Chapter 2
Classical Clocks in General Relativity

This chapter discusses the regime of point-like systems with internal degrees of freedom in general relativity. First, a heuristic derivation of the framework is given and the focus is on the consistency and general properties of the formalism. It is shown that the framework has a well-behaved action, its own symmetry group and that in the non-relativistic limit it reduces to the Newtonian theory. The interpretation of the framework as describing ideal clocks is then presented. The second part of the chapter sketches a derivation of the composite particle framework. It shows how the framework naturally arises as a description of a generic many particle system that is sufficiently well-localised. Crucial approximations involved in the derivation are quantitatively discussed. Finally, relevance of the framework for tests of general relativity is reviewed in the light of time dilation and redshift experiments.

For simplicity, space-time is assumed to be static, with metric tensor $g_{\mu\nu}$ satisfying $g_{0i} = g_{i0} = 0$ and $g_{ij} = g_{ji}$ for $i, j = 1, 2, 3$, metric is dimensionless with signature $(- + ++)$.

2.1 Point Particles with Internal Degrees of Freedom

2.1.1 Hamiltonian

In general relativity a point particle following a world line $x^\mu(t)$ (where, for now, t shall be considered an arbitrary parameterisation of the world line) can be described by a covariant momentum four-vector $p_\mu(x)$, where $x \equiv x^\mu$, and this short notation will be used hereafter (for coordinates as well as for the momenta). Energy of the

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Section 2.1.5 includes excerpts from *Quantum formulation of the Einstein Equivalence Principle*, M. Zych and Č. Brukner, arXiv:1502.00971 (2015).

M. Zych, *Quantum Systems Under Gravitational Time Dilation*,
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particle is described by the component p_0 and the three-momentum—by components p_j, $j = 1, 2, 3$. From the covariance of the momentum four-vector p_μ follows that the scalar product $p_\mu p^\mu = \sum_{\mu\nu} p_\mu(x) g^{\mu\nu}(x) p_\nu(x)$ is a coordinate invariant quantity. In particular, in the coordinates in which the particle is at rest $p'_j = \frac{\partial x^\nu}{\partial x'_j} p_\nu = 0$ for $j = 1, 2, 3$ this invariant reads:

$$p_\mu p^\mu = p'_0 g'^{00} p'_0 \tag{2.1}$$

(the x-dependence of the quantities will hereafter be omitted whenever it does not lead to ambiguities). The time component x'^0 is not fixed by the requirement that the particle is at rest with respect to this new coordinates. Taking this component to be the proper time along the particle's world line results in $g'^{00} = \eta^{00} = -1$, where $\eta_{\mu\nu}$ is the Minkowski metric. This natural choice of the time parameter will be adopted hereafter.

The total rest frame energy is given by cp'_0. It comprises not only the energy stemming from the rest mass mc^2 of the system but also any binding or kinetic energies of the internal degrees of freedom and thus also the particle's internal Hamiltonian H_{int}. It can thus be written as

$$cp'_0 = mc^2 + H_{int} \equiv H_{rest}.$$

On the other hand, p_0 describes dynamics of the particle with respect to the initial, arbitrary, time coordinate and includes energy of the internal as well as the external degrees of freedom. It constitutes the total Hamiltonian of the system relative to the arbitrary but fixed coordinates and will be denoted by $H \equiv cp_0$. From the Eq. (2.1) the total Hamiltonian of a point-like relativistic particle with internal degrees of is found to be

$$H = \sqrt{\frac{-c^2 p_j p^j + g'^{00} H_{rest}^2}{g^{00}}}$$

Incorporating that $g'^{00} = -1$ in the "primed" coordinate system, and that $g_{00} = (g^{00})^{-1}$ for a static metric, the Hamiltonian takes the form

$$H = \sqrt{-g_{00}(c^2 p_j p^j + H_{rest}^2)}. \tag{2.2}$$

The Hamiltonian in Eq. (2.2) generalises that of an elementary massive particle of mass m, which reads $H_m = \sqrt{-g_{00}(c^2 p_j p^j + mc^2)}$, by including the total internal mass-energy $mc^2 + H_{int}$ rather than just the rest mass parameter mc^2. Systems that can be described with the formalism of point-like particles with internal degrees of freedom include atoms or molecules (see also Sect. 2.2), for which large contribution to the rest mass is given by binding energies between atoms, nucleons, quarks, etc. (from the perspective of the standard model of particle physics all masses reduce to the energy of the Higgs field). Note, that in general relativity, there is no unique distinction between the “mass” and “energy” and the above can be seen as an expression of the mass-energy equivalence. It will be discussed in the Sect. 2.1.5 how the definition of the mass and the split between mass and energy arise in the non-relativistic limit. Such a split depends on the context in which the system is discussed (energy scale, type and precision of measurements, etc.) and entails that the lowest energy of the internal Hamiltonian H_{int} is zero.

2.1.2 Lagrangian

The regime of point-like composed particles is as an effective approximation to the underlying elementary physics (and the next section will discuss this in detail), therefore, it is worth showing that such an effective description presents a fully consistent theory. Below the Lagrangian for a point-like relativistic system with internal degrees of freedom is derived from the Hamiltonian in Eq. (2.2). It is then shown that such a Lagrangian gives rise to a scalar action and thus to covariantly conserved energy-momentum tensor, consistent with the Hamiltonian derived in Eq. (2.2).

Hamiltonian of a system with N degrees of freedom is defined over a phase space $(x_1, .., x_N; p_1, \ldots, p_N)$, whereas Lagrangian is described in a configuration space $(x_1, .., x_N; \dot{x}_1, .., \dot{x}_N)$, where $\dot{x}_n^\mu \equiv \frac{dx_n^\mu}{dt}$, $n = 1, \ldots, N$. Given a Hamiltonian $H(x_1, .., x_N; p_1, \ldots, p_N)$ its corresponding Lagrangian $L(x_1, .., x_N; \dot{x}_1, .., \dot{x}_N)$, giving fully equivalent descriptions of the system, is obtained via a Legendre transform:

$$L = \sum_n p_{ni}\dot{x}_n^i - H; \tag{2.3}$$

where $\dot{x}_n^i$ and p_{ni} are 3-velocities and 3-momenta, respectively, $i = 1, 2, 3$, and

$$\dot{x}_n^i := \frac{\partial H}{\partial p_{ni}}. \tag{2.4}$$

Velocity $\dot{x}_n^i$ is called canonically conjugate to the momentum p_{ni} and by the virtue of Eq. (2.4) is defined with respect to the same time parameter with respect to which the Hamiltonian H is defined.

The Hamiltonian of a point-like particle with internal degrees of freedom $H = \sqrt{\frac{-c^2 p_j p^j + g'^{00} H_{rest}^2}{g^{00}}}$, in Eq. (2.2), describes a system with an external degree of freedom, with momentum p_i, and with N internal degrees of freedom, whose coordinates and momenta will be denoted by q_k and w_k, respectively, $k = 1, \ldots, N$. The dynamics of these internal degrees of freedom is given by internal energy, $H_{rest} \equiv H_{rest}(q_k, w_k)$, which implies that such defined w_k are defined with respect to the *rest frame* time coordinate. From Eq. (2.4), velocity $\dot{x}^j$ conjugate to the external momentum p_j of the Hamiltonian H above reads

$$\dot{x}^j = \frac{-c^2 p^j}{g^{00} H} \tag{2.5}$$

and velocities $\dot{q}_k$ canonically conjugate to w_k read

$$\dot{q}_k = \frac{g'^{00} H_{rest}}{g^{00} H} \frac{\partial H_{rest}}{\partial w_k}. \tag{2.6}$$

From Eqs. (2.3), (2.5) and (2.6) follows

$$L = -\frac{c^2 p^j p_j}{H g^{00}} + \sum_k \frac{g'^{00} H_{rest}}{g^{00} H} \frac{\partial H_{rest}}{\partial w_k} w_k - H.$$

Note, that the Legendre transform of the rest frame Hamiltonian H_{rest} with respect to internal momenta w_k is by definition the rest frame Lagrangian

$$L_{rest} = \sum_k \frac{\partial H_{rest}}{\partial w_k} w_k - H_{rest} \tag{2.7}$$

and thus

$$L = \frac{g'^{00} H_{rest}}{g^{00} H} L_{rest}$$

From the above, Eq. (2.5) and from the explicit form of H follows that

$$c \frac{g'^{00} H_{rest}}{g^{00} H} = \sqrt{g'^{00}(g_{00} + \dot{x}^j \dot{x}_j)} = \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}. \tag{2.8}$$

where the last equality incorporates that $g'^{00} = -1$ and a time coordinate $x^0 = ct$ has been introduced such that $\dot{x}^0 = \frac{\partial H}{\partial p_0} = c$, which guarantees that the two pictures are indeed an equivalent description of the same physics and with respect to the same coordinates. Collecting the two formulas above, the final form of the Lagrangian follows:

$$L = L_{rest} \frac{1}{c} \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \equiv L_{rest} \dot{\tau}, \tag{2.9}$$

where it has been used that $\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}/c$ is nothing but the derivative of the proper time element $d\tau := \sqrt{-g_{\mu\nu} dx^\mu dx^\nu}/c$ along the system's world line $x^\mu(t)$ with respect to the time parameter t.

Lagrangian L in Eq. (2.9) generalises that of a structureless massive particle with mass m, which reads $L_m = -mc^2 \dot{\tau}$. In particular, the rest frame Lagrangian L_{rest} has the form

$$L_{rest} = -mc^2 + L_{int},$$

which, follows from Eq. (2.7) with $H_{rest} = mc^2 + H_{int}$ and L_{int} arises as the Legendre transform of H_{int}. In a full analogy to the Hamiltonian description in Sect. 2.1.1 (see Eq. (2.2) and comments thereafter) the internal dynamics of a composed particle comprises the static part $-mc^2$ and the dynamical part L_{int}.

The action S corresponding to the Lagrangian L reads $S = \int L dt$ and using Eq. (2.9) it can be written as

$$S = \int L_{rest} d\tau. \tag{2.10}$$

The rest frame Lagrangian L_{rest} defined in Eq. (2.7) is a scalar (by definition, since the rest energy H_{rest} was defined as scalar) as well as the proper time element—and thus so is S in Eq. (2.10). This entails that the energy-momentum tensor $T^{\mu\nu}(x)$ of the system is covariantly conserved and that equations of motion resulting from such an action are generally covariant, see e.g. [1] for a general reference.

2.1.3 Energy-Momentum Tensor

Below an explicit form of the energy-momentum tensor of a point-like relativistic particle with internal dynamics is derived. The modern definition of the energy momentum tensor of a system with the Lagrangian L reads

$$T^{\mu\nu}(x) := \frac{-2}{\sqrt{-g}} \frac{\delta L}{\delta g_{\mu\nu}(x)}, \tag{2.11}$$

where $\frac{\delta}{\delta g_{\mu\nu}(x)}$ denotes a functional derivative with respect to the field $g_{\mu\nu}(x)$ and $g := \mathrm{Det} g_{\mu\nu}$ (determinant of the matrix $g_{\mu\nu}$).

In order to obtain $T^{\mu\nu}$ for the Lagrangian $L = L_{rest}\dot{\tau}$ note, that the internal Lagrangian depends on the internal variables q_k and on their velocities with respect to the proper time τ: $L_{rest} = L_{rest}(q_k, \frac{q_k}{d\tau})$. This is evident from its definition in Eq. (2.7) and from the fact that $\frac{\partial H_{rest}}{\partial w_k} = \frac{dq_k}{d\tau}$ (which can be seen by substituting $\dot{\tau} = \frac{g'^{00} H_{rest}}{g^{00} H}$, Eq. (2.8), into Eq. (2.6)). In order to compute the variation of the internal Lagrangian $\frac{\delta L_{rest}(q_k, dq_k/d\tau)}{\delta g_{\mu\nu}(x)}$—which is not independent of the metric since $c\dot{\tau} = \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$—it is convenient to use the relation $\frac{dq_k}{d\tau} = \frac{\dot{q}_k}{\dot{\tau}}$, which yields

$$\frac{\delta L_{rest}}{\delta g_{\mu\nu}(x)} = -\sum_k \frac{\partial L_{rest}}{\partial \frac{dq_k}{d\tau}} \frac{dq_k}{d\tau} \frac{1}{\dot{\tau}} \frac{\delta \dot{\tau}}{\delta g_{\mu\nu}(x)}.$$

The energy momentum tensor of the system is given by $T^{\mu\nu}(x) = \frac{-2}{\sqrt{-g}}(\frac{\delta L_{rest}}{\delta g_{\mu\nu}(x)}\dot{\tau} + L_{rest}\frac{\delta \dot{\tau}}{\delta g_{\mu\nu}(x)})$ and takes the form

$$T^{\mu\nu}(x) = \frac{2}{\sqrt{-g}}\left(\sum_k \frac{\partial L_{rest}}{\partial \frac{dq_k}{d\tau}}\frac{dq_k}{d\tau} - L_{rest}\right)\frac{\delta\dot{\tau}}{\delta g_{\mu\nu}(x)}$$

Note, that inverse Legendre transform, applied to a Lagrangian, gives the corresponding Hamiltonian, therefore

$$H_{rest} = \sum_k \frac{\partial L_{rest}}{\partial \frac{dq_k}{d\tau}}\frac{dq_k}{d\tau} - L_{rest}. \tag{2.12}$$

The variation of $\dot{\tau}$ with respect to the metric reads

$$\frac{\delta\dot{\tau}}{\delta g_{\mu\nu}(x)} = \frac{-\dot{x}^\mu\dot{x}^\nu}{2c^2\dot{\tau}},$$

and the final form of the energy-momentum tensor for the considered system takes the form

$$T^{\mu\nu}(x) = \frac{-1}{\sqrt{-g}}H_{rest}\frac{\dot{x}^\mu\dot{x}^\nu}{c^2\dot{\tau}}. \tag{2.13}$$

Such defined energy-momentum tensor is also metric energy-momentum. The corresponding four-momentum is given by $p_\mu = \sqrt{-g}T^0{}_\mu$ and in particular $\sqrt{-g}T^0{}_0 = -H_{rest}\frac{\dot{x}^0\dot{x}^0 g_{00}}{\dot{\tau}}$ is the energy component, which is equal[1] to the Hamiltonian H, as can be checked by substituting $\dot{\tau} = \frac{g'^{00}H_{rest}}{g^{00}H}$ (Eq. (2.8)), $x^0 = ct$ and $g'^{00} = -1$.

The energy-momentum tensor in Eq. (2.13) generalises that of a point particle with mass m, which reads $T_m^{\mu\nu}(x) = \frac{-1}{\sqrt{-g}}m\frac{\dot{x}^\mu\dot{x}^\nu}{\dot{\tau}}$, analogously as for the case of Lagrangian and Hamiltonian, discussed in Sects. 2.1.2 and 2.1.1.

2.1.4 Routhian or Point Particles as Ideal Clocks

The relevance of here discussed physical regime lies in the fact that point-like systems with internal dynamics represent ideal clocks in relativity. Lagrangian of a point-like composed relativistic system obtained in Sect. 2.1.2 has a general, product form $L = L_{rest}\dot{\tau}$ (cf. 2.9) which already has an intuitive interpretation in terms of the time dilation of the internal dynamics of the system. However, in the context of time dilation experiments the frequency of clocks is often given in terms of energy while

[1]For scalar particles the metric energy-momentum tensor coincides with the canonically defined via Noether's theorem (whose energy component coincides with the energy obtained from the Legendre transform of the Lagrangian). For other than spin zero scalar particles the two tensor differ (the canonical tensor is e.g. not symmetric), see e.g. [2] for further discussion. From the perspective of general relativity the metric energy-momentum is much more convenient: it is by definition covariantly conserved and constitutes the source term in the Einstein equations derived from the variational principle.

time dilation is expressed in terms of relative velocity and position in the gravitational potential between the rest reference frames of the clocks. Similar observation holds for redshift experiments. It is therefore convenient use a formalism where internal degrees of freedom are described by a Hamiltonian, whereas the external ones– by a Lagrangian (in configuration space). Such a formalism is provided by a Routhian.

Routhian is defined as a partial Legendre transform of the Lagrangian—with respect to a subset of degrees of freedom; it is therefore a Hamiltonian with respect to the Legendre-transformed ones and a Lagrangian with respect to the remaining ones. In particular, the Routhian considered here is

$$R := \sum_k \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k - L, \tag{2.14}$$

and the notation is the same as in Sect. 2.1.2. The internal momentum w_k in terms of the configuration space variables reads

$$w_k = \frac{\partial L_{rest}}{\partial (dq_k/d\tau)} \equiv \frac{\partial L}{\partial \dot{q}_k}$$

and Eq. (2.14) takes the form

$$R = (\sum_k \frac{\partial L_{rest}}{\partial \frac{q_k}{d\tau}} \frac{q_k}{d\tau} - L_{rest})\dot{\tau}.$$

Recall, that the expression in the bracket is the internal Hamiltonian, Eq. (2.12), and the Routhian can be written as:

$$R = H_{rest}\dot{\tau}. \tag{2.15}$$

The Routhian in Eq. (2.15) acts like a Hamiltonian on the internal degree of freedom and describes dynamics of a system with respect to a time parameter t. The speed of the time evolution of any internal observable a_i with respect to t is thus given by a Poisson bracket $\dot{a}_i = \{a_i, R\}$. Equivalently, using Eq. (2.15), one finds

$$\dot{a}_i = \{a_i, H_{rest}\}\dot{\tau}. \tag{2.16}$$

It immediately follows from Eq. (2.16) that $v_{rest} := \{a_i, H_{rest}\} = \frac{da_i}{d\tau}$, is the speed of the time evolution of a_i with respect to the proper time τ. Thus, internal degrees of freedom of the system moving along a world line γ will evolve as if the time elapsed during this evolution was the proper length of the world line $\tau_\gamma = \int_\gamma d\tau$:

$$a_i(t_f) - a_i(t_{in}) = \int_\gamma \dot{a}_i dt = \int_\gamma v_{rest}\dot{\tau} dt = v_{rest}\tau_\gamma, \tag{2.17}$$

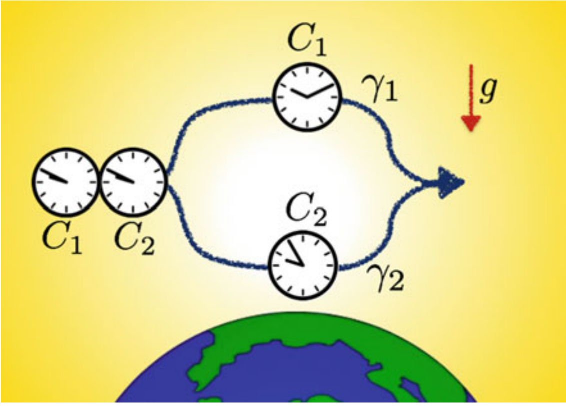

Fig. 2.1 Time dilation between two systems C_1 and C_2 moving along world lines γ_1, γ_2, respectively. The two systems have time evolving internal degrees of freedom and can thus be used as "clocks". Prepared in the same internal configuration, the systems C_1, C_2 evolve into different final states if the proper lengths of γ_1 and γ_2 differ, Eq. (2.18) (e.g. due to earth's gravitational field g)—initially synchronised "clocks" taken along different paths generally show different times when brought together. Universality of this prediction gave rise to the formulation of the "twin paradox"—a Gedankenexperiment in which instead of synchronised clocks one considers twins, who set up on separate journeys and meet at a predefined future event, when it transpires that each of them has aged differently—by the amount of proper time which elapsed along their world line

where t_f and t_{in} are the values of the parameter t that correspond to the end points of γ, and for simplicity a constant speed of the internal dynamics has been assumed. Any two systems moving along different world lines γ_1 and γ_2, e.g. as in Fig. 2.1, will thus exhibit a universal difference in their internal time evolution, equal to the proper time difference between their trajectories. Indeed, from Eq. (2.17) follows

$$\int_{\gamma_1} \dot{a}_i dt - \int_{\gamma_2} \dot{a}_i dt = v_{rest} \Delta\tau \tag{2.18}$$

where $\Delta\tau := \tau_{\gamma_1} - \tau_{\gamma_2}$.

Finally, note that for an observer using a time parameter t the system will evolve by a time interval $\int_\gamma dt$ and this observer will describe the system's internal evolution as time dilated by a universal factor $\dot{\tau}$. Consequently, the internal Hamiltonian of the system will in such coordinates be "red-shifted" by the factor $\dot{\tau}$ and will read $H_{rest}\dot{\tau}$. Thus, time dilation is tantamount to the presence of interactions between the internal degrees of freedom (described by H_{rest}) and the external ones (described by $\dot{\tau}$), which originate from the specific form of general relativistic description of composed systems.

Dynamics of relativistic point-like systems allows an interoperation that such systems "measure" proper time along their world lines. Their internal evolution exhibits a universal time dilation when the systems are taken along different trajectories in space-time—only depending on the proper time difference between the trajectories and not on the systems themselves. In this sense point-like systems with internal degrees of freedom can be seen as ideal clocks.

2.1.5 Low Energy and Non-relativistic Limits and Symmetries of the Framework

Point particles with internal degrees of freedom in the low energy and non-relativistic limits. In order to consider the non-relativistic limit of the framework, valid in some earth-based scenario, the Schwarzschild space time will be considered. To lowest order the metric components read [1]: $g_{00} \approx -(1 + 2\frac{\Phi(x)}{c^2})$, $g_{ij} \approx \delta_{ij}$, where $\Phi(x) = -\frac{Gm_e}{x}$ is the earth's gravitational potential (with G the Newton constant and m_e—the mass of earth) which immediately gives the low-energy limit of the Routhian, R_{le}:

$$H_{rest}\dot{\tau} \approx H_{rest}\left(1 - \frac{\vec{v}^{\,2}}{2c^2} + \frac{\Phi(x)}{c^2}\right) =: R_{le}, \tag{2.19}$$

with $\vec{v} := (\dot{x}^1, \dot{x}^2, \dot{x}^3)$ denoting the centre of mass velocity with respect to the laboratory reference frame. The corresponding Hamiltonian reads

$$H_{le} = H_{rest} + \frac{\vec{p}^{\,2}c^2}{2H_{rest}} + H_{rest}\frac{\Phi(x)}{c^2} \tag{2.20}$$

and one could think that the Eq. (2.19) already gives the non-relativistic limit of the dynamics of a particle with a total mass $M := H_{rest}/c^2$. However, if the internal degrees of freedom are not stationary, Eqs (2.19), (2.20) will feature relativistic time dilation effects. In the Eq. (2.19) the terms $H_{rest}(1 - \frac{\vec{v}^{\,2}}{2c^2})$ and $H_{rest}(1 + \frac{\Phi(x)}{c^2})$ give rise to the lowest order special relativistic and gravitational time dilation of internal dynamics, respectively: time evolution of an internal observable a_i under R_{le} reads $v_{rest}(1 - \frac{\vec{v}^2}{2c^2} + \frac{\Phi(x)}{c^2})$ with $v_{rest} := \{a_i, H_{rest}\}$, cf. Sect. 2.1.4. From an operational perspective the theory of Newtonian particles with "dynamical mass" H_{rest}/c^2—Eq. (2.20)—is not a consistent non-relativistic theory, in particular it does not give rise to the Euclidean space-time with absolute time. In order to find the consistent non-relativistic limit it is convenient to split the internal energy H_{rest} into a constant part E_{rest}—such that $\{a_i, E_{rest}\} = 0$—and the remaining part which drives the dynamics of the internal degrees of freedom. The Routhian in Eq. (2.19) can equivalently be written as $R_{le} = H_{rest} + E_{rest}\left(-\frac{\vec{v}^{\,2}}{2c^2} + \frac{\Phi(x)}{c^2}\right) + (H_{rest} - E_{rest})\left(-\frac{\vec{v}^{\,2}}{2c^2} + \frac{\Phi(x)}{c^2}\right)$. The first term, H_{rest}, results in the universal rate of internal evolution with no time dilation effects, as well as the second term proportional to E_{rest} and it is the last term proportional to $H_{rest} - E_{rest}$ which introduces time dilation. Therefore, the consistent non-relativistic limit is given by $H_{rest} + E_{rest}\left(-\frac{\vec{v}^{\,2}}{2c^2} + \frac{\Phi(x)}{c^2}\right)$ and arises when the dynamical part of the internal energy is small, i.e. internal dynamics is slow. Moreover, the constant E_{rest} defines the mass parameter of the system $m := E_{rest}/c^2$ and $H_{rest} - E_{rest}$ is equivalent to H_{int} introduced earlier in this Chapter. The non-relativistic limit can thus equivalently be written as

$$R_{nr} = mc^2 + H_{int} - m\frac{\vec{v}^2}{2} + m\Phi(x). \tag{2.21}$$

Under R_{nr} the rate of the internal evolution is the same in any coordinates $\{a, R_{nr}\} = \{a, H_{int}\}$ and such a dynamical theory is consistent with the Euclidean space-time. Moreover, this shows that the non-relativistic limit of point particles with internal dynamics represents ideal clocks of the non-relativistic mechanics, as expected. With the definition of the mass parameter as $m = E_{rest}/c^2$ one also recovers the usual understanding of the non-relativistic limit as the zeroth order in $1/c^2$. The low-energy Hamiltonian in terms of $H_{rest} = mc^2 + H_{int}$ reads $H_{le} = mc^2 + H_{int} + \frac{\vec{p}^2}{2(m+H_{int}/c^2)} + (m + H_{int}/c^2)\Phi(x)$ and taking the non-relativistic limit gives the familiar Hamiltonian of the Newtonian dynamics

$$H_{nr} = mc^2 + H_{int} + \frac{\vec{p}^2}{2m} + m\Phi(x).$$

In summary, for composed systems the correct non-relativistic limit is described not only by small kinetic and potential energies of the external degree of freedom, but also by slow internal evolution—i.e. small *dynamical* part of the internal energy. The reason is that, in such a case the contributions of the dynamical part of the internal energy are relevant only in the rest energy term but can be neglected in the kinetic and potential energy terms (since these are already small in the considered limit), to which only the static part of the internal energy effectively contributes. This can be seen as the origin of the split between the mass and energy in the non-relativistic physics, which are fully equivalent in the relativistic theory. From this perspective the mass parameter of a non-elementary system can be *defined* as the static part of its internal energy, in appropriate units.[2]

Finally, note that for a non-relativistic limit of a theory to be at all meaningful, one needs to assume that the relevant measurements have finite precision. If one could measure internal states arbitrary precisely, time dilation of their evolution could never be neglected. The regime of *approximate* validity of the non-relativistic mechanics is thus operationally defined in the light of the available measurement precision—in this context *slow* internal dynamics means: slow enough that (over the considered, finite, time scales) the presence of the couplings that give rise to time dilation, leads to non-resolvable difference in the evolution of the internal states, which can thus be neglected.

[2]Of course the constant term mc^2 can be omitted as it has no physical meaning for the dynamics of the particle, neither in classical nor in the quantum theory. Such a constant energy term only acquires physical meaning when the gravitational potential produced by the system is considered.

Symmetries of the framework The symmetry group of a massive elementary relativistic particle is the Poincaré group and the mass parameter is just a property of the system. Configuration space of a point-particle whose internal coordinates are denoted q_k, $k = 1, .., N$ and the centre of mass coordinate is x will be denoted $(\{q_k\}_{k=1}^N; x^\mu)$. Transformations of x are generated by the Poincaré group, and they leave the dynamics invariant. In the framework of point particles the translations of the internal coordinates are generated by the internal energy operator (and are the only internal transformations considered here). Thus, by construction, the symmetry group of the framework of relativistic point particles with internal dynamics is a central extension of the Poincaré group: mass is generalised from a parameter to a generator, H_{rest}/c^2, which commutes with all other group generators—it lies in the centre of the new group. The algebra of the Poincaré group is otherwise unchanged—this central extensions is thus called trivial as it is a product of the Poincaré group and of the internal symmetry group (here only generated by the internal Hamiltonian). Analogously, in the non-relativistic limit, Eq. (2.21), the symmetry group of the framework is a product of the Galilei group and of the symmetry group of the internal dynamics (as well generated by internal Hamiltonian).

A non-trivial and physically relevant is the case of the low-energy regime, described by H_{le}, Eq. (2.20) or, equivalently, by the R_{le}, Eq. (2.19). This regime arises when the only observable relativistic effects are the time dilation effects on the internal evolution and the dynamics in such a regime is invariant under the corresponding limit of the relativistic transformations. In particular, the boost by w acts on the coordinates as $x'^i \approx x + wt$, $t' \approx (t + \frac{xw}{c^2})(1 - \frac{w^2}{2c^2})^{-1}$ and leaves the dynamics invariant. This can be seen directly by applying the transformation above to the equation of motion for the internal coordinate, which in this limit reads $\dot{q}_k = \{q_k, H_{rest}\}(1 - \frac{v^2}{2c^2})$. Transformed to the "primed coordinates" defined above it reads $\frac{dq_k}{dt'} = \dot{q}_k \frac{dt}{dt'} \approx \{q_k, H_{rest}\}(1 - \frac{(v+w)^2}{2c^2})$ and finally note that, up to considered order, $H_{rest}(1 - \frac{(v+w)^2}{2c^2}) = R'_{le}$.

On the other hand H_{le} has the same form as the non-relativistic, Newtonian, Hamiltonian but with a dynamical mass, formally defined as $M := H_{rest}/c^2$ (compare the paragraph above) $H_{le} = Mc^2 + \frac{\vec{p}}{2M}$ (the gravitational interaction will be skipped in this section). Velocity of the internal coordinate under H_{le} reads

$$\dot{q}_k = v_0 c^2 (1 - \frac{\dot{\vec{x}}^2}{2c^2}), \tag{2.22}$$

where $\dot{\vec{x}}$ is canonically conjugate to $\vec{p}$, $v_0 := \{q_k, M\}$, and c^2 appears here merely for adjusting the units. Note, that Galilei group elements: translations and rotations of the external coordinate as well as translation of the time parameter leave the above invariant. Galilei boost by $\vec{w}$: $\vec{x}' = \vec{x} + \vec{w}t$ transforms Eq. (2.22) into

$$\dot{q}'_k = \dot{q}_k - v_0 \dot{\vec{x}} \vec{w} - v_0 \frac{1}{2} \vec{w}^2 \tag{2.23}$$

which gives [3]:

$$q_k' = q_k - v_0 \vec{w}\vec{x} - v_0 \frac{1}{2}\vec{w}^2 t + a_w \tag{2.24}$$

where a_w is a constant. The symmetry group $\tilde{\mathcal{G}}$ of the entire configuration space $(\{q_k\}, \vec{x}, t)$ thus comprises an element α of the internal symmetry group and an element g of the Galilei group $\mathcal{G}$. The latter comprises: spatial translations, parametrised by $\vec{a}$; temporal translations b; boosts $\vec{w}$; and rotations R. An element $g \in \mathcal{G}$ will be denoted $g = (R, \vec{w}, \vec{a}, b)$. An element of the total symmetry group $\tilde{\mathcal{G}}$ is thus denoted by $\tilde{g} = (\alpha, g)$ and its action on of the configuration space of the system is

$$\tilde{g}(\{q_k\}, \vec{x}, t) = (\{q_k + \alpha - v_0 \vec{w}\vec{x} - v_0 \frac{1}{2}\vec{w}^2 t\}, R\vec{x} + \vec{w} + \vec{a}, t + b). \tag{2.25}$$

The above general rule allows to consider the following combination of transformations from $\tilde{\mathcal{G}}$: a boost by $\vec{w}$ then shift by $\vec{a}$, boost by $-\vec{w}$ and shift by $-\vec{a}$:

$$\begin{aligned}
(\{q_k\}, \vec{x}, t) &\xrightarrow{\tilde{g}_{\vec{w}}} (\{q_k - v_0 \vec{w}\vec{x} - v_0 \frac{1}{2}\vec{w}^2 t\}, \vec{x} + \vec{w}t, t) \\
&\xrightarrow{\tilde{g}_{\vec{a}}} (\{q_k - v_0 \vec{w}\vec{x} - v_0 \frac{1}{2}\vec{w}^2 t\}, \vec{x} + \vec{w}t + \vec{a}, t) \\
&\xrightarrow{\tilde{g}_{-\vec{w}}} (\{q_k + v_0 \vec{w}\vec{a}\}, \vec{x} + \vec{a}, t) \\
&\xrightarrow{\tilde{g}_{-\vec{a}}} (\{q_k + v_0 \vec{w}\vec{a}\}, \vec{x}, t).
\end{aligned} \tag{2.26}$$

The transformation $\tilde{g} = \tilde{g}_{-\vec{a}}\tilde{g}_{-\vec{w}}\tilde{g}_{\vec{a}}\tilde{g}_{\vec{w}} = (\vec{w}\vec{a}, g_{-\vec{a}}g_{-\vec{w}}g_{\vec{a}}g_{\vec{w}}) = (\vec{w}\vec{a}, id_{\mathcal{G}})$ is not identity in $\tilde{\mathcal{G}}$ although the Galilei group elements themselves add up to the identity of the Galilei group $id_{\mathcal{G}}$. This means that the group $\tilde{\mathcal{G}}$ does not factor into a product of the two groups, as it was in the non-relativistic limit. Furthermore, the above chain of transformations shows that in $\tilde{\mathcal{G}}$ the generator of spatial translations P and of the boosts K satisfy

$$PK - KP \propto M \tag{2.27}$$

(since M is the generator of internal shifts). In the Galilei group itself the right hand side of Eq. (2.27) vanishes. The remaining commutators of the algebra $\tilde{\mathcal{G}}$ are the same as those in $\mathcal{G}$ and M commutes with other elements of $\tilde{\mathcal{G}}$, which immediately follows from the Eq. (2.25). Thus, $\tilde{\mathcal{G}}$ is a non-trivial central extension of the Galilei group. The above results agree with Ref. [3], which showed that assuming dynamical mass in Galilei invariant physics results in dynamics which is invariant under the central extension of the Galilei group. This is fully consistent with the fact that the limit of the Poincaré group (with mass as a parameter) is central extension of the Galilei group (with mass parameter on the right hand side of the Eq. (2.27)) and not the Galilei group itself [4, 5].

Note, that from the perspective of the group structure the operationally well defined non-relativistic limit corresponds to the system whose symmetry group is a product of the Galilei and the internal symmetry group. On the other hand, the discussion of the previous paragraph shows that the non-relativistic limit corresponds to the case when the internal energy (or the "dynamical mass") is effectively static (in the kinetic and potential energy terms) and can be understood as a parameter. However, this still leaves the right hand side of Eq. (2.27) non-zero: $PK - KP \propto M \to const. \equiv m$, which formally is again a structure of a central extension of the Galilei group and not the Galilei group. This is only an apparent discrepancy: with $PK - KP \propto m$ the transformations of the external coordinate would not induce any non-trivial transformations on the internal coordinates—a constant generator m has vanishing Poisson brackets with any function.[3] In other words, the limit $M \to m$ in the commutator above entails that the transformation law for internal coordinates, Eq. (2.24), reduces to $q'_k = q_k + \alpha$. The Hamiltonian leading to such a transformation law is $H_{rest} + \frac{\vec{p}^2}{2m}$—i.e. the fully non-relativistic one. As long as the generator m is (approximately) a constant, transformations of the configuration space resulting from a central extension of the Galilei group are indistinguishable from those originating from the group with $PK - KP = 0$ i.e. from the Galilei group. Thus, also from the symmetry-group perspective a consistent non-relativistic limit for a theory of "Newtonian" particles with a "dynamical mass" is obtained when the dynamical contribution to the mass can be neglected (everywhere apart from the rest energy term).

The Hamiltonian H_{le} viewed as the low energy limit of a relativistic system with internal degrees of freedom is invariant under the Poincaré group (to lowest order) and viewed as the theory of "Newtonian" particles with "dynamical mass"—is invariant under a non-trivial central extension of the Galilei group.[4] These two approaches are operationally undistinguishable. The framework of "Newtonian" particles with "dynamical mass" and its corresponding symmetry group—non-trivial central extensions of the Galilei group with central element M—thus have a natural physical interpretation, in terms of low-energy relativistic systems with internal degrees of freedom. In particular the "dynamical mass" M can be interpreted as the total internal mass-energy of the system, in appropriate units. Central extensions of the Galilei group and the corresponding models of "Newtonian" particles with "dynamical mass" were studied before in the literature—mostly from a formal perspective in the context of quantum mechanics [3, 6, 7], or in the context of discussing mass as independent degree of freedom [8–11].

From the operational point of view adopted in this thesis, central extensions of the Galilei group do not describe the non-relativistic physics. Recall, that the non-

[3]With the $PK - KP \propto const.$ the v_0 which appears in the "kinetic terms" of Eq. (2.25) satisfies $v_0 = \{q_k, const.\} \to 0$, whereas the v_0 in the rest energy term is not constrained by requirement imposed on the commutator.

[4]Note that the transformation law for internal coordinates induced by requiring Galilei invariance, Eq. (2.23), matches the low-velocity limit of the relativistic transformation, which reads $\dot{q}'_k = \dot{q}_k \frac{dt}{dt'} \approx \dot{q}_k - v_0 \dot{x} w - v_0 \frac{w^2}{2}$, where $\frac{dt}{dt'} \approx (1 - \frac{\dot{x} w}{c^2})(1 - \frac{w^2}{2c^2})$ for $\dot{q}_k$ is given by Eq. (2.22).

relativistic limit is given by the condition that the mass effectively becomes static and can thus be understood as a parameter $m = E_{rest}/c^2$. From the perspective of the group structure, the non-relativistic limit is obtained when the central element M is approximately *constant*. This condition is valid not only in classical mechanics—the quantum case is directly analogous. This sheds a new light on the physical meaning of the superselection rule for the mass in the non-relativistic quantum mechanics [3, 6, 7] and can be seen as its classical counterpart. These topics will be discussed in some more detail in the next Chapter.

2.2 Derivation of the Point-Particle Framework

2.2.1 Effective Dynamics of Relativistic N-Particle Systems

Consider a system comprising N particles with masses m_n, $n = 1, .., N$, charges e_n and coordinates $x_n^\mu(s)$, parametrised by a common parameter s and interacting via a four-potential A_μ. Lagrangian of such a system reads, [1]:

$$L_N = \sum_n -m_n c^2 \frac{d\tau_n}{ds} + e_n A_\mu(x_n) \frac{dx_n^\mu(s)}{ds} \tag{2.28}$$

where $c\frac{d\tau_n}{ds} = \sqrt{-g_{\mu\nu}(x_n)\frac{dx_n^\mu}{ds}\frac{dx_n^\nu}{ds}}$ and x_n denotes a four-vector, $x_n \equiv x_n^\nu$. The components x_n^0 can be chosen to be the same for all the particles and given by the common parameter s, which can be chosen to be time t as measured by a clock stationary and in the origin of the reference frame with respect to which the coordinates are defined. For all n one can thus set $x_n^0 = s \equiv ct$.

Consider a world line $Q^\mu(t)$ and coordinates where $\dot{Q}'^i = 0$ and where Q'^0 is chosen to be the proper time along this world line, $c\dot{\tau} = \sqrt{-g_{\mu\nu}(Q)\dot{Q}^\mu\dot{Q}^\nu}$. Since the quantities defining the Lagrangian are coordinate invariants, (i.e. $g_{\mu\nu}\frac{dx_n^\mu}{dt}\frac{dx_n^\nu}{dt} = g'_{\mu\nu}\frac{dx_n'^\mu}{dt}\frac{dx_n'^\nu}{dt}$ and $A_\mu\frac{dx_n^\mu(t)}{dt} = A'_\mu\frac{dx_n'^\mu(t)}{dt}$) changing the parametrisation in Eq. (2.28) into τ one can equivalently write the Lagrangian L_N as

$$L_N = \sum_n \left(-m_n c\sqrt{-g'_{\mu\nu}\frac{dx_n'^\mu}{d\tau}\frac{dx_n'^\nu}{d\tau}} + e_n A'_\mu \frac{dx_n'^\mu}{d\tau}\right)\frac{1}{c}\sqrt{-g_{\mu\nu}(Q)\dot{Q}^\mu\dot{Q}^\nu}. \tag{2.29}$$

No approximations were introduced thus far and Eq. (2.29) is fully equivalent to Eq. (2.28). Equation (2.29) has a general structure $L_N = f(x_n'^\nu, dx_n'^\nu/d\tau)\dot{\tau}$, however, this alone does not imply that it can be considered to describe a composed particle whose internal dynamics is given by $f(x_n'^\nu, dx_n'^\nu/d\tau)$. For such an interpretation to be consistent Q^μ shall represent the world line of the particle, e.g. its centre of mass, and the coordinates where $Q'^i = 0$ shall correspond to the centre of momentum frame, where the total momentum of the system vanishes. The canonical momentum conjugate to x_n is defined as

$$p_{ni} := \frac{\partial L_N}{\partial \dot{x}_n^i} \tag{2.30}$$

and reads $p_{ni}(x_n) = \frac{m_n \dot{x}_{in}}{\dot{\tau}_n} + e_n A_i(x_n)$. In general the vectors p_{in} belong to different tangent spaces and there is no unique way to sum them up and define the total linear momentum. However, such a definition is possible in case where the metric components are approximately constant in the region occupied by the particles:

$$\forall_{n,m} g_{\mu\nu}(x_n) \approx g_{\mu\nu}(x_m), \tag{2.31}$$

which means that the region is approximately flat. For well-behaved metrics, the condition in Eq. (2.31) is satisfied if the particles are following sufficiently close-by[5] world lines

$$\forall_{n,m} x_n \approx x_m. \tag{2.32}$$

The linear momenta are given by $P_{ni}(x_n) := \sum_n (p_{ni}(x_n) - e_n A_i(x_n))$ and the total linear momentum under the approximation (2.31) can meaningfully be defined as a sum of the individual linear momenta $P_i = \sum_n P_{ni}(x_n)$. The centre of mass frame is defined by the condition $P'_i = \sum_n P'_{ni}(x_n) = 0$, where $P'_{ni} = \frac{\partial x^\mu}{\partial x'^i} P_{n\nu}$. Note, that the validity of the requirement in Eq. (2.31) also guarantees that the condition $P'_i = 0$ (and P_i itself) is generally covariant. Indeed, with $P^i = \sum_n g^{ij}(x_n) P_{nj}$ one finds

$$P'^i = \sum_n \frac{\partial x'^i}{\partial x^\mu} g^{\mu\nu}(x_n) P_{n\nu} = \sum_n \frac{\partial x'^i}{\partial x^\mu} g^{\mu\nu}(x_n) \frac{\partial x'^\alpha}{\partial x^\nu} P_{n\alpha} = \sum_n g'^{ij}(x_n) P'_{nj}, \tag{2.33}$$

and if Eq. (2.31) holds:

$$P'^i \approx g'^{ij}(x_N) \sum_n P'_{nj} = 0. \tag{2.34}$$

More generally, the condition in Eq. (2.31) guarantees that a single coordinate chart can be used to cover all the region occupied by the particles, and in which the metric takes the Minkowski form. This allows defining trajectories on the space-time manifold for the parallel transport of the tangent vectors to some fixed point and or defining such quantities like the total momentum (without assuming Eq. (2.32)). Such a construction of the centroid for an N-particle system has been first given by Dixon [12] and leads to a generally covariant (and independent of the choice of the fixed point) notion of the centre of mass.

If the approximation in Eq. (2.32) holds, one can identify the world line $Q^\mu(t)$ in Eq. (2.29) with the world line of one of the constituent particles, say x_N, which

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[5] Quantitative estimations will be discussed in more detail further in this section.

is done hereafter. $Q^\mu(t) \approx x_N$ thus represents the position of the composite system and the "primed" coordinates, defined by the condition $Q'^i = 0$, can be identified with the centre of momentum frame. Condition in Eq. (2.32) also guarantees that the resulting description of the system is independent of which of the N world lines one choses to represent the system. Note, that this condition does not require that the particles follow exactly the same world lines and have exactly the same velocities—it is required to hold only to the extent that there is no considerable time-dilation between the different world lines. In such a case the quantity

$$L_{rest} = \sum_n \left(-m_n c \sqrt{-g'_{\mu\nu} \frac{dx_n'^\mu}{d\tau} \frac{dx_n'^\nu}{d\tau}} + e_n A'_\mu \frac{dx_n'^\mu}{d\tau} \right) \tag{2.35}$$

can consistently be used to represents internal Lagrangian of a composite system following the world line $Q^\mu(t)$. The total Lagrangian takes the form

$$L_N \approx L_{rest} \dot{\tau} =: L_{tot}, \tag{2.36}$$

with L_{rest} given by Eq. (2.35). Thus, a system comprising N relativistic particles moving along a narrow "world tube" approximates that of a composite point particle, discussed in the previous sections.

For completeness, derivation of the Hamiltonian corresponding to the Lagrangian L_{tot}, Eq. (2.36), is given below. The total Hamiltonian H_{tot} is obtained from the Legendre transform of L_{tot} and is a function of the momentum P—canonically conjugate to the external degree of freedom Q—and of the momenta w_n—conjugate to the internal degrees of freedom x'_n:

$$H_{tot}(Q, P; w_1, \dots, w_n) = P_i \dot{Q}^i + \sum_n w_{in} \dot{x}_n'^i - L_{tot}; \tag{2.37}$$

the canonical momentum of the external degree of freedom is defined as

$$P_i = \frac{\partial L_{tot}}{\partial \dot{Q}^i} = \frac{\partial L_{rest}}{\partial \dot{Q}^i} \dot{\tau} + L_{rest} \frac{d\dot{\tau}}{d\dot{Q}^i}$$

and the internal momenta:

$$w_{in} = \frac{\partial L_{tot}}{\partial \dot{x}_k'^i} = \frac{\partial L_{rest}}{\partial \frac{dx_k'^i}{d\tau}},$$

see also Sect. 2.1.4.

In order to compute $\frac{\partial L_{rest}}{\partial \dot{Q}^i}$ recall that quantities such as $\frac{dx^i}{d\tau}$ can equivalently be written as $\frac{\dot{x}^i}{\dot{\tau}}$ and thus $\frac{L_{rest}}{d\dot{Q}^i} = \sum_n \frac{\partial L_{rest}}{\partial \frac{dx_n^i}{d\tau}} \dot{x}_n^i \frac{-1}{\dot{\tau}^2} \frac{d\dot{\tau}}{d\dot{Q}^i}$, see also Sect. 2.1.3. The canonical momentum P_i of the composed particle thus reads

$$P_i = \left(\sum_n - \frac{\partial L_{rest}}{\partial \frac{x'_n}{d\tau}} \frac{dx'_n}{d\tau} + L_{rest} \right) \frac{d\dot{\tau}}{d\dot{Q}^i} = H_{rest} \frac{\dot{Q}_i}{c^2 \dot{\tau}}, \tag{2.38}$$

where the definition of the internal Hamiltonian given in Eq. (2.12) has been used.

First, the general structure of H_{tot} will be obtained. Substituting into Eq. (2.37) the expression for P_i from Eq. (2.38) and the definition of w_n above one finds

$$H_{tot} = H_{rest} \frac{\dot{Q}^i \dot{Q}_i}{c^2 \dot{\tau}} + H_{rest} \dot{\tau} = -H_{rest} \frac{g_{00} \dot{Q}^0 \dot{Q}^0}{c^2 \dot{\tau}} = -H_{rest} \frac{g_{00}}{\dot{\tau}}, \tag{2.39}$$

where the definition of H_{rest} as Legendre transform of L_{rest} has been used, as well as $c\dot{\tau} = \sqrt{-\dot{Q}^\mu \dot{Q}_\nu}$ and $Q^0 = ct$. Using again Eq. (2.38) one finds

$$c^2 P_i P^i = H^2_{rest} \frac{\dot{Q}^i \dot{Q}_i}{c^2 \dot{\tau}^2} = H^2_{rest} (-1 + \frac{-g_{00}}{\dot{\tau}^2})$$

and after substitution into the Eq. (2.39) a final form of H_{tot} is found:

$$H_{tot} = \sqrt{-g_{00}(c^2 P_i P^2 + H^2_{rest})}, \tag{2.40}$$

and as expected the structure of H_{tot} above is that of the Hamiltonian in Eq. (2.2), describing an idealised point-like particle with internal degrees of freedom.

For the particular system defined by the Lagrangian L_{tot} in Eq. (2.36) with L_{rest} given by Eq. (2.35) the internal momenta explicitly read $w^i_n = m_n \frac{\dot{x}'_{ni}}{\dot{\tau}_n} + e_n A'_i$. Analogous calculation to the one which led to the total Hamiltonian results in

$$H_{rest} = \sum_n \sqrt{-g'_{00}[(w_{ni} - e_n A'_i)(w^i_n - e_n A'^i) + m^2_n]} - e_n A'_0. \tag{2.41}$$

2.2.2 *Quantitative Discussion of the Approximations*

A relativistic N-particle system in the regime where Eq. (2.31) holds can be described as an effectively point-like, composed particle with a generally covariant external four-momentum vector P_i and with the internal energy H_{rest}. The latter is defined as the total energy in the zero-momentum frame, i.e. where $P'_i = 0$. The error made in describing an extended many particle system as a point-like "composed" particle on a world line $x^\mu(t)$ can be quantified e.g. by the difference between $P^\mu = \sum_n g^{\mu\nu}(x_n) P_{n\nu}(x_n)$ and $g^{\mu\nu}(x) P_\nu \equiv g^{\mu\nu}(x) \sum_n P_{n\nu}(x_n)$:

$$P^\mu - g^{\mu\nu}(x) P_\nu = \sum_n (g^{\mu\nu}(x_n) - g^{\mu\nu}(x)) P_{n\nu}. \tag{2.42}$$

Thus, the scale for the approximations is set by the variation of the metric across the distances between the constituent particles and by the total energy-momentum. Consider a region $\mathcal{U} := \bigcup_t \mathcal{U}_t$, with $\mathcal{U}_t$ such that $\forall_n x_n(t) \in \mathcal{U}_t$. Assumption that the variation of the metric in $\mathcal{U}$ is bounded can be expressed as

$$\exists_K \forall_{\mu\nu,n,m} |g^{\mu\nu}(x_n) - g^{\mu\nu}(x_m)| < K/(4N) \tag{2.43}$$

where $K > 0$ is a constant. Furthermore, if the energy scale of the particles in the region is bounded by some $\tilde{P}$, defined as

$$\tilde{P} := \max\{|P_{n\mu}(x_n)| : n \in \{1, \ldots, N\}, x_n \in \mathcal{U}, \mu = 0, \ldots, 3\}, \tag{2.44}$$

then, using Eqs. (2.43) and (2.44), the absolute value of the error, Eq. (2.42), satisfies

$$|P^\mu - g^{\mu\nu} P_\nu| = |\sum_{n,\nu} (g^{\mu\nu}(x_n) - g^{\mu\nu}(x)) P_{n\nu}| < K\tilde{P} \tag{2.45}$$

for any μ. An example of a metric with the required property is the Schwarzschild metric in isotropic coordinates: Its components restricted to any compact region of space-time with no singularity are Lipschitz functions.

On the other hand, Eq. (2.45) implies that given an energy bound and a finite measurement precision for any relativistic composite system on a well-behaved metric there exist a bound on the system's size, such that if the system is smaller than this bound, the error made by using the approximation (2.31) is below the resolution. The system can then be consistently treated as point-like particle with internal dynamical degrees of freedom, and is described by a Hamiltonian of a form $H = \sqrt{-g_{00}(c^2 p_j p^j + H_{rest}^2)}$, introduced in Sect. 2.1.1 or, equivalently, by a corresponding Lagrangian $L = L_{rest}\dot{\tau}$, Sect. 2.1.2, or a Routhian, $R = H_{rest}\dot{\tau}$, Sect. 2.1.4.

2.3 Relevance of the Framework for Experiments

The framework of point-like particles with internal dynamics naturally arises as a description of sufficiently well-localised (see Sect. 2.2.2) many-particle systems. This conclusion is important because the formalism can be interpreted as a theory of ideal clocks in relativity, see Sect. 2.1.4.

First, note that validity of such an effective description of composed systems underlies many experimental tests of special or general relativistic time-dilation. Such experiments are performed with various physical systems that are used to measure the elapsed time, such as Caesium atoms in the experiment of Hafele and Keating [13]. In this experiment one atomic clock was stationary on Earth and other were flown on board of a plane, east and westwards around the Earth. The experiment was consistent with a prediction that clocks will show the proper time along their world line, and that these proper times will differ for the different trajectories—due to special relativistic and gravitational time dilation. Results of this and similar experiments are conveniently described as if the passage of time was directly compared between the different world lines—one speaks about the elapsed proper time along some path. Such an account of the time dilation experiments is only possible because the dynamics of physical systems used as clocks effectively takes the form $H_{rest} \int_\gamma d\tau$, where γ can be treated as the system's world line. It is this universality of general relativistic description of composed systems, which allows us to abstract from the details of the systems' dynamics and makes it meaningful to speak about the elapsed proper time $\tau_\gamma = \int_\gamma d\tau$ itself.

Another important feature of the regime of composite particles is that it shows how relativistic effects can be probed without the need for high centre of mass velocities. The systems can be slow or even stationary in the laboratory reference frame and are effectively described by the Eq. (2.19). The rest frame rate of their internal evolution υ_{rest} according to Eq. (2.19) is time dilated by a factor $(1 - \frac{\vec{v}^2}{2c^2})$ if the system is moving with the centre of mass velocity $\vec{v}$ (see also Sect. 2.1.4) with respect to the laboratory frame. In a recent experiment realised in the group of David Wineland [14] such an effect stemming from velocities as slow as few meters per second has been measured. This has been possible due to fast (of order 10^{15} Hz) and stable (up to 10^{-17}) internal dynamics of the utilised system—an Aluminium ion, and due to high precision of the measurement. Note, that the above effect is the regime described by H_{le}, formally equivalent to the framework of "Newtonian" particles with "dynamical mass", which supports relativistic interpretation of that framework. In the experiment [14] also gravitational time dilation between two ion clocks was measured, with the clocks separated by ≈ 30 cm. Rate of internal dynamics of two identical systems stationary in a laboratory reference frame and located at different heights x_1 and $x_2 = x_1 + h$ reads: $\upsilon_{int}\left(1 + \frac{\Phi(x_1)}{c^2}\right)$ and $\upsilon_{int}\left(1 + \frac{\Phi(x_2)}{c^2}\right)$, respectively, see Eq. (2.19). Up to the linear terms in h the difference between these two rates is $\upsilon_{rest}\frac{gh}{c^2}$ with $g = GM/x_1^2$ denoting the gravitational acceleration on the surface of the earth, approximately $10\,\text{m/s}^2$. For $h \approx 30$ cm the so-called redshift factor $\frac{gh}{c^2} \approx 10^{-17}$. Such a minute effect could be resolved on short time scales due to high rate and stability of internal dynamics. (For a single data point only about 2 hours of clocks comparison were sufficient, during which the clocks accrued about 10^{-14}s of a difference.) Similarly, redshift experiments—tests verifying the shift in internal energies due to the relativistic factors explained above—operate in a regime where centre of mass itself would not require relativistic description. For example, if the system's internal Hamiltonian has some spectral line E_0 as measured in its rest frame, it will be measured to be $E = E_0(1 + \frac{\Phi(x_2)}{c^2})$, if the system is placed in a gravitational potential $\Phi(x)$. This effect has been probed by Pound and Rebka [15] in an experiment where the very narrow Mössbauer transition in iron was compared between an iron foil placed at the bottom and at the top of a 22.5 m high tower at Harvard University's Jefferson laboratory. The experiment was consistent with the prediction that the Mössbauer transitions in identical iron foils placed at different heights in the gravitational potential will differ by the gravitational redshift factor $\frac{gH}{c^2} \approx 10^{-15}$, for $H = 22.5\,\text{m}$.

The above discussed features of the regime where composed relativistic systems can be described as point particles with internal structure are crucial for the design and analysis of time dilation experiments. Interestingly, the perspective taken here—that relativistic effects concerning properties of time and space stem from the specific form of interactions present in the relativistic dynamics of systems used as rods and clocks—has been mostly discussed in philosophy of physics, in the context of the "clock hypothesis" (which is the statement that clocks measure proper time along their world lines) see e.g. Ref. [16] for an interesting discussion. Considering explicitly this dynamical underpinning of time dilation will turn out to be particularly beneficial for studying time dilation effects in quantum theory, which is the main scope of this work.

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