From Discord to Entanglement

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Abstract Two prominent and widely studied notions of quantum correlations are discord and entanglement, with the latter occupying a central place in quantum information theory, while the former being regarded of marginal significance and even being criticized by some researchers, although the deep relations between them have been revealed in recent years. Discord and entanglement, being indistinguishable for pure states, only differ for mixed states. The aim of this work is to subsume entanglement under discord by identifying entanglement as the minimal shadow of discord over extended systems. For this purpose, we first present a brief and concise review of some historical aspects of discord and entanglement, emphasizing the ideas leading to them and the intimate relations between them. Then by exploiting an intrinsic connection between classicality and separability of correlations, we derive entanglement from discord in terms of state extensions, and put discord in a more primitive place than entanglement in this context. We comment that the entanglement of pure states studied by EPR and Schrödinger can actually also be well understood as discord, only with the emergence of nonlocality characterized by the Bell inequalities involving mixed states rather than pure states, the LOCC paradigm for mixed-state entanglement becomes significant and attracts great interests. Discord and entanglement are different manifestations of the same global quantum substrate, with discord conceptually more ubiquitous in quantum information and more deeply rooted in quantum measurements.

1 Introduction

Correlations permeate our interpretation and understanding of the physical world. To extract correlation information from physical systems, whether classical or quantum,

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one has to perform measurements. A key deviation of the quantum from the classical is the fundamental different characteristics of measurements: While a classical measurement, by definition, can extract information without disturbance in principle, a quantum measurement often causes unavoidable disturbance to the measured system. Actually, quantum measurements lie at the very heart of quantum mechanics [1], and are the central characters in both theoretical and experimental investigations of quantum information. The early work of EPR "disproving" completeness of quantum mechanics and state steering [2], Bohr's response to the EPR argument [3], as elaborated by Wiseman [4], and the discussion of probability relations of bipartite states by Schrödinger [5–7], all depend crucially on quantum measurements. The quantum-to-classical transition in decoherence is essentially a consequence of quantum measurements [8–10].

Discord arises from the loss of information caused by quantum measurements, and was explicitly introduced by Ollivier and Zurek [11], and Henderson and Vedral [12], to quantify the quantumness of correlations. Its early roots, although implicit, may be traced back to the EPR-Bohr argument on completeness of quantum mechanics [2–4], to Everett's thesis on universal wavefunction and relative-state formulation of quantum mechanics [13], to Lindblad's investigations of entropy and quantum measurements [14, 15], etc. Its various aspects, including calculation, operational meaning, ramifications, and applications, are widely studied in the last decade [16–49].

Entanglement is the underpinning of many fundamental quantum tasks [50, 51], and is often regarded as a synonym of quantum correlations in early studies, although now it is recognized that the notion of quantum correlations has a much wide scope, and entanglement is a particular, albeit most important, kind of quantum correlations, i.e., entanglement can be identified as nonlocal quantum correlations. The detection and quantification of entanglement are extremely complicated and difficult for mixed states, and there are extensive and intensive studies of these issues in the last two decades [52–59]. The study of entanglement dated back explicitly, as the discord implicitly, to the seminal works of Einstein, Podolsky and Rosen [2], and Schrödinger [5–7], as early as 1930s. Now entanglement is regarded as a key resource in quantum information and is often intertwined with quantum nonlocality [52, 58–61].

Discord and entanglement actually have the same historical as well as theoretical origin. The present work is to clarify this, and to identify entanglement as the minimal discord over state extensions. The work is arranged as follows. In Sect. 2, we recall various notions of correlations, including total correlations, classical correlations, discord, entanglement, as well as their interplay, in order to set up the context of our investigation in Sect. 3, which is devoted to the study of entanglement in terms of discord. We demonstrate that entanglement is actually a kind of shadow (irreducible residue) of discord over extended systems, and suggest some interesting problems for further investigations. Finally, we conclude with discussions in Sect. 4.

2 Classical Versus Quantum Correlations

Correlations are always encoded in physical systems and can be mathematically synthesized by states (density operators) of composite systems. In the information-theoretic description of the classical world, correlations are usually quantified by the Shannon mutual information of bivariate probability distributions [62, 63]. More precisely, the amount of correlations of a bivariate discrete probability distribution $p^{ab} = \{p^{ab}_{ij}\}$, shared between parties a and b, is well quantified by the Shannon mutual information [62, 63]

$$I(p^{ab}) := H(p^a) + H(p^b) - H(p^{ab}),$$

where $p^a = \{p^a_i := \sum_j p^{ab}_{ij}\}$ and $p^b = \{p^b_j := \sum_i p^{ab}_{ij}\}$ are the marginal probability distributions, $H(p^a) := -\sum_i p^a_i \log p^a_i$ is the Shannon entropy. The Shannon mutual information is dominated by the marginal entropies in the sense that [63]

$$I(p^{ab}) \leq \min\{H(p^a), H(p^b)\}.$$

In particular, for perfect correlations $p^{ab} = \{p^a_i \delta_{ij}\}$, it holds that $I(p^{ab}) = H(p^a) = H(p^b)$, which saturates the above upper bound and shows that the marginal entropy is fully employed to establish correlations in such a case. However, the above inequality fails in general for the quantum cases, as we will see shortly.

The Shannon mutual information for bivariate probability distributions can be straightforwardly extended to the quantum case as a measure of total correlations [64, 65]: For any bipartite quantum state (pure or mixed) ρ^{ab}, the amount of total correlations is well quantified by the quantum mutual information [12, 50, 66–69]

$$I(\rho^{ab}) := S(\rho^a) + S(\rho^b) - S(\rho^{ab}),$$

where $\rho^a := \mathrm{tr}_b \rho^{ab}$ and $\rho^b := \mathrm{tr}_a \rho^{ab}$ are the marginal states, and $S(\rho^a) := -\mathrm{tr}\rho^a \log \rho^a$ is the von Neumann entropy. However, unlike the classical case, the quantum mutual information is not dominated by the marginal entropies in general, but rather is dominated by twice of the marginal entropies, as shown by the celebrated Araki-Lieb inequality [65]

$$I(\rho^{ab}) \leq 2 \min\{S(\rho^a),\ S(\rho^b)\}.$$

This subtle factor 2 is really the origin of the difference between the classical and the quantum, and indicates the presence of quantum correlations, i.e., while the correlations in a classical bivariate probability distribution are always classical, there may exist both classical and quantum correlations in bipartite quantum states, which together constitute the total correlations, as quantified by the quantum mutual information. This can be most strikingly illustrated in terms of perfect correlations: In the classical case, the strongest correlations that party a with fixed marginal entropy $H(p^a)$ can possibly establish with another party b are described by the perfect correlations in the bivariate probability distribution $p^{ab} = \{p^{ab}_{ij} = p^a_i \delta_{ij}\}$, or in its quantum formalism, $\rho^{ab} = \sum_i p^a_i |i\rangle_a\langle i|_a \otimes |i\rangle_b\langle i|$ with $\{|i\rangle_a\}$ and $\{|i\rangle_b\}$ orthonormal bases for parties a and b, respectively. The amount of total correlations coincides with the marginal entropy, i.e.,

$$I(\rho^{ab}) = I(p^{ab}) = H(p^a) = S(\rho^a).$$

This is also the amount of classical correlations, and there are no quantum correlations here. In contrast, for the quantum case, consider the quantum pure state $\sigma^{ab} = |\Psi^{ab}\rangle\langle\Psi^{ab}|$ with the Schmidt decomposition $|\Psi^{ab}\rangle = \sum_i \sqrt{p^a_i}|i\rangle_a \otimes |i\rangle_b$ and the marginal $\sigma^a = \mathrm{tr}_b|\Psi^{ab}\rangle\langle\Psi^{ab}| = \rho^a$, the amount of total correlations is

$$I(\sigma^{ab}) = 2S(\sigma^a) = 2H(p^a) = 2S(\rho^a).$$

The extra amount of correlations in the quantum case, $I(\sigma^{ab}) - I(\rho^{ab}) = H(p^a)$, is the root lurking in the EPR argument and the state steering [2, 5–7].

The total correlations in a classically correlated state can be fully extracted by certain measurements, but this is not true for genuinely quantum correlated states. To see this and to facilitate the comparison between the classical and the quantum, we cast the classical bivariate probability distribution $p^{ab} = \{p^{ab}_{ij}\}$ in the quantum formalism as

$$\rho^{ab} = \sum_{ij} p^{ab}_{ij}|i\rangle_a\langle i| \otimes |j\rangle_b\langle j|$$

where $\{|i\rangle_a\}$ and $\{|j\rangle_b\}$ are orthonormal bases for parties a and b, respectively. The amount of total correlations in this state, as quantified by the quantum mutual information $I(\rho^{ab})$, coincides with the Shannon mutual information $I(p^{ab})$ in the bivariate probability distribution p^{ab}, i.e., $I(\rho^{ab}) = I(p^{ab})$. This can be interpreted as that all correlations in $\rho^{ab} = \sum_{ij} p^{ab}_{ij}|i\rangle_a\langle i| \otimes |j\rangle_b\langle j|$ are classical, and there are no quantum correlations in this state. Indeed, the state ρ^{ab} is left undisturbed after the local von Neumann measurements $\Pi^a = \{\Pi^a_i := |i\rangle_a\langle i|\}$ and $\Pi^b = \{\Pi^b_j := |j\rangle_b\langle j|\}$ by parties a and b, respectively, in the sense that $\rho^{ab} = \Pi(\rho^{ab})$, where

$$\Pi(\rho^{ab}) := \sum_{ij}(\Pi^a_i \otimes \Pi^b_j)\rho^{ab}(\Pi^a_i \otimes \Pi^b_j)$$

is the post-measurement state. All the correlations in this state are extracted by these measurements.

A characteristic feature of classicality is the invariance under certain quantum measurements. In contrast, disturbance under quantum measurements signifies quantumness. In the context of correlations, one may define a bipartite state to be classically correlated if it is left undisturbed under certain von Neumann measurement [70]. More precisely, for a bipartite state σ^{ab}, if there exist local von Neumann measurements $\{\Pi^a_i\}$ and $\{\Pi^b_j\}$ such that $\sigma^{ab} = \Pi(\sigma^{ab}) := \sum_{ij}(\Pi^a_i \otimes \Pi^b_j)\sigma^{ab}(\Pi^a_i \otimes \Pi^b_j)$, then σ^{ab} can be considered as a classically correlated state, and the correlations therein can be fully extracted without loss. In this case, σ^{ab} can be identified with the classical bivariate probability distribution $p^{ab} = \{p^{ab}_{ij} := \mathrm{tr}(\Pi^a_i \otimes \Pi^b_j)\sigma^{ab}\}$.

We have the following equivalent characterizations for the classically correlated states, which justify the notion of classicality of correlations [70–72]:

(1) σ^{ab} is classically correlated.
(2) σ^{ab} can be represented as $\sigma^{ab} = \sum_{ij} p^{ab}_{ij} \Pi^a_i \otimes \Pi^b_j$, where $p^{ab} = \{p^{ab}_{ij}\}$ is a bivariate probability distribution, Π^a_i and Π^b_j are orthogonal projections for parties a and b, respectively [70].
(3) σ^{ab} commutes with each $\Pi^a_i \otimes \Pi^b_j$, where Π^a_i and Π^b_j are the spectral projections of the reduced states $\sigma^a = \mathrm{tr}_b \sigma^{ab}$ and $\sigma^b = \mathrm{tr}_a \sigma^{ab}$, respectively [70].
(4) The correlations in σ^{ab} can be locally broadcast [71].
(5) Both parties a and b can establish perfect correlations with other systems [72].

Although a state σ^{ab} may not be classically correlated, the post-measurement state $\Pi(\sigma^{ab}) := \sum_{ij}(\Pi^a_i \otimes \Pi^b_j)\sigma^{ab}(\Pi^a_i \otimes \Pi^b_j)$ is always a classical state after any local von Neumann measurement $\Pi = \{\Pi^a_i \otimes \Pi^b_j\}$. By the monotonicity of quantum relative entropy [65],

$$I(\Pi(\sigma^{ab})) \leq I(\sigma^{ab}),$$

and the difference $I(\sigma^{ab}) - I(\Pi(\sigma^{ab}))$ signifies the loss caused by the measurements and captures quantumness of correlations.

Similarly, one may also define classicality of correlations with respect to one party. More precisely, one defines σ^{ab} to be classical-quantum if there exists a local von Neumann measurement $\Pi^a = \{\Pi^a_i\}$ for party a which leaves the state undisturbed in the sense that $\sigma^{ab} = \Pi^a(\sigma^{ab})$, where

$$\Pi^a(\sigma^{ab}) := \sum_i (\Pi^a_i \otimes \mathbf{1}^b)\sigma^{ab}(\Pi^a_i \otimes \mathbf{1}^b)$$

is the post-measurement state after party a performs the quantum measurement Π^a. Analogously, the following characterizations of classical-quantum states are equivalent [70, 73]:

(1) σ^{ab} is classical-quantum.
(2) σ^{ab} can be represented as $\sigma^{ab} = \sum_i p_i \Pi^a_i \otimes \sigma^b_i$, where $\{p_i\}$ is a probability distribution, Π^a_i are orthogonal projections for party a, and σ^b_i are local states for party b.
(3) σ^{ab} commutes with each $\Pi^a_i \otimes \mathbf{1}^b$, where Π^a_i are the spectral projections of $\sigma^a := \mathrm{tr}_b \sigma^{ab}$.
(4) The correlations in σ^{ab} can be locally broadcast by party a [73].

In general, a classical-quantum state may not be classically correlated due to the non-commutativity of σ^b_i for party b, and it is impossible to identify such a state with a classical bivariate probability distribution in general.

All the above characterizations are intimately related to (and actually equivalent to) the celebrated quantum no-broadcasting theorem [73]: A family of quantum states can be broadcast if and only if the states commute [74].

Motivated by the idea that classical correlations are those that can be extracted via quantum measurements, i.e., the maximum amount of correlations extractable by local von Neumann measurements, a straightforward measure of classical correlations in a bipartite quantum state may be defined as [11, 12]

$$C^a(\rho^{ab}) := \max_{\Pi^a} I(\Pi^a(\rho^{ab})),$$

where the maximization is over all local von Neumann measurements Π^a for party a. One can similarly define $C^b(\rho^{ab})$ with the measurement performed on party b, or in a symmetric fashion [15, 21],

$$C(\rho^{ab}) = \max_{\Pi} I(\Pi(\rho^{ab}))$$

with the maximization over all local von Neumann measurements $\Pi = \{\Pi_i^a \otimes \Pi_j^b\}$. In general, $C^a(\rho^{ab}) \neq C^b(\rho^{ab})$ and by the monotonicity of quantum relative entropy,

$$C(\rho^{ab}) \le C^a(\rho^{ab}) \le I(\rho^{ab}), \quad C^{ab}(\rho^{ab}) \le C^a(\rho^{ab}) \le S(\rho^a), \quad C^{ab}(\rho^{ab}) \le C^b(\rho^{ab}) \le S(\rho^b).$$

However, it may happen that $C^a(\rho^{ab}) > S(\rho^b)$ [75].

The original discord of a bipartite state ρ^{ab} is defined as [11]

$$Q^a(\rho^{ab}) := I(\rho^{ab}) - C^a(\rho^{ab}),$$

which is asymmetric with respect to the two parties. It is known that $Q^a(\rho^{ab}) = 0$ if and only if ρ^{ab} is classical-quantum. A symmetric version of discord in a bipartite state ρ^{ab} is defined as the difference [21]

$$Q(\rho^{ab}) := I(\rho^{ab}) - C(\rho^{ab})$$

between the amounts of total correlations and classical correlations, and thus summarizes quantum correlations in a state. Clearly, $Q(\rho^{ab}) = 0$ if and only if ρ^{ab} is classically correlated.

In general, discord and classical correlations can be defined with respect to other general distance-like measures [23, 70], which yield the relative entropy of quantumness [23], the geometric discord based on Hilbert-Schmidt distance (or the trace distance, or the Bures distance) [24, 25, 43, 45, 47, 48], etc. Here we recall that the relative entropy of quantumness, which will be used late, is defined as [23]

$$Q_{\mathrm{rel}}(\rho^{ab}) := \min_{\Pi} D(\rho^{ab}|\Pi(\rho^{ab})),$$

where the minimization is over all local von Neumann measurements $\Pi = \{\Pi_i^a \otimes \Pi_j^b\}$, i.e., the relative entropy of quantumness is defined as the minimal distance between ρ^{ab} and the set of classically correlated states, with the (pseudo-)distance being the quantum relative entropy $D(\rho^{ab}|\sigma^{ab}) := \mathrm{tr}\rho^{ab}(\log\rho^{ab} - \log\sigma^{ab})$.

Now, we come to the separability/entanglement paradigm. A state ρ^{ab} shared between two parties a and b is called separable if it has a decomposition [52]

$$\rho^{ab} = \sum_i p_i \rho_i^a \otimes \rho_i^b$$

with local states ρ_i^a and ρ_i^b for parties a and b, respectively, and $p_i \geq 0$, $\sum_i p_i = 1$. Otherwise it is called entangled (nonseparable). Various entanglement measures, such as the entanglement of formation, entanglement cost, distillable entanglement, squashed entanglement, robustness of entanglement, etc., have been introduced to quantify different aspects of entanglement [57, 58]. In particular, the relative entropy of entanglement is defined as [23, 54]

$$E_{\mathrm{rel}}(\rho^{ab}) := \min_{\sigma^{ab}} D(\rho^{ab}|\sigma^{ab})$$

where the minimization is over all separable states σ^{ab}. Thus the relative entropy of entanglement is the minimal distance between ρ^{ab} and the set of separable (rather than classically correlated) states. Accordingly, the relative entropy of entanglement is always dominated by the relative entropy of quantumness, i.e., $E_{\mathrm{rel}}(\rho^{ab}) \leq Q_{\mathrm{rel}}(\rho^{ab})$, since the set of classically correlated states is a strict subset of the set of separable states.

Discord and entanglement are both measures of quantum correlations beyond classical ones. They coincide for pure states but differ for mixed states. Discord and entanglement have similarities as well as radical difference. On one hand, discord and entanglement are quite different: The phenomenon of discord is a manifestation of quantum correlations due to non-commutativity rather than nonlocality. Classically correlated states are separable, but the converse is not true. Separable state may possess non-zero discord, although their entanglement vanish. In this sense, discord can be regarded as a more general type of quantum correlations than entanglement. On the other hand, separable states may be helpful in distributing and manipulating entanglement [76–79], and entanglement can be indirectly linked to discord created in quantum measurements [34–36]. Furthermore, there are quantitative relations connecting entanglement between two parties a and b with the discord between party a and a third party c which serves to purify the state possessed by ab [80, 81]. More precisely, the Koasi-Winter formula $\mathcal{C}^b(\rho^{ab}) + E_f(\rho^{ac}) = S(\rho^a)$ implies that [80]

$$E_f(\rho^{ac}) = \mathcal{Q}^b(\rho^{ab}) + S(\rho^{ab}|\rho^b),$$

where $|\Psi^{abc}\rangle$ is a purification of ρ^{ab} with $\rho^{ab} = \mathrm{tr}_c|\Psi^{abc}\rangle\langle\Psi^{abc}|$, $\rho^{bc} = \mathrm{tr}_a|\Psi^{abc}\rangle\langle\Psi^{abc}|$, $\rho^a = \mathrm{tr}_{bc}|\Psi^{abc}\rangle\langle\Psi^{abc}|$, $E_f(\cdot)$ is the entanglement of formation, and

$S(\rho^{ab}|\rho^b) := S(\rho^{ab}) - S(\rho^b)$ is the quantum conditional entropy, $\mathcal{C}^b(\rho^{ab})$ and $\mathcal{Q}^b(\rho^{ab})$, similar to $C^b(\rho^{ab})$ and $Q^b(\rho^{ab}) = I(\rho^{ab}) - C^b(\rho^{ab})$, are the measures of classical correlations and discord defined in terms of general POVMs rather than von Neumann measurements [12].

3 Entanglement as Discord

A remarkable relation between the two classification schemes for correlations, classical/quantum [70, 71] and separable/entanglement [52], is that on one hand, a classically correlated state is always separable, on the other hand, any separable state can be imbedded into a classically correlated state in the sense that for any separable state ρ^{ab}, there is a classically correlated state $\rho^{a'a:bb'}$ shared between aa' and bb' such that

$$\rho^{ab} = \mathrm{tr}_{a'b'}\rho^{a'a:bb'},$$

where a' and b' are two ancillary systems [82]. Any entangled state does not admit such an extension. Phrased alternatively, a bipartite state is separable if and only if it admits an extension which is classically correlated with the natural bipartition, i.e., with a and b in different parties. This identifies entanglement as truly nonlocal quantum correlations, and has some interesting consequences [83–85]. Here we will exploit it to define entanglement in terms of discord. More precisely, for any reasonable measure of discord $\mathcal{Q}(\cdot)$, not necessary defined in terms of the quantum mutual information as the original one, we define

$$\mathcal{E}(\rho^{ab}) := \min_{\mathrm{tr}_{a'b'}\rho^{a'a:bb'}=\rho^{ab}} \mathcal{Q}(\rho^{a'a:bb'}),$$

where the minimization is over all state extensions $\rho^{a'a:bb'}$ of ρ^{ab} (i.e., $\rho^{ab} = \mathrm{tr}_{a'b'}\rho^{a'a:bb'}$), including the cases when a' or b' is trivial (one dimensional), and the discord $\mathcal{Q}(\rho^{a'a:bb'})$ is taken with respect to the bipartition $a'a : bb'$. This renders entanglement to a kind of shadow of discord, i.e., the minimal discord over state extensions.

Clearly, $\mathcal{E}(\rho^{ab}) = 0$ for separable ρ^{ab}. This follows from the theorem in Li and Luo [82] concerning the relation between separable states and classical states: A bipartite state ρ^{ab} is separable if and only if it can be extended to a certain classical state $\rho^{a'a:bb'}$ (with respect to the bipartition $a'a : bb'$).

The entanglement measure $\mathcal{E}(\cdot)$ has the nice property that it is automatically dominated by the discord in the sense that

$$\mathcal{E}(\rho^{ab}) \le \mathcal{Q}(\rho^{ab})$$

since $\rho^{a'a:bb'} = \rho^{ab}$ with the a' and b' being trivial (one-dimensional) can be regarded as a state extension of ρ^{ab} itself.

With the above property, we may decompose the total correlations, as quantified by the quantum mutual information $I(\rho^{ab})$, into classical correlations $C(\rho^{ab})$ plus dissonance $\mathcal{D}(\rho^{ab})$ plus entanglement $\mathcal{E}(\rho^{ab})$:

$$I(\rho^{ab}) = C(\rho^{ab}) + \mathcal{D}(\rho^{ab}) + \mathcal{E}(\rho^{ab}),$$

where the difference

$$\mathcal{D}(\rho^{ab}) := \mathcal{Q}(\rho^{ab}) - \mathcal{E}(\rho^{ab})$$

is interpreted as a measure of dissonance as termed by Kavan et al. [23].

$\mathcal{E}(\cdot)$ is locally unitary invariant in the sense that

$$\mathcal{E}((U^a \otimes U^b)\rho^{ab}(U^a \otimes U^b)^\dagger) = \mathcal{E}(\rho^{ab})$$

for any unitary operators U^a and U^b on parties a and b, respectively, as long as the discord is invariant under local unitary transformations.

Since any pure state $\rho^{ab} = |\Psi^{ab}\rangle\langle\Psi^{ab}|$ has only trivial extensions of the form $\rho^{a'b'} \otimes |\Psi^{ab}\rangle\langle\Psi^{ab}|$, it follows that the entanglement $\mathcal{E}(\rho^{ab})$ coincides with the discord $\mathcal{Q}(\rho^{ab})$, i.e., $\mathcal{E}(\rho^{ab}) = \mathcal{Q}(\rho^{ab})$, for any pure state ρ^{ab}, as long as the discord has the decreasing property $\mathcal{Q}(\rho^{a'b'} \otimes |\Psi^{ab}\rangle\langle\Psi^{ab}|) \geq \mathcal{Q}(|\Psi^{ab}\rangle\langle\Psi^{ab}|)$.

Since any state extension $\rho^{a''a'a:bb'b''}$ of $\rho^{a'a:bb'}$ is necessarily a state extension of the reduced state $\rho^{ab} = \mathrm{tr}_{a'b'}\rho^{a'a:bb'}$, it follows from

$$\begin{aligned}\mathcal{E}(\rho^{ab}) &\leq \min_{\mathrm{tr}_{a''a'b'b''}\rho^{a''a'a:bb'b''}=\rho^{ab}} \mathcal{Q}(\rho^{a''a'a:bb'b''}) \\ &\leq \min_{\mathrm{tr}_{a''b''}\rho^{a''a'a:bb'b''}=\rho^{a'a:bb'}} \mathcal{Q}(\rho^{a''a'a:bb'b''}) \\ &= \mathcal{E}(\rho^{a'a:bb'})\end{aligned}$$

that $\mathcal{E}(\cdot)$ is non-increasing under local partial trace (state reduction) in the sense that

$$\mathcal{E}(\rho^{ab}) \leq \mathcal{E}(\rho^{a'a:bb'})$$

for any state extension $\rho^{a'a:bb'}$ of ρ^{ab}.

We list some important and interesting problems requiring further investigations:

(1) Classify the discord measures such that the induced entanglement measures are convex in the sense that

$$\mathcal{E}(\sum_i p_i\rho_i^{ab}) \leq \sum_i p_i\mathcal{E}(\rho_i^{ab}),$$

where ρ_i^{ab} are bipartite states shared by parties a and b, and $p_i \geq 0,\ \sum_i p_i = 1$. We remark that this may be related to the direct sum property of the discord measures.

(2) More generally, classify the discord measures such that the induced entanglement measures are entanglement monotones.

(3) How to evaluate the entanglement measures? On may try to find some analytical formulas for some highly symmetric states, and establish some bounds for general cases.

(4) What are the relations between the relative entropy of entanglement and the relative entropy of quantumness? If one defines an entanglement measure induced by the relative entropy of quantumness $Q_{\mathrm{rel}}(\cdot)$ as

$$\mathcal{E}_{\mathrm{rel}}(\rho^{ab}) := \min_{\mathrm{tr}_{a'b'}\rho^{a'a:bb'}=\rho^{ab}} Q_{\mathrm{rel}}(\rho^{a'a:bb'}),$$

where the minimization is over all state extensions $\rho^{a'a:bb'}$ of ρ^{ab} (i.e., $\rho^{ab} = \mathrm{tr}_{a'b'}\rho^{a'a:bb'}$), then an interesting question arises as the relation between this induced entanglement measure $\mathcal{E}_{\mathrm{rel}}(\cdot)$ and the original relative entropy of entanglement $E_{\mathrm{rel}}(\cdot)$: Does it hold that

$$E_{\mathrm{rel}}(\rho^{ab}) = \mathcal{E}_{\mathrm{rel}}(\rho^{ab})?$$

Since

$$E_{\mathrm{rel}}(\rho^{ab}) \leq E_{\mathrm{rel}}(\rho^{a'a:bb'}) \leq Q(\rho^{a'a:bb'}),$$

we have

$$E_{\mathrm{rel}}(\rho^{ab}) \leq \mathcal{E}_{\mathrm{rel}}(\rho^{ab}),$$

thus it remains to establish the reversed inequality.

4 Discussions

Discord stems directly from the pivotal and ubiquitous notion of quantum measurements, while entanglement is widely regarded as a key feature of quantum information. We have reviewed briefly several aspects of discord and entanglement with emphasis on their intertwining, and have illustrated that discord is not only a kind of quantum correlations beyond entanglement, but also that quantum discord contracts to entanglement, i.e., entanglement can be interpreted as the irreducible residue, as the minimal shadow, of discord over all state extensions. This puts discord, conceptually, in a more primitive place than entanglement, sheds lights on the fundamental importance of quantumness in characterizing quantum correlations, and highlights the significance of the interplay between quantum measurements and state extensions in quantum information science. We have outlined some problems for further investigations.

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