2

Lagrange Problems

In this chapter we study the structure of approximate solutions of an autonomous nonconcave discrete-time optimal control system with a compact metric space of states. This control system is described by a bounded upper semicontinuous objective function which determines an optimality criterion. In the turnpike theory, it is known that approximate solutions are determined mainly by the objective function, and are essentially independent of the choice of time intervals and data, except in regions close to the endpoints of the time intervals. In this chapter our main goal is to analyze the structure of approximate solutions in regions close to the endpoints of the time intervals.

2.1 Discrete-Time Optimal Control Systems

Let (X, ρ) be a compact metric space and Ω be a nonempty closed subset of $X \times X$.

A sequence $\{x_t\}_{t=0}^{\infty} \subset X$ is called an (Ω)-program if $(x_t, x_{t+1}) \in \Omega$ for all integers $t \geq 0$. A sequence $\{x_t\}_{t=T_1}^{T_2} \subset X$, where integers T_1, T_2 satisfy $0 \leq T_1 < T_2$, is called an (Ω)-program if $(x_t, x_{t+1}) \in \Omega$ for all integers $t \in [T_1, T_2 - 1]$.

We consider the problem

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) \to \max, \ \{(x_t, x_{t+1})\}_{t=0}^{T-1} \subset \Omega, \ x_0 = z_1, \ x_T = z_2 \qquad (P1)$$

studied in [72], the problem

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) \to \max, \ \{(x_t, x_{t+1})\}_{t=0}^{T-1} \subset \Omega, \ x_0 = z_1 \qquad (P2)$$

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studied in [71], and the problem

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) \to \max, \ \{(x_t, x_{t+1})\}_{t=0}^{T-1} \subset \Omega, \tag{P3}$$

considered in [83], where T is a natural number, $z_1, z_2 \in X$, and $v : \Omega \to R^1$ is a bounded upper semicontinuous objective function.

In models of economic growth the set X is the space of states, v is a utility function, and $v(x_t, x_{t+1})$ evaluates consumption at moment t. The interest in discrete-time optimal problems of types (P1)–(P3) also stems from the study of various optimization problems which can be reduced to it [8, 37–39, 46, 70].

In [71–73, 75–77, 83, 84] we were interested in a turnpike property of the approximate solutions of problems (P1)–(P3) which is independent of the length of the interval T, for all sufficiently large intervals. In [71, 72, 83, 84] we studied the problems (P1), (P2), and (P3) and showed under certain assumptions that the turnpike property holds and that the turnpike $\bar{x}$ is a unique solution of the maximization problem $v(x, x) \to \max$, $(x, x) \in \Omega$. More precisely, we considered a class of (v)-good programs which are approximate solutions of the corresponding infinite horizon optimal control problem associated with the objective function v. It was shown that the turnpike property holds and $\bar{x}$ is the turnpike if the following asymptotic turnpike property holds: the all (v)-good programs converge to $\bar{x}$.

In this chapter we show that the asymptotic turnpike property holds for most objective functions in the sense of Baire category. In other words, the asymptotic turnpike property holds for a generic (typical) objective function. This result was obtained in [83].

In this chapter we also study the structure of approximate solutions of the problems $(P2)$ and $(P3)$ in regions close to the endpoints of the time intervals. We show that in regions close to the right endpoint T of the time interval these approximate solutions are determined only by the objective function, and are essentially independent of the choice of interval and endpoint value z_1. For the problems $(P3)$, approximate solutions are determined only by the objective function also in regions close to the left endpoint 0 of the time interval.

More precisely, we define $\bar{\Omega} = \{(y, x) \in X \times X : \ (x, y) \in \Omega\}$ and $\bar{v}(y, x) = v(x, y)$ for all $(x, y) \in \Omega$ and consider the set $\mathcal{P}(\bar{v})$ of all solutions of a corresponding infinite horizon optimal control problem associated with the pair $(\bar{v}, \bar{\Omega})$. For given a real positive number ϵ and a natural number τ, we show that if T is large enough and $\{x_t\}_{t=0}^{T}$ is an approximate solution of the problem $(P2)$, then $\rho(x_{T-t}, y_t) \le \epsilon$ for all integers $t \in [0, \tau]$, where $\{y_t\}_{t=0}^{\infty} \in \mathcal{P}(\bar{v})$.

Moreover, using the Baire category approach, we show that for most objective functions v the set $\mathcal{P}(\bar{v})$ is a singleton.

These results were obtained in [86, 90].

2.2 The Turnpike Results

Let (X, ρ) be a compact metric space and Ω be a nonempty closed subset of $X \times X$. Denote by $\mathcal{M}(\Omega)$ the set of all bounded functions $u : \Omega \to R^1$. For each $w \in \mathcal{M}(\Omega)$ set

$$\|w\| = \sup\{|w(x,y)| : \ (x,y) \in \Omega\}. \tag{2.1}$$

For each $x, y \in X$, each integer $T \geq 1$, and each $u \in \mathcal{M}(\Omega)$ set

$$\sigma(u,T,x) = \sup\{\sum_{i=0}^{T-1} u(x_i, x_{i+1}) : \ \{x_i\}_{i=0}^{T} \text{ is an } (\Omega) - \text{program and } x_0 = x\}, \tag{2.2}$$

$$\sigma(u,T,x,y) = \sup\{\sum_{i=0}^{T-1} u(x_i, x_{i+1}) : \tag{2.3}$$

$$\{x_i\}_{i=0}^{T} \text{ is an } (\Omega) - \text{program and } x_0 = x, \ x_T = y\},$$

$$\sigma(u,T) = \sup\{\sum_{i=0}^{T-1} u(x_i, x_{i+1}) : \ \{x_i\}_{i=0}^{T} \text{ is an } (\Omega) - \text{program}\}. \tag{2.4}$$

(Here we use the convention that the supremum of an empty set is $-\infty$).

For each $x, y \in X$, each pair of integers T_1, T_2 satisfying $0 \leq T_1 < T_2$, and each sequence $\{u_t\}_{t=T_1}^{T_2-1} \subset \mathcal{M}(\Omega)$ set

$$\sigma(\{u_t\}_{t=T_1}^{T_2-1}, T_1, T_2, x) = \sup\{\sum_{t=T_1}^{T_2-1} u_t(x_t, x_{t+1}) :$$

$$\{x_t\}_{t=T_1}^{T_2} \text{ is an } (\Omega) - \text{program and } x_{T_1} = x\}, \tag{2.5}$$

$$\sigma(\{u_t\}_{t=T_1}^{T_2-1}, T_1, T_2, x, y) = \sup\{\sum_{t=T_1}^{T_2-1} u_t(x_t, x_{t+1}) :$$

$$\{x_t\}_{t=T_1}^{T_2} \text{ is an } (\Omega) - \text{program and } x_{T_1} = x, \ x_{T_2} = y\}, \tag{2.6}$$

$$\sigma(\{u_t\}_{t=T_1}^{T_2-1}, T_1, T_2) = \sup\{\sum_{t=T_1}^{T_2-1} u_t(x_t, x_{t+1}) : \ \{x_t\}_{t=T_1}^{T_2} \text{ is an } (\Omega) - \text{program}\}, \tag{2.7}$$

$$\widehat{\sigma}(\{u_t\}_{t=T_1}^{T_2-1}, T_1, T_2, y) = \sup\{\sum_{t=T_1}^{T_2-1} u_t(x_t, x_{t+1}) :$$

$$\{x_t\}_{t=T_1}^{T_2} \text{ is an } (\Omega) - \text{program and } x_{T_2} = y\}. \tag{2.8}$$

Assume that $v \in \mathcal{M}(\Omega)$ is an upper semicontinuous function. We suppose that there exist $\bar{x}_v \in X$ and constants $\bar{c}_v > 0$ and $\bar{r}_v > 0$ such that the following assumptions hold.

(A1) $\{(x,y) \in X \times X : \ \rho(x,\bar{x}_v),\ \rho(y,\bar{x}_v) \le \bar{r}_v\} \subset \Omega$ and v is continuous at $(\bar{x}_v, \bar{x}_v)$.

(A2) $\sigma(v,T) \le Tv(\bar{x}_v,\bar{x}_v) + \bar{c}_v$ for all integers $T \ge 1$.

It is easy to see that for each natural number T and each (Ω)-program $\{x_t\}_{t=0}^{T}$,

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) \le \sigma(v,T) \le Tv(\bar{x}_v,\bar{x}_v) + \bar{c}_v. \tag{2.9}$$

Inequality (2.9) implies the following result.

Proposition 2.1. *For each (Ω)-program $\{x_t\}_{t=0}^{\infty}$ either the sequence*

$$\{\sum_{t=0}^{T-1} v(x_t, x_{t+1}) - Tv(\bar{x}_v,\bar{x}_v)\}_{T=1}^{\infty}$$

is bounded or $\lim_{T\to\infty}[\sum_{t=0}^{T-1} v(x_t, x_{t+1}) - Tv(\bar{x}_v,\bar{x}_v)] = -\infty$.

An (Ω)-program $\{x_t\}_{t=0}^{\infty}$ is called (v,Ω)-good if the sequence

$$\{\sum_{t=0}^{T-1} v(x_t, x_{t+1}) - Tv(\bar{x}_v,\bar{x}_v)\}_{T=1}^{\infty}$$

is bounded.

We suppose that the following assumption holds:

(A3) (the asymptotic turnpike property or, briefly, (ATP)) For any (v,Ω)-good program $\{x_t\}_{t=0}^{\infty}$, $\lim_{t\to\infty} \rho(x_t,\bar{x}_v) = 0$.

In view of (A1) and (A3), if $(\bar{x}_v,\bar{x}_v)$ is not an isolated point of $X \times X$, then $\|v\| > 0$.

Examples of optimal control problems satisfying (A1)–(A3) are given in [71, 72, 84].

Denote by $\mathrm{Card}(A)$ the cardinality of a set A and suppose that the sum over empty set is zero.

It is clear that for each pair of integers T_1, T_2 satisfying $0 \le T_1 < T_2$, each sequence $\{w_t\}_{t=T_1}^{T_2-1} \subset \mathcal{M}(\Omega)$, and each pair of points $x, y \in X$ satisfying $\rho(x,\bar{x}_v), \rho(y,\bar{x}_v) \le \bar{r}_v$ the value $\sigma(\{w_t\}_{t=T_1}^{T_2-1}, T_1, T_2, x, y)$ is finite.

Definition 2.2. *Let $T \ge 1$ be an integer. We denote by Y_T the set of all points $x \in X$ for which there exists an (Ω)-program $\{x_t\}_{t=0}^{T}$ satisfying $x_0 = \bar{x}_v$ and $x_T = x$ and denote by $\bar{Y}_T$ the set of all points $x \in X$ for which there exists an (Ω)-program $\{x_t\}_{t=0}^{T}$ such that $x_0 = x$ and $x_T = \bar{x}_v$.*

The following two theorems obtained in [76] and presented in [84] establish the turnpike property for approximate solutions of the optimal control problems of the types (P1) and (P2) with objective functions u_t, $t = 0, \dots, T-1$ which belong to a small neighborhood of v.

Theorem 2.3. *Let a positive number $\epsilon < \bar{r}_v$, $L_0 \geq 1$ be an integer and $M_0 > 0$. Then there exist an integer $L \geq 1$ and $\delta \in (0, \epsilon)$ such that for each integer $T > 2L$, each $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$ satisfying*

$$\|u_t - v\| \leq \delta,\ t = 0 \dots T-1,$$

and each (Ω)-program $\{x_t\}_{t=0}^{T}$ which satisfies

$$x_0 \in \bar{Y}_{L_0},\ x_T \in Y_{L_0},$$

$$\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=0}^{T-1}, 0, T, x_0, x_T) - M_0$$

and

$$\sum_{t=\tau}^{\tau+L-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=\tau}^{\tau+L-1}, \tau, \tau+L, x_\tau, x_{\tau+L}) - \delta$$

for each integer $\tau \in [0, T-L]$ there exist integers $\tau_1 \in [0, L]$, $\tau_2 \in [T-L, T]$ such that

$$\rho(x_t, \bar{x}_v) \leq \epsilon,\ t = \tau_1, \dots, \tau_2.$$

Moreover, if $\rho(x_0, \bar{x}_v) \leq \delta$, then $\tau_1 = 0$ and if $\rho(x_T, \bar{x}_v) \leq \delta$, then $\tau_2 = T$.

Theorem 2.4. *Let a positive number $\epsilon < \bar{r}_v$, $L_0 \geq 1$ be an integer and $M_0 > 0$. Then there exist an integer $L \geq 1$ and a number $\delta \in (0, \epsilon)$ such that for each integer $T > 2L$, each $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$ satisfying*

$$\|u_t - v\| \leq \delta,\ t = 0 \dots, T-1$$

and each (Ω)-program $\{x_t\}_{t=0}^{T}$ which satisfies

$$x_0 \in \bar{Y}_{L_0},\ \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=0}^{T-1}, 0, T, x_0) - M_0$$

and

$$\sum_{t=\tau}^{\tau+L-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=\tau}^{\tau+L-1}, \tau, \tau+L, x_\tau, x_{\tau+L}) - \delta$$

for each integer $\tau \in [0, T-L]$ there exist integers $\tau_1 \in [0, L]$, $\tau_2 \in [T-L, T]$ such that

$$\rho(x_t, \bar{x}_v) \leq \epsilon,\ t = \tau_1, \dots, \tau_2.$$

Moreover if $\rho(x_0, \bar{x}_v) \leq \delta$, then $\tau_1 = 0$.

In this chapter we prove the following theorem obtained in [83] which establishes the turnpike property for approximate solutions of the optimal control problems of the type (P3).

Theorem 2.5. *Let $\epsilon \in (0, \bar{r}_v)$ and $M > 0$. Then there exist a positive number $\delta < \min\{1, M\}$ and a natural number L such that the following assertions hold.*

1. Assume that an integer $T \geq L$, $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$ and an (Ω)-program $\{x_t\}_{t=0}^{T}$ satisfy

$$\|u_t - v\| \leq \delta,\ t = 0, \ldots, T-1, \tag{2.10}$$

$$\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=0}^{T-1}, 0, T) - M. \tag{2.11}$$

Then the inequality

$$Card(\{t \in \{0, \ldots, T\} :\ \rho(x_t, \bar{x}_v) > \epsilon\}) < L$$

holds.

2. Assume that an integer $T \geq 2L$, $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$ and an (Ω)-program $\{x_t\}_{t=0}^{T}$ satisfy (2.10), (2.11), and

$$\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=0}^{T-1}, 0, T, x_0, x_T) - \delta. \tag{2.12}$$

Then there exist integers $\tau_1 \in [0, L]$ and $\tau_2 \in [T-L, T]$ such that

$$\rho(x_t, \bar{x}_v) \leq \epsilon,\ t = \tau_1, \ldots, \tau_2.$$

Moreover, if $\rho(x_0, \bar{x}_v) \leq \delta$, then $\tau_1 = 0$ and if $\rho(x_T, \bar{x}_v) \leq \delta$, then $\tau_2 = T$.

3. Assume that $\{u_t\}_{t=0}^{\infty} \subset \mathcal{M}(\Omega)$ and an (Ω)-program $\{x_t\}_{t=0}^{\infty}$ satisfy

$$\|u_t - v\| \leq \delta \text{ for all integers } t \geq 0,$$

$$\limsup_{T \to \infty} [\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) - \sigma(\{u_t\}_{t=0}^{T-1}, 0, T)] > -M.$$

Then

$$Card(\{t \text{ is a nonnegative integer such that } \rho(x_t, \bar{x}_v) > \epsilon\}) < L.$$

4. Assume that an integer $T > 0$, $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$, an (Ω)-program $\{x_t\}_{t=0}^{T}$ satisfy (2.10), (2.11), (2.12), and integers T_1, T_2 satisfy $0 \leq T_1 < T_2 \leq T$. Then

$$\sum_{t=T_1}^{T_2-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=T_1}^{T_2-1}, T_1, T_2) - (4L+2)(2\|v\| + 2) - M - 1.$$

Assertions 1 and 2 establish turnpike properties for approximate solutions of the problem (P3) while Assertion 3 establish the turnpike property for

approximate solutions of the corresponding infinite horizon problem. Moreover, they also show the stability of the turnpike phenomenon under small perturbations of the objective function v.

It is clear that $(\mathcal{M}(\Omega), \|\cdot\|)$ is a Banach space. Denote by $\mathcal{M}_0(\Omega)$ the set of all upper semicontinuous functions $v \in \mathcal{M}(\Omega)$ for which there exist $\bar{x}_v \in X$, $\bar{c}_v > 0$ and $\bar{r}_v \in (0,1)$ such that

$$\{(x,y) \in X \times X : \ \rho(x, \bar{x}_v),\ \rho(y, \bar{x}_v) \le \bar{r}_v\} \subset \Omega, \tag{2.13}$$

$$v \text{ is continuous at } (\bar{x}_v, \bar{x}_v), \tag{2.14}$$

$$\sigma(v,T) \le Tv(\bar{x}_v, \bar{x}_v) + \bar{c}_v \text{ for all integers } T \ge 1. \tag{2.15}$$

In other words, $\mathcal{M}_0(\Omega)$ is the set of all upper semicontinuous functions $v \in \mathcal{M}(\Omega)$ which satisfy assumptions (A1) and (A2) with some $\bar{x}_v \in X$, $\bar{r}_v \in (0,1)$, $\bar{c}_v > 0$.

Denote by $\mathcal{M}_{c,0}(\Omega)$ the set of all continuous functions $v \in \mathcal{M}_0(\Omega)$. Denote by $\bar{\mathcal{M}}_{c,0}(\Omega)$ and $\bar{\mathcal{M}}_0(\Omega)$ the closure of subspaces $\mathcal{M}_{c,0}(\Omega)$ and $\mathcal{M}_0(\Omega)$ in $\mathcal{M}(\Omega)$, respectively.

We equip the sets $\bar{\mathcal{M}}_{c,0}(\Omega)$ and $\bar{\mathcal{M}}_0(\Omega)$ with the metric d induced by the norm $\|\cdot\|$: $d(u_1,u_2) = \|u_1 - u_2\|$, $u_1, u_2 \in \bar{\mathcal{M}}_0(\Omega)$.

For each $u \in \bar{\mathcal{M}}_0(\Omega)$ and each $r > 0$ set

$$B_d(u,r) = \{w \in \bar{\mathcal{M}}_0(\Omega) : \ \|u - w\| < r\}.$$

We associate with any $v \in \mathcal{M}_0(\Omega)$ the triplet $(\bar{x}_v, \bar{c}_v, \bar{r}_v)$ satisfying (2.13), (2.14), and (2.15).

Denote by $\mathcal{M}_*(\Omega)$ the set of all $v \in \mathcal{M}_0(\Omega)$ such that for any (v,Ω)-good program $\{x_i\}_{i=0}^{\infty}$,

$$\lim_{i \to \infty} \rho(x_i, \bar{x}_v) = 0.$$

Set

$$\mathcal{M}_{c*}(\Omega) = \mathcal{M}_*(\Omega) \cap \mathcal{M}_c(\Omega).$$

In this chapter we prove the following theorem established in [83].

Theorem 2.6. *$\mathcal{M}_*(\Omega)$ contains a set which is a countable intersection of open everywhere dense subsets of $\bar{\mathcal{M}}_0(\Omega)$ and $\mathcal{M}_{c*}(\Omega)$ contains a set which is a countable intersection of open everywhere dense subsets of $\bar{\mathcal{M}}_{c,0}(\Omega)$.*

2.3 Auxiliary Results for Theorem 2.5

Lemma 2.7. *Let ϵ and M_0 be positive numbers. Then there exists an integer $T \ge 1$ such that for each (Ω)-program $\{x_t\}_{t=0}^{T}$ which satisfies*

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) \ge Tv(\bar{x}_v, \bar{x}_v) - M_0$$

the inequality $\min\{\rho(x_i, \bar{x}_v) : \ i = 1, \dots, T\} \le \epsilon$ holds.

Proof. Assume the contrary. Then for each integer $k \geq 1$ there exists an (Ω)-program $\{x_t^{(k)}\}_{t=0}^{k}$ such that

$$\sum_{t=0}^{k-1} v(x_t^{(k)}, x_{t+1}^{(k)}) \geq kv(\bar{x}_v, \bar{x}_v) - M_0, \tag{2.16}$$

$$\rho(x_t^{(k)}, \bar{x}_v) > \epsilon \text{ for all integers } t = 1, \ldots, k. \tag{2.17}$$

Let k be a natural number. Relations (2.16) and (2.4) and assumption (A2) imply that for each natural number $j < k$,

$$\sum_{t=0}^{j-1} v(x_t^{(k)}, x_{t+1}^{(k)}) = \sum_{t=0}^{k-1} v(x_t^{(k)}, x_{t+1}^{(k)}) - \sum_{t=j}^{k-1} v(x_t^{(k)}, x_{t+1}^{(k)})$$

$$\geq kv(\bar{x}_v, \bar{x}_v) - M_0 - \sum_{t=j}^{k-1} v(x_t^{(k)}, x_{t+1}^{(k)})$$

$$\geq kv(\bar{x}_v, \bar{x}_v) - M_0 - \sigma(v, k-j) \geq kv(\bar{x}_v, \bar{x}_v) - M_0 - (k-j)v(\bar{x}_v, \bar{x}_v) - \bar{c}_v.$$

Combined with (2.16) this inequality implies that for each natural number k and each integer $j \in \{1, \ldots, k\}$ we have

$$\sum_{t=0}^{j-1} v(x_t^{(k)}, x_{t+1}^{(k)}) \geq jv(\bar{x}_v, \bar{x}_v) - \bar{c}_v - M_0. \tag{2.18}$$

There exists a strictly increasing sequence of natural numbers $\{k_i\}_{i=1}^{\infty}$ such that for each nonnegative integer t there exists a limit

$$x_t = \lim_{i \to \infty} x_t^{(k_i)}. \tag{2.19}$$

It is easy to see that the sequence $\{x_t\}_{t=0}^{\infty}$ is an (Ω)-program. In view of relations (2.19) and (2.17), we have

$$\rho(x_t, \bar{x}_v) \geq \epsilon \text{ for all natural numbers } t. \tag{2.20}$$

By relations (2.19) and (2.18), for each natural number T,

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) \geq Tv(\bar{x}_v, \bar{x}_v) - M_0 - \bar{c}_v.$$

This inequality implies that the sequence $\{x_t\}_{t=0}^{\infty}$ is a (v, Ω)-good program. It follows from assumption (A3) that the equality $\lim_{t \to \infty} \rho(x_t, \bar{x}_v) = 0$ holds. This equality contradicts (2.20). The contradiction we have reached proves the lemma. □

Lemma 2.8 (Lemma 3.3 of [72]). *Let M_0, ϵ be positive numbers. Then there exists a natural number T_0 such that for each integer $T \geq T_0$, each (Ω)-program $\{x_t\}_{t=0}^T$ which satisfies*

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) \geq Tv(\bar{x}_v, \bar{x}_v) - M_0 \tag{2.21}$$

and each integer $S \in [0, T - T_0]$ the inequality

$$\min\{\rho(x_t, \bar{x}_v) : \ t = S + 1, \dots, S + T_0\} \leq \epsilon$$

holds.

Proof. Lemma 2.7 implies that there exists an integer $T_0 \geq 1$ such that the following property holds:

(i) For each (Ω)-program $\{x_t\}_{t=0}^{T_0}$ which satisfies

$$\sum_{t=0}^{T_0-1} v(x_t, x_{t+1}) \geq T_0 v(\bar{x}_v, \bar{x}_v) - M_0 - 2\bar{c}_v$$

the inequality

$$\min\{\rho(x_i, \bar{x}_v) : \ i = 1, \dots, T_0\} \leq \epsilon$$

holds.

Let an integer $T \geq T_0$, let an (Ω)-program $\{x_t\}_{t=0}^T$ satisfy (2.21), and let an integer $S \in [0, T - T_0]$. By relations (2.21) and (2.9), we have

$$\sum_{t=S}^{S+T_0-1} v(x_t, x_{t+1}) - T_0 v(\bar{x}_v, \bar{x}_v) \geq -M_0 - 2\bar{c}_v.$$

It follows from this inequality and property (i) that

$$\min\{\rho(x_i.\bar{x}_v) : \ i = S + 1, \dots, S + T_0\} \leq \epsilon.$$

Lemma 2.8 is proved. □

Lemma 2.9. *Let $\epsilon > 0$. Then there exists $\delta \in (0, \bar{r}_v)$ such that for each integer $T \geq 1$ and each (Ω)-program $\{x_t\}_{t=0}^T$ satisfying*

$$\rho(x_0, \bar{x}_v), \ \rho(x_T, \bar{x}_v) \leq \delta, \tag{2.22}$$

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) \geq \sigma(v, T, x_0, x_T) - \delta \tag{2.23}$$

the inequality $\rho(x_t, \bar{x}_v) \leq \epsilon$ holds for all $t = 0, \dots, T$.

Proof. Since the function v is continuous at the point $(\bar{x}_v, \bar{x}_v)$ for any integer $k \geq 1$ there exists a number

$$\delta_k \in (0, 2^{-k}\bar{r}_v) \tag{2.24}$$

such that

$$|v(x,y) - v(\bar{x}_v, \bar{x}_v)| \leq 2^{-k} \tag{2.25}$$

for each pair of points $x, y \in X$ satisfying

$$\rho(x, \bar{x}_v),\ \rho(y, \bar{x}_v) \leq \delta_k. \tag{2.26}$$

Assume that the lemma is wrong. Then for each integer $k \geq 1$ there exist a natural number T_k and an (Ω)-program $\{x_t^{(k)}\}_{t=0}^{T_k}$ such that

$$\rho(x_0^{(k)}, \bar{x}_v),\ \rho(x_{T_k}^{(k)}, \bar{x}_v) \leq \delta_k, \tag{2.27}$$

$$\sum_{t=0}^{T_k-1} v(x_t^{(k)}, x_{t+1}^{(k)}) \geq \sigma(v, x_0^{(k)}, x_{T_k}^{(k)}, T_k) - \delta_k, \tag{2.28}$$

$$\max\{\rho(x_t^{(k)}, \bar{x}_v) :\ t = 0, \ldots, T_k\} > \epsilon. \tag{2.29}$$

Let a natural number k be given. Define a sequence $\{z_t\}_{t=0}^{T_k} \subset X$ as follows:

$$z_0 = x_0^{(k)},\ z_{T_k} = x_{T_k}^{(k)},\ z_t = \bar{x}_v,\ t \in \{0, \ldots, T_k\} \setminus \{0, T_k\}. \tag{2.30}$$

In view of relations (2.30), (2.27), (2.24), and (A1), the sequence $\{z_t\}_{t=0}^{T_k}$ is a program. By (2.28) and (2.30), we have

$$\sum_{t=0}^{T_k-1} v(x_t^{(k)}, x_{t+1}^{(k)}) \geq \sigma(v, x_0^{(k)}, x_{T_k}^{(k)}, T_k) - \delta_k \geq \sum_{t=0}^{T_k-1} v(z_t, z_{t+1}) - \delta_k. \tag{2.31}$$

It follows from relations (2.27), (2.30), and the choice of δ_k (see (2.24)–(2.26)) that

$$|v(z_0, z_1) - v(\bar{x}_v, \bar{x}_v)| \leq 2^{-k},\ |v(z_{T_k-1}, z_{T_k}) - v(\bar{x}_v, \bar{x}_v)| \leq 2^{-k},$$

$$v(z_t, z_{t+1}) = v(\bar{x}_v, \bar{x}_v),\ t \in \{0, \ldots, T_k - 1\} \setminus \{0, T_k - 1\}. \tag{2.32}$$

In view of (2.32) and (2.31),

$$\sum_{t=0}^{T_k-1} v(x_t^{(k)}, x_{t+1}^{(k)}) \geq T_k v(\bar{x}_v, \bar{x}_v) - 2 \cdot 2^{-k} - \delta_k. \tag{2.33}$$

Put

$$S_0 = 0,\ S_k = \sum_{i=1}^{k} (T_i + 1) - 1 \text{ for all natural numbers } k. \tag{2.34}$$

Define a sequence $\{x_t\}_{t=0}^{\infty} \subset X$ as follows:

$$x_t = x_t^{(1)},\ t = 0, \dots, T_1,\ x_t = x_i^{(k+1)} \tag{2.35}$$

for each natural number k, each $i \in \{0, \dots, T_{k+1}\}$ and $t = S_k + i + 1$.

By relations (2.35), (2.27), (2.24), and (A1), the sequence $\{x_t\}_{t=0}^{\infty}$ is an (Ω)-program. It follows from (2.34), (2.35), (2.27), and (2.24) that for each natural number k we have

$$|v(x_{S_k}, x_{S_k+1}) - v(\bar{x}_v, \bar{x}_v)| \le 2 \cdot 2^{-k}. \tag{2.36}$$

In view of relations (2.34), (2.35), (2.33), (2.30), and the choice of δ_j, $j = 1, 2, \dots$ (see (2.24)–(2.27)) for any natural number $k \ge 2$, we have

$$\sum_{t=0}^{S_k-1} v(x_t, x_{t+1}) - S_k v(\bar{x}_v, \bar{x}_v) = \sum_{j=1}^{k} \left(\sum_{t=0}^{T_j-1} [v(x_t^{(j)}, x_{t+1}^{(j)}) - v(\bar{x}_v, \bar{x}_v)] \right)$$

$$+ \sum_{j=1}^{k-1} [v(x_{T_j}^{(j)}, x_0^{(j+1)}) - v(\bar{x}_v, \bar{x}_v)] \ge -\sum_{j=1}^{k} (2 \cdot 2^{-j} + \delta_j) - 2 \sum_{j=1}^{k-1} 2^{-j}.$$

Together with inclusion (2.24) this relation implies that for any natural number $k \ge 2$,

$$\sum_{t=0}^{S_k-1} v(x_t, x_{t+1}) - S_k v(\bar{x}_v, \bar{x}_v) \ge -5 \sum_{j=1}^{k} 2^{-j} \ge -10.$$

It follows from this inequality and Proposition 2.1 that the sequence $\{x_t\}_{t=0}^{\infty}$ is a (v, Ω)-good program. By assumption (A3), we have

$$\lim_{t \to \infty} \rho(x_t, \bar{x}_v) = 0.$$

On the other hand it follows from relations (2.29), (2.34), and (2.35) that $\limsup_{t \to \infty} \rho(x_t, \bar{x}_v) \ge \epsilon$. The contradiction we have reached proves Lemma 2.9. □

Lemma 2.10. *Let $\epsilon \in (0, \bar{r}_v)$ and $M_0 > 0$. Then there exist a number $\delta_0 \in (0, 1)$ and an integer $L_0 > 4$ such that for each integer $T \ge L_0$, each finite sequence of functions $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$, each (Ω)-program $\{x_t\}_{t=0}^{T}$ which satisfy*

$$\|u_t - v\| \le \delta_0,\ t = 0, \dots, T-1, \tag{2.37}$$

$$\min\{\rho(x_t, \bar{x}_v) :\ t = 1, \dots, T\} > \epsilon \tag{2.38}$$

and each pair of points $z_0, z_1 \in X$ satisfying

$$\rho(\bar{x}_v, z_i) \le \bar{r}_v,\ i = 0, 1, \tag{2.39}$$

the following relation is valid:

$$\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \le u_0(z_0, \bar{x}_v) + \sum_{t=1}^{T-2} u_t(\bar{x}_v, \bar{x}_v) + u_{T-1}(\bar{x}_v, z_1) - M_0.$$

Proof. Fix a real number

$$M_1 > 8 + 4\|v\| + M_0. \tag{2.40}$$

In view of Lemma 2.8 there exists an integer $L_0 > 4$ such that the following property holds:

(i) for each integer $T \ge L_0$, each (Ω)-program $\{x_t\}_{t=0}^T$ which satisfies

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) \ge Tv(\bar{x}_v, \bar{x}_v) - M_1$$

and each integer $S \in [0, T - L_0]$ the inequality

$$\min\{\rho(x_t, \bar{x}_v) : \ t = S+1, \dots, S+L_0\} \le \epsilon$$

is valid.

Fix a positive number

$$\delta_0 < (4L_0)^{-1}. \tag{2.41}$$

Assume that an integer $T \ge L_0$, $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}$, an (Ω)-program $\{x_t\}_{t=0}^T$ satisfy (2.37) and (2.38) and that a pair of points $z_0, z_1 \in X$ satisfy (2.39). There exists an integer $k \ge 1$ such that

$$kL_0 \le T < (k+1)L_0. \tag{2.42}$$

In view of (2.39) the sequence $z_0, \bar{x}, \dots, \bar{x}$ $(T-2$ times$), z_1$ is an (Ω)-program. Property (i) and (2.38) imply that for every integer i satisfying $0 < i \le k-1$,

$$\sum_{t=(i-1)L_0}^{iL_0-1} v(x_t, x_{t+1}) < L_0 v(\bar{x}_v, \bar{x}_v) - M_1, \tag{2.43}$$

$$\sum_{t=(k-1)L_0}^{T-1} v(x_t, x_{t+1}) < (T - (k-1)L_0)v(\bar{x}_v, \bar{x}_v) - M_1. \tag{2.44}$$

It follows from (2.37), (2.41), (2.43), and (2.44) that for every integer i satisfying $0 < i \le k-1$, we have

$$\sum_{t=(i-1)L_0}^{iL_0-1} u_t(x_t, x_{t+1}) \le \sum_{t=(i-1)L_0}^{iL_0-1} v(x_t, x_{t+1}) + \delta_0 L_0$$

$$\le L_0 v(\bar{x}_v, \bar{x}_v) - M_1 + \delta_0 L_0$$

$$\leq \sum_{t=(i-1)L_0}^{iL_0-1} u_t(\bar{x}_v, \bar{x}_v) + \delta_0 L_0 - M_1 + \delta_0 L_0$$

$$\leq \sum_{t=(i-1)L_0}^{iL_0-1} u_t(\bar{x}_v, \bar{x}_v) - M_1 + 1 \tag{2.45}$$

and

$$\sum_{t=(k-1)L_0}^{T-1} u_t(x_t, x_{t+1}) \leq \sum_{t=(k-1)L_0}^{T-1} v(x_t, x_{t+1}) + 2\delta_0 L_0$$

$$\leq (T-(k-1)L_0)v(\bar{x}_v, \bar{x}_v) - M_1 + 2\delta_0 L_0$$

$$\leq \sum_{t=(k-1)L_0}^{T-1} u_t(\bar{x}_v, \bar{x}_v) + 2\delta_0 L_0 - M_1 + 2\delta_0 L_0$$

$$\leq \sum_{t=(k-1)L_0}^{T-1} u_t(\bar{x}_v, \bar{x}_v) - M_1 + 1. \tag{2.46}$$

It follows from (2.37), (2.40), (2.42), (2.45), and (2.46) that

$$\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \leq \sum_{t=0}^{T-1} u_t(\bar{x}_v, \bar{x}_v) - M_1 + 1$$

$$\leq u_0(z_0, \bar{x}_v) + \sum_{t=1}^{T-2} u_t(\bar{x}_v, \bar{x}_v) + u_{T-1}(\bar{x}_v, z_1) + 2\|u_0\| + 2\|u_T\| - M_1 + 1$$

$$\leq u_0(z_0, \bar{x}_v) + \sum_{t=1}^{T-2} u_t(\bar{x}_v, \bar{x}_v) + u_{T-1}(\bar{x}_v, z_1) - M_0.$$

Lemma 2.10 is proved. □

Lemma 2.11. *Let $\epsilon \in (0, \bar{r}_v)$, M_0 be a positive number and let $\delta_0 \in (0,1)$ and an integer $L_0 > 4$ be as guaranteed by Lemma 2.10.*

Assume that an integer $T \geq L_0$, $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}$ and that an (Ω)-program $\{x_t\}_{t=0}^{T}$ satisfy

$$\|u_t - v\| \leq \delta_0,\ t = 0, \ldots, T-1 \tag{2.47}$$

and

$$\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) > \sigma(\{u_t\}_{t=0}^{T-1}, 0, T) - M_0 + 2\|v\| + 2. \tag{2.48}$$

Then there exists a finite strictly increasing sequence of natural numbers $\{T_j\}_{j=1}^{q}$ *where* q *is a natural number such that*

$$T_1 \in [1, L_0],\ T_q \in [T - L_0 + 1, T], \tag{2.49}$$

$$T_{j+1} - T_j \le L_0 \text{ for all natural numbers } j < q, \tag{2.50}$$

$$\rho(\bar{x}_v, x_{T_j}) \le \epsilon,\ j =, 1, \dots, q. \tag{2.51}$$

Proof. We claim that there exists a natural number $T_1 \in [1, L_0]$ such that $\rho(\bar{x}_v, x_{T_1}) \le \epsilon$.

Assume the contrary. Then

$$\rho(x_t, \bar{x}_v) > \epsilon,\ t = 1, \dots, L_0. \tag{2.52}$$

There are two cases:

$$\rho(x_t, \bar{x}_v) > \epsilon,\ t = 1, \dots, T; \tag{2.53}$$

$$\min\{\rho(x_t, \bar{x}_v) :\ t = 1, \dots, T\} \le \epsilon. \tag{2.54}$$

Assume that (2.53) is valid. It follows from (2.47), (2.53), the inequality $T \ge L_0$, the choice of δ_0, L_0, and Lemma 2.10 that

$$\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \le \sum_{t=0}^{T-1} u_t(\bar{x}_v, \bar{x}_v) - M_0 \le \sigma(\{u_t\}_{t=0}^{T-1}, 0, T) - M_0.$$

This contradicts (2.48). The contradiction we have reached proves (2.54). In view of (2.52) and (2.54) there exists an integer

$$S \in (L_0, T] \tag{2.55}$$

for which

$$\rho(x_S, \bar{x}_v) \le \epsilon,\ \rho(x_t, \bar{x}_v) > \epsilon,\ t = 1, \dots, S-1. \tag{2.56}$$

Set

$$z_t = \bar{x}_v,\ t = 0, \dots, S-1,\ z_t = x_t,\ t = S, \dots, T. \tag{2.57}$$

Relations (2.56) and (2.57) imply that $\{z_t\}_{t=0}^{T}$ is an (Ω)-program. By (2.47), (2.55), (2.56), (2.57), the choice of δ_0, and L_0 and Lemma 2.10 (with $T = S - 1$), we have

$$\sum_{t=0}^{S-2} u_t(x_t, x_{t+1}) \le \sum_{t=0}^{S-2} u_t(\bar{x}_v, \bar{x}_v) - M_0 = \sum_{t=0}^{S-2} u_t(z_t, z_{t+1}) - M_0. \tag{2.58}$$

It follows from (2.57) and (2.58) that

$$\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) - \sum_{t=0}^{T-1} u_t(z_t, z_{t+1})$$

$$= \sum_{t=0}^{S-2} u_t(x_t, x_{t+1}) + u_{S-1}(x_{S-1}, x_S) - \sum_{t=0}^{S-2} u_t(\bar{x}_v, \bar{x}_v) - u_{S-1}(z_{S-1}, z_S)$$

$$\leq -M_0 + 2\|u_{S-1}\| \leq -M_0 + 2\|v\| + 2.$$

This implies that

$$\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \leq \sigma(\{u_t\}_{t=0}^{T-1}, 0, T) - M_0 + 2\|v\| + 2.$$

The inequality above contradicts (2.48). The contradiction we have reached proves that there exists a natural number

$$T_1 \in [1, L_0] \tag{2.59}$$

for which

$$\rho(\bar{x}_v, x_{T_1}) \leq \epsilon. \tag{2.60}$$

Assume now that k is a natural number and we defined a strictly increasing sequence of positive integers $\{T_i\}_{i=1}^k$ such that

$$T_1 \in [1, L_0],\ T_k \leq T, \tag{2.61}$$

$$\rho(\bar{x}_v, x_{T_i}) \leq \epsilon,\ i =, 1, \ldots, k, \tag{2.62}$$

for every natural number $i < k$, we have

$$T_{i+1} - T_i \leq L_0. \tag{2.63}$$

(It is clear that for $k = 1$ our assumption is true.)

If $T - T_k < L_0$, then the construction is completed. Assume that

$$T - T_k \geq L_0. \tag{2.64}$$

We claim that there exists an integer T_{k+1} such that

$$T_{k+1} - T_k \leq [1, L_0],\ \rho(\bar{x}_v, x_{T_{k+1}}) \leq \epsilon. \tag{2.65}$$

Assume the contrary. Then

$$\rho(\bar{x}_v, x_t) > \epsilon,\ t = T_k + 1, \ldots, T_k + L_0. \tag{2.66}$$

There are two cases:

$$\rho(\bar{x}_v, x_t) > \epsilon,\ t = T_k + 1, \ldots, T; \tag{2.67}$$

$$\min\{\rho(\bar{x}_v, x_t) :\ t = T_k + 1, \ldots, T\} \leq \epsilon. \tag{2.68}$$

Assume that (2.67) is true. It follows from (2.47), (2.62), (2.64), (2.67), the choice of δ_0, L_0, and Lemma 2.10 that

$$\sum_{t=T_k}^{T-1} u_t(x_t, x_{t+1}) \leq u_{T_k}(x_{T_k}, \bar{x}_v) + \sum_{t=T_k+1}^{T-1} u_t(\bar{x}_v, \bar{x}_v) - M_0. \tag{2.69}$$

Define

$$z_t = x_t,\ t = 0, \ldots, T_k,\ z_t = \bar{x}_v,\ t = T_k + 1, \ldots, T. \tag{2.70}$$

In view of (2.70) and (2.62), $\{z_t\}_{t=0}^{T}$ is an (Ω)-program. By (2.70) and (2.69), we have

$$\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) - \sigma(\{u_t\}_{t=0}^{T-1}, 0, T)$$

$$\leq \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) - \sum_{t=0}^{T-1} u_t(z_t, z_{t+1})$$

$$= \sum_{t=T_k}^{T-1} u_t(x_t, x_{t+1}) - u_{T_k}(x_{T_k}, \bar{x}_v) - \sum_{t=T_k+1}^{T-1} u_t(\bar{x}_v, \bar{x}_v) \leq -M_0.$$

The relation above contradicts (2.48). The contradiction we have reached proves (2.68). In view of (2.66) and (2.68), there exists an integer S such that

$$T_k + L_0 < S \leq T,\ \rho(x_S, \bar{x}_v) \leq \epsilon, \tag{2.71}$$

$$\rho(x_t, \bar{x}_v) > \epsilon,\ t = T_k + 1, \ldots, S - 1. \tag{2.72}$$

Set

$$z_t = x_t,\ t = 0, \ldots, T_k,\ z_t = \bar{x}_v,\ t = T_k + 1, \ldots, S - 1,\ z_t = x_t,\ t = S, \ldots, T. \tag{2.73}$$

It follows from (2.62), (2.71), and (2.73) that $\{z_t\}_{t=0}^{T}$ is an (Ω)-program. By (2.37), (2.71), (2.72), the choice of δ_0 and L_0 and Lemma 2.10, we have

$$\sum_{t=T_k}^{S-2} u_t(x_t, x_{t+1})$$

$$\leq u_{T_k}(x_{T_k}, \bar{x}_v) + \sum_{t=T_k+1}^{S-2} u_t(\bar{x}_v, \bar{x}_v) - M_0$$

$$= \sum_{t=T_k}^{S-2} u_t(z_t, z_{t+1}) - M_0. \tag{2.74}$$

In view of (2.73) and (2.74), we have

$$\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) - \sigma(\{u_t\}_{t=0}^{T-1}, 0, T) \leq \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) - \sum_{t=0}^{T-1} u_t(z_t, z_{t+1})$$

$$= \sum_{t=T_k}^{T-1} u_t(x_t, x_{t+1}) - \sum_{t=T_k}^{T-1} u_t(z_t, z_{t+1})$$

$$= \sum_{t=T_k}^{S-1} u_t(x_t, x_{t+1}) - u_{T_k}(x_{T_k}, \bar{x}_v) - \sum_{t=T_k+1}^{S-2} u_t(\bar{x}_v, \bar{x}_v) - u_{S-1}(\bar{x}_v, x_S)$$

$$\leq -M_0 + 2\|u_{S-1}\| \leq -M_0 + 2\|v\| + 2.$$

The relation above contradicts (2.48). The contradiction we have reached proves that there is an integer T_{k+1} satisfying (2.65).

It is clear that the assumption made for k also holds for $k+1$. Thus by induction we construct the sequence T_i, $i = 1, \ldots, q$ which is obtained by a finite number of steps and T_q is its last element. It follows from the construction that (2.49)–(2.51) are true. This completes the proof of Lemma 2.11. □

2.4 Proof of Theorem 2.5

Lemma 2.9 implies that there exists a positive number

$$\delta_0 < \min\{\bar{r}_v, \epsilon, 1\}$$

such that the following property holds:

(i) for every natural number T and every (Ω)-program $\{x_t\}_{t=0}^{T}$ which satisfy

$$\rho(x_0, \bar{x}_v),\ \rho(x_T, \bar{x}_v) \leq \delta_0$$

and

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) \geq \sigma(v, T, x_0, x_T) - \delta_0$$

the inequality $\rho(x_t, \bar{x}_v) \leq \epsilon$ is true for all integers $t = 0, \ldots, T$.

Lemma 2.11 implies that there exist $\delta_1 \in (0, 1)$ and an integer $L_1 > 4$ such that the following property holds:

(ii) for every natural number $T \geq L_1$, every finite sequence of functions $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$ and every (Ω)-program $\{x_t\}_{t=0}^{T}$ which satisfy

$$\|u_t - v\| \leq \delta_1,\ t = 0, \ldots, T-1,$$

and

$$\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) > \sigma(\{u_t\}_{t=0}^{T-1}, 0, T) - M - 4$$

there exists a strictly increasing sequence of natural numbers $\{T_j\}_{j=1}^{q}$ where q is a natural number such that

$$T_1 \in [1, L_1],\ T_q \in [T - L_1 + 1, T], \tag{2.75}$$

$$T_{j+1} - T_j \leq L_1 \text{ for all natural numbers } j < q, \tag{2.76}$$

$$\rho(\bar{x}_v, x_{T_j}) < \delta_0,\ j =, 1, \ldots, q. \tag{2.77}$$

Choose a positive number δ and an integer $L \geq 1$ such that

$$\delta < \min\{\delta_1, M, 8^{-1}L_1^{-1}\delta_0\}, \tag{2.78}$$

$$L > 8L_1 + 2L_1(M+1)\delta_0^{-1}.$$

Let us prove Assertion 1. Property (ii), (2.10), (2.11), and (2.78) imply that there exists a strictly increasing sequence of natural numbers $\{T_j\}_{j=1}^{q}$, where q is a natural number, such that (2.75), (2.76), and (2.77) are true. In view of (2.75), (2.76), and (2.78),

$$q > 2. \tag{2.79}$$

Let

$$E = \{j \in \{1, \ldots, q-1\} :$$

$$\sum_{t=T_j}^{T_{j+1}-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=T_j}^{T_{j+1}-1}, T_j, T_{j+1}, x_{T_j}, x_{T_{j+1}}) - \delta_0/2. \tag{2.80}$$

By (2.11) and (2.80), we have

$$-M \leq \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) - \sigma(\{u_t\}_{t=0}^{T-1}, 0, T)$$

$$\leq \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) - \sigma(\{u_t\}_{t=0}^{T-1}, 0, T, x_0, x_T)$$

$$\leq \sum \{ \sum_{t=T_j}^{T_{j+1}-1} u_t(x_t, x_{t+1}) - \sigma(\{u_t\}_{t=T_j}^{T_{j+1}-1}, T_j, T_{j+1}, x_{T_j}, x_{T_{j+1}}) :$$

$$j \in \{1, \ldots, q-1\} \setminus E\}$$

$$\leq (-\delta_0/2)\mathrm{Card}(\{1, \ldots, q-1\} \setminus E\}$$

and

$$\mathrm{Card}(\{1, \ldots, q-1\} \setminus E\}) \leq 2M\delta_0^{-1}. \tag{2.81}$$

Let

$$j \in E. \tag{2.82}$$

It follows from (2.10), (2.80), and (2.82) that

$$\sum_{t=T_j}^{T_{j+1}-1} v(x_t, x_{t+1}) \geq \sum_{t=T_j}^{T_{j+1}-1} u_t(x_t, x_{t+1}) - \delta L_1$$

$$\geq \sigma(\{u_t\}_{t=T_j}^{T_{j+1}-1}, T_j, T_{j+1}, x_{T_j}, x_{T_{j+1}}) - \delta_0/2 - \delta L_1$$

$$\geq \sigma(v, T_{j+1} - T_j, x_{T_j}, x_{T_{j+1}}) - \delta L_1 - \delta_0/2 - \delta L_1$$

$$\geq \sigma(v, T_{j+1} - T_j, x_{T_j}, x_{T_{j+1}}) - \delta_0. \tag{2.83}$$

In view of (2.77), (2.83), and property (i),

$$\rho(x_t, \bar{x}_v) \le \epsilon,\ t = T_j, \dots, T_{j+1} \text{ and all } j \in E.$$

Combined with (2.77) the relation above implies that

$$\{t \in \{0, \dots, T\} :\ \rho(x_t, \bar{x}_v) > \epsilon\}$$

$$\subset \{0, \dots, T_1 - 1\} \cup \{T - L_1 + 1, \dots, T\}$$

$$\cup \{\{t \text{ is an integer and } T_j < t < T_{j+1}\} :\ j \in \{1, \dots, q-1\} \setminus E\}.$$

Combined with (2.75), (2.76), (2.78), and (2.81) this implies that

$$\text{Card}(\{t \in \{0, \dots, T\} :\ \rho(x_t, \bar{x}_v) > \epsilon\})$$

$$\le 2L_1 + L_1 \text{Card}(\{1, \dots, q-1\} \setminus E\}) \le 2L_1 + 2ML_1\delta_0^{-1} < L.$$

This completes the proof of Assertion 1.

Let us prove Assertion 2. It follows from (2.10), (2.11), (2.78), and property (ii) that there exists a strictly increasing sequence of natural numbers $\{T_j\}_{j=1}^q$, where $q \ge 1$ is an integer, such that (2.76), (2.77) are true and that

$$T_q \in [T - L_1 + 1, T],\ T_1 \in [0, L_1].$$

Evidently, if $\rho(x_0, \bar{x}_v) \le \delta$, then we may assume that $T_1 = 0$ and if $\rho(x_T, \bar{x}_v) \le \delta$, then we may assume that $T_q = T$.

Let an integer $j \in \{1, \dots, q-1\}$. In view of (2.10), (2.12), (2.76), and (2.78), we have

$$\sum_{t=T_j}^{T_{j+1}-1} v(x_t, x_{t+1}) \ge \sum_{t=T_j}^{T_{j+1}-1} u_t(x_t, x_{t+1}) - \delta L_1$$

$$\ge \sigma(\{u_t\}_{t=T_j}^{T_{j+1}-1}, T_j, T_{j+1}, x_{T_j}, x_{T_{j+1}}) - \delta(L_1 + 1)$$

$$\ge \sigma(v, T_{j+1} - T_j, x_{T_j}, x_{T_{j+1}}) - \delta L_1 - \delta(L_1 + 1)$$

$$\ge \sigma(v, T_{j+1} - T_j, x_{T_j}, x_{T_{j+1}}) - \delta_0.$$

Combined with (2.77) and property (i) this implies that

$$\rho(x_t, \bar{x}_v) \le \epsilon \text{ for all integers } t \in [T_j, T_{j+1}] \text{ and all } j = 1, \dots, q-1.$$

This implies that

$$\rho(x_t, \bar{x}_v) \le \epsilon \text{ for all integers } t \in [T_1, T_q].$$

This completes the proof of Assertion 2.

Assertion 3 easily follows from Assertion 1.

Let us prove Assertion 4. There are two cases: $T_2 - T_1 \le 4L$; $T_2 - T_1 > 4L$.

Evidently, if $T_2 - T_1 \le 4L$, then by (2.10), Assertion 4 holds. Assume that

$$T_2 - T_1 > 4L. \tag{2.84}$$

There exists an (Ω)-program $\{y_t\}_{t=T_1}^{T_2}$ such that

$$\sum_{t=T_1}^{T_2-1} u_t(y_t, y_{t+1}) \ge \sigma(\{u_t\}_{t=T_1}^{T_2-1}, T_1, T_2) - \delta. \tag{2.85}$$

It follows from (2.10), (2.11), (2.12), (2.84), (2.85), and Assertion 2 that

$$\rho(x_t, \bar{x}_v) \le \epsilon \text{ for all integers } t = L, \dots, T - L, \tag{2.86}$$

$$\rho(y_t, \bar{x}_v) \le \epsilon \text{ for all integers } t = T_1 + L, \dots, T_2 - L. \tag{2.87}$$

Define

$$z_t = x_t,\ t = 0, \dots, T_1 + L,\ z_t = y_t,\ t = T_1 + L + 1, \dots, T_2 - L - 1,$$

$$z_t = x_t,\ t = T_2 - L, \dots, T. \tag{2.88}$$

It follows from (2.88), (2.86), (2.87), and (A1) that $\{z_t\}_{t=0}^{T}$ is an (Ω)-program. In view of (2.10), (2.11), and (2.88), we have

$$-M \le \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) - \sigma(\{u_t\}_{t=0}^{T-1}, 0, T) \le \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) - \sum_{t=0}^{T-1} u_t(z_t, z_{t+1})$$

$$= \sum_{t=T_1+L}^{T_2-L-1} u_t(x_t, x_{t+1}) - \sum_{t=T_1+L}^{T_2-L-1} u_t(z_t, z_{t+1})$$

$$\le \sum_{t=T_1}^{T_2-1} u_t(x_t, x_{t+1}) + 2L(\|v\| + 1) - \sum_{t=T_1+L}^{T_2-L-1} u_t(y_t, y_{t+1}) + 4(\|v\| + 1)$$

$$\le \sum_{t=T_1}^{T_2-1} u_t(x_t, x_{t+1}) + 2(\|v\| + 1)(L + 2) - \sum_{t=T_1}^{T_2-1} u_t(y_t, y_{t+1}) + 2L(\|v\| + 1).$$

Together with (2.85) this implies that

$$\sum_{t=T_1}^{T_2-1} u_t(x_t, x_{t+1}) \ge -M - 2(\|v\| + 1)(2L + 2) + \sum_{t=T_1}^{T_2-1} u_t(y_t, y_{t+1})$$

$$\ge -M - 2(\|v\| + 1)(2L + 2) + \sigma(\{u_t\}_{t=T_1}^{T_2-1}, T_1, T_2) - 1.$$

This implies Assertion 4. This completes the proof of Theorem 2.5. □

2.5 Proof of Theorem 2.6

Lemma 2.12. *The set $\mathcal{M}_*(\Omega)$ is an everywhere dense subset of $\bar{\mathcal{M}}_0(\Omega)$ and $\mathcal{M}_{c*}(\Omega)$ is an everywhere dense subset of $\bar{\mathcal{M}}_{c,0}(\Omega)$.*

Proof. Let $v \in \mathcal{M}_0(\Omega)$ and $\gamma \in (0,1)$. Define

$$v_\gamma(x,y) = v(x,y) - \gamma\rho(x,\bar{x}_v) - \gamma\rho(y,\bar{x}_v).$$

It is not difficult to see that $v_\gamma \in \mathcal{M}_*(\Omega)$, if $v \in \mathcal{M}_{c,0}(\Omega)$, then $v_\gamma \in \mathcal{M}_{c*}(\Omega)$ and $\lim_{\gamma\to 0+} v_\gamma = v$. Lemma 2.12 is proved.

Let a function $v \in \mathcal{M}(\Omega)_*$ and $n \geq 1$ be an integer. In view of (A1) and (2.14), there exists a number

$$r_1(v,n) \in (0, 4^{-1}\bar{r}_v) \tag{2.89}$$

such that

$$|v(\bar{x}_v,\bar{x}_v) - v(z_1,z_2)| < 8^{-1}\bar{r}_v n^{-1} \tag{2.90}$$

for every pair of points $z_1, z_2 \in X$ which satisfy $\rho(z_i,\bar{x}_v) \leq r_1(v,n)$, $i = 1,2$.

Theorem 2.5 implies that there exist a positive number

$$\delta(v,n) < (4n)^{-1}$$

and an integer $L(v,n) \geq 1$ such that for every function

$$u \in B_d(v,\delta(v,n)) \tag{2.91}$$

the following properties hold:

(i) for every integer $T \geq L(v,n)$ and every (Ω)-program $\{x_t\}_{t=0}^T$ which satisfies

$$\sum_{t=0}^{T-1} u(x_t,x_{t+1}) \geq \sigma(u,T) - 2n \tag{2.92}$$

the relation

$$\mathrm{Card}(\{t \in \{0,\dots,T\} :\ \rho(x_t,\bar{x}_v) > (4n)^{-1}r_1(v,n)\}) < L(v,n)$$

is true;

(ii) for every (Ω)-program $\{x_t\}_{t=0}^\infty$ which satisfies

$$\limsup_{T\to\infty}[\sum_{t=0}^{T-1} u(x_t,x_{t+1}) - \sigma(u,T)] > -2n$$

the inequality

$$\mathrm{Card}(\{t \text{ is a nonnegative integer such that}$$
$$\rho(x_t,\bar{x}_v) > (4n)^{-1}r_1(v,n)\}) < L(v,n)$$

is true;

(iii) for every integer $T \geq 2L(v,n)$ and every (Ω)-program $\{x_t\}_{t=0}^{T}$ which satisfies (2.92) and

$$\sum_{t=0}^{T-1} u(x_t, x_{t+1}) \geq \sigma(u, T, x_0, x_T) - \delta(v,n) \tag{2.93}$$

the inequality

$$\rho(x_t, \bar{x}_v) \leq (4n)^{-1} r_1(v,n)$$

is true for all $t = L(v,n), \ldots, T - L(v,n)$;

(iv) for every natural number T, every (Ω)-program $\{x_t\}_{t=0}^{T}$ which satisfies (2.92) and (2.93), and every pair of integers T_1, T_2 satisfying $0 \leq T_1 < T_2 \leq T$ the inequality

$$\sum_{t=T_1}^{T_2-1} u(x_t, x_{t+1}) \geq \sigma(u, T_2 - T_1) - (4L(v,n)+2)(2\|v\|+2) - 2n - 1$$

is true.

Set

$$\mathcal{F} = \cap_{n=1}^{\infty} \cup \{B_d(v, \delta(v,n)) : \ v \in \mathcal{M}_*(\Omega), \ n \text{ is a natural number}\}, \tag{2.94}$$

$$\mathcal{F}_c = \cap_{n=1}^{\infty} \cup \{B_d(v, \delta(v,n)) \cap \bar{\mathcal{M}}_{c,0}(\Omega) : \ v \in \mathcal{M}_{c*}(\Omega), n \text{ is a natural number}\}. \tag{2.95}$$

Lemma 2.12 implies that $\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $\bar{\mathcal{M}}_0(\Omega)$ and $\mathcal{F}_c$ is a countable intersection of open everywhere dense subsets of $\bar{\mathcal{M}}_{c,0}(\Omega)$. Let

$$u \in \mathcal{F}. \tag{2.96}$$

In order to complete the proof it is sufficient to show that $u \in \mathcal{M}_*(\Omega)$. It is clear that for every integer $T \geq 1$ there exists an (Ω)-program $\{x_i^{(T)}\}_{i=0}^{T}$ for which

$$\sum_{t=0}^{T-1} u(x_t^{(T)}, x_{t+1}^{(T)}) = \sigma(u, T). \tag{2.97}$$

Extracting subsequences and using diagonalization process we obtain that there exists a strictly increasing sequence of natural numbers $\{T_k\}_{k=1}^{\infty}$ such that for every nonnegative integer t there exists

$$x_t^* = \lim_{k \to \infty} x_t^{(T_k)}. \tag{2.98}$$

In view of (2.94) and (2.96), for every integer $n \geq 1$ there exists

$$v_n \in \mathcal{M}_*(\Omega) \tag{2.99}$$

such that

$$u \in B_d(v_n, \delta(v_n, n)). \tag{2.100}$$

Let $n \geq 1$ be an integer. It follows from (2.97), (2.99), (2.100), and property (iii) that for every integer $T \geq 2L(v_n, n)$ we have

$$\rho(x_t^{(T)}, \bar{x}_{v_n}) \leq (4n)^{-1} r_1(v_n, n),\ t = L(v_n, n), \dots, T - L(v_n, n). \tag{2.101}$$

In view of (2.98) and (2.101), for every integer $T \geq L(v_n, n)$, we have

$$\rho(x_t^*, \bar{x}_{v_n}) \leq (4n)^{-1} r_1(v_n, n). \tag{2.102}$$

Property (iv), (2.97), (2.99), and (2.100) imply that for every natural number T and every pair of integers T_1, T_2 satisfying $0 \leq T_1 < T_2 \leq T$ the inequality

$$\sum_{t=T_1}^{T_2-1} u(x_t^{(T)}, x_{t+1}^{(T)}) \geq \sigma(u, T_2 - T_1) - (4L(v_n, n) + 2)(2\|v_n\| + 2) - 2n - 1 \tag{2.103}$$

holds. In view of (2.98) and (2.103), for every pair of integers $T_2 > T_1 \geq 0$, we have

$$\sum_{t=T_1}^{T_2-1} u(x_t^*, x_{t+1}^*) \geq \sigma(u, T_2 - T_1) - (4L(v_n, n) + 2)(2\|v_n\| + 2) - 2n - 1. \tag{2.104}$$

Since n is any natural number it follows from (2.102) that $\{x_t^*\}_{t=0}^{\infty}$ is a Cauchy sequence and there exists

$$x^* = \lim_{t \to \infty} x_t^*. \tag{2.105}$$

Relations (2.102) and (2.105) imply that for every natural number n, we have

$$\rho(x^*, \bar{x}_{v_n}) \leq (4n)^{-1} r_1(v_n, n). \tag{2.106}$$

In view of (2.104) and (2.105), for every natural number n and every natural number T, we have

$$Tu(x^*, x^*) \geq \sigma(u, T) - (4L(v_n, n) + 2)(2\|v_n\| + 4) - 2n - 1. \tag{2.107}$$

Let $n \geq 1$ be an integer and points $y, z \in X$ satisfy

$$\rho(y, x^*),\ \rho(z, x^*) \leq (4n)^{-1} r_1(v_n, n). \tag{2.108}$$

By (2.106) and (2.108),

$$\rho(y, \bar{x}_{v_n}) \leq \rho(y, x^*) + \rho(x^*, \bar{x}_{v_n}) \leq (2n)^{-1} r_1(v_n, n), \tag{2.109}$$

$$\rho(z, \bar{x}_{v_n}) \leq \rho(z, x^*) + \rho(x^*, \bar{x}_{v_n}) \leq (2n)^{-1} r_1(v_n, n). \tag{2.110}$$

Relations (2.89), (2.90), (2.99), (2.109), and (2.110) imply that

$$(y, z) \in \Omega,\ (x^*, x^*) \in \Omega. \tag{2.111}$$

It follows from (2.89), (2.90), (2.100), (2.106), (2.109), and (2.110) that

$$|u(y,z) - u(x^*,x^*)| \le |v_n(y,z) - v_n(x^*,x^*)| + 2\delta(v_n,n)$$
$$\le |v_n(y,z) - v_n(\bar{x}_{v_n},\bar{x}_{v_n})| + |v_n(\bar{x}_{v_n},\bar{x}_{v_n}) - v_n(x^*,x^*)| + 2\delta(v_n,n)$$
$$\le (4n)^{-1}\bar{r}_{v_n} + 2n^{-1} < 1/n.$$

Thus for every pair of points $y, z \in X$ satisfying (2.108), we have $(y,z) \in \Omega$ and

$$|u(y,z) - u(x^*,x^*)| < n^{-1}.$$

Since n is any natural number we conclude that the function u is continuous at the point (x^*,x^*) and together with (2.107) this implies that $u \in \mathcal{M}_0(\Omega)$.

In order to complete the proof it is sufficient to show that for every (u,Ω)-good program $\{x_i\}_{i=0}^{\infty}$, we have

$$\lim_{i\to\infty} \rho(x_i,x^*) = 0.$$

Assume that a program $\{x_i\}_{i=0}^{\infty}$ is (u,Ω)-good. Proposition 2.1 implies that there exists a natural number n_0 such that for all natural numbers T,

$$\sum_{t=0}^{T-1} u(x_t,x_{t+1}) - \sigma(u,T) > -n_0. \tag{2.112}$$

Let an integer $n > n_0$. Property (ii), (2.99), (2.100), and (2.112) imply that there exists a natural number $j(n)$ such that for every integer $t \ge j(n)$, we have

$$\rho(x_t,\bar{x}_{v_n}) \le (4n)^{-1} r_1(v_n,n).$$

Combined with (2.116) the inequality above implies that for all integers $t \ge j(n)$,

$$\rho(x_t,x^*) \le \rho(x_t,\bar{x}_{v_n}) + \rho(\bar{x}_{v_n},x^*) \le (2n)^{-1} r_1(v_n,n) \le (2n)^{-1}.$$

Since n is any natural number we conclude that

$$\lim_{i\to\infty} \rho(x_t,x^*) = 0$$

and $u \in \mathcal{M}_*(\Omega)$. This completes the proof of Theorem 2.6. □

2.6 Overtaking Optimal Programs

In the sequel we use a notion of an overtaking optimal program [70, 84].

An (Ω)-program $\{x_t\}_{t=0}^{\infty}$ is called (v,Ω)-overtaking optimal if for each (Ω)-program $\{y_t\}_{t=0}^{\infty}$ satisfying $y_0 = x_0$ the inequality

$$\limsup_{T\to\infty} \sum_{t=0}^{T-1} [v(y_t,y_{t+1}) - v(x_t,x_{t+1})] \le 0$$

holds.

The following result obtained in [71] and presented in Chap. 2 of [84] establishes the existence of an overtaking optimal program.

Theorem 2.13. *Assume that $x \in X$ and that there exists a (v, Ω)-good program $\{x_t\}_{t=0}^{\infty}$ such that $x_0 = x$. Then there exists a (v, Ω)-overtaking optimal program $\{x_t^*\}_{t=0}^{\infty}$ such that $x_0^* = x$.*

The next result obtained in [71] and presented in Chap. 2 of [84] provides necessary and sufficient conditions for overtaking optimality.

Theorem 2.14. *Assume that $\{x_t\}_{t=0}^{\infty}$ is an (Ω)-program and that there exists a (v, Ω)-good program $\{y_t\}_{t=0}^{\infty}$ such that $y_0 = x_0$. Then the program $\{x_t\}_{t=0}^{\infty}$ is (v, Ω)-overtaking optimal if and only if the following conditions hold:*

(i) $\lim_{t\to\infty} \rho(x_t, \bar{x}_v) = 0$;

(ii) for each natural number T and each (Ω)-program $\{y_t\}_{t=0}^{T}$ satisfying $y_0 = x_0$, $y_T = x_T$ the inequality $\sum_{t=0}^{T-1} v(y_t, y_{t+1}) \le \sum_{t=0}^{T-1} v(x_t, x_{t+1})$ holds.

The following two theorems which establish the uniform convergence of overtaking optimal programs to $\bar{x}_v$, are proved in Sect. 2.9. They were obtained in [86].

Theorem 2.15. *Let L_0 be a natural number and ϵ be a positive number. Then there exists an integer $T_0 \ge 1$ such that for each (v, Ω)-overtaking optimal program $\{x_t\}_{t=0}^{\infty}$ satisfying $x_0 \in \bar{Y}_{L_0}$ the inequality $\rho(x_t, \bar{x}_v) \le \epsilon$ holds for all integers $t \ge T_0$.*

Theorem 2.16. *Let ϵ be a positive number. Then there exists a positive number δ such that for each (v, Ω)-overtaking optimal program $\{x_t\}_{t=0}^{\infty}$ satisfying $\rho(x_0, \bar{x}_v) \le \delta$ the inequality $\rho(x_t, \bar{x}_v) \le \epsilon$ holds for all nonnegative integers t.*

2.7 Auxiliary Results

Let $v \in \mathcal{M}(\Omega)$ be an upper semicontinuous function. Suppose that $\bar{x}_v \in X$, $\bar{r}_v \in (0, 1)$, $\bar{c}_v > 0$ and that assumptions (A1), (A2), and (A3) hold.

For each positive number M denote by X_M the set of all points $x \in X$ for which there exists a (Ω)-program $\{x_t\}_{t=0}^{\infty}$ such that $x_0 = x$ and that for all natural numbers T the following inequality holds:

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) - Tv(\bar{x}_v, \bar{x}_v) \ge -M.$$

It is not difficult to see that $\cup\{X_M : M \in (0, \infty)\}$ is the set of all points $x \in X$ such that there exists a (v, Ω)-good program $\{x_t\}_{t=0}^{\infty}$ satisfying $x_0 = x$.

The boundedness of v implies the following result.

Proposition 2.17. *Let T be a natural number. Then there exists a positive number M such that $\bar{Y}_T \subset X_M$.*

The next result follows from Lemma 2.8.

Proposition 2.18. *Let M be positive number. Then there exists an integer $T \geq 1$ such that the inclusion $X_M \subset \bar{Y}_T$ holds.*

We define a function $\pi^v(x)$, $x \in X$ which plays an important role in our study. For all $x \in X \setminus \cup\{X_M : M \in (0,\infty)\}$ set

$$\pi^v(x) = -\infty.$$

Let

$$x \in \cup\{X_M : M \in (0,\infty)\}. \tag{2.113}$$

Set

$$\pi^v(x) = \sup\{\limsup_{T\to\infty} \sum_{t=0}^{T-1} (v(x_t, x_{t+1}) - v(\bar{x}_v, \bar{x}_v)) :$$

$$\{x_t\}_{t=0}^{\infty} \text{ is an } (\Omega) - \text{program such that } x_0 = x\}. \tag{2.114}$$

In view of (A2), (2.113), and (2.114),

$$-\infty < \pi^v(x) \leq \bar{c}_v. \tag{2.115}$$

By (2.113), (2.114), and Proposition 2.1,

$$\pi^v(x) = \sup\{\limsup_{T\to\infty} \sum_{t=0}^{T-1} (v(x_t, x_{t+1}) - v(\bar{x}_v, \bar{x}_v)) :$$

$$\{x_t\}_{t=0}^{\infty} \text{ is a } (v,\Omega) - \text{program such that } x_0 = x\}. \tag{2.116}$$

Denote by $\mathcal{P}(v,x)$ the set of all (v,Ω)-overtaking optimal programs $\{x_t\}_{t=0}^{\infty}$ satisfying $x_0 = x$. By Theorem 2.13, the set $\mathcal{P}(v,x)$ is nonempty. Definition (2.114) implies the following result.

Proposition 2.19. *1. Let $\{x_t\}_{t=0}^{\infty}$ be a (v,Ω)-good program. Then for each integer $t \geq 0$,*

$$\pi^v(x_t) \geq v(x_t, x_{t+1}) - v(\bar{x}_v, \bar{x}_v) + \pi^v(x_{t+1}). \tag{2.117}$$

2. Let $T \geq 1$ be an integer and $\{x_t\}_{t=0}^{T}$ be an (Ω)-program such that $\pi^v(x_T) > -\infty$. Then (2.117) holds for all integers $t = 0, \ldots, T-1$.

The next result follows from the definition of (v,Ω)-overtaking optimal programs.

Proposition 2.20. *Let $x \in \cup\{X_M : M \in (0,\infty)\}$ and $\{x_t\}_{t=0}^{\infty}$ be a (v,Ω)-overtaking optimal program satisfying $x_0 = x$. Then*

$$\pi^v(x) = \limsup_{T\to\infty} \sum_{t=0}^{T-1} (v(x_t, x_{t+1}) - v(\bar{x}_v, \bar{x}_v)).$$

Corollary 2.21. *Let $\{x_t\}_{t=0}^{\infty}$ be a (v, Ω)-overtaking optimal and (v, Ω)-good program. Then for any integer $t \geq 0$,*

$$\pi^v(x_t) = v(x_t, x_{t+1}) - v(\bar{x}_v, \bar{x}_v) + \pi^v(x_{t+1}).$$

Set

$$\sup(\pi^v) = \sup\{\pi^v(z) : \ z \in \cup\{X_M : \ M \in (0, \infty)\}\}, \tag{2.118}$$

$$X_v = \{x \in \cup\{X_M : \ M \in (0, \infty)\} : \ \pi^v(x) \geq \sup(\pi^v) - 1\}. \tag{2.119}$$

Proposition 2.22. $\pi^v(\bar{x}_v) = 0$.

Proof. Set $x_t = \bar{x}_v$ for all integers $t \geq 0$. By Theorem 2.14 and (A2), the program $\{x_t\}_{t=0}^{\infty}$ is a (v, Ω)-overtaking optimal. In view of Proposition 2.20, $\pi^v(\bar{x}_v) = 0$. □

Theorem 2.16, Proposition 2.20, and (A1) imply the following result.

Proposition 2.23. *The function π^v is finite in a neighborhood of $\bar{x}_v$ and continuous at $\bar{x}_v$.*

Proposition 2.24. *Assume that $x_0 \in \cup\{X_M : \ M \in (0, \infty)\}$ and $\{x_t\}_{t=0}^{\infty} \in \mathcal{P}(v, x_0)$. Then*

$$\pi^v(x_0) = \lim_{T \to \infty} \sum_{t=0}^{T-1} (v(x_t, x_{t+1}) - v(\bar{x}_v, \bar{x}_v)).$$

Proof. It follows from (A3), Proposition 2.23, and Corollary 2.21 that

$$\pi^v(x_0) = \lim_{T \to \infty} (\pi^v(x_0) - \pi^v(x_T)) = \lim_{T \to \infty} \sum_{t=0}^{T-1} (v(x_t, x_{t+1}) - v(\bar{x}_v, \bar{x}_v)).$$

Proposition 2.24 is proved. □

The next result is proved in Sect. 2.10.

Proposition 2.25. *There exists an integer $L_v \geq 1$ such that $X_v \subset \bar{Y}_{L_v}$.*

The following result is proved in Sect. 2.11.

Proposition 2.26. *The function $\pi^v : X \to R^1 \cup \{-\infty\}$ is upper semicontinuous.*

Set

$$\mathcal{D}(v) = \{x \in X : \ \pi^v(x) = \sup(\pi^v)\}. \tag{2.120}$$

By Proposition 2.26 and (2.115) the set $\mathcal{D}(v)$ is nonempty and closed subset of X. The following proposition is proved in Sect. 2.11.

Proposition 2.27. *Let $\{x_t\}_{t=0}^{\infty}$ be a (v,Ω)-good program such that for all integers $t \geq 0$,*

$$v(x_t, x_{t+1}) - v(\bar{x}_v, \bar{x}_v) = \pi^v(x_t) - \pi^v(x_{t+1}).$$

Then $\{x_t\}_{t=0}^{\infty}$ is a (v,Ω)-overtaking optimal program.

The next result easily follows from Proposition 2.25, Theorem 2.15, (2.119), and (2.120).

Proposition 2.28. *For each $\epsilon > 0$ there exists a natural number T_ϵ such that for each $z \in \mathcal{D}(v)$ and each (Ω)-program $\{x_t\}_{t=0}^{\infty} \in \mathcal{P}(v,z)$ the inequality $\rho(x_t, \bar{x}_v) \leq \epsilon$ holds for all integers $t \geq T_\epsilon$.*

In order to study the structure of solutions of the problems (P2) and (P3) we introduce the following notation and definitions.

Set

$$\bar{\Omega} = \{(x,y) \in X \times X : \ (y,x) \in \Omega\}. \tag{2.121}$$

Clearly, $\bar{\Omega}$ is a nonempty closed subset of $X \times X$ and

$$\{(x,y) \in X \times X : \ \rho(x, \bar{x}_v), \ \rho(y, \bar{x}_v) \leq \bar{r}_v\} \subset \bar{\Omega}. \tag{2.122}$$

Then $\mathcal{M}(\bar{\Omega})$ is the set of all bounded functions $u : \bar{\Omega} \to R^1$ with $\|u\| = \sup\{|u(z)| : \ z \in \bar{\Omega}\}$.

For each $u \in \mathcal{M}(\Omega)$ define $\bar{u} \in \mathcal{M}(\bar{\Omega})$ by

$$\bar{u}(x,y) = u(y,x), \ (x,y) \in \bar{\Omega}. \tag{2.123}$$

Clearly, $u \to \bar{u}, \ u \in \mathcal{M}(\Omega)$ is a linear invertible isometry operator.

Let $0 \leq T_1 < T_2$ be integers and let $\{x_t\}_{t=T_1}^{T_2}$ be an (Ω)-program. Define $\{\bar{x}_t\}_{t=T_1}^{T_2} \subset X$ by

$$\bar{x}_t = x_{T_2 - t + T_1}, \ t = T_1, \dots, T_2. \tag{2.124}$$

Clearly, $\{\bar{x}_t\}_{t=T_1}^{T_2}$ is an $(\bar{\Omega})$-program.

Assume that $\{u_t\}_{t=T_1}^{T_2-1} \subset \mathcal{M}(\Omega)$. It is easy to see that

$$\sum_{t=T_1}^{T_2-1} \bar{u}_{T_2-t+T_1-1}(\bar{x}_t, \bar{x}_{t+1}) = \sum_{t=T_1}^{T_2-1} u_{T_2-t+T_1-1}(x_{T_2-t+T_1-1}, x_{T_2-t+T_1})$$

$$= \sum_{t=T_1}^{T_2-1} u_t(x_t, x_{t+1}). \tag{2.125}$$

The next result easily follows from (2.125).

Proposition 2.29. *Let $T \geq 1$ be an integer, $M \geq 0$, $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$ and $\{x_t^{(i)}\}_{t=0}^{T}$, $i = 1, 2$ are (Ω)-programs. Then*

$$\sum_{t=0}^{T-1} u_t(x_t^{(1)}, x_{t+1}^{(1)}) \geq \sum_{t=0}^{T-1} u_t(x_t^{(2)}, x_{t+1}^{(2)}) - M$$

if and only if

$$\sum_{t=0}^{T-1} \bar{u}_{T-t-1}(\bar{x}_t^{(1)}, \bar{x}_{t+1}^{(1)}) \geq \sum_{t=0}^{T-1} \bar{u}_{T-t-1}(x_t^{(2)}, x_{t+1}^{(2)}) - M.$$

Proposition 2.29 implies the following result.

Proposition 2.30. *Let $T \geq 1$ be an integer, $M \geq 0$, $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$ and $\{x_t\}_{t=0}^{T}$ be an (Ω)-program. Then $\{\bar{x}_t\}_{t=0}^{T}$ is an $(\bar{\Omega})$-program and the following assertions hold:*

if $\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=0}^{T-1}, 0, T) - M$, then

$$\sum_{t=0}^{T-1} \bar{u}_{T-t-1}(\bar{x}_t, \bar{x}_{t+1}) \geq \sigma(\{\bar{u}_{T-t-1}\}_{t=0}^{T-1}, 0, T) - M;$$

if $\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=0}^{T-1}, 0, T, x_0, x_T) - M$, then

$$\sum_{t=0}^{T-1} \bar{u}_{T-t-1}(\bar{x}_t, \bar{x}_{t+1}) \geq \sigma(\{\bar{u}_{T-t-1}\}_{t=0}^{T-1}, 0, T, \bar{x}_0, \bar{x}_T) - M;$$

if $\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \widehat{\sigma}(\{u_t\}_{t=0}^{T-1}, 0, T, x_T) - M$, then

$$\sum_{t=0}^{T-1} \bar{u}_{T-t-1}(\bar{x}_t, \bar{x}_{t+1}) \geq \sigma(\{\bar{u}_{T-t-1}\}_{t=0}^{T-1}, 0, T, \bar{x}_0) - M;$$

if $\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=0}^{T-1}, 0, T, x_0) - M$, then

$$\sum_{t=0}^{T-1} \bar{u}_{T-t-1}(\bar{x}_t, \bar{x}_{t+1}) \geq \widehat{\sigma}(\{\bar{u}_{T-t-1}\}_{t=0}^{T-1}, 0, T, \bar{x}_T) - M.$$

The next result is proved in Sect. 2.11.

Proposition 2.31. *Let $v \in \mathcal{M}(\Omega)$ be an upper semicontinuous function. Suppose that $\bar{x}_v \in X$, $\bar{r}_v > 0$, $\bar{c}_v > 0$ and that assumptions (A1), (A2), and (A3) hold. Then the function $\bar{v}$ is upper semicontinuous,*

$$\{(x, y) \in X \times X : \ \rho(x, \bar{x}_v), \ \rho(y, \bar{x}_v) \leq \bar{r}_v\} \subset \bar{\Omega}, \tag{2.126}$$

the function $\bar{v}$ is continuous at $(\bar{x}_v, \bar{x}_v)$,

$$\sigma(\bar{v}, T) \le T\bar{v}(\bar{x}_v, \bar{x}_v) + \bar{c}_v \text{ for all integers } T \ge 1 \tag{2.127}$$

and for all $(\bar{v}, \bar{\Omega})$-good programs $\{x_t\}_{t=0}^{\infty}$,

$$\lim_{t\to\infty} \rho(x_t, \bar{x}_v) = 0.$$

In view of Proposition 2.31, if $v \in \mathcal{M}(\Omega)$ is upper semicontinuous and satisfies (A1)–(A3), then $\bar{v}$ is also upper semicontinuous and satisfies (A1)–(A3). Therefore all the results presented above for the pair (v, Ω) are also true for the pair $(\bar{v}, \bar{\Omega})$.

2.8 Structure of Solutions in the Regions Close to the Endpoints

In Sect. 2.13 we prove the following result which describes the structure of approximate solutions of the problems of the type (P2) in the regions close to the right endpoints.

Theorem 2.32. *Suppose that $v \in \mathcal{M}(\Omega)$ is an upper semicontinuous function, $\bar{x}_v \in X$, $\bar{r}_v > 0$, $\bar{c}_v > 0$ and that assumptions (A1), (A2), and (A3) hold. Let $L_0 \ge 1$, $\tau_0 \ge 1$ be integers and $\epsilon > 0$. Then there exist $\delta > 0$ and an integer $T_0 \ge \tau_0$ such that for each integer $T \ge T_0$, each $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$ satisfying*

$$\|u_t - v\| \le \delta,\ t = 0 \dots, T-1$$

and each (Ω)-program $\{x_t\}_{t=0}^{T}$ which satisfies

$$x_0 \in \bar{Y}_{L_0},\ \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \ge \sigma(\{u_t\}_{t=0}^{T-1}, 0, T, x_0) - \delta$$

there exists an $(\bar{\Omega})$-program

$$\{x_t^*\}_{t=0}^{\infty} \in \cup\{\mathcal{P}(\bar{v}, z) :\ z \in \mathcal{D}(\bar{v})\}$$

such that

$$\rho(x_{T-t}, x_t^*) \le \epsilon,\ t = 0, \dots, \tau_0.$$

Recall that $\cup\{\mathcal{P}(\bar{v}, z) :\ z \in \mathcal{D}(\bar{v})\}$ is the set of all $(\bar{v}, \bar{\Omega})$-overtaking optimal programs $\{x_t^*\}_{t=0}^{\infty}$ such that x_0^* is the point of maximum of the function $\pi^{\bar{v}}$.

In Sect. 2.14 we prove the following result which describes the structure of approximate solutions of the problems of the type (P3) in the regions close to the endpoints.

Theorem 2.33. *Suppose that $v \in \mathcal{M}(\Omega)$ is an upper semicontinuous function, $\bar{x}_v \in X$, $\bar{r}_v > 0$, $\bar{c}_v > 0$ and that assumptions (A1), (A2), and (A3) hold. Let $\tau_0 \geq 1$ be an integer and $\epsilon > 0$. Then there exist $\delta > 0$ and an integer $T_0 \geq \tau_0$ such that for each integer $T \geq T_0$, each $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$ satisfying*

$$\|u_t - v\| \leq \delta,\ t = 0 \ldots, T-1$$

and each (Ω)-program $\{x_t\}_{t=0}^{T}$ which satisfies

$$\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=0}^{T-1}, 0, T) - \delta$$

there exist an (Ω)-program

$$\{y_t^*\}_{t=0}^{\infty} \in \cup\{\mathcal{P}(v, z) :\ z \in \mathcal{D}(v)\}$$

and an $(\bar{\Omega})$-program

$$\{x_t^*\}_{t=0}^{\infty} \in \cup\{\mathcal{P}(\bar{v}, z) :\ z \in \mathcal{D}(\bar{v})\}$$

such that for all integers $t = 0, \ldots, \tau_0$,

$$\rho(x_{T-t}, x_t^*) \leq \epsilon,\ \rho(x_t, y_t^*) \leq \epsilon.$$

The next result is proved in Sect. 2.15.

Proposition 2.34. *Suppose that $v \in \mathcal{M}(\Omega)$ is an upper semicontinuous function, $\bar{x}_v \in X$, $\bar{r}_v > 0$, $\bar{c}_v > 0$ and that assumptions (A1), (A2), and (A3) hold. Let $\tau_0 \geq 1$ be an integer and $\epsilon > 0$. Then there exist $\delta > 0$ and an integer $T_0 \geq \tau_0$ such that for each $u \in B_d(v, \delta) \cap \mathcal{M}_*(\Omega)$ the following properties hold:*
for each $\{x_t\}_{t=0}^{\infty} \in \cup\{\mathcal{P}(u, z) :\ z \in \mathcal{D}(u)\}$ there exists

$$\{y_t\}_{t=0}^{\infty} \in \cup\{\mathcal{P}(v, z) :\ z \in \mathcal{D}(v)\}$$

such that $\rho(x_t, y_t) \leq \epsilon$ for all integers $t = 0, \ldots, \tau_0$;
for each $\{x_t\}_{t=0}^{\infty} \in \cup\{\mathcal{P}(\bar{u}, z) :\ z \in \mathcal{D}(\bar{u})\}$ there exists

$$\{y_t\}_{t=0}^{\infty} \in \cup\{\mathcal{P}(\bar{v}, z) :\ z \in \mathcal{D}(\bar{v})\}$$

such that $\rho(x_t, y_t) \leq \epsilon$ for all integers $t = 0, \ldots, \tau_0$.

We have already mentioned that the mapping $v \to \bar{v}$, $v \in \mathcal{M}(\Omega)$ is a linear isometry which has the inverse. It is not difficult to see that

$$\bar{v} \in \mathcal{M}_0(\bar{\Omega}) \text{ for all } v \in \mathcal{M}_0(\Omega),$$

$$\bar{v} \in \mathcal{M}_{c,0}(\bar{\Omega}) \text{ for all } v \in \mathcal{M}_{c,0}(\Omega),$$

$$\bar{v} \in \mathcal{M}_*(\bar{\Omega}) \text{ for all } v \in \mathcal{M}_*(\Omega),$$

$$\bar{v} \in \mathcal{M}_{c*}(\bar{\Omega}) \text{ for all } v \in \mathcal{M}_{c*}(\Omega),$$
$$\bar{v} \in \bar{\mathcal{M}}_0(\bar{\Omega}) \text{ if and only if } v \in \bar{\mathcal{M}}_0(\Omega),$$
$$\bar{v} \in \bar{\mathcal{M}}_{c,0}(\bar{\Omega}) \text{ if and only if } v \in \bar{\mathcal{M}}_{c,0}(\Omega).$$

The next theorem which is proved in Sect. 2.16 shows that for most objective functions v (in the sense of the Baire category) the sets $\cup\{\mathcal{P}(v,z) : z \in \mathcal{D}(v)\}$ and $\cup\{\mathcal{P}(\bar{v},z) : z \in \mathcal{D}(\bar{v})\}$ are singletons. In this case approximate solutions of the problems of the types (P2) and (P3) in the regions close to the endpoints have a simple structure.

Theorem 2.35. *Let $\mathfrak{M}$ be either $\bar{\mathcal{M}}_0(\Omega)$ or $\bar{\mathcal{M}}_{c,0}(\Omega)$. Then there exists a set $\mathcal{F} \subset \mathfrak{M} \cap \mathcal{M}_*(\Omega)$ which is a countable intersection of open everywhere dense subsets of $\mathfrak{M}$ such that for each $v \in \mathcal{F}$ there exists a unique pair of points $z, \bar{z} \in X$ such that*

$$\pi^v(z) = \sup(\pi^v),\ \pi^{\bar{v}}(\bar{z}) = \sup(\pi^{\bar{v}})$$

and there exist a unique (v,Ω)-overtaking optimal program $\{z_t\}_{t=0}^{\infty}$ satisfying $z_0 = z$ and a unique $(\bar{v},\bar{\Omega})$-overtaking optimal program $\{\widehat{z}_t\}_{t=0}^{\infty}$ satisfying $\widehat{z}_0 = \bar{z}$.

The results of this section were obtained in [86].

2.9 Proof of Theorems 2.15 and 2.16

We prove Theorems 2.15 and 2.16 simultaneously. In the case of Theorem 2.16 set $L_0 = 1$. By Theorem 2.3, there exist an integer $L \geq 1$ and $\delta \in (0, \min\{\epsilon, \bar{r}_v/2\})$ such that for the following property holds:

(i) for each integer $T > 2L$ and each (Ω)-program $\{x_t\}_{t=0}^{T}$ which satisfies

$$x_0 \in \bar{Y}_{L_0},\ x_T \in Y_{L_0},$$
$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) \geq \sigma(v, T, x_0, x_T) - \delta$$

there exist integers $\tau_1 \in [0, L]$, $\tau_2 \in [T-L, T]$ such that $\rho(x_t, \bar{x}_v) \leq \epsilon,\ t = \tau_1, \ldots, \tau_2$; moreover, if $\rho(x_0, \bar{x}_v) \leq \delta$, then $\tau_1 = 0$ and if $\rho(x_T, \bar{x}_v) \leq \delta$, then $\tau_2 = T$.

Assume that $\{x_t\}_{t=0}^{\infty}$ is an (v,Ω)-overtaking optimal program such that

$$x_0 \in \bar{Y}_{L_0}. \tag{2.128}$$

This implies that it is (v,Ω)-good and in view of (A3), $\lim_{t\to\infty} x_t = \bar{x}_v$. Then for all sufficiently large natural numbers T, $\rho(x_T, \bar{x}_v) \leq \delta$ and $x_T \in Y_{L_0}$. Since the program $\{x_t\}_{t=0}^{\infty}$ is (v,Ω)-overtaking optimal it follows from the

inclusion above, (2.128) and the property (i) that for all sufficiently large natural numbers T, $\rho(x_t, \bar{x}_v) \le \epsilon,\ t = L, \dots, T$ and if $\rho(x_0, \bar{x}_v) \le \delta$, then

$$\rho(x_t, \bar{x}_v) \le \epsilon,\ t = 0, \dots .T$$

This completes the proof of Theorems 2.15 and 2.16. □

2.10 Proof of Proposition 2.25

By Assertion 1 of Theorem 2.5 and (A2) there exists a natural number L_v such that for each integer $T \ge L_v$ and each (Ω)-program $\{x_t\}_{t=0}^T$ satisfying

$$\sum_{t=0}^{T-1} [v(x_t, x_{t+1}) - v(\bar{x}_v, \bar{x}_v)] > -2 + \sup(\pi^v) \tag{2.129}$$

we have

$$\mathrm{Card}(\{t \in \{0, \dots, T\} :\ \rho(x_t, \bar{x}_v) > \bar{r}_v/8\}) < L_v. \tag{2.130}$$

Assume that

$$x \in X_v. \tag{2.131}$$

By (2.119), (2.131), and Proposition 2.24, there exists a (v, Ω)-overtaking optimal program $\{x_t\}_{t=0}^\infty$ such

$$x_0 = x,$$

$$\sup(\pi^v) - 1 \le \pi^v(x) = \lim_{T \to \infty} \sum_{t=0}^{T-1} (v(x_t, x_{t+1}) - v(\bar{x}_v, \bar{x}_v)).$$

There exists a natural number T_0 such that for all integers $T \ge T_0$,

$$\sum_{t=0}^{T-1} (v(x_t, x_{t+1}) - v(\bar{x}_v, \bar{x}_v)) \ge \sup(\pi^v) - 2.$$

Thus

$$\sum_{t=0}^{T_0 + L_v} (v(x_t, x_{t+1}) - v(\bar{x}_v, \bar{x}_v)) \ge \sup(\pi^v) - 2.$$

By the inequality above and the choice of L_v (see (2.129) and (2.130)),

$$\mathrm{Card}(\{t \in \{0, \dots, T_0 + L_v + 1\} :\ \rho(x_t, \bar{x}_v) > \bar{r}_v/8\}) < L_v.$$

Thus there exists an integer $\tau \in [0, L_v - 1]$ such that $\rho(x_\tau, \bar{x}_v) \le \bar{r}_v/8$. This implies that $x_0 \in \bar{Y}_{\tau+1} \subset \bar{Y}_{L_v}$. Proposition 2.25 is proved. □

2.11 Proofs of Propositions 2.26, 2.27, and 2.31

Proof of Proposition 2.26. Assume that $\{x^{(i)}\}_{i=1}^{\infty} \subset X$, $x \in X$ and

$$\lim_{i\to\infty} x^{(i)} = x.$$

We show that

$$\pi^v(x) \geq \limsup_{i\to\infty} \pi^v(x^{(i)}).$$

We may assume without loss of generality that

$$-\infty < \limsup_{i\to\infty} \pi^v(x^{(i)}) = \lim_{i\to\infty} \pi^v(x^{(i)}), \tag{2.132}$$

$$\pi^v(x^{(i)}) > -\infty \text{ for all integers } i \geq 1. \tag{2.133}$$

By (2.133), Theorem 2.13, and Proposition 2.24, for each integer $i \geq 1$, there exists a (v, Ω)-overtaking optimal program $\{x_t^{(i)}\}_{t=0}^{\infty}$ such

$$x_0^{(i)} = x^{(i)},$$

$$\pi^v(x^{(i)}) = \lim_{T\to\infty} \sum_{t=0}^{T-1} (v(x_t^{(i)}, x_{t+1}^{(i)}) - v(\bar{x}_v, \bar{x}_v)). \tag{2.134}$$

Extracting a subsequence and re-indexing, if necessary, we may assume without loss of generality that for each integer $t \geq 0$ there exists

$$x_t = \lim_{i\to\infty} x_t^{(i)}. \tag{2.135}$$

In view of (2.135), $\{x_t\}_{t=0}^{\infty}$ is an (Ω)-program.

Let $\epsilon > 0$. By Propositions 2.22 and 2.23, there exists $\delta \in (0, \min\{\epsilon, \bar{r}_v/4\})$ such that for each $x \in X$ satisfying $\rho(x, \bar{x}_v) \leq \delta$,

$$|\pi^v(x)| \leq \epsilon/2. \tag{2.136}$$

It follows from (2.132), (2.134), and (A2) that for all sufficiently large natural numbers i there exists an integer $S_i \geq 1$ such that for all integers $T \geq S_i$,

$$\sum_{t=0}^{T-1} v(x_t^{(i)}, x_{t+1}^{(i)}) \geq T v(\bar{x}_v, \bar{x}_v) + \lim_{j\to\infty} \pi^v(x^{(j)}) - 4$$

$$\geq \sigma(v, T) - \bar{c}_v + \lim_{j\to\infty} \pi^v(x^{(j)}) - 4.$$

Together with Assertion 1 of Theorem 2.5 this implies the existence of a natural number L_1 such that for all sufficiently large natural numbers i,

$$\min\{\rho(x_t^{(i)}, \bar{x}_v) : \ t = 0, \dots, L_1 - 1\} \leq \delta.$$

This implies that for all sufficiently large natural numbers i,

$$x_0^{(i)} \in \bar{Y}_{L_1}. \tag{2.137}$$

Since $\{x_t^{(i)}\}_{t=0}^{\infty}$, $i = 1, 2, \ldots$ are (v, Ω)-overtaking optimal programs it follows from (2.137) and Theorem 2.15 that there exists a natural number L_2 such that for all sufficiently large natural numbers i,

$$\rho(x_t^{(i)}, \bar{x}_v) \le \delta \text{ for all integers } t \ge L_2. \tag{2.138}$$

We may assume without loss of generality that (2.138) holds for all integers $i \ge 1$. Let $T \ge L_2$ be an integer. By Corollary 2.21, (2.138), and the choice of δ (see (2.136)), for all integers $i \ge 1$,

$$\sum_{t=0}^{T-1} (v(x_t^{(i)}, x_{t+1}^{(i)}) - v(\bar{x}_v, \bar{x}_v)) = \pi^v(x_0^{(i)}) - \pi^v(x_T^{(i)}) \ge \pi^v(x_0^{(i)}) - \epsilon/2.$$

In view of the relation above and the upper semicontinuity of v, for all integers $T \ge L_2$,

$$\sum_{t=0}^{T-1} (v(x_t, x_{t+1}) - v(\bar{x}_v, \bar{x}_v)) \ge \limsup_{i\to\infty} \sum_{t=0}^{T-1} (v(x_t^{(i)}, x_{t+1}^{(i)}) - v(\bar{x}_v, \bar{x}_v))$$

$$\ge \limsup_{i\to\infty} \pi^v(x_0^{(i)}) - \epsilon/2.$$

By (2.134), (2.135), and the relation above,

$$\pi^v(x) \ge \limsup_{T\to\infty} \sum_{t=0}^{T-1} (v(x_t, x_{t+1}) - v(\bar{x}_v, \bar{x}_v)) \ge \limsup_{i\to\infty} \pi^v(x^{(i)}) - \epsilon/2.$$

Since ϵ is any positive number we conclude that $\pi^v(x) \ge \limsup_{i\to\infty} \pi^v(x^{(i)})$. Proposition 2.26 is proved. □

Proof of Proposition 2.27. In view of (A3),

$$\lim_{t\to\infty} x_t = \bar{x}_v. \tag{2.139}$$

By Theorem 2.13 there exists an (v, Ω)-overtaking optimal program $\{y_i\}_{i=0}^{\infty}$ such that

$$y_0 = x_0. \tag{2.140}$$

Proposition 2.24 and (2.140) imply that

$$\pi^v(x_0) = \lim_{T\to\infty} \sum_{t=0}^{T-1} (v(y_t, y_{t+1}) - v(\bar{x}_v, \bar{x}_v)). \tag{2.141}$$

On the other hand it follows from (2.139) and Propositions 2.22 and 2.23 that for any integer $T \geq 1$,

$$\sum_{t=0}^{T-1}(v(x_t, x_{t+1}) - v(\bar{x}_v, \bar{x}_v)) = \pi^v(x_0) - \pi^v(x_T) \to \pi^v(x_0) \text{ as } T \to \infty. \quad (2.142)$$

By (2.141) and (2.142),

$$\lim_{T\to\infty} \sum_{t=0}^{T-1}(v(x_t, x_{t+1}) - v(y_t, y_t)) = 0.$$

This implies that $\{x_i\}_{i=0}^{\infty}$ is an (v, Ω)-overtaking optimal program. Proposition 2.27 is proved. □

Proof of Proposition 2.31. It is clear that the function $\bar{v}$ is upper semicontinuous and continuous at the point $(\bar{x}_v, \bar{x}_v)$ and that (2.126) and (2.127) hold. Let $\{x_i\}_{i=0}^{\infty}$ be a $(\bar{v}, \bar{\Omega})$-good program. Then there exists a number $M_1 > 0$ such that for each pair of integers $T_2 > T_1 \geq 0$,

$$|\sum_{t=T_1}^{T_2-1}(\bar{v}(x_t, x_{t+1}) - v(\bar{x}_v, \bar{x}_v))| \leq M_1. \quad (2.143)$$

In order to complete the proof of the proposition it is sufficient to show that

$$\lim_{t\to\infty} x_t = \bar{x}_v.$$

Let $\epsilon > 0$. By Lemma 2.9 and Proposition 2.30, there exists $\delta \in (0, \bar{r}_v)$ such that the following property holds:

(ii) for each natural number T and each $(\bar{\Omega})$-program $\{z_t\}_{t=0}^{T}$ which satisfies $\rho(z_0, \bar{x}_v),\ \rho(z_T, \bar{x}_v) \leq \delta$,

$$\sum_{t=0}^{T-1} \bar{v}(z_t, z_{t+1}) \geq \sigma(\bar{v}, z_0, z_T, T) - \delta$$

the inequality $\rho(z_t, \bar{x}_v) \leq \epsilon$ holds for all integers $t = 0, \ldots, T$.

By Lemma 2.8 and Proposition 2.30, there exists an integer $T_0 \geq 1$ such that the following property holds:

(iii) for each $(\bar{\Omega})$-program $\{z_t\}_{t=0}^{T_0}$ which satisfies

$$\sum_{t=0}^{T_0-1} \bar{v}(z_t, z_{t+1}) \geq T_0\bar{v}(\bar{x}_v, \bar{x}_v) - M_1$$

we have

$$\min\{\rho(z_t, \bar{x}_v) :\ t = 0, \ldots, T_0\} \leq \delta.$$

Since $\{x_i\}_{i=0}^{\infty}$ is a $(\bar{v}, \bar{\Omega})$-good program there exists a natural number τ_0 such that for each pair of integers $T_2 > T_1 \geq \tau_0$,

$$\sum_{t=T_1}^{T_2-1} \bar{v}(x_t, x_{t+1}) \geq \sigma(\bar{v}, x_{T_1}, x_{T_2}, T_2 - T_1) - \delta. \tag{2.144}$$

It follows from the choice of M_1 (see (2.143)) and the property (iii) that there exists a strictly increasing sequence of natural numbers $\{S_k\}_{k=1}^{\infty}$ such that $S_1 \geq \tau_0$ and $\rho(x_{S_k}, \bar{x}_v) \leq \delta$ for all integers $k \geq 1$. By the relations above, (2.144) and the property (ii), for every natural number k, $\rho(x_t, \bar{x}_v) \leq \epsilon$, $t = S_k, \ldots, S_{k+1}$. Therefore $\rho(x_t, \bar{x}_v) \leq \epsilon$ for all integers $t \geq S_1$ and $\lim_{t\to\infty} x_t = \bar{x}_v$. Proposition 2.31 is proved. □

2.12 A Basic Lemma for Theorem 2.32

Lemma 2.36. *Suppose that $v \in \mathcal{M}(\Omega)$ is an upper semicontinuous function, $\bar{x}_v \in X$, $\bar{r}_v > 0$, $\bar{c}_v > 0$ and that assumptions (A1), (A2), and (A3) hold. Let $T_0 \geq 1$ be an integer and $\epsilon \in (0, 1)$. Then there exists $\delta \in (0, \epsilon)$ such that for each (Ω)-program $\{x_t\}_{t=0}^{T_0}$ which satisfies*

$$\pi^v(x_0) \geq \sup(\pi^v) - \delta, \tag{2.145}$$

$$\sum_{t=0}^{T_0-1} (v(x_t, x_{t+1}) - v(\bar{x}_v, \bar{x}_v)) - \pi^v(x_0) + \pi^v(x_T) \geq -\delta \tag{2.146}$$

there exists an (Ω)-program

$$\{z_t\}_{t=0}^{\infty} \in \cup\{\mathcal{P}(v, z) : \ z \in \mathcal{D}(v)\} \tag{2.147}$$

such that

$$\rho(z_t, x_t) \leq \epsilon, \ t = 0, \ldots, T_0. \tag{2.148}$$

Proof. Assume that the lemma does not hold. Then there exist a sequence $\{\delta_k\}_{k=1}^{\infty} \subset (0, 1]$ and a sequence of (Ω)-programs $\{x_t^{(k)}\}_{t=0}^{T_0}$, $k = 1, 2, \ldots$ such that

$$\lim_{k\to\infty} \delta_k = 0 \tag{2.149}$$

and that for each integer $k \geq 1$ and each (Ω)-program $\{z_t\}_{t=0}^{\infty}$ satisfying (2.147),

$$\pi^v(x_0^{(k)}) \geq \sup(\pi^v) - \delta_k, \tag{2.150}$$

$$\sum_{t=0}^{T_0-1} (v(x_t^{(k)}, x_{t+1}^{(k)}) - v(\bar{x}_v, \bar{x}_v)) - \pi^v(x_0^{(k)}) + \pi^v(x_{T_0}^{(k)}) \geq -\delta_k \tag{2.151}$$

we have

$$\max\{\rho(z_t, x_t^{(k)}) : \ t = 0, \ldots, T_0\} > \epsilon. \tag{2.152}$$

In view of (2.150) and (2.151), for each integer $k \geq 1$, $\pi^v(x_0^{(k)})$ and $\pi^v(x_{T_0}^{(k)})$ are finite. Extracting a subsequence and re-indexing, if necessary, we may assume without loss of generality that for each integer $t \in [0, T_0]$ there exists

$$x_t = \lim_{k\to\infty} x_t^{(k)}. \tag{2.153}$$

Proposition 2.26, (2.150), and (2.153) imply that

$$\sup(\pi^v) \geq \pi^v(x_0) \geq \limsup_{k\to\infty} \pi^v(x_0^{(k)}) \geq \sup(\pi^v). \tag{2.154}$$

By upper semicontinuity of v and π^v (see Proposition 2.26), (2.149)–(2.151), (2.153), and (2.154),

$$\sum_{t=0}^{T_0-1} (v(x_t, x_{t+1}) - v(\bar{x}_v, \bar{x}_v)) - \pi^v(x_0) + \pi^v(x_{T_0})$$

$$\geq \limsup_{k\to\infty} (\sum_{t=0}^{T_0-1} (v(x_t^{(k)}, x_{t+1}^{(k)}) - v(\bar{x}_v, \bar{x}_v)) - \pi^v(x_0^{(k)}) + \pi^v(x_{T_0}^{(k)}))$$

$$\geq \limsup_{k\to\infty} (-\delta_k) = 0.$$

Thus

$$\sum_{t=0}^{T_0-1} (v(x_t, x_{t+1}) - v(\bar{x}_v, \bar{x}_v)) - \pi^v(x_0) + \pi^v(x_{T_0}) \geq 0. \tag{2.155}$$

In view of (2.154) and (2.155), $\pi^v(x_{T_0})$ is finite. Together with Proposition 2.19 and (2.155) this implies that for all integers $t = 0, \ldots, T_0 - 1$,

$$v(x_t, x_{t+1}) - v(\bar{x}_v, \bar{x}_v) = \pi^v(x_t) - \pi^v(x_{t+1}). \tag{2.156}$$

Since $\pi^v(x_{T_0})$ is finite Theorem 2.13 implies that there is a (v, Ω)-overtaking optimal and (v, Ω)-good program $\{\tilde{x}_t\}_{t=0}^{\infty}$ satisfying

$$\tilde{x}_0 = x_{T_0}. \tag{2.157}$$

For all integers $t > T_0$ set

$$x_t = \tilde{x}_{t-T_0}. \tag{2.158}$$

It is clear that $\{x_t\}_{t=0}^{\infty}$ is an (Ω)-program. Since $\{\tilde{x}_t\}_{t=0}^{\infty}$ is (v, Ω)-overtaking optimal program Corollary 2.21 and (2.158) imply that (2.156) holds for all integers $t \geq 0$. Since $\{\tilde{x}_t\}_{t=0}^{\infty}$ is (v, Ω)-good program it follows from (2.156), (2.158), and Proposition 2.27 that $\{x_t\}_{t=0}^{\infty}$ is (v, Ω)-overtaking optimal program. In view of (2.120) and (2.154), $x_0 \in \mathcal{D}(v)$ and

$$\{x_t\}_{t=0}^{\infty} \in \cup\{\mathcal{P}(v, z) : \ z \in \mathcal{D}(v)\}.$$

In view of (2.153), for all sufficiently large natural numbers k, $\rho(x_t, x_t^{(k)}) \leq \epsilon/4$, $t = 0, \ldots, T_0$. This contradicts (2.147) and (2.152). The contradiction we have reached proves Lemma 2.36. □

2.13 Proof of Theorem 2.32

We may assume that $\bar{r}_v \in (0,1)$. Recall that

$$\{(x,y) \in X \times X : \ \rho(x,\bar{x}_v),\ \rho(y,\bar{x}_v) \le \bar{r}_v\} \subset \Omega. \tag{2.159}$$

By Lemma 2.36 applied to the function $\bar{v}$ there exists $\delta_1 \in (0, \min\{\epsilon, \bar{r}_v/2\})$ such that the following property holds:

(Pi) for each $(\bar{\Omega})$-program $\{y_t\}_{t=0}^{\tau_0}$ which satisfies

$$\pi^{\bar{v}}(y_0) \ge \sup(\pi^{\bar{v}}) - \delta_1, \tag{2.160}$$

$$\sum_{t=0}^{\tau_0-1} (\bar{v}(y_t, y_{t+1}) - \bar{v}(\bar{x}_v, \bar{x}_v)) - \pi^{\bar{v}}(y_0) + \pi^{\bar{v}}(y_T) \ge -\delta_1 \tag{2.161}$$

there exists an $(\bar{\Omega})$-program

$$\{z_t\}_{t=0}^{\infty} \in \cup\{\mathcal{P}(\bar{v}, z) : \ z \in \mathcal{D}(\bar{v})\} \tag{2.162}$$

such that

$$\rho(z_t, y_t) \le \epsilon,\ t = 0, \dots, \tau_0. \tag{2.163}$$

By Propositions 2.22 and 2.23 and (A2), there exists $\delta_2 \in (0, \delta_1)$ such that for each $z \in X$ satisfying $\rho(z, \bar{x}_v) \le 2\delta_2$,

$$|\pi^{\bar{v}}(z)| = |\pi^{\bar{v}}(z) - \pi^{\bar{v}}(\bar{x}_v)| \le \delta_1/8 \tag{2.164}$$

and for each $(x,y) \in X \times X$ satisfying $\rho(x, \bar{x}_v) \le 2\delta_2$, $\rho(y, \bar{x}_v) \le 2\delta_2$,

$$|v(x,y) - \bar{v}(\bar{x}_v, \bar{x}_v)| \le \delta_1/8. \tag{2.165}$$

By Theorem 2.4, there exist an integer $L \ge 1$ and a number $\delta_3 > 0$ such that the following property holds:

(Pii) for each integer $T > 2L$, each $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$ satisfying $\|u_t - v\| \le \delta_3$, $t = 0 \dots, T-1$ and each (Ω)-program $\{z_t\}_{t=0}^{T}$ which satisfies

$$z_0 \in \bar{Y}_{L_0},\ \sum_{t=0}^{T-1} u_t(z_t, z_{t+1}) \ge \sigma(\{u_t\}_{t=0}^{T-1}, 0, T, z_0) - \delta_3$$

we have $\rho(z_t, \bar{x}_v) \le \delta_2$, $t = L, \dots, T-L$.

By Proposition 2.25, (2.119), (2.120), and Theorem 2.15 applied to the function $\bar{v}$ there exists a natural number τ_1 such that for each $(\bar{\Omega})$-program

$$\{z_t\}_{t=0}^{\infty} \in \cup\{\mathcal{P}(\bar{v}, z) : \ z \in \mathcal{D}(\bar{v})\} \tag{2.166}$$

we have

$$\rho(z_t, \bar{x}_v) \le \delta_2 \text{ for all integers } t \ge \tau_1. \tag{2.167}$$

Choose a positive number δ and an integer T_0 such that

$$\delta < (16(L+\tau_1+\tau_0+6))^{-1}\min\{\delta_1,\delta_2,\delta_3\}, \tag{2.168}$$

$$T_0 > 2L + 2\tau_0 + 2\tau_1 + 4. \tag{2.169}$$

Assume that an integer $T \geq T_0$, $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$ satisfies

$$\|u_t - v\| \leq \delta,\ t = 0\ldots, T-1 \tag{2.170}$$

and $\{x_t\}_{t=0}^{T}$ is an (Ω)-program which satisfies

$$x_0 \in \bar{Y}_{L_0},\ \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=0}^{T-1}, 0, T, x_0) - \delta. \tag{2.171}$$

By (2.168)–(2.171) and the property (Pii),

$$\rho(x_t, \bar{x}_v) \leq \delta_2,\ t = L, \ldots, T-L. \tag{2.172}$$

In view of (2.169),

$$[T-L-\tau_0-\tau_1-4, T-L-\tau_0-\tau_1] \subset [L, T-L-\tau_0-\tau_1]. \tag{2.173}$$

Relations (2.172) and (2.173) imply that

$$\rho(x_t, \bar{x}_v) \leq \delta_2,\ t \in \{T-L-\tau_0-\tau_1-4, T-L-\tau_0-\tau_1\}. \tag{2.174}$$

Fix an $(\bar{\Omega})$-program $\{z_t\}_{t=0}^{\infty}$ satisfying (2.166). Then (2.167) is true. Define a sequence $\{\tilde{x}_t\}_{t=0}^{T}$ by

$$\tilde{x}_t = x_t,\ t = 0, \ldots, T-L-\tau_0-\tau_1-4,$$

$$\tilde{x}_t = z_{T-t},\ t = T-L-\tau_0-\tau_1-3, \ldots, T. \tag{2.175}$$

In view of (2.167) and (2.175),

$$\rho(\tilde{x}_{T-L-\tau_0-\tau_1-3}, \bar{x}_v) = \rho(z_{L+\tau_0+\tau_1+3}, \bar{x}_v) \leq \delta_2. \tag{2.176}$$

By (2.159) and (2.174)–(2.176), $\{\tilde{x}_t\}_{t=0}^{T}$ is an (Ω)-program. It follows from (2.170), (2.171), and (2.175) that

$$\delta \geq \sum_{t=0}^{T-1} (u_t(\tilde{x}_t, \tilde{x}_{t+1}) - u_t(x_t, x_{t+1}))$$

$$= \sum_{t=T-L-\tau_0-\tau_1-4}^{T-1} (u_t(\tilde{x}_t, \tilde{x}_{t+1}) - u_t(x_t, x_{t+1}))$$

$$\geq \sum_{t=T-L-\tau_0-\tau_1-4}^{T-1} v(\tilde{x}_t, \tilde{x}_{t+1}) - \sum_{t=T-L-\tau_0-\tau_1-4}^{T-1} v(x_t, x_{t+1}) - 2\delta(L+\tau_0+\tau_1+4).$$

Together with (2.175) this implies that

$$\sum_{t=T-L-\tau_0-\tau_1-4}^{T-1} v(x_t, x_{t+1}) \geq \sum_{t=T-L-\tau_0-\tau_1-4}^{T-1} v(\tilde{x}_t, \tilde{x}_{t+1}) - 2\delta(L+\tau_0+\tau_1+5)$$

$$= v(x_{T-L-\tau_0-\tau_1-4}, z_{L+\tau_0+\tau_1+3}) + \sum_{t=0}^{L+\tau_0+\tau_1+2} \bar{v}(z_t, z_{t+1}) - 2\delta(L+\tau_0+\tau_1+5). \tag{2.177}$$

By (2.165), (2.167), and (2.174),

$$|\bar{v}(z_{L+\tau_0+\tau_1+3}, x_{T-L-\tau_0-\tau_1-4}) - \bar{v}(z_{L+\tau_0+\tau_1+3}, z_{L+\tau_0+\tau_1+4})| \leq \delta_1/4.$$

Together with (2.177) this implies that

$$\sum_{t=T-L-\tau_0-\tau_1-4}^{T-1} v(x_t, x_{t+1}) \geq \sum_{t=0}^{L+\tau_0+\tau_1+3} \bar{v}(z_t, z_{t+1}) - \delta_1/4 - 2\delta(L+\tau_0+\tau_1+5). \tag{2.178}$$

Set

$$y_t = x_{T-t},\ t = 0, \ldots, L+\tau_0+\tau_1+4. \tag{2.179}$$

It follows from (2.125), (2.168), (2.178), and (2.179) that

$$\sum_{t=0}^{L+\tau_0+\tau_1+3} \bar{v}(y_t, y_{t+1}) \geq \sum_{t=0}^{L+\tau_0+\tau_1+3} \bar{v}(z_t, z_{t+1}) - \delta_1/4 - \delta_1/8. \tag{2.180}$$

By (2.166), (2.174), (2.179), (2.180), Proposition 2.19, and Corollary 2.21,

$$\pi^{\bar{v}}(y_0) - \sup(\pi^{\bar{v}}) + \sum_{t=0}^{\tau_0-1} (\bar{v}(y_t, y_{t+1}) - \bar{v}(\bar{x}_v, \bar{x}_v)) - \pi^{\bar{v}}(y_0) + \pi^{\bar{v}}(y_{\tau_0})$$

$$\geq \pi^{\bar{v}}(y_0) - \pi^{\bar{v}}(z_0)$$

$$+ \sum_{t=0}^{L+\tau_0+\tau_1+3} (\bar{v}(y_t, y_{t+1}) - \bar{v}(\bar{x}_v, \bar{x}_v)) - \pi^{\bar{v}}(y_0) + \pi^{\bar{v}}(y_{L+\tau_0+\tau_1+4})$$

$$\geq \pi^{\bar{v}}(y_0) - \pi^{\bar{v}}(z_0) + \sum_{t=0}^{L+\tau_0+\tau_1+3} (\bar{v}(z_t, z_{t+1}) - \bar{v}(\bar{x}_v, \bar{x}_v))$$

$$-\pi^{\bar{v}}(y_0) + \pi^{\bar{v}}(y_{L+\tau_0+\tau_1+4}) - 3\delta_1/8$$

$$\geq \pi^{\bar{v}}(y_0) - \pi^{\bar{v}}(z_0) + \pi^{\bar{v}}(z_0) - \pi^{\bar{v}}(z_{L+\tau_0+\tau_1+3}) - \pi^{\bar{v}}(y_0) - \pi^{\bar{v}}(y_{L+\tau_0+\tau_1+4}) - 3\delta_1/8$$

$$= \pi^{\bar{v}}(y_{L+\tau_0+\tau_1+4}) - \pi^{\bar{v}}(z_{L+\tau_0+\tau_1+3}) - 3\delta_1/8. \tag{2.181}$$

By (2.164) and (2.167),

$$\pi^{\bar{v}}(z_{L+\tau_0+\tau_1+3}) \leq \delta_1/8. \tag{2.182}$$

In view of (2.164), (2.179), and (2.174),

$$|\pi^{\bar{v}}(y_{L+\tau_0+\tau_1+4})| = |\pi^{\bar{v}}(x_{T-L-\tau_0-\tau_1-4})| \le \delta_1/8. \tag{2.183}$$

It follows from (2.181) to (2.183),

$$\pi^{\bar{v}}(y_0) - \sup(\pi^{\bar{v}}) + \sum_{t=0}^{\tau_0-1} (\bar{v}(y_t, y_{t+1}) - \bar{v}(\bar{x}_v, \bar{x}_v)) - \pi^{\bar{v}}(y_0) + \pi^{\bar{v}}(y_{\tau_0}) \ge -\delta_1.$$

Together with (2.174), (2.179), and Proposition 2.19 this implies that

$$\pi^{\bar{v}}(y_0) - \sup(\pi^{\bar{v}}) \ge -\delta_1, \tag{2.184}$$

$$\sum_{t=0}^{\tau_0-1} (\bar{v}(y_t, y_{t+1}) - \bar{v}(\bar{x}_v, \bar{x}_v)) - \pi^{\bar{v}}(y_0) + \pi^{\bar{v}}(y_{\tau_0}) \ge -\delta_1. \tag{2.185}$$

By (2.184), (2.185), and the property (Pi), there exists

$$\{\xi_t\}_{t=0}^{\infty} \in \cup\{\mathcal{P}(\bar{v}, z) : \ z \in \mathcal{D}(\bar{v})\}$$

such that $\rho(\xi_t, x_{T-t}) = \rho(\xi_t, y_t) \le \epsilon, \ t = 0, \dots, \tau_0$. Theorem 2.32 is proved. □

2.14 Proof of Theorem 2.33

Theorem 2.32 and Assertion 1 of Theorem 2.5 imply the following result.

Proposition 2.37. *Suppose that $v \in \mathcal{M}(\Omega)$ is an upper semicontinuous function, $\bar{x}_v \in X$, $\bar{r}_v > 0$, $\bar{c}_v > 0$ and that assumptions (A1), (A2), and (A3) hold. Let $\tau_0 \ge 1$ be an integer and $\epsilon > 0$. Then there exist $\delta > 0$ and an integer $T_0 \ge \tau_0$ such that for each integer $T \ge T_0$, each $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$ satisfying $\|u_t - v\| \le \delta, \ t = 0 \dots, T-1$ and each (Ω)-program $\{x_t\}_{t=0}^{T}$ which satisfies*

$$\sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \ge \sigma(\{u_t\}_{t=0}^{T-1}, 0, T) - \delta$$

there exists an $(\bar{\Omega})$-program $\{x_t^*\}_{t=0}^{\infty} \in \cup\{\mathcal{P}(\bar{v}, z) : \ z \in \mathcal{D}(\bar{v})\}$ such that for all integers $t = 0, \dots, \tau_0$, $\rho(x_{T-t}, x_t^*) \le \epsilon$.

Since the mapping $v \to \bar{v}$, $v \in \mathcal{M}(\Omega)$ is an isometry Theorem 2.33 follows from Propositions 2.30, 2.31, and 2.37.

2.15 Proof of Proposition 2.34

Since the mapping $v \to \bar{v}$, $v \in \mathcal{M}(\Omega)$ is an isometry Proposition 2.34 follows from Propositions 2.30, 2.31 and the following result.

Proposition 2.38. *Suppose that $v \in \mathcal{M}_*(\Omega)$, $\tau_0 \geq 1$ is an integer and $\epsilon > 0$. Then there exist $\delta > 0$ and an integer $T_0 \geq \tau_0$ such that for each $u \in B_d(v, \delta) \cap \mathcal{M}_*(\Omega)$ and each $\{x_t\}_{t=0}^{\infty} \in \cup\{\mathcal{P}(\bar{u}, z) : \ z \in \mathcal{D}(\bar{u})\}$ there exists*

$$\{y_t\}_{t=0}^{\infty} \in \cup\{\mathcal{P}(\bar{v}, z) : \ z \in \mathcal{D}(\bar{v})\}$$

such that $\rho(x_t, y_t) \leq \epsilon$ for all integers $t = 0, \ldots, \tau_0$.

Proof. By (A2), for each $u \in \mathcal{M}_0(\Omega)$ and each integer $T \geq 1$,

$$\sigma(u, T, \bar{x}_u, \bar{x}_u) = Tu(\bar{x}_u, \bar{x}_u). \tag{2.186}$$

Together with Assertion 1 of Theorem 2.5 this implies that there exists $\delta_0 \in (0, \epsilon)$ such that for each $u \in B_d(v, \delta_0) \cap \mathcal{M}_0(\Omega)$,

$$\rho(\bar{x}_u, \bar{x}_v) \leq \bar{r}_v/4. \tag{2.187}$$

By Theorem 2.32, there exist $\delta \in (0, \delta_0)$ and a natural number $T_0 \geq \tau_0$ such that the following property holds:

(Piii) for each integer $T \geq T_0$, each $u \in \mathcal{M}_0(\Omega)$ satisfying $\|u - v\| \leq \delta$ and each (Ω)-program $\{z_t\}_{t=0}^{T}$ which satisfies

$$\rho(z_0, \bar{x}_v) \leq \bar{r}_v/2, \ \sum_{t=0}^{T-1} u(z_t, z_{t+1}) \geq \sigma(u, T, z_0) - \delta$$

there exists an $(\bar{\Omega})$-program

$$\{x_t^*\}_{t=0}^{\infty} \in \cup\{\mathcal{P}(\bar{v}, z) : \ z \in \mathcal{D}(\bar{v})\} \tag{2.188}$$

such that

$$\rho(z_{T-t}, x_t^*) \leq \epsilon, \ t = 0, \ldots, \tau_0. \tag{2.189}$$

Assume that

$$u \in \mathcal{M}_*(\Omega), \ \|u - v\| \leq \delta, \tag{2.190}$$

$$\{x_t\}_{t=0}^{\infty} \in \cup\{\mathcal{P}(\bar{u}, z) : \ z \in \mathcal{D}(\bar{u})\}. \tag{2.191}$$

In view of (2.190),

$$\lim_{t \to \infty} x_t = \bar{x}_u. \tag{2.192}$$

By (2.192), there exists an integer $S_0 > T_0$ such that

$$\rho(x_{S_0}, \bar{x}_u) \leq \bar{r}_v/4. \tag{2.193}$$

By (2.190) and the choice of δ_1, (2.187) holds. In view of (2.187) and (2.193),

$$\rho(x_{S_0}, \bar{x}_v) \le \bar{r}_v/2. \tag{2.194}$$

Set

$$z_t = x_{S_0 - t},\ t = 0, \dots, S_0. \tag{2.195}$$

By (2.194) and (2.195),

$$\rho(z_0, \bar{x}_v) \le \bar{r}_v/2. \tag{2.196}$$

It is clear that $\{z_t\}_{t=0}^{S_0}$ is an (Ω)-program. We show that

$$\sum_{t=0}^{S_0-1} u(z_t, z_{t+1}) = \sigma(u, S_0, z_0). \tag{2.197}$$

Let $\{y_t\}_{t=0}^{S_0}$ be an (Ω)-program satisfying

$$y_0 = z_0. \tag{2.198}$$

In order to prove (2.197) it is sufficient to show that

$$\sum_{t=0}^{S_0-1} u(z_t, z_{t+1}) \ge \sum_{t=0}^{S_0-1} u(y_t, y_{t+1}).$$

It follows from (2.195) that

$$\sum_{t=0}^{S_0-1} u(z_t, z_{t+1}) = \sum_{t=0}^{S_0-1} \bar{u}(x_{S_0-t-1}, x_{S_0-t}). \tag{2.199}$$

Set

$$\bar{y}_t = y_{S_0 - t},\ t = 0, \dots, S_0. \tag{2.200}$$

In view of (2.200),

$$\sum_{t=0}^{S_0-1} u(y_t, y_{t+1}) = \sum_{t=0}^{S_0-1} \bar{u}(\bar{y}_t, \bar{y}_{t+1}). \tag{2.201}$$

By (2.190), (2.191), (2.193), (2.196), (2.198), (2.199), (2.201), and Corollary 2.21,

$$\sum_{t=0}^{S_0-1} u(z_t, z_{t+1}) - \sum_{t=0}^{S_0-1} u(y_t, y_{t+1}) = \sum_{t=0}^{S_0-1} \bar{u}(x_t, x_{t+1}) - \sum_{t=0}^{S_0-1} \bar{u}(\bar{y}_t, \bar{y}_{t+1})$$

$$= \sum_{t=0}^{S_0-1} [\bar{u}(x_t, x_{t+1}) - u(\bar{x}_u, \bar{x}_u) - \pi^{\bar{u}}(x_t) + \pi^{\bar{u}}(x_{t+1})] + \pi^{\bar{u}}(x_0) - \pi^{\bar{u}}(x_{S_0})$$

$$- \sum_{t=0}^{S_0-1} [\bar{u}(\bar{y}_t, \bar{y}_{t+1}) - u(\bar{x}_u, \bar{x}_u) - \pi^{\bar{u}}(\bar{y}_t) + \pi^{\bar{u}}(\bar{y}_{t+1})] - \pi^{\bar{u}}(\bar{y}_0) + \pi^{\bar{u}}(\bar{y}_{S_0})$$

$$= \pi^{\bar{u}}(x_0) - \pi^{\bar{u}}(\bar{y}_0) - \sum_{t=0}^{S_0-1} [\bar{u}(\bar{y}_t, \bar{y}_{t+1}) - u(\bar{x}_u, \bar{x}_u) - \pi^{\bar{u}}(\bar{y}_t) + \pi^{\bar{u}}(\bar{y}_{t+1})]$$

$$\geq \pi^{\bar{u}}(x_0) - \pi^{\bar{u}}(\bar{y}_0) \geq 0.$$

Thus (2.197) holds. By (2.190), (2.195)–(2.197), and the property (Pii) applied to $\{z_t\}_{t=0}^{S_0}$ there exists an $(\bar{\Omega})$-program $\{x_t^*\}_{t=0}^{\infty}$ satisfying (2.188) such that for all $t = 0, \dots, \tau_0$,

$$\epsilon \geq \rho(z_{S_0-t}, x_t^*) = \rho(x_t^*, x_t).$$

Proposition 2.38 is proved. □

2.16 Proof of Theorem 2.35

In the sequel we use the following auxiliary results.

Proposition 2.39. *Let $v \in \mathcal{M}_*(\Omega)$, $\{x_t\}_{t=0}^{\infty}$ be a (v, Ω)-overtaking optimal and (v, Ω)-good program and $t_2 > t_1$ be nonnegative integers such that $x_{t_1} = x_{t_2}$. Then $x_t = \bar{x}_v$ for all integers $t = t_1, \dots, t_2$.*

Proof. By Corollary 2.21, for all integers $t \geq 0$,

$$v(x_t, x_{t+1}) - v(\bar{x}_v, \bar{x}_v) = \pi^v(x_t) - \pi^v(x_{t+1}). \tag{2.202}$$

We may assume without loss of generality that $t_1 = 0$. There exists a sequence $\{\tilde{x}_t\}_{t=0}^{\infty}$ such that

$$\tilde{x}_t = x_t,\ t = 0, \dots, t_2,\ \tilde{x}_{t+t_2} = \tilde{x}_t,\ t = 0, 1, \dots. \tag{2.203}$$

In view of (2.202) and (2.203), for all integers $t \geq 0$,

$$v(\tilde{x}_t, \tilde{x}_{t+1}) - v(\bar{x}_v, \bar{x}_v) = \pi^v(\tilde{x}_t) - \pi^v(\tilde{x}_{t+1}). \tag{2.204}$$

Together with (2.203) this implies that $\{\tilde{x}_t\}_{t=0}^{\infty}$ is a (v, Ω)-good program. By (A3), $\lim_{t\to\infty} \tilde{x}_t = \bar{x}_v$. Combined with (2.203) this implies that $x_t = \bar{x}_v$ for all integers $t = 0, \dots, t_2$. Proposition 2.39 is proved. □

The next result follows from Theorem 2.16.

Proposition 2.40. *Let $v \in \mathcal{M}_*(\Omega)$, $\{x_t\}_{t=0}^{\infty}$ be a (v, Ω)-overtaking optimal program such that $x_0 = \bar{x}_v$. Then $x_t = \bar{x}_v$ for all integers $t \geq 0$.*

For any $(x, y) \in X \times X$ and any nonempty set $D \subset X \times X$ put

$$\rho((x, y), D) = \inf\{\rho(x, z_1) + \rho(y, z_2) :\ (z_1, z_2) \in D\}.$$

Since the mapping $v \to \bar{v}$, $v \in \mathcal{M}(\Omega)$ is an isometry Theorem 2.35 follows from Propositions 2.30, 2.31 and the following result.

Proposition 2.41. *Let $\mathfrak{M}$ be either $\bar{\mathcal{M}}_0(\Omega)$ or $\bar{\mathcal{M}}_{c,0}(\Omega)$. Then there exists a set $\mathcal{F} \subset \mathfrak{M} \cap \mathcal{M}_*(\Omega)$ which is a countable intersection of open everywhere dense subsets of $\mathfrak{M}$ such that for each $v \in \mathcal{F}$ there exists a unique point $z_v \in X$ such that $\pi^v(z_v) = \sup(\pi^v)$ and there exists a unique (v, Ω)-overtaking optimal program $\{z_t^v\}_{t=0}^\infty$ satisfying $z_0^v = z_v$.*

Proof. By Theorem 2.6, there exists a set $\mathcal{F}_0 \subset \mathfrak{M} \cap \mathcal{M}_*(\Omega)$ which is a countable intersection of open everywhere dense subsets of $\mathfrak{M}$. Denote by E the set of all $v \in \mathfrak{M} \cap \mathcal{M}_*(\Omega)$ for which the following property holds:

(Piv) there exists a unique point $z_v \in X$ such that $\pi^v(z_v) = \sup(\pi^v)$ and there exists a unique (v, Ω)-overtaking optimal program $\{z_t^v\}_{t=0}^\infty$ satisfying $z_0^v = z_v$.

We show that E is an everywhere dense subset of $\mathfrak{M}$. Let $v \in \mathfrak{M} \cap \mathcal{M}_*(\Omega)$. It is sufficient to show that for any neighborhood $\mathcal{U}$ of v in $\mathfrak{M}$, $\mathcal{U} \cap E \neq \emptyset$. There are two cases:

$$\pi^v(\bar{x}_v) = \sup(\pi^v); \tag{2.205}$$

$$\pi^v(\bar{x}_v) < \sup(\pi^v); \tag{2.206}$$

Assume that (2.205) holds. Let $\gamma \in (0, 1)$. Define

$$v_\gamma(x, y) = v(x, y) - \gamma(\rho(x, \bar{x}_v) + \rho(y, \bar{x}_v)),\ (x, y) \in \Omega. \tag{2.207}$$

It is not difficult to see that

$$v_\gamma \in \mathfrak{M} \cap \mathcal{M}_0(\Omega) \text{ with } \bar{x}_{v_\gamma} = \bar{x}_v. \tag{2.208}$$

By (2.207), any (v_γ, Ω)-good program $\{x_t\}_{t=0}^\infty$ is (v, Ω)-good and $\lim_{t\to\infty} x_t = \bar{x}_v$. Thus

$$v_\gamma \in \mathcal{M}_*(\Omega). \tag{2.209}$$

It follows from (2.207) to (2.209) and Proposition 2.22 that

$$v_\gamma(\bar{x}_v, \bar{x}_v) = v(\bar{x}_v, \bar{x}_v),\ \pi^{v_\gamma}(y) \le \pi^v(y),\ y \in X,\ \pi^{v_\gamma}(\bar{x}_v) = \pi^v(\bar{x}_v) = 0. \tag{2.210}$$

Assume that $z \in X$ satisfies

$$\pi^{v_\gamma}(z) = \sup(\pi^{v_\gamma}) \tag{2.211}$$

and that $\{z_t\}_{t=0}^\infty$ is a (v_γ, Ω)-overtaking optimal program satisfying

$$z_0 = z. \tag{2.212}$$

By (2.205), (2.207)–(2.212), and Proposition 2.24,

$$0 = \pi^{v_\gamma}(z) = \lim_{T\to\infty} \sum_{t=0}^{T-1} [v_\gamma(z_t, z_{t+1}) - v_\gamma(\bar{x}_v, \bar{x}_v)]$$

$$= \lim_{T\to\infty}[\sum_{t=0}^{T-1} v(z_t, z_{t+1}) - Tv(\bar{x}_v, \bar{x}_v) - \gamma \sum_{t=0}^{T-1}(\rho(z_t, \bar{x}_v) + \rho(z_{t+1}, \bar{x}_v))]$$

$$\leq \limsup_{T\to\infty} \sum_{t=0}^{T-1}(v(z_t, z_{t+1}) - v(\bar{x}_v, \bar{x}_v)) - \gamma \sum_{t=0}^{\infty}(\rho(z_t, \bar{x}_v) + \rho(z_{t+1}, \bar{x}_v))$$

$$\leq \sup(\pi^v) - \gamma \sum_{t=0}^{\infty}(\rho(z_t, \bar{x}_v) + \rho(z_{t+1}, \bar{x}_v)) = -\gamma \sum_{t=0}^{\infty}(\rho(z_t, \bar{x}_v) + \rho(z_{t+1}, \bar{x}_v)).$$

This implies that $z_t = \bar{x}_v$ for all integers $t \geq 0$ and $v_\gamma \in E$.

Assume that (2.206) holds. There exist $z_* \in X$ such that

$$\pi^v(z_*) = \sup(\pi^v) \tag{2.213}$$

and a (v, Ω)-overtaking optimal program $\{z_t^*\}_{t=0}^{\infty}$ such that

$$z_0^* = z_*.$$

Proposition 2.24 implies that

$$\pi^v(z_*) = \lim_{T\to\infty} \sum_{t=0}^{T-1}[v(z_t^*, z_{t+1}^*) - v(\bar{x}_v, \bar{x}_v)]. \tag{2.214}$$

In view of (A3),

$$\lim_{t\to\infty} z_t^* = \bar{x}_v. \tag{2.215}$$

It follows from (2.206), (2.213), (2.215), and Proposition 2.23 that

$$\pi^v(z_t^*) < \pi^v(z_0^*) \text{ for all large enough integers } t \geq 1. \tag{2.216}$$

By (2.216), there exists an integer $\tau_0 \geq 0$ such that $\pi^v(z_{\tau_0}^*) = \pi^v(z^*)$ and that $\pi^v(z_t^*) < \pi^v(z^*)$ for all integers $t > \tau_0$. We may assume without loss of generality $\tau_0 = 0$. Thus

$$\pi^v(z_0^*) = \sup(\pi^v),\ \pi^v(z_t^*) < \pi^v(z_0^*) \text{ for all integers } t \geq 1. \tag{2.217}$$

Let $\gamma \in (0, 1)$. For all $(x, y) \in \Omega$ define

$$v_\gamma(x, y) = v(x, y) - \gamma\rho((x, y), (\{(z_t^*, z_{t+1}^*) :\ t = 0, 1, \dots\} \cup \{(\bar{x}_v, \bar{x}_v)\})). \tag{2.218}$$

In follows from (2.218) that

$$v_\gamma \in \mathfrak{M} \cap \mathcal{M}_0(\Omega),\ v_\gamma(\bar{x}_v, \bar{x}_v) = v(\bar{x}_v, \bar{x}_v),$$

any (v_γ, Ω)-good program is (v, Ω)-good and converges to $\bar{x}_v$. Thus

$$v_\gamma \in \mathcal{M}_*(\Omega),\ \bar{x}_{v_\gamma} = \bar{x}_v. \tag{2.219}$$

By (2.213), (2.214), (2.218), (2.219), and the equality $z_0^* = z_*$,

$$\pi^{v_\gamma}(y) \le \pi^v(y),\ y \in X,\ \pi^{v_\gamma}(z_*) = \pi^v(z_*). \tag{2.220}$$

In view of (2.219), for all integers $T \ge 1$,

$$\sigma(v_\gamma, T, \bar{x}_v, \bar{x}_v) = Tv(\bar{x}_v, \bar{x}_v). \tag{2.221}$$

Proposition 2.22 and (2.219) imply that

$$\pi^{v_\gamma}(\bar{x}_v) = 0. \tag{2.222}$$

Set

$$K = \{(z_t^*, z_{t+1}^*) :\ t = 0, 1, \dots\} \cup \{(\bar{x}_v, \bar{x}_v)\}. \tag{2.223}$$

Assume that $y \in X$ satisfies

$$\pi^{v_\gamma}(y) = \sup(\pi^{v_\gamma}) \tag{2.224}$$

and that $\{y_t\}_{t=0}^\infty$ is a (v_γ, Ω)-overtaking optimal program satisfying

$$y_0 = y. \tag{2.225}$$

By (2.213), (2.218)–(2.220), (2.223)–(2.225), Proposition 2.24 and the inclusion $\{y_t\}_{t=0}^\infty \in \mathcal{P}(v_\gamma, y)$,

$$\pi^v(z_*) = \pi^{v_\gamma}(z_*) = \pi^{v_\gamma}(y) = \lim_{T\to\infty} \sum_{t=0}^{T-1} [v_\gamma(y_t, y_{t+1}) - v_\gamma(\bar{x}_v, \bar{x}_v)]$$

$$= \lim_{T\to\infty} [\sum_{t=0}^{T-1} (v(y_t, y_{t+1}) - v(\bar{x}_v, \bar{x}_v) - \gamma\rho((y_t, y_{t+1}), K))]$$

$$\le \pi^v(y) - \gamma \sum_{t=0}^{\infty} \rho((y_t, y_{t+1}), K).$$

Together with (2.221) this implies that

$$\pi^v(y) = \pi^v(z_*), \tag{2.226}$$

$$(y_t, y_{t+1}) \in K \text{ for all integers } t \ge 0. \tag{2.227}$$

In view of (2.206), (2.214), (2.217), (2.223), (2.225)–(2.227), and the equality $z_0^* = z_*$,

$$y = y_0 = z_*. \tag{2.228}$$

We show by induction that $y_t = z_t^*$ for all integers $t \ge 0$. There are two cases:

$$z_t^* \ne \bar{x}_v \text{ for all integers } t \ge 0; \tag{2.229}$$

$$\bar{x}_v \in \{z_t^* :\ t = 0, 1, \dots\}. \tag{2.230}$$

Assume that (2.229) holds. By Proposition 2.39, (2.213), (2.214), and (2.229),

$$z^*_{t_1} \neq z^*_{t_2} \text{ for all integers } t_2 > t_1 \geq 0. \tag{2.231}$$

Assume that $T \geq 0$ is an integer and that

$$y_t = z^*_t,\ t = 0, \dots, T. \tag{2.232}$$

(Note that in view of (2.228) and the equality $z^*_0 = z_*$ our assumption holds for $T = 0$.) By (2.223), (2.227), (2.229), (2.231), (2.232),

$$(z^*_T, y_{T+1}) = (y_T, y_{T+1}) \in K = \{(z^*_t, z^*_{t+1}) :\ t = 0, 1, \dots\} \cup \{(\bar{x}_v, \bar{x}_v)\}$$

and $y_{T+1} = z^*_{T+1}$. Thus $y_t = z^*_t$ for all integers $t \geq 0$.

Assume that (2.230) holds. By (2.206), (2.213), and the equality $z^*_0 = z_*$, there is a natural number S such that

$$z^*_S = \bar{x}_v,\ z^*_t \neq \bar{x}_v \text{ for all integers } t \in [0, S). \tag{2.233}$$

Propositions 2.39 and 2.40 imply that,

$$z^*_t = \bar{x}_v \text{ for all integers } t \geq S, \tag{2.234}$$

$$z^*_{t_2} \neq z^*_{t_1} \text{ for all integers } t_1, t_2 \in [0, S] \text{ such that } t_1 < t_2. \tag{2.235}$$

Assume that $T \geq 0$ is an integer and that

$$y_t = z^*_t,\ t = 0, \dots, T. \tag{2.236}$$

(Note that in view of (2.228), our assumption holds for $T = 0$.) If $T < S$, then by (2.223), (2.227), (2.233)–(2.236),

$$(z^*_T, y_{T+1}) = (y_T, y_{T+1}) \in \{(z^*_t, z^*_{t+1}) :\ t = 0, 1, \dots\} \cup \{(\bar{x}_v, \bar{x}_v)\}$$

and $y_{T+1} = z^*_{T+1}$. If $T \geq S$, then by (2.223), (2.227), (2.232), (2.233), (2.234),

$$(\bar{x}_v, y_{T+1}) = (y_T, y_{T+1}) \in \{(z^*_t, z^*_{t+1}) :\ t = 0, 1, \dots\} \cup \{(\bar{x}_v, \bar{x}_v)\}$$

and $y_{T+1} = \bar{x}_v = z^*_{T+1}$. Thus $y_t = z^*_t$ for all integers $t \geq 0$ in both the cases (see (2.229), (2.230)). This implies that $v_\gamma \in E$. Therefore the inclusion above holds in both the cases (see (2.205), (2.206)). Since $v_\gamma \to v$ as $\gamma \to 0^+$ in $\mathfrak{M}$ we conclude that for any neighborhood $\mathcal{U}$ of v in $\mathfrak{M}$, $\mathcal{U} \cap E \neq \emptyset$. Thus E is an everywhere dense subset of $\mathfrak{M}$.

By definition, for every $v \in E$, there exists a unique (v, Ω)-overtaking optimal program $\{z^v_t\}_{t=0}^\infty$ satisfying $\pi^v(z^v_0) = \sup(\pi^v)$. Let $v \in E$ and $k \geq 1$ be an integer. By Proposition 2.34, there exist an open neighborhood $\mathcal{U}(v, k)$ of v in $\mathfrak{M}$ and an integer $T(v, k) \geq k$ such that the following property holds:

(Pv) for each $u \in \mathcal{U}(v, k) \cap \mathcal{M}_*(\Omega)$ and each $\{x_t\}_{t=0}^\infty \in \cup\{\mathcal{P}(u, z) :\ z \in \mathcal{D}(u)\}$ we have $\rho(x_t, z^v_t) \leq k^{-1}$, $t = 0, \dots, k$.

Set

$$\mathcal{F}_1 = \cap_{p=1}^{\infty} \cup \{\mathcal{U}(v,k) : \ v \in E, \ k \geq p\}, \ \mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_0. \tag{2.237}$$

Clearly, $\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $\mathfrak{M}$ and $\mathcal{F} \subset \mathcal{F}_0 \subset \mathcal{M}_*(\Omega)$.

Let $u \in \mathcal{F}$, $p \geq 1$ be an integer and $\{x_t^{(i)}\}_{t=0}^{\infty}$, $i = 1, 2$ be (u, Ω)-overtaking optimal programs such that

$$\pi^v(x_0^{(i)}) = \sup(\pi^v), \ i = 1, 2. \tag{2.238}$$

By (2.237), there exist $v_p \in E$ and an integer $k_p \geq p$ such that

$$u \in \mathcal{U}(v_p, k_p). \tag{2.239}$$

In view of (2.239) and (Pv), $\rho(x_t^{(i)}, z_t^{v_p}) \leq k_p^{-1} \leq p^{-1}$, $t = 0, \ldots, p$, $i = 1, 2$. This implies that $\rho(x_t^{(1)}, x_t^{(2)}) \leq 2p^{-1}$, $t = 0, \ldots, p$. Since p is any natural number we conclude that $x_t^{(1)} = x_t^{(2)}$ for all integers $t \geq 0$. Proposition 2.41 is proved. □

2.17 Structure of Solutions of the Problem ($P1$)

Suppose that $v \in \mathcal{M}_*(\Omega)$, $\bar{x}_v \in X$, $\bar{r}_v > 0$, $\bar{c}_v > 0$ and that assumptions (A1), (A2), and (A3) hold.

We prove the following result obtained in [86] which describes the structure of approximate solutions of the problems of the type (P1) in the regions close to the right and the left endpoints.

Theorem 2.42. *Let $L_0 \geq 1$, $\tau_0 \geq 1$ be integers, $\epsilon > 0$, $x \in \bar{Y}_{L_0}$ and $y \in Y_{L_0}$. Then there exist $\delta \in (0, \epsilon)$ and an integer $T_0 \geq \tau_0$ such that for each integer $T \geq T_0$, each $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$ satisfying $\|u_t - v\| \leq \delta$, $t = 0 \ldots, T-1$ and each (Ω)-program $\{x_t\}_{t=0}^{T}$ which satisfies*

$$x_0 = x, \ x_T = y, \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=0}^{T-1}, 0, T, x_0, x_T) - \delta$$

there exist a (v, Ω)-overtaking optimal program $\{y_t\}_{t=0}^{\infty}$ and a $(\bar{v}, \bar{\Omega})$-overtaking optimal program $\{z_t\}_{t=0}^{\infty}$ such that $y_0 = x$, $z_0 = y$ and for all $t = 0, \ldots, \tau_0$,

$$\rho(x_t, y_t) \leq \epsilon, \ \rho(x_{T-t}, z_t) \leq \epsilon.$$

Since the mapping $v \to \bar{v}$, $v \in \mathcal{M}(\Omega)$ is an isometry Theorem 2.42 follows from Propositions 2.30, 2.31 and the following result.

Theorem 2.43. *Let $L_0 \geq 1$, $\tau_0 \geq 1$ be integers, $\epsilon > 0$ and $x \in \bar{Y}_{L_0}$. Then there exist $\delta \in (0, \epsilon)$ and an integer $T_0 \geq \tau_0$ such that for each integer $T \geq T_0$, each $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$ satisfying $\|u_t - v\| \leq \delta$, $t = 0 \ldots, T-1$ and each (Ω)-program $\{x_t\}_{t=0}^{T}$ which satisfies*

$$x_0 = x,\ x_T \in Y_{L_0},\ \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=0}^{T-1}, 0, T, x_0, x_T) - \delta$$

there exists a (v, Ω)-overtaking optimal program $\{y_t\}_{t=0}^{\infty}$ such that $y_0 = x$ and for all $t = 0, \ldots, \tau_0$, $\rho(x_t, y_t) \leq \epsilon$.

Theorems 2.43 and 2.4 imply the following result.

Theorem 2.44. *Let $L_0 \geq 1$, $\tau_0 \geq 1$ be integers, $\epsilon > 0$ and $x \in \bar{Y}_{L_0}$. Then there exist $\delta \in (0, \epsilon)$ and an integer $T_0 \geq \tau_0$ such that for each integer $T \geq T_0$, each $\{u_t\}_{t=0}^{T-1} \subset \mathcal{M}(\Omega)$ satisfying $\|u_t - v\| \leq \delta$, $t = 0 \ldots, T-1$ and each (Ω)-program $\{x_t\}_{t=0}^{T}$ which satisfies*

$$x_0 = x,\ \sum_{t=0}^{T-1} u_t(x_t, x_{t+1}) \geq \sigma(\{u_t\}_{t=0}^{T-1}, 0, T, x_0) - \delta$$

there exists a (v, Ω)-overtaking optimal program $\{y_t\}_{t=0}^{\infty}$ such that $y_0 = x$ and for all $t = 0, \ldots, \tau_0$, $\rho(x_t, y_t) \leq \epsilon$.

2.18 Proof of Theorem 2.43

Assume that Theorem 2.43 does not hold. Then for each natural number k there exist an integer

$$T_k \geq \tau_0 + k, \tag{2.240}$$

$\{u_t^{(k)}\}_{t=0}^{T_k-1} \subset \mathcal{M}(\Omega)$ satisfying

$$\|u_t^{(k)} - v\| \leq k^{-1},\ t = 0 \ldots, T_k - 1, \tag{2.241}$$

an (Ω)-program $\{x_t^{(k)}\}_{t=0}^{T_k}$ such that

$$x_0^{(k)} = x,\ x_{T_k}^{(k)} \in Y_{L_0}, \tag{2.242}$$

$$\sum_{t=0}^{T_k-1} u_t^{(k)}(x_t^{(k)}, x_{t+1}^{(k)}) \geq \sigma(\{u_t^{(k)}\}_{t=0}^{T_k-1}, 0, T_k, x_0^{(k)}, x_{T_k}^{(k)}) - k^{-1} \tag{2.243}$$

and that for each (Ω)-program

$$\{y_t\}_{t=0}^{\infty} \in \mathcal{P}(v, x) \tag{2.244}$$

we have

$$\max\{\rho(x_t^{(k)}, y_t) : \ t = 0 \dots, \tau_0\} > \epsilon. \tag{2.245}$$

Extracting a subsequence and re-indexing, if necessary, we may assume without loss of generality that for any integer $t \geq 0$ there exists

$$x_t = \lim_{k \to \infty} x_t^{(k)}. \tag{2.246}$$

Clearly, $\{x_t\}_{t=0}^{\infty}$ is an (Ω)-program. We show that for each integer $T \geq 1$,

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) = \sigma(v, 0, T, x_0, x_T).$$

Assume the contrary. Then there exist a natural number S and $\Delta > 0$ such that

$$\sum_{t=0}^{S-1} v(x_t, x_{t+1}) < \sigma(v, 0, S, x_0, x_S) - 7\Delta.$$

Clearly, there exists an (Ω)-program $\{\tilde{x}_t\}_{t=0}^{S}$ such that

$$\tilde{x}_0 = x, \ \tilde{x}_S = x_S, \ \sum_{t=0}^{S-1} v(x_t, x_{t+1}) < \sum_{t=0}^{S-1} v(\tilde{x}_t, \tilde{x}_{t+1}) - 6\Delta. \tag{2.247}$$

By (A1), there exists $\delta \in (0, \bar{r}_v/4)$ such that for each $x, y \in X$ satisfying $\rho(x, \bar{x}_v), \ \rho(y, \bar{x}_v) \leq \delta$,

$$|v(x, y) - v(\bar{x}_v, \bar{x}_v)| \leq \Delta/2. \tag{2.248}$$

In view of (2.240)–(2.243), the inclusion $x \in \bar{Y}_{L_0}$ and Theorem 2.3, there exist natural numbers $k_1 > L_1$ such that for each integer $k \geq k_1$,

$$\rho(x_t^{(k)}, \bar{x}_v) \leq \delta, \ t = L_1, \dots, T_k - L_1. \tag{2.249}$$

It follows from (2.246) and (2.249) that

$$\rho(x_t, \bar{x}_v) \leq \delta \text{ for all integers } t \geq L_1. \tag{2.250}$$

Choose a natural number

$$k_2 > k_1 + S + 2L_1 + 8. \tag{2.251}$$

Relations (2.240) and (2.251) imply that for each integer $k \geq k_2$,

$$T_k - L_1 \geq k - L_1 > k_1 + S + L_1 + 8. \tag{2.252}$$

By (2.241), (2.246), and the upper semicontinuity of v,

$$\limsup_{k\to\infty} \sum_{t=0}^{k_1+S+L_1+7} u_t^{(k)}(x_t^{(k)}, x_{t+1}^{(k)}) = \limsup_{k\to\infty} \sum_{t=0}^{k_1+S+L_1+7} v(x_t^{(k)}, x_{t+1}^{(k)})$$

$$\le \sum_{t=0}^{k_1+S+L_1+7} v(x_t, x_{t+1}). \tag{2.253}$$

For each integer $k \ge k_2$ set

$$\widehat{x}_t^{(k)} = \tilde{x}_t,\ t = 0, \dots, S,\ \widehat{x}_t^{(k)} = x_t,\ t = S+1, \dots, k_1+S+L_1+8,$$

$$\widehat{x}_{k_1+S+L_1+9}^{(k)} = x_{k_1+S+L_1+9}^{(k)}. \tag{2.254}$$

In view of (2.247), (2.249), (2.250), (2.252), and (2.254), for each integer $k \ge k_2$, $\{\widehat{x}_t^{(k)}\}_{t=0}^{k_1+S+L_1+9}$ is an (Ω)-program. By (2.242), (2.247), and (2.254), for each integer $k \ge k_2$,

$$\widehat{x}_0^{(k)} = x = x_0^{(k)},\ \widehat{x}_{k_1+S+L_1+9}^{(k)} = x_{k_1+S+L_1+9}^{(k)}. \tag{2.255}$$

By (2.253), there exists an integer $k \ge k_2$ such that

$$\sum_{t=0}^{k_1+S+L_1+7} u_t^{(k)}(x_t^{(k)}, x_{t+1}^{(k)}) \le \sum_{t=0}^{k_1+S+L_1+7} v(x_t, x_{t+1}) + \Delta, \tag{2.256}$$

$$k^{-1}(k_1 + S + L_1 + 10) < \Delta. \tag{2.257}$$

Relations (2.241), (2.247)–(2.250), (2.254), (2.256), and (2.257) imply that

$$\sum_{t=0}^{k_1+S+L_1+8} u_t^{(k)}(x_t^{(k)}, x_{t+1}^{(k)}) - \sum_{t=0}^{k_1+S+L_1+8} u_t^{(k)}(\widehat{x}_t^{(k)}, \widehat{x}_{t+1}^{(k)})$$

$$= \sum_{t=0}^{k_1+S+L_1+7} u_t^{(k)}(x_t^{(k)}, x_{t+1}^{(k)}) - \sum_{t=0}^{k_1+S+L_1+7} u_t^{(k)}(\widehat{x}_t^{(k)}, \widehat{x}_{t+1}^{(k)})$$

$$+ u_{k_1+S+L_1+8}^{(k)}(x_{k_1+S+L_1+8}^{(k)}, x_{k_1+S+L_1+9}^{(k)})$$

$$- u_{k_1+S+L_1+8}^{(k)}(\widehat{x}_{k_1+S+L_1+8}^{(k)}, \widehat{x}_{k_1+S+L_1+9}^{(k)})$$

$$\le \sum_{t=0}^{k_1+S+L_1+7} v(x_t, x_{t+1}) + \Delta - \sum_{t=0}^{k_1+S+L_1+7} v(\widehat{x}_t^{(k)}, \widehat{x}_{t+1}^{(k)})$$

$$+ (k_1 + S + L_1 + 8)k^{-1} + 2k^{-1}$$

$$+ v(x_{k_1+S+L_1+8}^{(k)}, x_{k_1+S+L_1+9}^{(k)}) - v(x_{k_1+S+L_1+8}, x_{k_1+S+L_1+9}^{(k)})$$

$$\leq \Delta + \sum_{t=0}^{S-1} v(x_t, x_{t+1}) - \sum_{t=0}^{S-1} v(\tilde{x}_t, \tilde{x}_{t+1}) + k^{-1}(k_1 + S + L_1 + 10) + \Delta$$

$$\leq -6\Delta + 2\Delta + k^{-1}(k_1 + S + L_1 + 10) \leq -3\Delta_1 \leq -3k^{-1}.$$

Together with (2.255) this contradicts (2.243). The contradiction we have reached proves that for each integer $T > 0$,

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) = \sigma(v, 0, T, x, x_T). \tag{2.258}$$

Let $\epsilon > 0$. By (2.240)–(2.243), the inclusion $x \in \bar{Y}_{L_0}$ and Theorem 2.3, there exist natural numbers $k(\epsilon) > L(\epsilon)$ such that for each integer $k \geq k(\epsilon)$, the inequality $\rho(x_t^{(k)}, \bar{x}_v) \leq \epsilon$ holds for all $t = L(\epsilon), \ldots, T_k - L(\epsilon)$. Together with (2.256) this implies that $\rho(x_t, \bar{x}_v) \leq \epsilon$ for all integers $t \geq L(\epsilon)$. Since ϵ is any positive number we conclude that $\lim_{t\to\infty} x_t = \bar{x}_v$. Together with (2.242), (2.246), (2.258), and Theorem 2.14 this implies that $\{x_t\}_{t=0}^{\infty}$ is a (v, Ω)-overtaking optimal program and $\{x_t\}_{t=0}^{\infty} \in \mathcal{P}(v, x)$. By (2.246), for all sufficiently large natural numbers k, $\rho(x_t, x_t^{(k)}) \leq \epsilon/2$ for all $t = 0, \ldots, L_0$, a contradiction. The contradiction we have reached proves Theorem 2.43. □



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