Chapter 2
Second Quantization of Light

In this chapter we discuss the quantization of electromagnetic waves, which we also denoted as light (visible or invisible to human eyes). After reviewing classical and quantum properties of the light in the vacuum, we discuss the so-called second quantization of the light field showing that this electromagnetic field can be expressed as a infinite sum of harmonic oscillators. These oscillators, which describe the possible frequencies of the radiation field, are quantized by introducing creation and annihilation operators acting on the Fock space of number representation. We analyze the Fock states of the radiation field and compare them with the coherent states. Finally, we consider two enlightening applications: the Casimir effect and the radiation field at finite temperature.

2.1 Electromagnetic Waves

The light is an electromagnetic field characterized by the coexisting presence of an electric field $\mathbf{E}(\mathbf{r}, t)$ and a magnetic field $\mathbf{B}(\mathbf{r}, t)$. From the equations of James Clerk Maxwell in vacuum and in the absence of sources, given by

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0 \,, \tag{2.1}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \,, \tag{2.2}$$

$$\boldsymbol{\nabla} \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \,, \tag{2.3}$$

$$\boldsymbol{\nabla} \wedge \mathbf{B} = \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \,, \tag{2.4}$$

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one finds that the coupled electric and magnetic fields satisfy the wave equations

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{E} = \mathbf{0}\,, \tag{2.5}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{B} = \mathbf{0}\,, \tag{2.6}$$

where

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3 \times 10^8 \text{ m/s} \tag{2.7}$$

is the speed of light in the vacuum. Note that the dielectric constant (electric permittivity) ε_0 and the magnetic constant (magnetic permeability) μ_0 are respectively $\varepsilon_0 = 8.85 \times 10^{-12}\,\text{C}^2/(\text{N m}^2)$ and $\mu_0 = 4\pi \times 10^{-7}\,\text{V s}/(\text{A m})$. Equations (2.5) and (2.6), which are fully confirmed by experiments, admit monochromatic complex plane wave solutions

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\,, \tag{2.8}$$

$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\,, \tag{2.9}$$

where $\mathbf{k}$ is the wavevector and ω the angular frequency, such that

$$\omega = c\,k\,, \tag{2.10}$$

is the dispersion relation, with $k = |\mathbf{k}|$ is the wavenumber. From Maxwell's equations one finds that the vectors $\mathbf{E}$ and $\mathbf{B}$ are mutually orthogonal and such that

$$E = cB\,, \tag{2.11}$$

where $E = |\mathbf{E}|$ and $B = |\mathbf{B}|$. In addition they are transverse fields, i.e. orthogonal to the wavevector $\mathbf{k}$, which gives the direction of propagation of the wave. Notice that Eq. (2.11) holds for monochromatic plane waves but not in general. For completeness, let us remind that the wavelength λ is given by

$$\lambda = \frac{2\pi}{k}\,, \tag{2.12}$$

and that the linear frequency ν and the angular frequency $\omega = 2\pi\nu$ are related to the wavelength λ and to the wavenumber k by the formulas (Fig. 2.1)

$$\lambda\,\nu = \frac{\omega}{k} = c\,. \tag{2.13}$$

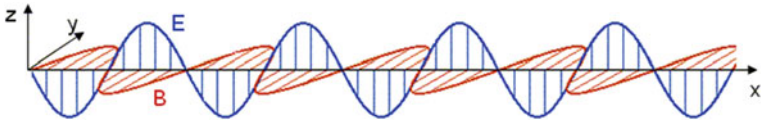


Fig. 2.1 Plot of a sinusoidal electromagnetic wave moving along the x axis

2.1.1 Photons

At the beginning of quantum mechanics Satyendra Nath Bose and Albert Einstein suggested that the light can be described as a gas of photons. A single photon of a monochromatic wave has the energy

$$\epsilon = h\nu = \hbar\omega \ , \tag{2.14}$$

where $h = 6.63 \times 10^{-34}$ J s is the Planck constant and $\hbar = h/(2\pi) = 1.05 \times 10^{-34}$ J s is the reduced Planck constant. The linear momentum of the photon is given by the de Broglie relations

$$\mathbf{p} = \frac{h}{\lambda}\mathbf{n} = \hbar\mathbf{k} \ , \tag{2.15}$$

where $\mathbf{n}$ is a unit vector in the direction of $\mathbf{k}$. Clearly, the energy of the photon can be written also as

$$\epsilon = p\, c \ , \tag{2.16}$$

which is the energy one obtains for a relativistic particle of energy

$$\epsilon = \sqrt{m^2c^4 + p^2c^2} \ , \tag{2.17}$$

setting to zero the rest mass, i.e. $m = 0$. The total energy H of monochromatic wave is given by

$$H = \sum_s \hbar\omega\, n_s \ , \tag{2.18}$$

where n_s is the number of photons with angular frequency ω and polarization s in the monochromatic electromagnetic wave. Note that in general there are two possible polarizations: $s = 1, 2$, corresponding to two linearly independent orthogonal unit vectors $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ in the plane perpendicular to the wavevector $\mathbf{k}$.

A generic electromagnetic field is the superposition of many monochromatic electromagnetic waves. Calling ω_k the angular frequency of the monochromatic wave with wavenumber $\mathbf{k}$, the total energy H of the electromagnetic field is

$$H = \sum_{\mathbf{k}} \sum_s \hbar\omega_k\, n_{\mathbf{k}s} \ , \tag{2.19}$$

where $n_{\mathbf{k}s}$ is the number of photons with wavevector $\mathbf{k}$ and polarization s. The results derived here for the electromagnetic field, in particular Eq. (2.19), are called semi-classical because they do not take into account the so-called "quantum fluctuations of vacuum", i.e. the following remarkable experimental fact: photons can emerge from the vacuum of the electromagnetic field. To justify this property of the electromagnetic field one must perform the so-called second quantization of the field.

2.1.2 Electromagnetic Potentials and Coulomb Gauge

In full generality the electric field $\mathbf{E}(\mathbf{r}, t)$ and the magnetic field $\mathbf{B}(\mathbf{r}, t)$ can be expressed in terms of a scalar potential $\phi(\mathbf{r}, t)$ and a vector potential $\mathbf{A}(\mathbf{r}, t)$ as follows

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} , \tag{2.20}$$

$$\mathbf{B} = \nabla \wedge \mathbf{A} . \tag{2.21}$$

Actually these equations do not determine the electromagnetic potentials uniquely, since for an arbitrary scalar function $\Lambda(\mathbf{r}, t)$ the so-called "gauge transformation"

$$\phi \to \phi' = \phi + \frac{\partial \Lambda}{\partial t} , \tag{2.22}$$

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} - \nabla\Lambda , \tag{2.23}$$

leaves the fields $\mathbf{E}$ and $\mathbf{B}$ unaltered. There is thus an infinite number of different electromagnetic potentials that correspond to a given configuration of measurable fields. We use this remarkable property to choose a gauge transformation such that

$$\nabla \cdot \mathbf{A} = 0 . \tag{2.24}$$

This condition defines the Coulomb (or radiation) gauge, and the vector field $\mathbf{A}$ is called transverse field. For a complex monochromatic plane wave

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A}_0 \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \tag{2.25}$$

the Coulomb gauge (2.24) gives

$$\mathbf{k} \cdot \mathbf{A} = 0 , \tag{2.26}$$

i.e. $\mathbf{A}$ is perpendicular (transverse) to the wavevector $\mathbf{k}$. In the vacuum and without sources, from the first Maxwell equation (2.1) and Eq. (2.20) one immediately finds

$$\nabla^2\phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = 0 , \tag{2.27}$$

and under the Coulomb gauge (2.24) one gets

$$\nabla^2 \phi = 0 \,. \tag{2.28}$$

Imposing that the scalar potential is zero at infinity, this Laplace's equation has the unique solution

$$\phi(\mathbf{r}, t) = 0 \,, \tag{2.29}$$

and consequently

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \,, \tag{2.30}$$

$$\mathbf{B} = \boldsymbol{\nabla} \wedge \mathbf{A} \,. \tag{2.31}$$

Thus, in the Coulomb gauge one needs only the electromagnetic vector potential $\mathbf{A}(\mathbf{r}, t)$ to obtain the electromagnetic field if there are no charges and no currents. Notice that here $\mathbf{E}$ and $\mathbf{B}$ are transverse fields like $\mathbf{A}$, which satisfy Eqs. (2.5) and (2.6). The electromagnetic field described by these equations is often called the radiation field, and also the vector potential satisfies the wave equation

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \mathbf{A} = \mathbf{0} \,. \tag{2.32}$$

We now expand the vector potential $\mathbf{A}(\mathbf{r}, t)$ as a Fourier series of monochromatic plane waves. The vector potential is a real vector field, i.e. $\mathbf{A} = \mathbf{A}^*$ and consequently we write

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{k}} \sum_{s} \left[A_{\mathbf{k}s}(t) \, \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{V}} + A^*_{\mathbf{k}s}(t) \, \frac{e^{-i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{V}} \right] \boldsymbol{\varepsilon}_{\mathbf{k}s} \,, \tag{2.33}$$

where $A_{\mathbf{k}s}(t)$ and $A^*_{\mathbf{k}s}(t)$ are the dimensional complex conjugate coefficients of the expansion, the complex plane waves $e^{i\mathbf{k}\cdot\mathbf{r}}/\sqrt{V}$ normalized in a volume V are the basis functions of the expansion, and $\boldsymbol{\varepsilon}_{\mathbf{k}1}$ and $\boldsymbol{\varepsilon}_{\mathbf{k}2}$ are two mutually orthogonal real unit vectors of polarization which are also orthogonal to $\mathbf{k}$ (transverse polarization vectors).

Taking into account Eqs. (2.30) and (2.31) we get

$$\mathbf{E}(\mathbf{r}, t) = -\sum_{\mathbf{k}} \sum_{s} \left[\dot{A}_{\mathbf{k}s}(t) \, \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{V}} + \dot{A}^*_{\mathbf{k}s}(t) \, \frac{e^{-i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{V}} \right] \boldsymbol{\varepsilon}_{\mathbf{k}s} \,, \tag{2.34}$$

$$\mathbf{B}(\mathbf{r}, t) = \sum_{\mathbf{k}} \sum_{s} \left[A_{\mathbf{k}s}(t) \, \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{V}} - A^*_{\mathbf{k}s}(t) \, \frac{e^{-i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{V}} \right] i\mathbf{k} \wedge \boldsymbol{\varepsilon}_{\mathbf{k}s} \,, \tag{2.35}$$

where both vector fields are explicitly real fields. Moreover, inserting Eq. (2.33) into Eq. (2.32) we recover familiar differential equations of decoupled harmonic oscillators:

$$\ddot{A}_{\mathbf{k}s}(t) + \omega_k^2 A_{\mathbf{k}s}(t) = 0 \,, \tag{2.36}$$

with $\omega_k = ck$, which have the complex solutions

$$A_{\mathbf{k}s}(t) = A_{\mathbf{k}s}(0)\, e^{-i\omega_k t} \,. \tag{2.37}$$

These are the complex amplitudes of the infinite harmonic normal modes of the radiation field.

A familiar result of electromagnetism is that the classical energy of the electromagnetic field in vacuum is given by

$$H = \int d^3\mathbf{r} \left(\frac{\varepsilon_0}{2} \mathbf{E}(\mathbf{r}, t)^2 + \frac{1}{2\mu_0} \mathbf{B}(\mathbf{r}, t)^2 \right) \,, \tag{2.38}$$

namely

$$H = \int d^3\mathbf{r} \left(\frac{\varepsilon_0}{2} \Big(\frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \Big)^2 + \frac{1}{2\mu_0} \big(\nabla \wedge \mathbf{A}(\mathbf{r}, t) \big)^2 \right) \tag{2.39}$$

by using the Maxwell equations in the Coulomb gauge (2.30) and (2.31). Inserting into this expression Eq. (2.33) or Eqs. (2.34) and (2.35) into Eq. (2.38) we find

$$H = \sum_{\mathbf{k}} \sum_{s} \epsilon_0 \omega_k^2 \, \left(A^*_{\mathbf{k}s} A_{\mathbf{k}s} + A_{\mathbf{k}s} A^*_{\mathbf{k}s} \right) \,. \tag{2.40}$$

It is now convenient to introduce dimensionless complex coefficients $a_{\mathbf{k}s}(t)$ and $a^*_{\mathbf{k}s}(t)$ related to the dimensional complex coefficients $A_{\mathbf{k}s}(t)$ and $A^*_{\mathbf{k}s}(t)$ by

$$A_{\mathbf{k}s}(t) = \sqrt{\frac{\hbar}{2\varepsilon_0 \omega_k}}\, a_{\mathbf{k}s}(t) \,. \tag{2.41}$$

$$A^*_{\mathbf{k}s}(t) = \sqrt{\frac{\hbar}{2\varepsilon_0 \omega_k}}\, a^*_{\mathbf{k}s}(t) \,. \tag{2.42}$$

In this way the energy H reads

$$H = \sum_{\mathbf{k}} \sum_{s} \frac{\hbar \omega_k}{2} \, \left(a^*_{\mathbf{k}s} a_{\mathbf{k}s} + a_{\mathbf{k}s} a^*_{\mathbf{k}s} \right) \,. \tag{2.43}$$

This energy is actually independent on time: the time dependence of the complex amplitudes $a^*_{\mathbf{k}s}(t)$ and $a_{\mathbf{k}s}(t)$ cancels due to Eq. (2.37). Instead of using the complex amplitudes $a^*_{\mathbf{k}s}(t)$ and $a_{\mathbf{k}s}(t)$ one can introduce the real variables

$$q_{\mathbf{k}s}(t) = \sqrt{\frac{2\hbar}{\omega_k}} \frac{1}{2} \left(a_{\mathbf{k}s}(t) + a^*_{\mathbf{k}s}(t) \right) \tag{2.44}$$

$$p_{\mathbf{k}s}(t) = \sqrt{2\hbar\omega_k} \frac{1}{2i} \left(a_{\mathbf{k}s}(t) - a^*_{\mathbf{k}s}(t) \right) \tag{2.45}$$

such that the energy of the radiation field reads

$$H = \sum_{\mathbf{k}} \sum_{s} \left(\frac{p^2_{\mathbf{k},s}}{2} + \frac{1}{2} \omega_k^2 \, q^2_{\mathbf{k}s} \right) . \tag{2.46}$$

This energy resembles that of infinitely many harmonic oscillators with unitary mass and frequency ω_k. It is written in terms of an infinite set of real harmonic oscillators: two oscillators (due to polarization) for each mode of wavevector $\mathbf{k}$ and angular frequency ω_k.

2.2 Second Quantization of Light

In 1927 Paul Dirac performed the quantization of the classical Hamiltonian (2.46) by promoting the real coordinates $q_{\mathbf{k}s}$ and the real momenta $p_{\mathbf{k}s}$ to operators:

$$q_{\mathbf{k}s} \to \hat{q}_{\mathbf{k}s} \, , \tag{2.47}$$

$$p_{\mathbf{k}s} \to \hat{p}_{\mathbf{k}s} \, , \tag{2.48}$$

satisfying the commutation relations

$$[\hat{q}_{\mathbf{k}s}, \, \hat{p}_{\mathbf{k}'s'}] = i\hbar \, \delta_{\mathbf{k},\mathbf{k}'} \, \delta_{s,s'} \, , \tag{2.49}$$

where $[\hat{A}, \, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$. The quantum Hamiltonian is thus given by

$$\hat{H} = \sum_{\mathbf{k}} \sum_{s} \left(\frac{\hat{p}^2_{\mathbf{k},s}}{2} + \frac{1}{2} \omega_k^2 \, \hat{q}^2_{\mathbf{k}s} \right) . \tag{2.50}$$

Following a standard approach for the canonical quantization of the Harmonic oscillator, we introduce annihilation and creation operators

$$\hat{a}_{\mathbf{k}s} = \sqrt{\frac{\omega_k}{2\hbar}} \left(\hat{q}_{\mathbf{k}s} + \frac{i}{\omega_k} \, \hat{p}_{\mathbf{k}s} \right) \, , \tag{2.51}$$

$$\hat{a}^+_{\mathbf{k}s} = \sqrt{\frac{\omega_k}{2\hbar}} \left(\hat{q}_{\mathbf{k}s} - \frac{i}{\omega_k} \, \hat{p}_{\mathbf{k}s} \right) \, , \tag{2.52}$$

which satisfy the commutation relations

$$[\hat{a}_{\mathbf{k}s}, \hat{a}^{+}_{\mathbf{k}'s'}] = \delta_{\mathbf{k},\mathbf{k}'}\,\delta_{s,s'}\ , \tag{2.53}$$

and the quantum Hamiltonian (2.50) becomes

$$\hat{H} = \sum_{\mathbf{k}} \sum_{s} \hbar\omega_k \left(\hat{a}^{+}_{\mathbf{k}s}\hat{a}_{\mathbf{k}s} + \frac{1}{2}\right)\ . \tag{2.54}$$

Obviously this quantum Hamiltonian can be directly obtained from the classical one, given by Eq. (2.43), by promoting the complex amplitudes $a_{\mathbf{k}s}$ and $a^{*}_{\mathbf{k}s}$ to operators:

$$a_{\mathbf{k}s} \to \hat{a}_{\mathbf{k}s}\ , \tag{2.55}$$

$$a^{*}_{\mathbf{k}s} \to \hat{a}^{+}_{\mathbf{k}s}\ , \tag{2.56}$$

satisfying the commutation relations (2.53).

The operators $\hat{a}_{\mathbf{k}s}$ and $\hat{a}^{+}_{\mathbf{k}s}$ act in the Fock space $\mathcal{F}$, i.e. the infinite dimensional Hilbert space of "number representation" introduced in 1932 by Vladimir Fock. A generic state of this Fock space $\mathcal{F}$ is given by

$$|\ \ldots\ n_{\mathbf{k}s}\ \ldots\ n_{\mathbf{k}'s'}\ \ldots\ n_{\mathbf{k}''s''}\ \ldots\rangle\ , \tag{2.57}$$

meaning that there are $n_{\mathbf{k}s}$ photons with wavevector $\mathbf{k}$ and polarization s, $n_{\mathbf{k}'s'}$ photons with wavevector $\mathbf{k}'$ and polarization s', $n_{\mathbf{k}''s''}$ photons with wavevector $\mathbf{k}''$ and polarization s'', et cetera. The Fock space $\mathcal{F}$ is given by

$$\mathcal{F} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus\ \cdots\ \oplus \mathcal{H}_\infty = \bigoplus_{n=0}^{\infty} \mathcal{H}_n\ , \tag{2.58}$$

where

$$\mathcal{H}_n = \mathcal{H} \otimes \mathcal{H} \otimes\ \cdots\ \otimes \mathcal{H} = \mathcal{H}^{\otimes n} \tag{2.59}$$

is the Hilbert space of n identical photons, which is n times the tensor product $\otimes$ of the single-photon Hilbert space $\mathcal{H} = \mathcal{H}^{\otimes 1}$. Thus, $\mathcal{F}$ is the infinite direct sum $\oplus$ of increasing n-photon Hilbert states $\mathcal{H}_n$, and we can formally write

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}\ . \tag{2.60}$$

Notice that in the definition of the Fock space $\mathcal{F}$ one must include the space $\mathcal{H}_0 = \mathcal{H}^{\otimes 0}$, which is the Hilbert space of 0 photons, containing only the vacuum state

$$|0\rangle = |\ \ldots\ 0\ \ldots\ 0\ \ldots\ 0\ \ldots\rangle\ , \tag{2.61}$$

and its dilatations $\gamma|0\rangle$ with γ a generic complex number. The operators $\hat{a}_{\mathbf{k}s}$ and $\hat{a}^+_{\mathbf{k}s}$ are called annihilation and creation operators because they respectively destroy and create one photon with wavevector $\mathbf{k}$ and polarization s, namely

$$\hat{a}_{\mathbf{k}s}|\ \ldots\ n_{\mathbf{k}s}\ \ldots\rangle = \sqrt{n_{\mathbf{k}s}}\ |\ \ldots\ n_{\mathbf{k}s} - 1\ \ldots\rangle\ , \tag{2.62}$$

$$\hat{a}^+_{\mathbf{k}s}|\ \ldots\ n_{\mathbf{k}s}\ \ldots\rangle = \sqrt{n_{\mathbf{k}s} + 1}\ |\ \ldots\ n_{\mathbf{k}s} + 1\ \ldots\rangle\ . \tag{2.63}$$

Note that these properties follow directly from the commutation relations (2.53). Consequently, for the vacuum state $|0\rangle$ one finds

$$\hat{a}_{\mathbf{k}s}|0\rangle = 0_F\ , \tag{2.64}$$

$$\hat{a}^+_{\mathbf{k}s}|0\rangle = |1_{\mathbf{k}s}\rangle = |\mathbf{k}s\rangle\ , \tag{2.65}$$

where 0_F is the zero of the Fock space (usually indicated with 0), and $|\mathbf{k}s\rangle$ is clearly the state of one photon with wavevector $\mathbf{k}$ and polarization s, such that

$$\langle\mathbf{r}|\mathbf{k}s\rangle = \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{V}}\boldsymbol{\varepsilon}_{\mathbf{k}s}\ . \tag{2.66}$$

From Eqs. (2.62) and (2.63) it follows immediately that

$$\hat{N}_{\mathbf{k}s} = \hat{a}^+_{\mathbf{k}s}\hat{a}_{\mathbf{k}s} \tag{2.67}$$

is the number operator which counts the number of photons in the single-particle state $|\mathbf{k}s\rangle$, i.e.

$$\hat{N}_{\mathbf{k}s}|\ \ldots\ n_{\mathbf{k}s}\ \ldots\rangle = n_{\mathbf{k}s}\ |\ \ldots\ n_{\mathbf{k}s}\ \ldots\rangle\ . \tag{2.68}$$

Notice that the quantum Hamiltonian of the light can be written as

$$\hat{H} = \sum_{\mathbf{k}}\sum_{s}\hbar\omega_k\left(\hat{N}_{\mathbf{k}s} + \frac{1}{2}\right)\ , \tag{2.69}$$

and this expression is very similar, but not equal, to the semiclassical formula (2.19). The differences are that $\hat{N}_{\mathbf{k}s}$ is a quantum number operator and that the energy E_{vac} of the vacuum state $|0\rangle$ is not zero but is instead given by

$$E_{vac} = \sum_{\mathbf{k}}\sum_{s}\frac{1}{2}\hbar\omega_k\ . \tag{2.70}$$

A quantum harmonic oscillator of frequency $\omega_{\mathbf{k}}$ has a finite minimal energy $\hbar\omega_{\mathbf{k}}$, which is called zero-point energy. In the case of the quantum electromagnetic field there is an infinite number of harmonic oscillators and the total zero-point energy, given by Eq. (2.70), is clearly infinite. The infinite constant E_{vac} is usually eliminated by simply shifting to zero the energy associated to the vacuum state $|0\rangle$.

The quantum electric and magnetic fields can be obtained from the classical expressions, Eqs. (2.34) and (2.35), taking into account the quantization of the classical complex amplitudes $a_{\mathbf{k}s}$ and $a^*_{\mathbf{k}s}$ and their time-dependence, given by Eq. (2.37) and its complex conjugate. In this way we obtain

$$\hat{\mathbf{E}}(\mathbf{r},t) = i \sum_{\mathbf{k}} \sum_{s} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}} \left[\hat{a}_{\mathbf{k}s}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} - \hat{a}^+_{\mathbf{k}s}\, e^{-(i\mathbf{k}\cdot\mathbf{r}-\omega_k t)}\right] \boldsymbol{\varepsilon}_{\mathbf{k}s} \,, \tag{2.71}$$

$$\hat{\mathbf{B}}(\mathbf{r},t) = \sum_{\mathbf{k}} \sum_{s} \sqrt{\frac{\hbar}{2\varepsilon_0 \omega_k V}} \left[\hat{a}_{\mathbf{k}s}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} - \hat{a}^+_{\mathbf{k}s}\, e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}\right] i\,\frac{\mathbf{k}}{|\mathbf{k}|} \wedge \boldsymbol{\varepsilon}_{\mathbf{k}s} \,. \tag{2.72}$$

It is important to stress that the results of our canonical quantization of the radiation field suggest a remarkable philosophical idea: there is a unique quantum electromagnetic field in the universe and all the photons we see are the massless particles associated to it. In fact, the quantization of the electromagnetic field is the first step towards the so-called quantum field theory or second quantization of fields, where all particles in the universe are associated to few quantum fields and their corresponding creation and annihilation operators.

2.2.1 *Fock versus Coherent States for the Light Field*

Let us now consider for simplicity a linearly polarized monochromatic wave of the radiation field. For instance, let us suppose that the direction of polarization is given by the vector $\boldsymbol{\varepsilon}$. From Eq. (2.71) one finds immediately that the quantum electric field can be then written in a simplified notation as

$$\hat{\mathbf{E}}(\mathbf{r},t) = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\, i\, \left[\hat{a}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} - \hat{a}^+\, e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right] \boldsymbol{\varepsilon} \tag{2.73}$$

where $\omega = \omega_k = c|\mathbf{k}|$. Notice that, to simplify the notation, we have removed the subscripts in the annihilation and creation operators $\hat{a}$ and $\hat{a}^+$. If there are exactly n photons in this polarized monochromatic wave the Fock state of the system is given by

$$|n\rangle = \frac{1}{\sqrt{n!}} \left(\hat{a}^+\right)^n |0\rangle \,. \tag{2.74}$$

It is then straightforward to show, by using Eqs. (2.62) and (2.63), that

$$\langle n|\hat{\mathbf{E}}(\mathbf{r},t)|n\rangle = \mathbf{0}, \tag{2.75}$$

for all values of the photon number n, no matter how large. This result holds for all modes, which means then that the expectation value of the electric field in any many-photon Fock state is zero. On the other hand, the expectation value of $\hat{E}(\mathbf{r}, t)^2$ is given by

$$\langle n|\hat{E}(\mathbf{r}, t)^2|n\rangle = \frac{\hbar\omega}{\varepsilon_0 V}\left(n + \frac{1}{2}\right) . \tag{2.76}$$

For this case, as for the energy, the expectation value is nonvanishing even when $n = 0$, with the result that for the total field (2.71), consisting of an infinite number of modes, the expectation value of $\hat{E}(\mathbf{r}, t)^2$ is infinite for all many-photon states, including the vacuum state. Obviously a similar reasoning applies for the magnetic field (2.72) and, as discussed in the previous section, the zero-point constant is usually removed.

One must remember that the somehow strange result of Eq. (2.75) is due to the fact that the expectation value is performed with the Fock state $|n\rangle$, which means that the number of photons is fixed because

$$\hat{N}|n\rangle = n|n\rangle . \tag{2.77}$$

Nevertheless, usually the number of photons in the radiation field is not fixed, in other words the system is not in a pure Fock state. For example, the radiation field of a well-stabilized laser device operating in a single mode is described by a coherent state $|\alpha\rangle$, such that

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle , \tag{2.78}$$

with

$$\langle\alpha|\alpha\rangle = 1 . \tag{2.79}$$

The coherent state $|\alpha\rangle$, introduced in 1963 by Roy Glauber, is thus the eigenstate of the annihilation operator $\hat{a}$ with complex eigenvalue $\alpha = |\alpha|e^{i\theta}$. $|\alpha\rangle$ does not have a fixed number of photons, i.e. it is not an eigenstate of the number operator $\hat{N}$, and it is not difficult to show that $|\alpha\rangle$ can be expanded in terms of number (Fock) states $|n\rangle$ as follows

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle . \tag{2.80}$$

From Eq. (2.78) one immediately finds

$$\bar{N} = \langle\alpha|\hat{N}|\alpha\rangle = |\alpha|^2 , \tag{2.81}$$

and it is natural to set

$$\alpha = \sqrt{\bar{N}}\, e^{i\theta} , \tag{2.82}$$

where $\bar{N}$ is the average number of photons in the coherent state, while θ is the phase of the coherent state. For the sake of completeness, we observe that

$$\langle\alpha|\hat{N}^2|\alpha\rangle = |\alpha|^2 + |\alpha|^4 = \bar{N} + \bar{N}^2 \tag{2.83}$$

and consequently

$$\langle\alpha|\hat{N}^2|\alpha\rangle - \langle\alpha|\hat{N}|\alpha\rangle^2 = \bar{N} \ , \tag{2.84}$$

while

$$\langle n|\hat{N}^2|n\rangle = n^2 \tag{2.85}$$

and consequently

$$\langle n|\hat{N}^2|n\rangle - \langle n|\hat{N}|n\rangle^2 = 0 \ . \tag{2.86}$$

The expectation value of the electric field $\hat{\mathbf{E}}(\mathbf{r}, t)$ of the linearly polarized monochromatic wave, Eq. (2.73), in the coherent state $|\alpha\rangle$ reads

$$\langle\alpha|\hat{\mathbf{E}}(\mathbf{r}, t)|\alpha\rangle = -\sqrt{\frac{2\bar{N}\hbar\omega}{\varepsilon_0 V}}\ \sin(\mathbf{k}\cdot\mathbf{r} - \omega t + \theta)\,\boldsymbol{\varepsilon} \ , \tag{2.87}$$

while the expectation value of $\hat{E}(\mathbf{r}, t)^2$ is given by

$$\langle\alpha|\hat{E}(\mathbf{r}, t)^2|\alpha\rangle = \frac{2\bar{N}\hbar\omega}{\varepsilon_0 V}\ \sin^2(\mathbf{k}\cdot\mathbf{r} - \omega t + \theta) \ . \tag{2.88}$$

These results suggest that the coherent state is indeed a useful tool to investigate the correspondence between quantum field theory and classical field theory. Indeed, in 1965 at the University of Milan Fortunato Tito Arecchi experimentally verified that a single-mode laser is in a coherent state with a definite but unknown phase. Thus, the coherent state gives photocount statistics that are in accord with laser experiments and has coherence properties similar to those of a classical field, which are useful for explaining interference effects.

To conclude this subsection we observe that we started the quantization of the light field by expanding the vector potential $\mathbf{A}(\mathbf{r}, t)$ as a Fourier series of monochromatic plane waves. But this is not the unique choice. Indeed, one can expand the vector potential $\mathbf{A}(\mathbf{r}, t)$ by using any orthonormal set of wavefunctions (spatio-temporal modes) which satisfy the Maxwell equations. After quantization, the expansion coefficients of the single mode play the role of creation and annihilation operators of that mode.

2.2.2 Linear and Angular Momentum of the Radiation Field

In addition to the total energy there are other interesting conserved quantities which characterize the classical radiation field. They are the total linear momentum

$$\mathbf{P} = \int d^3\mathbf{r}\, \varepsilon_0\, \mathbf{E}(\mathbf{r}, t) \wedge \mathbf{B}(\mathbf{r}, t) \tag{2.89}$$

and the total angular momentum

$$\mathbf{J} = \int d^3\mathbf{r}\, \varepsilon_0\, \mathbf{r} \wedge (\mathbf{E}(\mathbf{r}, t) \wedge \mathbf{B}(\mathbf{r}, t)) \ . \tag{2.90}$$

By using the canonical quantization one easily finds that the quantum total linear momentum operator is given by

$$\hat{\mathbf{P}} = \sum_{\mathbf{k}} \sum_{s} \hbar\mathbf{k} \left(\hat{N}_{\mathbf{k}s} + \frac{1}{2} \right) \ , \tag{2.91}$$

where $\hbar\mathbf{k}$ is the linear momentum of a photon of wavevector $\mathbf{k}$.

The quantization of the total angular momentum $\mathbf{J}$ is a more intricate problem. In fact, the linearly polarized states $|\mathbf{k}s\rangle$ (with $s = 1, 2$) are eigenstates of $\hat{H}$ and $\hat{\mathbf{P}}$, but they are not eigenstates of $\hat{\mathbf{J}}$, nor of $\hat{J}^2$, and nor of $\hat{J}_z$. The eigenstates of $\hat{J}^2$ and $\hat{J}_z$ do not have a fixed wavevector $\mathbf{k}$ but only a fixed wavenumber $k = |\mathbf{k}|$. These states, which can be indicated as $|kjm_j s\rangle$, are associated to vector spherical harmonics $\mathbf{Y}_{j,m_j}(\theta, \phi)$, with θ and ϕ the spherical angles of the wavevector $\mathbf{k}$. These states $|kjm_j s\rangle$ are not eigenstates of $\hat{\mathbf{P}}$ but they are eigenstates of $\hat{H}$, $\hat{J}^2$ and $\hat{J}_z$, namely

$$\hat{H}|kjm_j s\rangle = ck|kjm_j s\rangle \tag{2.92}$$

$$\hat{J}^2|kjm_j\rangle = j(j+1)\hbar^2|kjm_j s\rangle \tag{2.93}$$

$$\hat{J}_z|kjm_j s\rangle = m_j\hbar|kjm_j s\rangle \tag{2.94}$$

with $m_j = -j, -j+1, \ldots, j-1, j$ and $j = 1, 2, 3, \ldots$. In problems where the charge distribution which emits the electromagnetic radiation has spherical symmetry it is indeed more useful to expand the radiation field in vector spherical harmonics than in plane waves. Clearly, one can write

$$|\mathbf{k}s\rangle = \sum_{jm_j} c_{jm_j}(\theta, \phi)|kjm_j s\rangle \ , \tag{2.95}$$

where the coefficients $c_{jm_j}(\theta, \phi)$ of this expansion give the amplitude probability of finding a photon of fixed wavenumber $k = |\mathbf{k}|$ and orbital quantum numbers j and m_j with spherical angles θ and ϕ.

2.2.3 Zero-Point Energy and the Casimir Effect

There are situations in which the zero-point energy E_{vac} of the electromagnetic field oscillators give rise to a remarkable quantum phenomenon: the Casimir effect.

The zero-point energy (2.70) of the electromagnetic field in a region of volume V can be written as

$$E_{vac} = \sum_{\mathbf{k}} \sum_{s} \frac{1}{2} \hbar c \sqrt{k_x^2 + k_y^2 + k_z^2} = V \int \frac{d^3\mathbf{k}}{(2\pi)^3} \hbar c \sqrt{k_x^2 + k_y^2 + k_z^2} \tag{2.96}$$

by using the dispersion relation $\omega_k = ck = c\sqrt{k_y^2 + k_y^2 + k_z^2}$. In particular, considering a region with the shape of a parallelepiped of length L along both x and y and length a along z, the volume V is given by $V = L^2 a$ and the vacuum energy E_{vac} in the region is

$$\begin{aligned} E_{vac} &= \hbar c \int_{-\infty}^{+\infty} \frac{L\, dk_x}{2\pi} \int_{-\infty}^{+\infty} \frac{L\, dk_y}{2\pi} \int_{-\infty}^{+\infty} \frac{a\, dk_z}{2\pi} \sqrt{k_x^2 + k_y^2 + k_z^2} \\ &= \frac{\hbar c}{2\pi} L^2 \int_0^{\infty} dk_{\parallel} k_{\parallel} \Big[\int_0^{\infty} dn \sqrt{k_{\parallel}^2 + \frac{n^2\pi^2}{a^2}} \Big] , \end{aligned} \tag{2.97}$$

where the second expression is obtained setting $k_{\parallel} = \sqrt{k_x^2 + k_y^2}$ and $n = (a/\pi)k_z$.

Let us now consider the presence of two perfect metallic plates with the shape of a square of length L having parallel faces lying in the (x, y) plane at distance a. Along the z axis the stationary standing waves of the electromagnetic field vanishes on the metal plates and the k_z component of the wavevector $\mathbf{k}$ is no more a continuum variabile but it is quantized via

$$k_z = n \frac{\pi}{a} , \tag{2.98}$$

where now $n = 0, 1, 2, \ldots$ is an integer number, and not a real number as in Eq. (2.97). In this case the zero-point energy in the volume $V = L^2 a$ between the two plates reads

$$E'_{vac} = \frac{\hbar c}{2\pi} L^2 \int_0^{\infty} dk_{\parallel} k_{\parallel} \Big[\frac{k_{\parallel}}{2} + \sum_{n=1}^{\infty} \sqrt{k_{\parallel}^2 + \frac{n^2\pi^2}{a^2}} \Big] , \tag{2.99}$$

where for $k_z = 0$ the state $|(k_z, k_y, 0)1\rangle$ with polarization $\boldsymbol{\epsilon}_{\mathbf{k}1}$ parallel to the z axis (and orthogonal to the metallic plates) is bound, while the state $|(k_z, k_y, 0)2\rangle$ with polarization $\boldsymbol{\epsilon}_{\mathbf{k}2}$ orthogonal to the z axis (and parallel to the metallic plates) is not bound and it does not contribute to the discrete summation (Fig. 2.2).

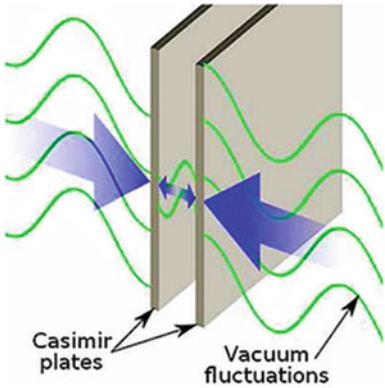


Fig. 2.2 Graphical representation of the parallel plates in the Casimir effect

The difference between E'_{vac} and E_{vac} divided by L^2 gives the net energy per unit surface area $\mathcal{E}$, namely

$$\mathcal{E} = \frac{E'_{vac} - E_{vac}}{L^2} = \frac{\hbar c}{2\pi}\left(\frac{\pi}{a}\right)^3 \left[\frac{1}{2}A(0) + \sum_{n=1}^{\infty} A(n) - \int_0^{\infty} dn\, A(n)\right], \quad (2.100)$$

where we have defined

$$A(n) = \int_0^{+\infty} d\zeta\, \zeta \sqrt{\zeta^2 + n^2} = \frac{1}{3}\left[(n^2+\infty)^{3/2} - n^2\right], \quad (2.101)$$

with $\zeta = (a/\pi)k_{\parallel}$. Notice that $A(n)$ is clearly divergent but Eq. (2.100) is not divergent due to the cancellation of divergences with opposite sign. In fact, by using the Euler-MacLaurin formula for the difference of infinite series and integrals

$$\frac{1}{2}A(0) + \sum_{n=1}^{\infty} A(n) - \int_0^{\infty} dn\, A(n) = -\frac{1}{6\cdot 2!}\frac{dA}{dn}(0) + \frac{1}{30\cdot 4!}\frac{d^3A}{dn^3}(0) - \frac{1}{42\cdot 6!}\frac{d^5A}{dn^5}(0) + \cdots \quad (2.102)$$

and Eq. (2.101) from which

$$\frac{dA}{dn}(0) = 0\,, \qquad \frac{d^3A}{dn^3}(0) = -2\,, \qquad \frac{d^5A}{dn^5}(0) = 0 \quad (2.103)$$

and all higher derivatives of $A(n)$ are zero, one eventually obtains

$$\mathcal{E} = -\frac{\pi^2}{720}\frac{\hbar c}{a^3}\,. \quad (2.104)$$

From this energy difference $\mathcal{E}$ one deduces that there is an attractive force per unit area $\mathcal{F}$ between the two plates, given by

$$\mathcal{F} = -\frac{d\mathcal{E}}{da} = -\frac{\pi^2}{240}\frac{\hbar c}{a^4} \ . \tag{2.105}$$

Numerically this result, predicted in 1948 by Hendrik Casimir during his research activity at the Philips Physics Laboratory in Eindhoven, is very small

$$\mathcal{F} = -\frac{1.30 \times 10^{-27}\text{N m}^2}{a^4} \ . \tag{2.106}$$

Nevertheless, it has been experimentally verified by Steven Lamoreaux in 1997 at the University of Washington and by Giacomo Bressi, Gianni Carugno, Roberto Onofrio, and Giuseppe Ruoso in 2002 at the University of Padua.

2.3 Quantum Radiation Field at Finite Temperature

Let us consider the quantum radiation field in thermal equilibrium with a bath at the temperature T. The relevant quantity to calculate all thermodynamical properties of the system is the grand-canonical partition function $\mathcal{Z}$, given by

$$\mathcal{Z} = Tr[e^{-\beta(\hat{H}-\mu\hat{N})}] \tag{2.107}$$

where $\beta = 1/(k_B T)$ with $k_B = 1.38 \times 10^{-23}$ J/K the Boltzmann constant,

$$\hat{H} = \sum_{\mathbf{k}}\sum_{s} \hbar\omega_k \, \hat{N}_{\mathbf{k}s} \ , \tag{2.108}$$

is the quantum Hamiltonian without the zero-point energy,

$$\hat{N} = \sum_{\mathbf{k}}\sum_{s} \hat{N}_{\mathbf{k}s} \tag{2.109}$$

is the total number operator, and μ is the chemical potential, fixed by the conservation of the particle number. For photons $\mu = 0$ and consequently the number of photons is not fixed. This implies that

$$\begin{aligned}
\mathcal{Z} &= \sum_{\{n_{\mathbf{k}s}\}} \langle \dots \, n_{\mathbf{k}s} \, \dots | e^{-\beta\hat{H}} | \dots \, n_{\mathbf{k}s} \, \dots \rangle = \sum_{\{n_{\mathbf{k}s}\}} \langle \dots \, n_{\mathbf{k}s} \, \dots | e^{-\beta\sum_{\mathbf{k}s}\hbar\omega_k \hat{N}_{\mathbf{k}s}} | \dots \, n_{\mathbf{k}s} \, \dots \rangle \\
&= \sum_{\{n_{\mathbf{k}s}\}} e^{-\beta\sum_{\mathbf{k}s}\hbar\omega_k n_{\mathbf{k}s}} = \sum_{\{n_{\mathbf{k}s}\}} \prod_{\mathbf{k}s} e^{-\beta\hbar\omega_k n_{\mathbf{k}s}} = \prod_{\mathbf{k}s}\sum_{n_{\mathbf{k}s}} e^{-\beta\hbar\omega_k n_{\mathbf{k}s}} = \prod_{\mathbf{k}s}\sum_{n=0}^{\infty} e^{-\beta\hbar\omega_k n} \\
&= \prod_{\mathbf{k}s} \frac{1}{1 - e^{-\beta\hbar\omega_k}} \ .
\end{aligned} \tag{2.110}$$

Quantum statistical mechanics dictates that the thermal average of any operator $\hat{A}$ is obtained as

$$\langle \hat{A} \rangle_T = \frac{1}{\mathcal{Z}} Tr[\hat{A}\, e^{-\beta(\hat{H}-\mu\hat{N})}] \,. \tag{2.111}$$

In our case the calculations are simplified because $\mu = 0$. Let us suppose that $\hat{A} = \hat{H}$, it is then quite easy to show that

$$\langle \hat{H} \rangle_T = \frac{1}{\mathcal{Z}} Tr[\hat{H}\, e^{-\beta\hat{H}}] = -\frac{\partial}{\partial\beta} \ln\left(Tr[e^{-\beta\hat{H}}]\right) = -\frac{\partial}{\partial\beta} \ln(\mathcal{Z}) \,. \tag{2.112}$$

By using Eq. (2.110) we immediately obtain

$$\ln(\mathcal{Z}) = -\sum_{\mathbf{k}} \sum_{s} \ln\left(1 - e^{-\beta\hbar\omega_k}\right) , \tag{2.113}$$

and finally from Eq. (2.112) we get

$$\langle \hat{H} \rangle_T = \sum_{\mathbf{k}} \sum_{s} \frac{\hbar\omega_k}{e^{\beta\hbar\omega_k} - 1} = \sum_{\mathbf{k}} \sum_{s} \hbar\omega_k \, \langle \hat{N}_{\mathbf{k}s} \rangle_T \,. \tag{2.114}$$

In the continuum limit, where

$$\sum_{\mathbf{k}} \to V \int \frac{d^3\mathbf{k}}{(2\pi)^3} \,, \tag{2.115}$$

with V the volume, and taking into account that $\omega_k = ck$, one can write the energy density $\mathcal{E} = \langle \hat{H} \rangle_T / V$ as

$$\mathcal{E} = 2 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{c\hbar k}{e^{\beta c\hbar k} - 1} = \frac{c\hbar}{\pi^2} \int_0^\infty dk \frac{k^3}{e^{\beta c\hbar k} - 1} \,, \tag{2.116}$$

where the factor 2 is due to the two possible polarizations ($s = 1, 2$). By using $\omega = ck$ instead of k as integration variable one gets

$$\mathcal{E} = \frac{\hbar}{\pi^2 c^3} \int_0^\infty d\omega \frac{\omega^3}{e^{\beta\hbar\omega} - 1} = \int_0^\infty d\omega \, \rho(\omega) \,, \tag{2.117}$$

where

$$\rho(\omega) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\hbar\omega} - 1} \tag{2.118}$$

is the energy density per frequency, i.e. the familiar formula of the black-body radiation, obtained for the first time in 1900 by Max Planck. The previous integral can be explicitly calculated and it gives

$$\mathcal{E} = \frac{\pi^2 k_B^4}{15 c^3 \hbar^3} T^4 \,, \tag{2.119}$$

which is nothing but the Stefan-Boltzmann law. In an similar way one determines the average number density of photons:

$$n = \frac{\langle \hat{N} \rangle_T}{V} = \frac{1}{\pi^2 c^3} \int_0^\infty d\omega \, \frac{\omega^2}{e^{\beta \hbar \omega} - 1} = \frac{2\zeta(3) k_B^3}{\pi^2 c^3 \hbar^3} T^3 \,. \tag{2.120}$$

where $\zeta(3) \simeq 1.202$. Notice that both energy density $\mathcal{E}$ and number density n of photons go to zero as the temperature T goes to zero. To conclude this section, we stress that these results are obtained at thermal equilibrium and under the condition of a vanishing chemical potential, meaning that the number of photons is not conserved when the temperature is varied.

2.4 Phase Operators

We have seen that the ladder operators $\hat{a}$ and $\hat{a}^+$ of a single mode of the electromagnetic field satisfy the fundamental relations

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle \,, \tag{2.121}$$

$$\hat{a}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle \,, \tag{2.122}$$

where $|n\rangle$ is the Fock state, eigenstate of the number operator $\hat{N} = \hat{a}^+\hat{a}$ and describing n photons in the mode. Ramarkably, these operator can be expressed in terms of the Fock states as follows

$$\hat{a} = \sum_{n=0}^{\infty} \sqrt{n+1}|n\rangle\langle n+1| = |0\rangle\langle 1| + \sqrt{2}|1\rangle\langle 2| + \sqrt{3}|2\rangle\langle 3| + \cdots \,, \tag{2.123}$$

$$\hat{a}^+ = \sum_{n=0}^{\infty} \sqrt{n+1}|n+1\rangle\langle n| = |1\rangle\langle 0| + \sqrt{2}|2\rangle\langle 1| + \sqrt{3}|3\rangle\langle 2| + \cdots \,. \tag{2.124}$$

It is in fact straightforward to verify that the expressions (2.123) and (2.124) imply Eqs. (2.121) and (2.122).

We now introduce the phase operators

$$\hat{f} = \hat{a}\,\hat{N}^{-1/2} \,, \qquad \hat{f}^+ = \hat{N}^{-1/2}\,\hat{a}^+ \,, \tag{2.125}$$

which can be expressed as

$$\hat{f} = \sum_{n=0}^{\infty} |n\rangle\langle n+1| = |0\rangle\langle 1| + |1\rangle\langle 2| + |2\rangle\langle 3| + \cdots \,, \tag{2.126}$$

$$\hat{f}^{+} = \sum_{n=0}^{\infty} |n+1\rangle\langle n| = |1\rangle\langle 0| + |2\rangle\langle 1| + |3\rangle\langle 2| + \cdots \,, \tag{2.127}$$

and satisfy the very nice formulas

$$\hat{f}|n\rangle = |n-1\rangle \,, \tag{2.128}$$

$$\hat{f}^{+}|n\rangle = |n+1\rangle \,, \tag{2.129}$$

showing that the phase operators act as lowering and raising operators of Fock states without the complication of coefficients in front of the obtained states.

The shifting property of the phase operators on the Fock states $|n\rangle$ resembles that of the unitary operator $e^{i\hat{p}/\hbar}$ acting on the position state $|x\rangle$ of a 1D particle, with $\hat{p}$ the 1D linear momentum which is canonically conjugated to the 1D position operator $\hat{x}$. In particular, one has

$$e^{i\hat{p}/\hbar}|x\rangle = |x-1\rangle \,, \tag{2.130}$$

$$e^{-i\hat{p}/\hbar}|x\rangle = |x+1\rangle \,. \tag{2.131}$$

Due to these formal analogies, in 1927 Fritz London and Paul Dirac independently suggested that the phase operator $\hat{f}$ can be written as

$$\hat{f} = e^{i\hat{\Theta}} \,, \tag{2.132}$$

where $\hat{\Theta}$ is the angle operator related to the number operator $\hat{N} = \hat{a}^{+}\hat{a}$. But, contrary to $e^{i\hat{p}/\hbar}$, the operator $\hat{f}$ is not unitary because

$$\hat{f}\hat{f}^{+} = \hat{1} \,, \qquad \hat{f}^{+}\hat{f} = \hat{1} - |0\rangle\langle 0| \,. \tag{2.133}$$

This implies that the angle operator $\hat{\Theta}$ is not Hermitian and

$$\hat{f}^{+} = e^{-i\hat{\Theta}^{+}} \,. \tag{2.134}$$

In addition, a coherent state $|\alpha\rangle$ is not an eigenstate of the phase operator $\hat{f}$. However, one easily finds

$$\langle\alpha|\hat{f}|\alpha\rangle = \alpha e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{\sqrt{n!(n+1)!}} \,, \tag{2.135}$$

and observing that for $x \gg 1$ one gets $\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n!(n+1)!}} \simeq \frac{e^x}{\sqrt{x}}$ this implies

$$\langle\alpha|\hat{f}|\alpha\rangle \simeq \frac{\alpha}{|\alpha|} = e^{i\theta} \ , \tag{2.136}$$

for $\bar{N} = |\alpha|^2 \gg 1$, with $\alpha = \sqrt{\bar{N}}e^{i\theta}$. Thus, for a large average number $\bar{N}$ of photons the expectation value of the phase operator $\hat{f}$ on the coherent state $|\alpha\rangle$ gives the phase factor $e^{i\theta}$ of the complex eigenvalue α of the coherent state.

2.5 Solved Problems

Problem 2.1
Show that the eigenvalues n of the operator $\hat{N} = \hat{a}^+\hat{a}$ are non negative.

Solution
The eigenvalue equation of $\hat{N}$ reads

$$\hat{a}^+\hat{a}|n\rangle = n|n\rangle \ .$$

We can then write

$$\langle n|\hat{a}^+\hat{a}|n\rangle = n\langle n|n\rangle = n \ ,$$

because the eigenstate $|n\rangle$ is normalized to one. On the other hand, we have also

$$\langle n|\hat{a}^+\hat{a}|n\rangle = (\hat{a}|n\rangle)^+(\hat{a}|n\rangle) = |(\hat{a}|n\rangle)|^2$$

Consequently, we get

$$n = |(\hat{a}|n\rangle)|^2 \geq 0 \ .$$

We stress that we have not used the commutations relations of $\hat{a}$ and $\hat{a}^+$. Thus we have actually proved that any operator $\hat{N}$ given by the factorization of a generic operator $\hat{a}$ with its self-adjunct $\hat{a}^+$ has a non negative spectrum.

Problem 2.2
Consider the operator $\hat{N} = \hat{a}^+\hat{a}$, where $\hat{a}$ and $\hat{a}^+$ satisfy the commutation rule $\hat{a}\hat{a}^+ - \hat{a}^+\hat{a} = 1$. Show that if $|n\rangle$ is an eigenstate of $\hat{N}$ with eigenvalue n then $a|n\rangle$ is an eigenstate of $\hat{N}$ with eigenvalue $n-1$ and $a^+|n\rangle$ is an eigenstate of $\hat{N}$ with eigenvalue $n+1$.

Solution
We have

$$\hat{N}\hat{a}|n\rangle = (\hat{a}^+\hat{a})\hat{a}|n\rangle \ .$$

The commutation relation between $\hat{a}$ and $\hat{a}^+$ can be written as

$$\hat{a}^+\hat{a} = \hat{a}\hat{a}^+ - 1 \ .$$

This implies that

$$\hat{N}\hat{a}|n\rangle = (\hat{a}\hat{a}^+ - 1)\hat{a}|n\rangle = \hat{a}(\hat{a}^+\hat{a} - 1)|n\rangle = \hat{a}(\hat{N} - 1)|n\rangle = (n-1)\hat{a}|n\rangle \ .$$

Finally, we obtain

$$\hat{N}\hat{a}^+|n\rangle = (\hat{a}^+\hat{a})\hat{a}^+|n\rangle = \hat{a}^+(\hat{a}\hat{a}^+)|n\rangle = \hat{a}^+(\hat{a}^+\hat{a} + 1)|n\rangle = \hat{a}^+(\hat{N} + 1)|n\rangle = (n+1)\hat{a}^+|n\rangle \ .$$

Problem 2.3
Taking into account the results of the two previous problems, show that the spectrum of the number operator $\hat{N} = \hat{a}^+\hat{a}$, where $\hat{a}$ and $\hat{a}^+$ satisfy the commutation rule $\hat{a}\hat{a}^+ - \hat{a}^+\hat{a} = 1$, is the set of integer numbers.

Solution
We have seen that $\hat{N}$ has a non negative spectrum. This means that $\hat{N}$ possesses a lowest eigenvalue n_0 with $|n_0\rangle$ its eigenstate. This eigenstate $|n_0\rangle$ is such that

$$\hat{a}|n_0\rangle = 0 \ .$$

In fact, on the basis of the results of the previous problem, $\hat{a}|n_0\rangle$ should be eigenstate of $\hat{N}$ with eigenvalue $n_0 - 1$ but this is not possible because n_0 is the lowest eigenvalue of $\hat{N}$. Consequently the state $\hat{a}|n_0\rangle$ is not a good Fock state and we set it equal to 0. In addition, due to the fact that

$$\begin{aligned} \hat{N}|n_0\rangle &= n_0|n_0\rangle \\ &= \hat{a}^+\hat{a}|n_0\rangle = \hat{a}^+\left(\hat{a}|n_0\rangle\right) = \hat{a}^+(0) = 0 \end{aligned}$$

we find that $n_0 = 0$. Thus, the state $|0\rangle$, called vacuum state, is the eigenstate of $\hat{N}$ with eigenvalue 0, i.e.

$$\hat{N}|0\rangle = 0|0\rangle = 0 \ ,$$

but also

$$\hat{a}|0\rangle = 0 \ .$$

Due to this equation, it follows that the eigenstates of $\hat{N}$ are only those generated by applying m times the operator $\hat{a}^+$ on the vacuum state $|0\rangle$, namely

$$|m\rangle = \frac{1}{\sqrt{m!}}(\hat{a}^+)^m|0\rangle \ ,$$

where the factorial is due to the normalization. Finally, we notice that it has been shown in the previous problem that the state $|m\rangle$ has integer eigenvalue m.

Problem 2.4
Consider the following quantum Hamiltonian of the one-dimensional harmonic oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \, .$$

By using the properties of the annihilation operator

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) \, ,$$

determine the eigenfunction of the ground state of the system.

Solution
Let us observe the following property of the annihilation operator

$$\langle x|\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)\langle x| \, .$$

In addition, from

$$\hat{a}|0\rangle = 0 \, ,$$

we find

$$\langle x|\hat{a}|0\rangle = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)\langle x|0\rangle = 0 \, .$$

Introducing the characteristic harmonic length

$$l_H = \sqrt{\frac{\hbar}{m\omega}} \, ,$$

the dimensionless coordinate

$$\bar{x} = \frac{x}{l_H} \, ,$$

and the dimensionless eigenfunction

$$\bar{\psi}_n(\bar{x}) = l_H \, \psi_n(\bar{x}l_H) \, ,$$

we obtain

$$\left(\bar{x} + \frac{\partial}{\partial \bar{x}}\right)\bar{\psi}_0(\bar{x}) = 0 \, .$$

The solution of this first order differential equation is found by separation of variables:

$$\bar{x}\, d\bar{x} = -\frac{d\bar{\psi}_0}{\bar{\psi}_0} ,$$

from which

$$\bar{\psi}_0(\bar{x}) = \frac{1}{\pi^{1/4}} \exp\left(\frac{-\bar{x}^2}{2}\right) ,$$

having imposed the normalization

$$\int d\bar{x}\, |\bar{\psi}_0(\bar{x})|^2 = 1 .$$

Problem 2.5
Consider the quantum Hamiltonian of the two-dimensional harmonic oscillator

$$\hat{H} = \frac{\hat{p}_1^2 + \hat{p}_2^2}{2m} + \frac{1}{2} m\omega^2 (\hat{x}_1^2 + \hat{x}_2^2) .$$

By using the properties of the creation operators

$$\hat{a}_k^+ = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x}_k - \frac{i}{m\omega}\hat{p}_k\right) , \quad k = 1, 2$$

determine the eigenfunctions of the quantum Hamiltonian with eigenvalue $6\hbar\omega$.

Solution
The eigenvalues of the two-dimensional harmonic oscillator are given by

$$E_{n_1 n_2} = \hbar\omega(n_1 + n_2 + 1) ,$$

where $n_1, n_2 = 0, 1, 2, 3, \ldots$ are the quantum numbers.
The eigenstates $|n_1 n_2\rangle$ corresponding to the eigenvalue $6\hbar\omega$ are:

$$|50\rangle , \quad |41\rangle , \quad |32\rangle , \quad |23\rangle , \quad |14\rangle , \quad |05\rangle .$$

Thus the eigenfunctions to be determined are

$$\phi_{50}(x_1, x_2) , \quad \phi_{41}(x_1, x_2) , \quad \phi_{32}(x_1, x_2) , \quad \phi_{23}(x_1, x_2) , \quad \phi_{14}(x_1, x_2) , \quad \phi_{05}(x_1, x_2) .$$

In general, the eigenfunctions $\phi_{n_1 n_2}(x_1, x_2) = \langle x_1 x_2 | n_1 n_2 \rangle$ can be factorized as follows

$$\phi_{n_1 n_2}(x_1, x_2) = \psi_{n_1}(x_1)\, \psi_{n_2}(x_2)$$

where $\psi_{n_j}(x_j) = \langle x_j | n_j \rangle$, $j = 1, 2$. It is now sufficient to calculate the following eigenfunctions of the one-dimensional harmonic oscillator:

$$\psi_0(x) \ , \quad \psi_1(x) \ , \quad \psi_2(x) \ , \quad \psi_3(x) \ , \quad \psi_4(x) \ , \quad \psi_5(x) \ .$$

We observe that the creation operator of the one-dimensional harmonic oscillator satisfies this property

$$\langle x | \hat{a}^+ = \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{\hbar}{m\omega} \frac{\partial}{\partial x} \right) \langle x | \ ,$$

and consequently

$$\langle x | \hat{a}^+ | n \rangle = \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{\hbar}{m\omega} \frac{\partial}{\partial x} \right) \langle x | n \rangle = \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{\hbar}{m\omega} \frac{\partial}{\partial x} \right) \psi_n(x) \ .$$

Reminding that

$$\hat{a}^+ | n \rangle = \sqrt{n+1} | n+1 \rangle \ ,$$

we get

$$\langle x | n+1 \rangle = \frac{1}{\sqrt{n+1}} \langle x | a^+ | n \rangle \ ,$$

and the iterative formula

$$\psi_{n+1}(x) = \frac{1}{\sqrt{n+1}} \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{\hbar}{m\omega} \frac{\partial}{\partial x} \right) \psi_n(x) \ .$$

Now we introduce the characteristic harmonic length

$$l_H = \sqrt{\frac{\hbar}{m\omega}} \ ,$$

the dimensionless coordinate

$$\bar{x} = \frac{x}{l_H} \ ,$$

and the dimensionless wavefunction

$$\bar{\psi}_n(\bar{x}) = l_H \ \psi_n(\bar{x} l_H) \ ,$$

finding

$$\bar{\psi}_n(\bar{x}) = \frac{1}{\sqrt{2^n \, n!}} \left(\bar{x} - \frac{\partial}{\partial \bar{x}} \right)^n \bar{\psi}_0(\bar{x}) \ .$$

The function $\bar{\psi}_0(\bar{x})$ of the ground state is given by (see exercise 1.4)

$$\bar{\psi}_0(\bar{x}) = \frac{1}{\pi^{1/4}} \exp\left(\frac{-\bar{x}^2}{2}\right) .$$

Let us now calculate the effect of the operator

$$\left(\bar{x} - \frac{\partial}{\partial \bar{x}}\right)^n .$$

For $n = 1$:

$$\left(\bar{x} - \frac{\partial}{\partial \bar{x}}\right) \exp\left(\frac{-\bar{x}^2}{2}\right) = 2\bar{x} \ \exp\left(\frac{-\bar{x}^2}{2}\right) .$$

For $n = 2$:

$$\left(\bar{x} - \frac{\partial}{\partial \bar{x}}\right)^2 \exp\left(\frac{-\bar{x}^2}{2}\right) = 2(2\bar{x}^2 - 1) \ \exp\left(\frac{-\bar{x}^2}{2}\right) .$$

For $n = 3$:

$$\left(\bar{x} - \frac{\partial}{\partial \bar{x}}\right)^3 \exp\left(\frac{-\bar{x}^2}{2}\right) = (8\bar{x}^3 - 12\bar{x}) \ \exp\left(\frac{-\bar{x}^2}{2}\right) .$$

For $n = 4$:

$$\left(\bar{x} - \frac{\partial}{\partial \bar{x}}\right)^4 \exp\left(\frac{-\bar{x}^2}{2}\right) = (16\bar{x}^4 - 48\bar{x}^2 + 12) \ \exp\left(\frac{-\bar{x}^2}{2}\right) .$$

For $n = 5$:

$$\left(\bar{x} - \frac{\partial}{\partial \bar{x}}\right)^5 \exp\left(\frac{-\bar{x}^2}{2}\right) = (32\bar{x}^5 - 160\bar{x}^3 + 120\bar{x}) \ \exp\left(\frac{-\bar{x}^2}{2}\right) .$$

Finally, the eigenfunctions of H with eigenvalue $6\hbar\omega$ are linear combinations of functions $\phi_{n_1 n_2}(x_1, x_2)$, namely

$$\Phi(x_1, x_2) = \sum_{n_1 n_2} c_{n_1 n_2} \ \phi_{n_1 n_2}(x_1, x_2) \ \delta_{n_1+n_2,5} \ ,$$

where the coefficients $c_{n_1 n_2}$ are such that

$$\sum_{n_1 n_2} |c_{n_1 n_2}|^2 \ \delta_{n_1+n_2,5} = 1 \ .$$

Problem 2.6
Show that the coherent state $|\alpha\rangle$, defined by the equation

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \ ,$$

can be written as

$$|\alpha\rangle = \sum_{n=0}^{\infty} e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \ .$$

Solution
The coherent state $|\alpha\rangle$ can be expanded as

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \ .$$

Then we have

$$\hat{a}|\alpha\rangle = \hat{a} \sum_{n=0}^{\infty} c_n |n\rangle = \sum_{n=0}^{\infty} c_n \hat{a}|n\rangle = \sum_{n=1}^{\infty} c_n \sqrt{n}|n-1\rangle = \sum_{n=0}^{\infty} c_{n+1}\sqrt{n+1}|n\rangle$$

$$\alpha|\alpha\rangle = \alpha \sum_{n=0}^{\infty} c_n |n\rangle = \sum_{n=0}^{\infty} \alpha\, c_n |n\rangle \ .$$

Since, by definition

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \ ,$$

it follows that

$$c_{n+1}\sqrt{n+1} = \alpha\, c_n$$

from which

$$c_{n+1} = \frac{\alpha}{\sqrt{n+1}} c_n \ ,$$

namely

$$c_1 = \frac{\alpha}{\sqrt{1}} c_0 \ , \ c_2 = \frac{\alpha^2}{\sqrt{2!}} c_0 \ , \ c_3 = \frac{\alpha^3}{\sqrt{3!}} c_0 \ , \ \ldots$$

and in general

$$c_n = \frac{\alpha^n}{\sqrt{n!}} c_0 \ .$$

Summarizing

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_0 \frac{\alpha^n}{\sqrt{n!}} |n\rangle \ ,$$

where the parameter c_0 is fixed by the normalization

$$1 = \langle\alpha|\alpha\rangle = |c_0|^2 \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{n!} = |c_0|^2 \ e^{|\alpha|^2} \ ,$$

and consequently

$$c_0 = e^{-|\alpha|^2/2} \ ,$$

up to a constant phase factor.

Problem 2.7
Calculate the probability of finding the Fock state $|n\rangle$ in the vacuum state $|0\rangle$.

Solution
The probability p is given by

$$p = |\langle 0|n\rangle|^2 = \delta_{0,n} = \begin{cases} 1 \text{ if } n = 0 \\ 0 \text{ if } n \neq 0 \end{cases} \ .$$

Problem 2.8
Calculate the probability of finding the coherent state $|\alpha\rangle$ in the vacuum state $|0\rangle$.

Solution
The probability is given by

$$p = |\langle 0|\alpha\rangle|^2 \ .$$

Because

$$\langle 0|\alpha\rangle = \langle 0| \sum_{n=0}^{\infty} e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = \sum_{n=0}^{\infty} e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} \langle 0|n\rangle = e^{-|\alpha|^2/2} \frac{1}{\sqrt{0!}} = e^{-|\alpha|^2/2} \ ,$$

we get

$$p = e^{-|\alpha|^2} \ .$$

Problem 2.9
Calculate the probability of finding the coherent state $|\alpha\rangle$ in the Fock state $|n\rangle$.

Solution
The probability is given by

$$p = |\langle n|\alpha\rangle|^2 \, .$$

Because

$$\langle n|\alpha\rangle = \langle n| \sum_{m=0}^{\infty} e^{-|\alpha|^2/2} \frac{\alpha^m}{\sqrt{m!}} |m\rangle = \sum_{m=0}^{\infty} e^{-|\alpha|^2/2} \frac{\alpha^m}{\sqrt{m!}} \langle n|m\rangle = e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} \, ,$$

we get

$$p = e^{-|\alpha|^2} \, \frac{|\alpha|^{2n}}{n!} \, ,$$

which is the familiar Poisson distribution.

Problem 2.10
Calculate the probability of finding the coherent state $|\alpha\rangle$ in the coherent state $|\beta\rangle$.

Solution
The probability is given by

$$p = |\langle \beta|\alpha\rangle|^2 \, .$$

Because

$$\begin{aligned} \langle \beta|\alpha\rangle &= \langle \beta| \sum_{m=0}^{\infty} e^{-|\alpha|^2/2} \frac{\alpha^m}{\sqrt{m!}} |m\rangle = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} e^{-|\beta|^2/2} e^{-|\alpha|^2/2} \frac{(\beta^*)^n}{\sqrt{n!}} \frac{\alpha^m}{\sqrt{m!}} \langle n|m\rangle \\ &= e^{-(|\alpha|^2+|\beta|^2)/2} \sum_{n=0}^{\infty} \frac{(\beta^*\alpha)^n}{n!} = e^{-(|\alpha|^2+|\beta|^2)/2} \, e^{\beta^*\alpha} = e^{-(|\alpha|^2+|\beta|^2-2\alpha\beta^*)/2} \, , \end{aligned}$$

we get

$$p = e^{-|\alpha-\beta|^2} \, .$$

This means that two generic coherent states $|\alpha\rangle$ and $|\beta\rangle$ are never orthogonal to each other.

Further Reading

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