(C_1-C_0)t,$$

where C_1 and C_0 are constant values of the function F at the ends of the arc along which the integral is calculated.

It follows that:

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \int \sum \left(2\frac{\mathrm{d}x_i}{\mathrm{d}t}\frac{\mathrm{d}y_i}{\mathrm{d}\alpha} + \frac{\mathrm{d}y_i}{\mathrm{d}t}\frac{\mathrm{d}x_i}{\mathrm{d}\alpha} + 2x_i\frac{\mathrm{d}^2 y_i}{\mathrm{d}t\mathrm{d}\alpha} + y_i\frac{\mathrm{d}^2 x_i}{\mathrm{d}t\mathrm{d}\alpha} \right)\mathrm{d}\alpha + 3(C_1 - C_0).$$

It follows that:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\mathrm{d}F}{\mathrm{d}y_i} = \frac{\mathrm{d}T}{\mathrm{d}y_i}, \quad \frac{\mathrm{d}y_i}{\mathrm{d}t} = -\frac{\mathrm{d}U}{\mathrm{d}x_i},$$
$$\frac{\mathrm{d}^2 x_i}{\mathrm{d}t\mathrm{d}\alpha} = \sum_k \frac{\mathrm{d}^2 T}{\mathrm{d}y_i\mathrm{d}y_k}\frac{\mathrm{d}y_k}{\mathrm{d}\alpha}, \quad \frac{\mathrm{d}^2 y_i}{\mathrm{d}t\mathrm{d}\alpha} = -\sum_k \frac{\mathrm{d}^2 U}{\mathrm{d}x_i\mathrm{d}x_k}\frac{\mathrm{d}x_k}{\mathrm{d}\alpha},$$

hence

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \int \sum\sum \left(2\frac{\mathrm{d}T}{\mathrm{d}y_i}\frac{\mathrm{d}y_i}{\mathrm{d}\alpha} + y_i\frac{\mathrm{d}^2 T}{\mathrm{d}y_i\mathrm{d}y_k}\frac{\mathrm{d}y_k}{\mathrm{d}\alpha} - \frac{\mathrm{d}U}{\mathrm{d}x_i}\frac{\mathrm{d}x_i}{\mathrm{d}\alpha} - 2x_i\frac{\mathrm{d}^2 U}{\mathrm{d}x_i\mathrm{d}x_k}\frac{\mathrm{d}x_k}{\mathrm{d}\alpha} \right)\mathrm{d}\alpha + 3(C_1 - C_0).$$

But because of the homogeneous function theorem we have:

$$\sum_i y_i\frac{\mathrm{d}^2 T}{\mathrm{d}y_i\mathrm{d}y_k} = \frac{\mathrm{d}T}{\mathrm{d}y_k}, \quad \sum_i x_i\frac{\mathrm{d}^2 U}{\mathrm{d}x_i\mathrm{d}x_k} = -2\frac{\mathrm{d}U}{\mathrm{d}x_k},$$

hence

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \int \sum \left(3\frac{\mathrm{d}T}{\mathrm{d}y_i}\frac{\mathrm{d}y_i}{\mathrm{d}\alpha} + 3\frac{\mathrm{d}U}{\mathrm{d}x_i}\frac{\mathrm{d}x_i}{\mathrm{d}\alpha} \right)\mathrm{d}\alpha + 3(C_1 - C_0)$$

or

$$\frac{\mathrm{d}J}{\mathrm{d}t} = 3\int (\mathrm{d}T + \mathrm{d}U) + 3(C_1 - C_0).$$

However, according to the definition of C_1 and C_0 we have

$$C_0 - C_1 = \int \mathrm{d}F = \int (\mathrm{d}T + \mathrm{d}U).$$

It therefore follows that:

$$\frac{\mathrm{d}J}{\mathrm{d}t} = 0.$$

Which was to be proved.

3 Transformation of Integral Invariants

Return to our differential equations:

$$\frac{dx_1}{dt} = X_1, \quad \frac{dx_2}{dt} = X_2, \quad \ldots \quad \frac{dx_n}{dt} = X_n \tag{1}$$

and assume that we have:

$$\frac{d(MX_1)}{dx_1} + \frac{d(MX_2)}{dx_2} + \cdots \frac{d(MX_n)}{dx_n} = 0, \tag{2}$$

such that the nth order integral

$$J = \int M dx_1 dx_2 \ldots dx_n$$

is an integral invariant.

Change variables by setting:

$$\begin{aligned} x_1 &= \psi_1(z_1, z_2, \ldots z_n), \\ x_2 &= \psi_2(z_1, z_2, \ldots z_n), \\ &\vdots \\ x_n &= \psi_n(z_1, z_2, \ldots z_n), \end{aligned} \tag{3}$$

and call Δ the Jacobian determinant of the n functions ψ relative to the n variables z.

After the change of variables we will have:

$$J = \int M \Delta dz_1 dz_2 \ldots dz_n.$$

If the invariant was positive before the change of variables, it will remain positive after this change, provided that Δ is always positive, finite, and one-to-one.

Since by permuting two of the variables z, the sign of Δ changes; it will be sufficient for us to assume that Δ always has the same sign or that it is never zero. It will additionally always need to be finite and one-to-one. This will happen if the change of variables (3) is bijective; meaning if, in the domain in consideration, the x are one-to-one functions of z and the z one-to-one functions of x.

Thus after a bijective change of variables, the positive invariants remain positive.

Here is an interesting specific case:

Suppose that an integral of Eq. (1) is known:

$$F(x_1, x_2, \ldots x_n) = C.$$

Take for new variables both $z_n = C$ and also $n - 1$ other variables $z_1, z_2, \ldots z_{n-1}$. It will often happen that $z_1, z_2, \ldots z_{n-1}$ can be chosen such that this change of variables is bijective in the domain in consideration.

After the change of variables, Eq. (1) becomes:

$$\frac{dz_1}{dt} = Z_1, \quad \frac{dZ_2}{dt} = z_2, \quad \ldots \quad \frac{dz_{n-1}}{dt} = Z_{n-1}, \quad \frac{dz_n}{dt} = Z_n = 0, \tag{4}$$

where $Z_1, Z_2, \ldots Z_{n-1}$ are known functions of $z_1, z_2, \ldots z_n$. If the constant $C = z_n$ is regarded as a given of the problem, the equations are reduced to order $n-1$ and are written:

$$\frac{dz_1}{dt} = Z_1, \quad \frac{dz_2}{dt} = Z_2, \quad \ldots \quad \frac{dz_{n-1}}{dt} = Z_{n-1}, \tag{4'}$$

the functions Z now depend only on $z_1, z_2, \ldots z_{n-1}$ because z_n was replaced there by its numeric value.

If there is a positive invariant of Eq. (1)

$$J = \int M dx_1 dx_2 \ldots dx_n,$$

then Eq. (4) will also have a positive invariant:

$$J = \int \mu dz_1 dz_2 \ldots dz_{n-1} dz_n.$$

I now state that Eq. (4′) which is of order $n-1$ also have a positive integral invariant which must be of order $n-1$.

In fact, stating that J is an integral invariant amounts to stating that

$$\frac{d(\mu Z_1)}{dz_1} + \frac{d(\mu Z_2)}{dz_2} + \cdots \frac{d(\mu Z_n)}{dz_n} = 0$$

or because Z_n is zero,

$$\frac{d(\mu Z_1)}{dz_1} + \frac{d(\mu Z_2)}{dz_2} + \cdots \frac{d(\mu Z_{n-1})}{dz_{n-1}} = 0,$$

which proves that the $n-1$ order integral

$$\int \mu dz_1 dz_2 \ldots dz_{n-1}$$

is an invariant for Eq. (4′).

Up till now we have applied the changes of variables to the unknown functions $x_1, x_2, \ldots x_n$, but we have kept time t which is our independent variable. We are now going to assume that we set:

$$t = \varphi(t_1)$$

and that we take t_1 as the new independent variable.

Equation (1) then become:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t_1} = X_i' = X_i \frac{\mathrm{d}\varphi}{\mathrm{d}t_1} = X_i \frac{\mathrm{d}t}{\mathrm{d}t_1} \quad (i = 1, 2, \ldots n) \tag{5}$$

If Eq. (1) has an nth order integral invariant

$$J = \int M \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n$$

then it will be true that

$$\sum \frac{\mathrm{d}}{\mathrm{d}x_i}(MX_i) = 0,$$

which can be written

$$\sum \frac{\mathrm{d}}{\mathrm{d}x_i}\left(M \frac{\mathrm{d}t_1}{\mathrm{d}t} X_i'\right) = 0.$$

Which shows that

$$\int M \frac{\mathrm{d}t_1}{\mathrm{d}t} \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n$$

is an integral invariant of Eq. (5).

For this transformation to be useful, it is necessary that t and t_1 be related such that $\mathrm{d}t_1/\mathrm{d}t$ can be regarded as a known, finite, continuous, and one-to-one function of $x_1, x_2, \ldots x_n$.

Suppose for example that we take for new independent variable:

$$x_n = t_1.$$

It then follows that

$$\frac{\mathrm{d}t_1}{\mathrm{d}t} = X_n$$

and Eq. (5) are written

$$\frac{\mathrm{d}x_1}{\mathrm{d}t_1} = \frac{X_1}{X_n}, \quad \frac{\mathrm{d}x_2}{\mathrm{d}t_1} = \frac{X_2}{X_n}, \quad \ldots \quad \frac{\mathrm{d}x_{n-1}}{\mathrm{d}t_1} = \frac{X_{n-1}}{X_n}, \quad \frac{\mathrm{d}x_n}{\mathrm{d}t_1} = 1,$$

and they allow as integral invariant:

$$\int MX_n \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n.$$

Similarly, if we take for new independent variable:

$$t_1 = \Theta(x_1, x_2, \ldots x_n),$$

where Θ is an arbitrary function of $x_1, x_2, \ldots x_n$, the new integral invariant will be written:

$$\int M\left(\frac{\mathrm{d}\Theta}{\mathrm{d}x_1}X_1 + \frac{\mathrm{d}\Theta}{\mathrm{d}x_2}X_2 + \cdots + \frac{\mathrm{d}\Theta}{\mathrm{d}x_n}X_n\right)\mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n.$$

It needs to be noted that the form and meaning of an integral invariant is changed much more significantly when the independent variable called time is changed then when the change of variables only involves the unknown functions $x_1, x_2, \ldots x_n$, because then the laws of motion for the representative point P become completely transformed.

Suppose $n = 3$ and consider x_1, x_2, x_3 as the spatial coordinates of a point P. The equation:

$$\Theta(x_1, x_2, x_3) = 0$$

will represent a surface. Consider an arbitrary portion of this surface and call this portion of surface S.

I will also suppose that at all points on S:

$$\frac{\mathrm{d}\Theta}{\mathrm{d}x_1}X_1 + \frac{\mathrm{d}\Theta}{\mathrm{d}x_2}X_2 + \frac{\mathrm{d}\Theta}{\mathrm{d}x_3}X_3 \neq 0.$$

It results from this that the portion of surface S is not tangent to any trajectory. I will thus state that S is a contactless surface.

Let P_0 be a point on S; a trajectory passes through this point. If the extension of this trajectory again crosses through S at a point P_1, I will state that P_1 is the *recurrence* of P_0. And in turn P_1 can have for recurrence P_2 which I will call the *second recurrence* of P_0 and so on.

If a curve C traced on S is considered, the n recurrences of the various points of this curve will form another curve C' that I will call the nth *recurrence* of C. In the same way, the area would be defined which is the nth *recurrence* of a given area which is part of S.

That stated, let there be a portion of contactless surface S with the equation $\Theta = 0$; let C be a closed curve traced on this surface and delimiting an area A; let C' and A' be the first recurrences, and C^n and A^n be the nth recurrences of C and A.

A trajectory passes through each of these points of C, and I extend this trajectory from its first meeting with C to its meeting with C'. The family of these trajectories will form a trajectory surface T.

I consider the volume V delimited by the trajectory surface T and by the two areas A and A'. Assume that there is a positive invariant:

$$J = \int M \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3.$$

I extend this invariant to the volume V and I state that $\mathrm{d}J/\mathrm{d}t$ is zero.

Let $\mathrm{d}\omega$ be an element of the surface S. Follow the normal to this element and on this normal take an infinitesimal length $\mathrm{d}n$. Let $\Theta + \frac{\mathrm{d}\Theta}{\mathrm{dn}}\mathrm{d}n$ be the value of Θ at the end of this length. If the normal was followed in the direction of increasing Θ, then:

$$\frac{\mathrm{d}\Theta}{\mathrm{d}n} > 0.$$

Set:

$$\frac{\frac{\mathrm{d}\Theta}{\mathrm{d}x_1}X_1 + \frac{\mathrm{d}\Theta}{\mathrm{d}x_2}X_2 + \frac{\mathrm{d}\Theta}{\mathrm{d}x_3}X_3}{\frac{\mathrm{d}\theta}{\mathrm{d}n}} = H,$$

we will then have

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \int\limits_{A'} MH\mathrm{d}\omega - \int\limits_{A} MH\mathrm{d}\omega,$$

where the first integral is extended to the area A' and the second to the area A.

The integral

$$\int MH\mathrm{d}\omega$$

retains the same value whether it is over the area A, or the area A', or consequently the area A^n. It is therefore an integral invariant of a specific kind which retains the same value for an arbitrary area or for one of its recurrences.

These invariants are additionally positive, because by assumption M and H and, as a consequence, MH are positive.

4 Using Integral Invariants

The following theorems are what make integral invariants interesting and we will make frequent use of them.

Above we defined stability by stating that the moving point P must remain at a finite distance; sometimes it will be given a different meaning. For there to be stability, after sufficiently long time the point P has to return if not to its initial position then at least to a position as close to this initial position as desired.

This latter meaning is how Poisson understood stability. When he proved that, if the second powers of the masses are considered, the major axes of the orbits do not change, he only looked at establishing that the series expansion of these major axes only contained periodic terms of the form $\sin \alpha t$ or $\cos \alpha t$ or mixed terms of the form $t \sin \alpha t$ or $t \cos \alpha t$, without including any secular term of the form t or t^2. Which does not mean that the major axes can never exceed a specific value, because a mixed term $t \cos \alpha t$ can grow beyond any limit; it only means that the major axes will go back through their initial value infinitely many times.

Can all the solutions be stable, in the meaning of Poisson? Poisson did not think so, because his proof expressly assumed that the mean motions are not commensurable; the proof therefore does not apply to arbitrary initial conditions of the motion.

The existence of asymptotic solutions, which we will establish later, is sufficient to show that if the initial position of the point P is chosen appropriately, then this point P will not return infinitely many times as close to this initial position as desired.

But I propose to establish that, in one of the specific cases of the three-body problem, the initial position of the point P can be chosen (and can be chosen infinitely many ways) such that this point P returns as close to its initial position as desired infinitely many times.

In other words, there will be infinitely many specific solutions to the problem which will not be stable in the second sense of the word—that is, in the meaning of Poisson; but, there will be infinitely many which are stable. I will add that the first can be regarded as exceptional and later I will seek to understand the precise meaning that I give to this word.

Assume $n = 3$ and consider x_1, x_2, x_3 as the spatial coordinates of a point P.

Theorem I *Assume that the point P remains at a finite distance and that the volume $\int \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3$ is an integral invariant; consider an arbitrary region r_0, however, small this region, there will be trajectories which will pass through it infinitely many times.*

In fact, since the point P remains at a finite distance, it will never leave a bounded region R. I call V the volume of this region R.

Now imagine a very small region r_0, I will call the volume of this region v. A trajectory passes through each of the points of r_0; this trajectory can be regarded as the path followed by a point moving according to the law defined by our differential equations. Therefore consider infinitely many moving points which at time zero fill the region r_0 and which then move according to this law. At time τ they will fill some region r_1, at time 2τ a region r_2, etc. and at time $n\tau$ a region r_n. I can assume that τ is large enough and that r_0 is small enough so that r_0 and r_1 have no point in common.

Since the volume is an integral invariant, these various regions $r_0, r_1, \ldots r_n$ will have the same volume v. If these regions have no point in common, then the total volume would not be larger than nv; on the other hand all these regions are inside R so the total volume is smaller than V. If therefore we have:

$$n > \frac{V}{v},$$

then it must be that at least two of our regions have a common portion. Let r_p and r_q be these two regions ($q > p$). If r_p and r_q have a common portion, it is clear that r_0 and r_{q-p} will have to have a common portion.

More generally, if k regions having a common portion can be found, no point in space could belong to more than $k-1$ of the regions. The total volume occupied by these regions would therefore be greater than $nv/(k-1)$. If therefore we have:

$$n > (k-1)\frac{V}{v},$$

then it must be possible to find k regions having a common portion. Let:

$$r_{p_1}, r_{p_2}, \ldots r_{p_k}$$

be these regions. Then

$$r_0, r_{p_2-p_1}, r_{p_3-p_1}, \ldots r_{p_k-p_1}$$

will also have a common portion.

But, let us take up the question again from a different perspective. By analogy with the nomenclature from the preceding section, we agree to state that the region r_n is the nth recurrence of r_0, and r_0 is the nth antecedent of r_n.

Suppose then that r_p is the first of the successive recurrences which has a common portion with r_0. Let this common portion be r'_0; let s'_0 be the pth antecedent of r'_0 which would also be part of r_0 because its pth recurrence is part of r_p.

Then let r'_{p_1} be the first of the recurrences of r'_0 which has a common portion with r'_0; let r''_0 be this common portion; its p_1th antecedent will be part of r'_0 and consequently of r_0, and its $p+p_1$th antecedent which I will call s''_0 will be part of s'_0 and consequently of r_0.

Thus s''_0 will be part of r_0 and so will its pth and p_1th recurrences.

And so on.

With r''_0 we will form r'''_0 as we formed r''_0 with r'_0 and r'_0 with r_0; we will then form $r^{IV}_0, \ldots r^n_0, \ldots$.

I will assume that the first of the successive recurrences of r^n_o which has a common portion with r^n_o is that of order p_n.

I will call s^n_0 the antecedent of r^n_o of order $p + p_1 + p_2 + \ldots p_{n-1}$.

Then s_0^n will be part of r_0 and also of its n recurrences of order:

$$p, p+p_1, p+p_1+p_2, \ldots p+p_1+p_2+\cdots p_{n-1}.$$

Additionally s_0^n will be part of s_0^{n-1}, s_0^{n-1} of s_0^{n-2}, etc.

There will then be points which belong at the same time to the regions $r_0, s_0', s_0'', \ldots s_0^n, s_o^{n+1}, \ldots$ ad infinitum. The set of these points will form a region σ which could additionally reduce to one or several points.

Then the region σ will be part of r_0 and also of its recurrences of order p, $p+p_1$, $p+p_1+p_2, \ldots, p+p_1+p_2+\cdots p_n, p+p_1+p_2+\cdots p_n+p_{n+1}, \ldots$ ad infinitum.

In other words, any trajectory coming from one of the points of σ will traverse the region r_0 infinitely many times.
Which was to be proved.

Corollary *It follows from the preceding that there exist infinitely many trajectories which traverse the region* r_0 *infinitely many times; but there can exist others which only traverse this region a finite number of times. I now propose to explain why the latter trajectories can be regarded as exceptional.*

Since this expression does not have any precise meaning in itself, I will need to start by filling-in the definition.

We agree to state that the ratio of the probability that the initial position of the moving point P belongs to a certain region r_0 to the probability that this initial position belongs to another region r_0' is equal to the ratio of the volume of r_0 to the volume of r_0'.

With the probabilities thus defined, I propose to establish that the probability is zero that a trajectory coming from a point in r_0 does not traverse this region more than k times, however large k is and however small the region r_0 is. This is what I mean when I state that the trajectories which only traverse r_0 a finite number of times are exceptional.

I assume that the initial position of the point P belongs to r_0 and I propose to calculate the probability that the trajectory coming from this point does not traverse the region r_0 $k+1$ times from the epoch O to the epoch $n\tau$.

We have seen that if the volume v of r_0 is such that:

$$n > \frac{kV}{v},$$

then $k+1$ regions can be found that I will call

$$r_0, r_{\alpha_1}, \ldots r_{\alpha_k}$$

and which will have a common portion. If s_{α_k} is this common portion, let s_0 be its antecedent of order α_k and designate the pth recurrence of s_0 by s_p.

I state that if the initial position of the point P belongs to s_0, then the trajectory coming from this point will cross the region r_0 at least $k+1$ times between the epoch 0 to the epoch $n\tau$.

In fact, the moving point which describes this trajectory will be found in the region s_0 at epoch 0, in the region s_p at epoch $p\tau$, and in the region s_n at the epoch $n\tau$. It will therefore necessarily traverse, between the epochs 0 and $n\tau$, the following regions:

$$s_0, s_{\alpha_k-\alpha_{k-1}}, s_{\alpha_k-\alpha_{k-2}}, \ldots s_{\alpha_k-\alpha_2}, s_{\alpha_k-\alpha_1}, s_{\alpha_k}.$$

Now I state that all these regions are part of r_0. In fact s_{α_k} is part of r_0 by definition; s_0 is part of r_0 because its α_kth recurrence s_{α_k} is part of r_{α_k}, and in general $s_{\alpha_k-\alpha_i}$ is part of r_0 because its α_ith recurrence s_{α_k} is part of r_{α_i}.

Therefore the moving point will pass through the region r_0 at least $k+1$ times. Which was to be proved.

Now let σ_0 be the portion of r_0 that does not belong either to s_0 or to any analogous region, such that the trajectories originating from the various points of σ_0 do not traverse the region r_0 at least $k+1$ times between the epochs 0 and $n\tau$. Let the volume of σ_0 be w.

The probability being sought, meaning the probability that our trajectory does not traverse r_0 $k+1$ times between these two epochs will then be w/v.

Now, by assumption, no trajectory originating from σ_0 traverses r_0, and especially not σ_0, $k+1$ times between these two epochs. We then have:

$$w < \frac{kV}{n}$$

and our probability will be smaller than

$$\frac{kV}{nv}.$$

However large k is and however small v is, n can always be taken large enough such that this expression is as small as we want. Therefore, there is a null probability that our trajectory, which we know originates from a point in r_0, does not traverse this region more than k times since the epoch 0 until the epoch $+\infty$. Which was to be proved.

Extension of theorem I. We assumed that:

(1) $n = 3$
(2) The volume is an integral invariant.
(3) The point P is constrained to remain within a finite distance.

The theorem is still true if the volume is not an integral invariant, provided that there exists an arbitrary positive invariant:

$$\int M \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3.$$

It is still true if $n > 3$, if there is a positive invariant:

$$\int M \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n$$

and if $x_1, x_2, \cdots x_n$, which are the coordinates of the point P in the n-dimensional space, are constrained to remain finite.

But there is more.

Suppose that $x_1, x_2, \cdots x_n$ are no longer constrained to remain finite but that the positive integral invariant

$$\int M \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n$$

over the entire n-dimensional space has a finite value. The theorem will still be true.

Here is a case which will come up more frequently.

Assume that an integral of Eq. (1) is known

$$F(x_1, x_2, \cdots x_n) = \text{const.}$$

If $F = \text{const.}$ is the general equation of a family of closed surfaces in n-dimensional space, if, in other words, F is a one-to-one function which becomes infinite when any one of the variables $x_1, x_2, \cdots x_n$ stops being finite, it is clear that $x_1, x_2, \cdots x_n$ will always remain finite, because F keeps a constant finite value; this is therefore within the conditions of the statement of the theorem.

But suppose that the surfaces $F = \text{const.}$ are not closed; it could nonetheless turn out that the positive integral invariant

$$\int M \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n$$

has a finite value over all the families of values of x such that:

$$C_1 < F < C_2;$$

the theorem will again be true.

This is what happens in particular in the following case.

In G. W. Hill's theory of the moon, in a first approximation he neglected the parallax of the sun, the eccentricity of the sun and the inclination of the orbits; he arrived at the following equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x', \quad \frac{\mathrm{d}x'}{\mathrm{d}t} = 2n'y' - x\left(\frac{\mu}{\sqrt{(x^2+y^2)^3}} - 3n'^2\right),$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = y', \quad \frac{\mathrm{d}y'}{\mathrm{d}t} = -2n'x' - \frac{\mu y}{\sqrt{(x^2+y^2)^3}},$$

which have the integral

$$F = \frac{x'^2+y'^2}{2} - \frac{\mu}{\sqrt{x^2+y^2}} - \frac{3}{2}n'^2x^2 = \text{const.}$$

and the integral invariant

$$\int \mathrm{d}x\mathrm{d}y\mathrm{d}x'\mathrm{d}y'.$$

If we regard x, y, x', and y' as the coordinates of a point in four-dimensional space, then the equation $F = \text{const.}$ represents a family of surfaces which are not closed. But the integral invariant over all points included between two of these surfaces is finite, as we will prove.

Theorem I is therefore still true; meaning that there exist trajectories which traverse any region of the four-dimensional space, however small this region might be, infinitely many times.

It remains to calculate the quadruple integral

$$J = \int \mathrm{d}x\mathrm{d}y\mathrm{d}x'dy',$$

where this integral is over all families of values such that

$$C_1 < F < C_2.$$

We change variables and transform our quadruple integral by setting:

$$x' = \cos\varphi\sqrt{2r}, \quad y' = \sin\varphi\sqrt{2r},$$
$$x = \rho\cos\omega, \quad y = \rho\sin\omega;$$

this integral becomes:

$$J = \int \rho\mathrm{d}\rho\mathrm{d}r\mathrm{d}\omega\mathrm{d}\varphi$$

and it also follows:

$$F = r - \frac{\mu}{\rho} - \frac{3}{2} n'^2 \rho^2 \cos^2 \omega.$$

We need to first integrate over φ between the limits 0 and 2π, which gives:

$$J = 2\pi \int \rho \mathrm{d}\rho \mathrm{d}r \mathrm{d}\omega$$

and the integration must be over all families of values of ρ, r, and ω which satisfy the inequalities:

$$\begin{aligned} r &> 0, \\ r &> C_1 + \frac{\mu}{\rho} + \frac{3}{2} n'^2 \rho^2 \cos^2 \omega, \\ r &< C_2 + \frac{\mu}{\rho} + \frac{3}{2} n'^2 \rho^2 \cos^2 \omega. \end{aligned} \tag{1}$$

The following can be deduced from these inequalities:

$$C_2 + \frac{\mu}{\rho} + \frac{3}{2} n'^2 \rho^2 \cos^2 \omega > 0.$$

Regard ρ and ω as polar coordinates of a point and construct the curve

$$C_2 + \frac{\mu}{\rho} + \frac{3}{2} n'^2 \rho^2 \cos^2 \omega = 0.$$

We will see that if C_2 is smaller than $-\frac{1}{2}(9n'\mu)^{2/3}$ this curve is composed of a closed oval located entirely inside the circle

$$\rho = \sqrt[3]{\frac{\mu}{3n'^2}}$$

and of two infinite branches located entirely outside the circle.

The reader will be able to do this construction easily; if the reader experiences any difficulty, I suggest they consult the original treatise of G.W. Hill in the *American Journal of Mathematics*, volume 1.

From this G. W. Hill concluded that if the point ρ, ω is inside this closed oval at the beginning of time, it will always remain there and consequently ρ will always remain smaller than $\sqrt[3]{\mu/3n'^2}$. Thus if the parallax of the sun, its eccentricity, and the inclinations are neglected, it will be possible to assign an upper limit to the radius vector of the moon. In fact as it relates to the moon, the constant C_2 is smaller than $-\frac{1}{2}(9n'\mu)^{2/3}$.

I propose to supplement this remarkable result from G. W. Hill by showing that, under these conditions, the moon would also experience stability in the meaning of Poisson; by that I mean that, if the motion's initial conditions are not exceptional, the moon would return as close as one wants to its initial position infinitely many times. That is why, as I explained above, I propose to prove that the integral J is finite.

Since ρ is smaller than $\sqrt[3]{\mu/3n'^2}$ and consequently bounded, the integral:

$$J = 2\pi \int \rho d\rho \mathrm{d}r \mathrm{d}\omega$$

can only become infinite if r increases indefinitely, and r cannot become infinite in light of the inequalities (1) unless ρ approaches zero.

Therefore set:

$$J = J' + J'',$$

where J' represents the integral over all families of values such that

$$r > 0, \quad \rho > \rho_0, \quad C_1 < F < C_2 \tag{2}$$

and J'' represents the integral over all families of values such that:

$$r > 0, \quad \rho < \rho_0, \quad C_1 < F < C_2. \tag{3}$$

When the inequalities (2) are satisfied ρ cannot become zero; therefore r cannot become infinite. Therefore the first integral, J' is finite.

Now examine J''. I can assume that ρ_0 was taken small enough that

$$C_1 + \frac{\mu}{\rho_0} > 0.$$

The inequalities $F > C_1$ and $\rho < \rho_0$ then lead to $r > 0$. We therefore need to integrate over r between the limits:

$$C_1 + \frac{\mu}{\rho} + \frac{3}{2} n'^2 \rho^2 \cos^2 \omega \quad \text{and} \quad C_2 + \frac{\mu}{\rho} + \frac{3}{2} n'^2 \rho^2 \cos^2 \omega.$$

It then follows:

$$J'' = 2\pi (C_2 - C_1) \int_0^{2\pi} \mathrm{d}\omega \int_0^{\rho_0} \rho \mathrm{d}\rho = 2\pi^2 \rho_0^2 (C_2 - C_1)$$

J'' is therefore finite and consequently also J.
Which was to be proved.

K. Bohlin generalized the result of G. W. Hill in the following way. We consider the following special case of the three-body problem. Let A be a body of mass $1-\mu$, B be a body of mass μ, and C a body of infinitesimal mass. Imagine that the two bodies A and B whose motion must be Keplerian, because it is not perturbed by the mass C, trace out around their mutual center of gravity, assumed to be fixed, two concentric circumferences, and that C moves in the plane of these two circumferences. I will take a constant distance AB as a unit of length, such that the radii of these two circumferences are $1-\mu$ and μ. I will assume that the unit of time has been selected such that the angular speed of the two points A and B on their circumferences is equal to 1 (or that the Gaussian gravitational constant is equal to 1, which amounts to the same thing).

We next select two moving axes with their origin at the center of gravity of the two masses A and B; the first of these axes will be the straight line AB and the second will be perpendicular to the first.

The coordinates of A relative to these two axes are $-\mu$ and 0; those of B are $1-\mu$ and 0; and those for C, I will call x and y; for the equations of motion I then have:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= x', \quad \frac{\mathrm{d}x'}{\mathrm{d}t} = 2y' + \frac{\mathrm{d}V}{\mathrm{d}x} + x, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= y', \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -2x' + \frac{\mathrm{d}V}{\mathrm{d}y} + y, \end{aligned}$$

by setting

$$V = \frac{1-\mu}{AC} + \frac{\mu}{BC}.$$

Additionally we have:

$$\overline{AC}^2 = (x+\mu)^2 + y^2, \quad \overline{BC}^2 = (x+\mu-1)^2 + y^2$$

These equations have an integral:

$$F = \frac{x'^2 + y'^2}{2} - V - \frac{x^2+y^2}{2} = K$$

and an integral invariant:

$$J = \int \mathrm{d}x\mathrm{d}y\mathrm{d}x'\mathrm{d}y'$$

K. Bohlin, in *Acta Mathematica* volume 10, generalized the result of G.W. Hill, by showing that if the constant K has a suitable value (which we will assume) and if the initial values of x and y are small enough, these quantities, x and y, will remain bounded.

I now propose to prove that the integral J over all families of values such that

$$K_1 < F < K_2$$

is finite; and from that we will be able to conclude, as we did above, that the stability in the meaning of Poisson pertains again in this case.

If the constants K_1 and K_2 are suitably chosen, the theorem from K. Bohlin shows that x and y will be bounded. As for x' and y', it will not be possible for them to become infinite unless V becomes infinite, meaning if AC approaches zero or if BC approaches zero.

Then set:

$$J = J' + J'' + J''',$$

where the integral J' is over all families of values such that:

$$K_1 < F < K_2, \quad \overline{AC}^2 > \rho_0^2, \quad \overline{BC}^2 > \rho_0^2, \quad \left(\rho_0 < \frac{1}{2}\right)$$

the integral J'' to all families of values such that:

$$K_1 < F < K_2, \quad \overline{AC}^2 < \rho_0^2, \quad \left(\text{hence } \overline{BC}^2 > \rho_0^2\right),$$

and the integral J''' to all families of values such that:

$$K_1 < F < K_2, \quad \overline{BC}^2 < \rho_0^2 \quad \left(\text{hence } \overline{AC}^2 > \rho_0^2\right).$$

Since for none of the families of values over which the integral J' extends do AC or BC become zero, this integral J' is finite.

Now examine the integral J''. I can assume that ρ_0 has been chosen small enough such that:

$$\frac{1-\mu}{\rho_0} + K_1 > 0, \quad \frac{\mu}{\rho_0} + K_1 > 0.$$

In this case $(x'^2 + y'^2)/2$ can vary between the bounds:

$$L_1 = K_1 + \frac{1-\mu}{AC} + \frac{\mu}{BC} + \frac{x^2+y^2}{2} \quad \text{and} \quad K_2 + \frac{1-\mu}{AC} + \frac{\mu}{BC} + \frac{x^2+y^2}{2} = L_2,$$

because the smaller of these two bounds is positive.

Then set as above:

$$x' = \sqrt{2r}\cos\varphi, \quad y' = \sqrt{2r}\sin\varphi, \quad \text{hence } r = \frac{x'^2 + y'^2}{2};$$

the integral will become

$$J'' = \int \mathrm{d}x\mathrm{d}y\mathrm{d}r\mathrm{d}\varphi$$

and it will be necessary to integrate over φ between the bounds 0 and 2π, and over r between the bounds L_1 and L_2; it will then become:

$$J'' = 2\pi(K_2 - K_1)\int \mathrm{d}x\mathrm{d}y.$$

The double integral $\int \mathrm{d}x\mathrm{d}y$ will then need to be over all families of values such that $\overline{AC}^2 < \rho_0^2$; it is therefore equal to $\pi\rho_0^2$; such that it becomes:

$$J'' = 2\pi^2\rho_0^2(K_2 - K_1).$$

J'' is therefore finite, and so are J''' and J.
Which was to be proved.

We therefore have to conclude that (if the initial conditions of motion are not exceptional in the meaning given to this word in the corollary to theorem I) the third body C will go back as close as one wants to its initial position infinitely many times.

In the general case of the three-body problem, it can no longer be affirmed that it will still be the same.

Theorem II *If $n = 3$ and x_1, x_2, x_3 represent the coordinates of a point in ordinary space, and if there is a positive invariant, there cannot be a closed contactless surface.*

In fact let

$$J = \int M\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$$

be a positive integral invariant. Assume that there is a closed and contactless surface, having the equation

$$F(x_1, x_2, x_3) = 0.$$

Let V be the volume delimited by this surface; we extend the invariant J to this entire volume.

Since the surface S is contactless, the expression:

$$\frac{dF}{dx_1}X_1 + \frac{dF}{dx_2}X_2 + \frac{dF}{dx_3}X_3$$

cannot become zero and consequently change sign; to be concrete, we will assume that it is positive.

Let $d\omega$ be a differential element of the surface S; take the normal to this element from the side of increasing F; take on this normal an infinitesimal segment dn. Let $\frac{dF}{dn}dn$ be the value of F at the end of this segment. We will then have:

$$\frac{dF}{dn} > 0.$$

since J is an invariant, we should have

$$\frac{dJ}{dt} = 0.$$

But we find

$$\frac{dJ}{dt} = \int M \frac{\frac{dF}{dx_1}X_1 + \frac{dF}{dx_2}X_2 + \frac{dF}{dx_3}X_3}{\frac{dF}{dn}} d\omega.$$

The integral on the right-hand side, over the entire surface S, is positive because the function within the integral sign is always positive.

We have arrived therefore at two contradictory results and we have to conclude that a closed contactless surface cannot exist.

Extension of Theorem II. It is easy to extend this theorem to the case of $n > 3$; to do that it is sufficient to translate it into analytical language, because geometric representation is no longer possible, and state:

If there is a positive integral invariant, there cannot exist a one-to-one function $F(x_1, x_2, \cdots x_n)$ which is positive, which becomes infinite each time one of the x stops being finite and which is such that

$$\frac{dF}{dt} = \frac{dF}{dx_1}X_1 + \frac{dF}{dx_2}X_2 + \cdots \frac{dF}{dx_n}X_n$$

always has the same sign when F is zero.

To make the importance of this theorem understood, I will limit myself to observing that it is a generalization of the one which I used for proving the legitimacy of Lindstedt's beautiful method.

However, with a perspective to subsequent applications, I prefer to give it a little bit different form by introducing into it a new concept: that of invariant curves.

At the end of the previous section we had considered a portion of surface S, defined by the equation

$$\Theta(x_1, x_2, x_3) = 0$$

and such that for all points of S it holds that

$$\frac{\mathrm{d}\Theta}{\mathrm{d}x_1}X_1 + \frac{\mathrm{d}\Theta}{\mathrm{d}x_2}X_2 + \frac{\mathrm{d}\Theta}{\mathrm{d}x_3}X_3 > 0,$$

such that S is a portion of contactless surface.

We have subsequently defined what was to be understood by the nth recurrence of a point from S or by the nth recurrence of a curve or an area belonging to S. I now understand and from now on I will understand the word recurrence in the meaning of the previous section and not in the meaning used above in the proof of Theorem I.

We have seen that if there is a positive invariant

$$\iiint M \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3,$$

there is also another integral

$$\int MHd\omega$$

which must be over all the elements $\mathrm{d}\omega$ of an area belonging to S and which has the following properties:

(1) The quantity under the integral sign, MH, is always positive.
(2) The integral has the same value for an arbitrary area belonging to S and for all areas of its recurrences which exist.

With that stated, I will call nth order *invariant curve* any curve traced on S and which coincides with its nth recurrence.

In most questions from dynamics, some very small parameters enter such that one is naturally led to develop solutions following increasing powers of these parameters. Such are the masses in celestial mechanics.

We will therefore imagine that our differential equations

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = X_1, \quad \frac{\mathrm{d}x_2}{\mathrm{d}t} = X_2, \quad \frac{\mathrm{d}x_3}{\mathrm{d}t} = X_3$$

depend on a parameter μ. We will suppose that X_1, X_2, X_3 are given functions of x_1, x_2, x_3 and μ which could be expanded in increasing powers of μ and that μ is very small.

Now consider an arbitrary function of μ; I assume that this function approaches 0 when μ approaches 0, such that the ratio of this function to μ^n approaches a finite limit. I will state that this function of μ is a very small quantity of nth order.

It needs to be indicated that it is not necessary for it to be possible to expand this function of μ in powers of μ.

With that established, let A_0 and B_0 be two points on a contactless surface S, and let A_1 and B_1 be their recurrences. If the position of A_0 and B_0 depends on μ according to an arbitrary law, then so will the position of A_1 and B_1. I am proposing to prove the following lemmas:

Lemma I *If a portion of contactless surface S passing through the point* a_0, b_0, c_0 *is considered; if* x_0, y_0, z_0 are *coordinates of a point on S and if* x_1, y_1, z_1 *are coordinates of its recurrence, then* x_1, y_1, z_1 *are expandable in powers of* $x_0 - a_0, y_0 - b_0, z_0 - c_0$ *and* μ *provided that these quantities are sufficiently small.*

I can always take for origin the point a_0, b_0, c_0 such that

$$a_0 = b_0 = c_0 = 0.$$

If then

$$z = \varphi(x, y)$$

is the equation of the surface S; this surface will pass through the origin O and one will have:

$$\varphi(0, 0) = 0.$$

I will additionally assume that the function $\varphi(x, y)$ is mapping at all the points on the portion of surface S considered. One trajectory passes through the origin O; imagine that when $\mu = 0$ this trajectory crosses the surface S at time $t = \tau$ at a point P whose coordinates will be:

$$x = a, \quad y = b, \quad z = c$$

According to the terminology that we have adopted, *when it is assumed that* $\mu = 0$, the point P will be the recurrence of the point O.

Now let x_0, y_0, z_0 be a point A very close to O and belonging to the surface S. If a trajectory passes through this point A, and if it is assumed that μ stops being zero but remains very small, it will be seen that this trajectory will come, at an epoch t only slightly different from τ, to cross the surface S at a point B very near P.

This point B, whose coordinates I will call x_1, y_1, z_1, will according to our terminology be the recurrence to the point A.

What I propose to prove is that x_1, y_1, z_1 are expandable in increasing powers of x_0, y_0, z_0 and μ.

In fact according to Theorem III from Sect. 2 of Chap. 1, if x, y, z are coordinates at time t of the moving point which describes the trajectory coming from point A and if additionally x_0, y_0, z_0, μ and $t-\tau$ are sufficiently small, then one will have:

$$\begin{aligned} x &= \psi_1(t-\tau, \mu, x_0, y_0, z_0), \\ y &= \psi_2(t-\tau, \mu, x_0, y_0, z_0), \\ z &= \psi_3(t-\tau, \mu, x_0, y_0, z_0), \end{aligned} \tag{4}$$

where ψ_1, ψ_2 and ψ_3 are series ordered in powers of $t-\tau, \mu, x_0, y_0$ and z_0.

These series will reduce, respectively, to a, b, c for

$$t-\tau = \mu = x_0 = y_0 = z_0 = 0.$$

Since $\varphi(x, y)$ is expandable in powers of $x-a$ and $y-b$, if $x-a$ and $y-b$ are small enough, we will also have:

$$\varphi(x, y) = \psi_4(t-\tau, \mu, x_0, y_0, z_0),$$

where ψ_4 is a series with the same form as ψ_1, ψ_2 and ψ_3.

We write that the point x, y, z is located on the surface S; we will have:

$$\psi_3 = \psi_4 \tag{5}$$

The relation (5) can be regarded as a relation between $t-\tau, \mu, x_0, y_0$ and z_0, and one can try to solve it for $t-\tau$.

For:

$$t-\tau = \mu = x_0 = y_0 = z_0 = 0$$

this relation is satisfied because one has

$$\psi_3 = \psi_4 = 0.$$

According to a theorem by Cauchy, which we proved in one of the preceding sections, one can draw $t-\tau$ from the relationship (5) in the following form:

$$t-\tau = \theta(\mu, x_0, y_0, z_0), \tag{6}$$

where θ is a series ordered in powers of μ, x_0, y_0 and z_0.

The only exception would be if for

$$t-\tau = \mu = x_0 = y_0 = z_0 = 0$$

it held that:

$$\frac{d\psi_3}{dt} = \frac{d\psi_4}{dt}.$$

Now this equation expresses that the trajectory starting from point O for $\mu = 0$ is going to *touch* the surface S at point P.

But it can not be that way, because we will always assume that S is a contactless surface or a portion of contactless surface.

In Eq. (4) we replace $t - \tau$ by θ and x, y, z by x_1, y_1, z_1; it follows:

$$\begin{aligned} x_1 &= \Theta_1(\mu, x_0, y_0, z_0), \\ y_1 &= \Theta_2(\mu, x_0, y_0, z_0), \\ z_1 &= \Theta_3(\mu, x_0, y_0, z_0), \end{aligned}$$

where Θ_1, Θ_2 and Θ_3 are expanded in powers of μ, x_0, y_0 and z_0.
Which was to be proved.

Lemma II *If the distance between two points A_0 and B_0 belonging to a portion of the contactless surface S is a very small quantity of order n, then so will the distance between their recurrences A_1 and B_1.*

In fact, let a_1, a_2, a_3 be the coordinates of a fixed point P_0 from S very near A_0 and B_0; and let a'_1, a'_2, a'_3 be the coordinates of its recurrence P_1.

Let x_1, x_2, x_3; x'_1, x'_2, x'_3; y_1, y_2, y_3; and y'_1, y'_2, y'_3 be the coordinates of A_0, A_1, B_0 and B_1.

According to Lemma I $x'_1 - a'_1, x'_2 - a'_2, x'_3 - a'_3$ are expandable in increasing powers of $x_1 - a_1, x_2 - a_2, x_3 - a_3$ and μ.

The expression for $y'_1 - a'_1, y'_2 - a'_2, y'_3 - a'_3$ as a function of $y_1 - a_1, y_2 - a_2, y_3 - a_3$ and μ will obviously be the same as that for $x'_1 - a'_1, x'_2 - a'_2, x'_3 - a'_3$ as a function of $x_1 - a_1, x_2 - a_2, x_3 - a_3$ and μ.

From that we conclude that it is possible to write:

$$\begin{aligned} x'_1 - y'_1 &= (x_1 - y_1)F_1 + (x_2 - y_2)F_2 + (x_3 - y_3)F_3, \\ x'_2 - y'_2 &= (x_1 - y_1)F'_1 + (x_2 - y_2)F'_2 + (x_3 - y_3)F'_3, \\ x'_3 - y'_3 &= (x_1 - y_1)F''_1 + (x_2 - y_2)F''_2 + (x_3 - y_3)F''_3, \end{aligned} \tag{7}$$

where the F are series expanded in powers of:

$$\mu, x_1 - a_1, x_2 - a_2, x_3 - a_3, y_1 - a_1, y_2 - a_2, y_3 - a_3.$$

The quantities F_1, F_2,, etc. are finite; therefore if $x_1 - y_1$, $x_2 - y_2$, and $x_3 - y_3$ are very small quantities of order n, then $x'_1 - y'_1$, $x'_2 - y'_2$, and $x'_3 - y'_3$ will be also.
Which was to be proved.

Theorem III *Let A_1AMB_1B be an invariant curve, such that A_1 and B_1 are the recurrences of A and B. I assume that the arcs AA_1 and BB_1 are very small (approach 0 with μ) but that their curvature is finite.*

I assume that this invariant curve and the position of points A and B depend on μ according to an arbitrary law. I assume that there exists a positive integral invariant. If the distance AB is very small of nth order, and the distance AA_1 is not very small of nth order, the arc AA_1 crosses the arc BB_1.

I can always join the points A and B by curve AB located entirely on the portion of contactless surface S and for which the total length is the same order of magnitude as the distance AB meaning a very small quantity of nth order. Let A_1B_1 be an arc of curve which is the recurrence of AB, it will thus be very small of nth order according to Lemma II.

Now here are the various scenarios that are conceivable:

Scenario 1. The two arcs AA_1 and BB_1 cross. I propose to establish that this is the actual scenario.
Scenario 2. The quadrilateral AA_1B_1B is such that the four arcs which are its sides have no other point in common than the four corners A, A_1, B, and B_1. This is the case from Fig. 1.
Scenario 3. The two arcs AB and A_1B_1 cross. This is the case from Fig. 2.
Scenario 4. One of the arcs AB or A_1B_1 crosses one of the arcs AA_1 or BB_1; but the arcs AA_1 and BB_1 do not cross nor do the two arcs AB and A_1B_1.

If there is a positive invariant, then according to the preceding section there will exist a certain integral

$$\int MH \mathrm{d}\omega$$

all the elements of which will be positive and which will have to have the same value for the area ABB_1MA and for its recurrence AA_1B_1MA.

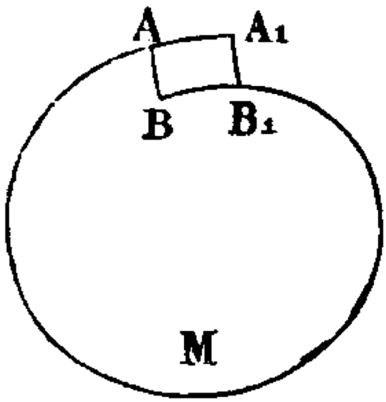

Fig. 1 The corners are the only common points of the four arcs

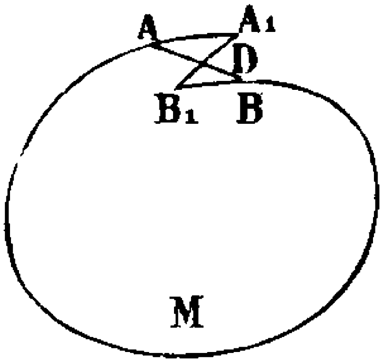

Fig. 2 The two arcs AB and A_1B_1 cross

This integral over the area

$$ABA_1B_1 = AA_1B_1MA - ABB_1MA$$

must therefore be zero and as all the elements of the integral are positive, the arrangement cannot be that from Fig. 1 where the area ABA_1B_1 is convex.

The second scenario must therefore be rejected.

In fact in the triangle ADA_1, the distances AD and A_1D are very small of nth order because they are smaller than the arcs AD and A_1D, which are smaller than the arcs AB or A_1B_1 which are of nth order. Furthermore it holds that:

$$AA_1 < AD + A_1D.$$

The distance AA_1 would therefore be a very small quantity of nth order which is contrary to the statement of the theorem.

The third scenario must therefore be rejected.

I state that the fourth scenario cannot be accepted either. Assume in fact for example that the arc AB crosses the arc AA_1 at a point A'. Let ANA' be the portion of the arc AB which goes from A to A'; let APA' be the portion of arc AA_1 which goes from A to A'.

I state that the arc $ANA'B$ can be replaced by the arc $APA'B$; and that the new arc $APA'B$ will be a very small quantity of nth order like the primitive arc $ANA'B$.

In fact the arc ANA' is smaller than AB and it is therefore of nth order;

the distance AA' is therefore itself of nth order; the arc APA' is smaller than AA_1 which is very small—meaning it approaches 0 with μ; the arc APA' is therefore very small and its curvature is finite; therefore a bound can be assigned to the ratio of the arc APA' to its chord AA'; this ratio is finite and AA' is of nth order; therefore APA' is of nth order, which was to be proved.

Furthermore the new arc $APA'B$ no longer crosses the arc AA_1 and it only has a common portion APA' with it.

This falls back on the second scenario which was already rejected.

The first scenario is therefore the only one acceptable and the theorem is therefore proved.

Remark In the statement of the theorem we have assumed that the arcs AA_1 and BB_1 are very small and that there curvature is finite. In reality we have only made use of this assumption for showing that if the chord AA' is very small of nth order, it is the same for the arc APA'.

The theorem will therefore still be true even if the arc AA_1 is no longer very small and its curvature finite, provided that it is possible to assign an upper bound to the ratio of an arbitrary arc (which is part of AA_1 or BB_1) to its chord.

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