

Polynomial Time Algorithms for Bichromatic Problems

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Abstract. In this article, we consider a collection of geometric problems involving points colored by two colors (red and blue), referred to as bichromatic problems. The motivation behind studying these problems is two fold; (i) these problems appear naturally and frequently in the fields like Machine learning, Data mining, and so on, and (ii) we are interested in extending the algorithms and techniques for single point set (monochromatic) problems to bichromatic case. For all the problems considered in this paper, we design low polynomial time exact algorithms. These algorithms are based on novel techniques which might be of independent interest.

1 Introduction

In discrete and computational geometry one of the most important classes of problems are those involving points colored by two colors (red and blue), referred to as bichromatic problems. These problems have vast applications in the fields of Machine learning, Data mining, Computer graphics, and so on. For example, one natural problem in learning and clustering is separability problem [19, 21], where given a red and a blue set of points, the goal is to separate as many red (desirable) points as possible from the blue (non-desirable) points using geometric objects. Another motivation to study bichromatic problems is to extend the algorithms and techniques in discrete and computational geometry for single point set problems to bichromatic case (see the survey by Kaneko and Kano [25] and some more recent works [2–4, 10, 11]).

In this paper we consider two bichromatic problems that arise naturally in practice. Throughout the paper as coloring we refer to a function that maps points to the range $\{\text{red}, \text{blue}\}$. In the first problem, we are given two finite disjoint sets of points R and B in the plane, colored by red and blue, respectively. Denote the respective cardinality of R and B by n and m . Also let $T = R \cup B$. In the second problem, we are given a finite collection Q of pairs of points in the plane and we need to find a coloring of the points using red and blue such that certain optimality criterion is satisfied. For the sake of simplicity of exposition we assume that all the points are in general position. Assuming this, we formally define the two problems mentioned before.

Maximum Red Rectangle Problem. A rectangle (of arbitrary orientation) is called *red* if it does not contain any blue points in its interior¹. The *size* of a red rectangle is defined as the number of red points contained in it. The *maximum red rectangle problem* (MRR) is to find a red rectangle of maximum size. Such a rectangle will be referred to as a *maximum red rectangle*. We note that the version where the rectangles are constrained to be axes parallel is a special case of MRR.

Liu and Nediak [27] considered the axes parallel version of MRR and designed an algorithm that runs in $O(n^2 \log n + nm + m \log m)$ time and $O(n)$ space. Backer and Keil [8] improved this time bound to $O((n+m)\log^3(n+m))$ using a divide-and-conquer approach due to Aggarwal and Suri [1]. However, their algorithm uses $O(n \log n)$ space. For axes parallel squares, they designed an $O((n+m)\log(n+m))$ time algorithm. As far as we are concerned the general version of MRR was not considered before. However, two related problems involving rectangles of arbitrary orientation have been studied before. The first one is the largest empty rectangle problem (LER) where given a point set P , the goal is to find a rectangle of the maximum area which does not contain any point of P in its interior [1, 14, 15, 28, 29]. This problem can be solved in $O(|P|^3)$ time with $O(|P|^2)$ space [14]. The second problem is a bichromatic problem, where the goal is to find the rectangle that contains all the red points, the minimum number of blue points and has the largest area. This problem can be solved in $O(m^3 + n \log n)$ time [5].

Several variants of MRR have been studied in the past. Aronov and Har-Peled [6] considered the problem of finding a maximum red disk and gave a $(1-\epsilon)$ -factor approximation algorithm with $O((n+m)\epsilon^{-2}\log^2(n+m))$ expected running time. Eckstein *et al.* [22] considered a variant of MRR for axes parallel hyperrectangles in high dimensions. They showed that, if the dimension of the space is not fixed, the problem is NP-hard. However, they presented an $O(n^{2d+1})$ time algorithm, for any fixed dimension $d \geq 3$. Later, Backer and Keil [7] have significantly improved this time bound to $O(n^d \log^{d-2} n)$. Bitner *et al.* [12] have studied the problem of computing all circles that contain the red points and as few blue points as possible in its interior. In [18] Cortes *et al.* have considered the following problem: find a largest subset of $R \cup B$ which can be enclosed by the union of two not necessarily disjoint, axes-aligned rectangles R_1 and R_2 such that R_1 (resp. R_2) contains only red (resp. blue) points.

Maximum Coloring Problem. We are given a collection $Q = \{(a_1, b_1), \dots, (a_n, b_n)\}$ of pairs of points. Let $P = \cup_{i=1}^n \{a_i, b_i\}$. A coloring of the points in P is *valid* if for each pair of points in Q , exactly one point is colored by blue and the other point is colored by red. Now consider any valid coloring C . For any halfplane h , we denote the set of red points and blue points in h by $h_r(C)$ and $h_b(C)$, respectively. Let $\mathcal{H}(C)$ be the set of all halfplanes h such that $|h_b(C)| = 0$. Define $\eta(C) = \max_{h \in \mathcal{H}(C)} |h_r(C)|$. The *Maximum Coloring Problem* (MaxCol) is to find a valid coloring C that maximizes $\eta(C)$.

¹ The red rectangle may contain blue points on its boundary.

Problems related to coloring of points have been studied in literature. Even *et al.* [23] defined Conflict-free coloring where given a set X of points and a set of geometric regions, the goal is to color the points with minimum number of colors so that for any region r , at least one of the points of X that lie in r has a unique color. This problem have further been studied in offline [16, 26] and online [9, 17] settings.

Our Results. We give exact algorithms for both of the problems considered in this paper. In Sect. 2, we design an algorithm for MRR that runs in $O(g(n, m) \log(m + n) + n^2)$ time with $O(n^2 + m^2)$ space, where $g(n, m) \in O(m^2(n + m))$ and $g(n, m) \in \Omega(m^2 + mn)$. To solve the problem in general case, we need to solve an interesting subproblem which can be of independent interest. The nontriviality of this problem arises mainly due to the arbitrary orientations of the rectangles.

In Sect. 3, we show that MaxCol can be solved in $O(n^{\frac{4}{3}+\epsilon} \log n)$ time. In particular, we show a linear time reduction from MaxCol to a problem and design an algorithm for the latter problem with the desired time complexity.

2 Maximum Red Rectangle Problem

Before stepping into the general case let us consider the axes parallel version. We note that the ideas mentioned here for the axes parallel case have been noted before in [8].

2.1 The Axes Parallel Case

Note that the number of red rectangles can be infinite. But for the sake of computing a maximum red rectangle we can focus on the following set. Consider the set S of red rectangles such that each side of any such rectangle contains a point of B or is unbounded. We note that the candidate set for the axes parallel version of Largest Empty Rectangle problem (LER) on B is exactly S . Using this connection we will use several results from the literature of LER. Namaad *et al.* [15] proved that $|S| = O(m^2)$ and in expected case $|S| = O(m \log m)$. Orlowski [29] has designed an algorithm that computes the set S in $O(|S|)$ time. The algorithm requires two sorted orderings of B , one with respect to x -coordinates and the other with respect to y -coordinates.

To compute a maximum red rectangle we use Orlowski's algorithm. In each iteration when the algorithm computes a rectangle, we make a query for the number of red points inside the rectangle. Lastly, we return the rectangle that contains the maximum number of red points. Now, using $O(n \log n)$ preprocessing time and space, one can create a data structure that can handle orthogonal rectangular query in $O(\log n)$ time. Hence the axes parallel version of MPP can be solved in $O(|S| \log n + n \log n + m \log m)$ time with $O(n \log n + m)$ space required.

2.2 The General Case

For a point p , we denote its x and y coordinate by x_p and y_p , respectively. A rectangle is said to be *anchored* by a point set P if it contains the points of P on its boundary.

Like in the axes parallel case, in general case also the number of red rectangles can be infinite. However, we will define a finite subset of those rectangles such that the subset contains a maximum red rectangle. We note that this is a key step based on which we design our algorithm. The approach is not that subtle like the one for axes parallel case.

Consider the set of red rectangles C with the following two properties. For any rectangle $T \in C$, (i) at least one side of T contains two points p, q such that $p \in B$ and $q \in R \cup B$; and (ii) each of the other sides of T either contains a point of B or is unbounded. In the full version of this paper, we prove the following two lemmas:

Lemma 1. $|C| = O(m^2(n + m))$ and $|C| = \Omega(m^2 + mn)$.

Lemma 2. *The set C as defined above contains a maximum red rectangle.*

We compute all the rectangles of C and return one that contains the maximum number of red points. Given two points p, q such that $p \in B$ and $q \in R \cup B$, we design a subroutine to compute the rectangles of C such that each of them contains p and q on a single side.

2.2.1 The Subroutine

We are given two points p, q such that $p \in B$ and $q \in R \cup B$. We would like to compute the set of rectangles C_{pq} anchored by p and q . Without loss of generality, suppose the line through p and q is on the x -axis. We show how to compute the subset C' of rectangles of C_{pq} whose interior are lying above the x axis. By symmetry, using a similar approach we can compute the rectangles of C_{pq} whose interior are lying below the x axis. We consider all the blue points strictly above the x axis, and we denote this set by B_1 .

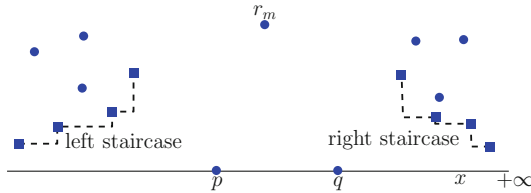


Fig. 1. Staircase points are shown by squares. (Color figure online)

Now consider the points of B_1 whose x coordinates are between x_p and x_q (if any). Let r_m be a point having minimum y coordinate among those points.

Let B' be the subset of points of B_1 having y coordinate less than y_{r_m} . Define the *left staircase* B_l to be the subset of B' , such that for any point $r_2 \in B'$, r_2 belongs to B_l if $x_{r_2} < x_p$ and $\nexists r_1 \in B'$ such that $x_{r_2} < x_{r_1} < x_p$ and $y_{r_2} \geq y_{r_1}$ (see Fig. 1). Similarly, we define the *right staircase* B_r as follows. For any r_2 of B' , $r_2 \in B_r$ if $x_q < x_{r_2}$ and $\nexists r_1 \in B'$ such that $x_q < x_{r_1} < x_{r_2}$ and $y_{r_1} \leq y_{r_2}$ (see Fig. 1). Also let $B_a = B_l \cup B_r \cup \{r_m\}$. Now we have the following observation.

Observation 3. *For any rectangle $T \in C'$, T contains p, q in one of its sides and each of the other sides of T either contains a point of B_a or is unbounded.*

Orlowski [29] designed a linear time algorithm for finding the set of rectangles that has one side on x -axis and each of the other sides either contains a point of a staircase or is unbounded. Our subroutine at first computes the rectangle anchored by p, q and r_m (if any). Then it uses Orlowski's algorithm to compute the remaining rectangles of C' . It also computes the number of red points inside each rectangle it scans by making a query and returns the rectangle that contains the maximum number of red points. The following theorem is due to Goswami *et al.* [24].

Theorem 4. [24] *For any set of n points, using $O(n^2)$ preprocessing time and space, one can create a data structure that can handle rectangular (of any arbitrary orientation) query in $O(\log n)$ time.*

We note, that given two sorted lists of blue points in non-decreasing order of x and y coordinates, respectively, by Theorem 4 the subroutine runs in $O(|C'| \log n)$ time. Hence we have the following lemma.

Lemma 5. *Using $O(n^2 + m \log m)$ preprocessing time and $O(n^2 + m)$ space a maximum red rectangle of C_{pq} can be computed in $O(|C_{pq}| \log n)$ time.*

Note that this subroutine can be trivially used for finding a maximum red rectangle in C by calling it for each $p \in B$ and $q \in R \cup B$, and by returning a maximum red rectangle among all such choices. But, as we need to compute the sorted lists in every rotated plane (defined by p and q) we need $O(m^2(n + m) \log m)$ time in total just for sorting. Thus this trivial algorithm runs in $O(m^2(n + m) \log m + n^2 + |C| \log n)$ time. In the next subsection, we improve this time complexity by using a careful observation that rescues us from the burden of sorting in every rotated plane.

2.2.2 The Improved Algorithm

The rectangles of C are oriented at angles with respect to the x -axis in the range $[0, 360)$. Moreover, there are $O(m(n + m))$ orientations (or angles) of these rectangles each defined by a point of B and a point of $R \cup B$. Note that we need the sorted lists of blue points in each such orientation. We use a novel technique for maintaining the sorted lists using some crucial observations. We note that this problem itself might be of independent interest.

Consider two points $a, b \in B$ and an angle θ . We want to find the ordering of these two points with respect to x coordinates, in the plane oriented counterclockwise at the angle θ . Let us denote this plane by P_θ . Also let B_x^θ (resp. B_y^θ) be the set of blue points in P_θ sorted in increasing order of x (resp. y) coordinates. The following lemma explains how the relative ordering of two points gets changed with changes in plane orientation.

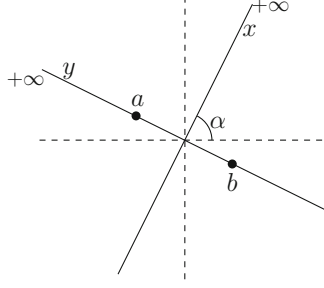


Fig. 2. The plane generated by rotation of axes by an angle α where $x_a = x_b$.

Lemma 6. *For any two points $a, b \in B$, there exists an angle $\phi < 180$ such that $x_a = x_b$ in P_ϕ and $P_{\phi+180}$, and exactly one of the following is true,*

- (i) $x_a < x_b$ in P_θ for $\theta \in [0, \phi) \cup (\phi + 180, 360)$ and $x_a > x_b$ in P_θ for $\theta \in (\phi, \phi + 180)$
- (ii) $x_a > x_b$ in P_θ for $\theta \in [0, \phi) \cup (\phi + 180, 360)$ and $x_a < x_b$ in P_θ for $\theta \in (\phi, \phi + 180)$

Proof. Denote the line segment connecting a and b (w.r.t P_0) by l . We translate a, b in a way so that the midpoint of l is now at the origin. Note that translation does not change the ordering of the points. Let f_a (resp. f_b) be the function such that $f_a(\theta)$ (resp. $f_b(\theta)$) is the x -coordinate value of a (resp. b) in P_θ . Now consider a continuous rotation process of the axes (or the plane) in counterclockwise direction with respect to the origin, keeping the points a, b fixed. If $f_a(0)$ is equal to $f_b(0)$, then a and b are on the y -axis in P_0 . Let $\alpha = 0$ in this case. Otherwise, a and b are on the opposite sides of the y -axis. In this case, as we rotate the axes the value of $|f_a(\theta) - f_b(\theta)|$ becomes zero at some angle $\theta = \alpha$ when both a and b lie on the y -axis (see Fig. 2). Note that if $f_a(0) < f_b(0)$, $f_a(\theta) < f_b(\theta)$ for all $\theta \in [0, \alpha)$, as a and b do not change their sides with respect to the y -axis during this rotation process. Similarly, if $f_a(0) > f_b(0)$, $f_a(\theta) > f_b(\theta)$ for all $\theta \in [0, \alpha)$. Now in both of the cases ($f_a(0) = f_b(0)$ or $f_a(0) \neq f_b(0)$), after a slight rotation a and b will be on the opposite sides of the y -axis. As we rotate further the value of $|f_a(\theta) - f_b(\theta)|$ again becomes zero at some angle $\theta = \beta$ when again both a and b lie on the y -axis. In the first case, either $f_a(\theta) > f_b(\theta)$ for all $\theta \in (\alpha, \beta)$, or $f_a(\theta) < f_b(\theta)$ for all $\theta \in (\alpha, \beta)$. In the second case, if a was on the left (resp. right) of the y -axis in P_θ for $\theta \in [0, \alpha)$, now a is on the right (resp. left) side of

the y -axis. Thus if $f_a(\theta) < f_b(\theta)$ for $\theta \in [0, \alpha)$, $f_a(\theta) > f_b(\theta)$ for all $\theta \in (\alpha, \beta)$. Similarly, if $f_a(\theta) > f_b(\theta)$ for $\theta \in [0, \alpha)$, $f_a(\theta) < f_b(\theta)$ for all $\theta \in (\alpha, \beta)$. As we rotate past β in both of the cases, a and b will again be on the opposite sides of the y -axis. If $f_a(\theta) < f_b(\theta)$ for $\theta \in (\alpha, \beta)$, $f_a(\theta) > f_b(\theta)$ for all $\theta \in (\beta, 360)$. Similarly, if $f_a(\theta) > f_b(\theta)$ for $\theta \in (\alpha, \beta)$, $f_a(\theta) < f_b(\theta)$ for all $\theta \in (\beta, 360)$.

We let $\phi = \alpha$. Note that $f_a(\theta)$ can be equal to $f_b(\theta)$ for $\theta \in [0, 360)$ only when y -axis is aligned with the line through a and b . Also note that during a complete 360° rotation of the axes this can happen exactly twice. Moreover, the difference between the corresponding angles should be 180 , i.e. $\beta = \phi + 180$. This completes the proof of the lemma. \square

Similarly, we get the following lemma for ordering of two points with respect to y -coordinates.

Lemma 7. *For any two points $a, b \in B$, there exists an angle $\phi < 180$ such that $y_a = y_b$ in P_ϕ and $P_{\phi+180}$, and exactly one of the following is true,*

- (i) $y_a < y_b$ in P_θ for $\theta \in [0, \phi) \cup (\phi + 180, 360)$ and $y_a > y_b$ in P_θ , for $\theta \in (\phi, \phi + 180)$
- (ii) $y_a > y_b$ in P_θ for $\theta \in [0, \phi) \cup (\phi + 180, 360)$ and $y_a < y_b$ in P_θ , for $\theta \in (\phi, \phi + 180)$

We refer to the angles ϕ and $\phi + 180$ in Lemmas 6 and 7 as *critical angles* with respect to a and b . Let E_1 (resp. E_2) be the set of all critical angles corresponding to pairs of blue points with respect to x (resp. y) coordinates. An element of E_1 (resp. E_2) will be referred to as an event point of first (resp. second) type. Let A be the set of angles (or orientations) defined by pairs of points (p, q) such that $p \in B$ and $q \in R \cup B$. We refer to an element of A as an event point of third type. We construct the set $E = E_1 \cup E_2 \cup A$ of event points. Also we add the angle 0 to E as an event point of both first and second type.

Lemmas 6 and 7 hint the outline of an algorithm. We sort the list E of event points in increasing order of their angles. We process the event points in this order. To start with we construct the set B_x^0 (resp. B_y^0) at the very first event point which is 0 . In case the algorithm encounters an event point of the first (resp. second) type it swaps the corresponding two points in the current sorted list B_x^ϕ (resp. B_y^ϕ), where $\phi \in [0, 360)$. At an event point of the third type it makes a call to the subroutine of Sect. 2.2.1. Lastly, a rectangle that contains the maximum red points over all orientations is returned as the solution.

The correctness of this algorithm depends on the correctness of maintaining the sorted lists in all the orientations. As the sorted order with respect to the x (resp. y) coordinates do not change in between two consecutive event points of the first (resp. second) type, it is sufficient to prove the following lemma.

Lemma 8. *At each event point ϕ of the first (resp. second) type, the correct sorted list B_x^ϕ (resp. B_y^ϕ) is maintained.*

Proof. We prove this lemma for the event points of the first type. The proof for the event points of the second type is similar. We use induction argument on the

event points. In the base case for $\phi = 0$ we correctly compute the set B_x^0 . Now assume that for $\alpha \in E_1$ and any $\phi \leq \alpha$ such that $\phi \in E_1$, the correct sorted list B_x^ϕ is maintained. Let β be the successor of α in E_1 . Let a and b be the two points corresponding to β . By Lemma 6, the ordering of x_a and x_b should be different in P_ψ and P_θ for $\psi \in (\alpha, \beta)$ and $\theta \in (\beta, \beta + 180)$. Thus our algorithm rightly swaps a and b in B_x^α to get B_x^β . The location of any other point remains same. Now if there is any point c in B_x^α in between a and b , then the order of a and c (resp. b and c) also gets changed and we end up computing a wrong list. Thus it is sufficient to prove the following claim.

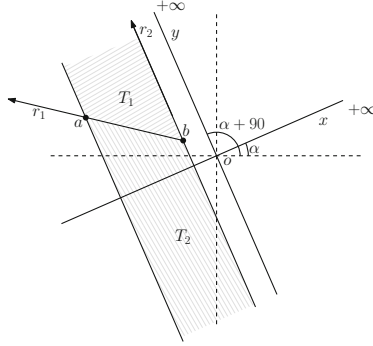


Fig. 3. The sets T_1 and T_2 . T_1 is contained in the wedge defined by r_1 and r_2 .

Claim 9. *a and b must be consecutive in B_x^α .*

Proof. We consider the case where $\beta < 180$. The other case is symmetric. Now consider the plane P_α . Without loss of generality assume that $x_a \leq x_b$ in this plane. Suppose there is a point c in B_x^α in between a and b . If $x_a = x_b$, then $x_a = x_c$. But this violates the general position assumption. Thus we consider the case where $x_a < x_b$. Let l_a (resp. l_b) be the line passing through a (resp. b) which is parallel to the y -axis. Let T be the strip between the two lines l_a and l_b , i.e. T contains all the points p such that $x_a \leq x_p \leq x_b$. Then c must be contained in T . Note that the line segment that connects a and b divides T into two sets T_1 and T_2 , above and below \overline{ab} respectively (see Fig. 3). Suppose c is in T_1 . Consider two rays r_1 and r_2 both originated at b that pass through the point a and $(x_b, y_b + 1)$, respectively. Note that r_1 and r_2 are oriented at angles $\beta + 90^\circ$ and $\alpha + 90^\circ$, respectively. Also the wedge defined by r_1 and r_2 with angle less than 180° contains T_1 (see Fig. 3). Thus as c is in T_1 there is a critical angle θ with respect to b and c such that $\alpha \leq \theta < \beta$. If $\alpha < \theta < \beta$, we get a contradiction to the assumption that β is the next event point in E_1 . If θ is equal to α , $x_c = x_b$. As $c \in T_1$, $y_c > y_b$. Thus in any plane P_θ with $\theta \in (\alpha, \beta)$, $x_b < x_c$. Hence if c appears before b in B_x^α , we get a contradiction to the assumption that B_x^α is the correct sorted list. In case c is in T_2 , one can get similar contradiction. Hence the claim follows. \square

Now consider the time complexity of our algorithm. Sorting of $O(m^2 + mn)$ event points takes $O((m^2 + mn) \log(m + n))$ time. Handling of any event point of first (resp. second) type takes constant time. By Lemma 5, handling of all the event points of third type takes in total $O(|C| \log n)$ time. As $|C| = \Omega(m^2 + mn)$ we get the following theorem.

Theorem 10. *MPP can be solved in $O(|C| \log(m+n) + n^2)$ time with $O(n^2 + m^2)$ space required.*

3 Maximum Coloring Problem

Recall, that we are given the collection $Q = \{(a_1, b_1), \dots, (a_n, b_n)\}$ of pairs of points and $P = \cup_{1 \leq i \leq n} \{a_i, b_i\}$. A coloring of the points in P is valid if for each pair of points in Q , exactly one point is colored by red and the other is colored by blue. For any such valid coloring C and any half plane h , we denote the number of red points and the number of blue points in h by $h_r(C)$ and $h_b(C)$ (see Fig. 4). For any valid coloring C , $\eta(C) = \max |h_r(C)|$, where the maximum is taken over all halfplanes h for which $|h_b(C)| = 0$.

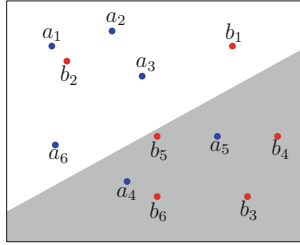


Fig. 4. An example of a valid coloring. For the halfplane h shown in grey, $h_r(C) = 4$ and $h_b(C) = 2$. (Color figure online)

In the Maximum Coloring Problem (MaxCol), the objective is to find a coloring C^* which maximizes $\eta()$ over all valid colorings. Next we show that a maximum valid coloring can be found by solving the following problem.

Problem 1. *Given a collection $Q = \{(a_1, b_1), \dots, (a_n, b_n)\}$ of pair of points find a halfplane that contains at most one point from each pair and contains maximum number of points from P where $P = \cup_{1 \leq i \leq n} \{a_i, b_i\}$.*

The following lemma shows that it is sufficient to solve Problem 1 to get a solution for MaxCol.

Lemma 11. *Suppose there exist an algorithm \mathcal{A} which finds an optimal solution to Problem 1 in time $T(n)$. Then MaxCol can be solved in $T(n) + O(n)$ time.*

Proof. Let the Algorithm \mathcal{A} outputs the halfplane h^* and $|h^* \cap P| = k$. We prove the claim that given h^* , we can find a valid coloring C^* , where $\eta(C^*) = k$, which maximizes $\eta()$ over all valid colorings.

We define C^* as follows. Now for each pair (a_i, b_i) such that $|h^* \cap \{a_i, b_i\}| = 1$, we color the point in $h^* \cap \{a_i, b_i\}$ by red and the point in $h^* \setminus \{a_i, b_i\}$ by blue. For each pair (a_i, b_i) such that $|h^* \cap \{a_i, b_i\}| = 0$, we arbitrarily color one point by red and the other point by blue. Given h^* such a coloring can be found in $O(n)$ time.

Clearly this coloring is a valid coloring. Next we show that this is maximum as well. Suppose C^* is not maximum and there exists a valid coloring C such that $\eta(C^*) < \eta(C)$. Let h_C be the halfplane corresponding to C that contains only red points and $|P \cap h_C| = \eta(C)$. But then h_C is a half plane such that h_C contains at most one point from each pair in Q and $|P \cap h_C| > |P \cap h^*|$ which contradicts the optimality of h^* . \square

Instead of directly solving Problem 1, we consider its dual. We use the standard point/line duality that preserves the above/below relationship between points and lines. Let $\mathbb{L} = \{(l_1, \bar{l}_1), \dots, (l_n, \bar{l}_n)\}$ be the collection of the corresponding n pairs of lines in the dual space and let L be the set of those $2n$ lines. Also let A be the set of points p such that for each pair in \mathbb{L} , at most one line of the pair lie below p . For any $p \in A$, let $\nu(p)$ denote the number of lines in L that lie below p . Similarly, let B be the set of points p such that for each pair in \mathbb{L} , at most one line of the pair lie above p . For any $p \in B$, let $\nu(p)$ denote the number of lines in L that lie above p . Using duality, we get the following problem.

Problem 2. *Given a collection \mathbb{L} of n pairs of lines in the plane, find a point in $A \cup B$ that maximizes the function $\nu()$.*

Here we describe how to find a point in A that maximizes $\nu()$. The point in B which maximizes $\nu()$ can be found similarly. More specifically we solve the following decision problem.

Problem 3. *Given a collection \mathbb{L} of n pairs of lines in the plane and an integer k , does there exist a point p in A such that $\nu(p) = k$.*

If one can solve this decision version in time $T'(n)$, then Problem 2 can be solved in $O(T'(n) \log n)$ time by using a simple binary search on the values of k . Observe, that for an “YES” instance of Problem 3, the point p must be on the k -level in the arrangement of those $2n$ lines (the k -level of an arrangement of a set of lines is the polygonal chain formed by the edges that have exactly k other lines strictly below them). Thus it is sufficient to compute the k -level and decide if there is a point p on it such that $p \in A$. Our approach is the following.

Let Γ_k be the k -level. We traverse the vertices of Γ_k in sorted order of their x -coordinates. Throughout the traversal we maintain a list $Arr[]$ of size n , where $Arr[j]$ denotes the number of lines from $\{l_j, \bar{l}_j\}$ that are currently below the k -level. Note that the value of $Arr[j]$ can be 0, 1 or 2. We also maintain an integer n_b which denotes the number of pairs of lines currently below Γ_k . In other words n_b is the number of 2's in $Arr[]$. We update $Arr[]$ and n_b at each vertex of Γ_k . If

a line belonged to a pair $\{l_j, \bar{l}_j\}$ leaves Γ_k , we reduce the value of $Arr[j]$ by one. Moreover, if the value of $Arr[j]$ was 2 before, we reduce n_b by one. Similarly, if a line belonged to a pair $\{l_j, \bar{l}_j\}$ enters Γ_k , we increase the value of $Arr[j]$ by one. If $Arr[j]$ becomes 2, we also increase n_b by one. At any point during the traversal if n_b becomes zero, we report yes. Otherwise we report no at the end.

Chan [13] designed an algorithm that computes the k -level in an arrangement of n lines in the plane in $O(n \log b + b^{1+\epsilon})$ time. From the result of Dey [20] we know $b = O(n(k+1)^{\frac{1}{3}})$. Thus k -level can be computed in $O(n \log n + nk^{\frac{1}{3}})$ time. Hence we have the following lemma.

Lemma 12. *Problem 3 can be solved in $O(n \log n + n^{1+\epsilon} k^{\frac{1+\epsilon}{3}})$ time.*

Hence by using a binary search on the values of k we have the following result.

Theorem 13. *Problems 1 and 2 can be solved in $O(n^{\frac{4}{3}+\epsilon} \log n)$ time.*

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