Parametric Transfer Matrices for Sampled-Data Control Systems with Linear Continuous Periodic Process and Control Delay

Bernhard P. Lampe and Efim N. Rosenwasser

Abstract For the class of multi-input multi-output systems being composed of a linear continuous periodic (LCP) process, pure delay, and a digital controller, the paper provides closed expressions for the parametric transfer matrix (PTM), even for the difficult, but practically important case, when the external excitations act on continuous system parts. In the same way as ordinary transfer matrices in the linear time-invariant (LTI) case, the PTM for LCP systems is a fundamental concept for analysis and design of those systems. The properties of the constructed parametric transfer matrices as functions of the real parameter and the complex variable are investigated. These properties are similar to those from ordinary transfer matrices, so that the PTM after some modifications, can be applied with similar tools. Moreover, formulae are derived that are appropriate for the practical computation of the PTM. An example demonstrates, how the formulae can be handled.

1 Introduction

The classical approach for analysis and design of sampled-data (SD) systems containing continuous LTI processes consists of the transfer to a discrete (-time) model [1, 2, 6, 28]. This approach only provides exact solutions in the case, when all continuous input signals are sampled before acting on the continuous system parts. Then it is sufficient to consider the problems from the viewpoint of the computer, i.e., computer-oriented models are adequate [2]. However, in most practical situations, this condition does not hold, because, e.g., continuous disturbances directly act on the continuous plant. In those cases, a rigorous solution needs the application of process-oriented models of the system, which are more complicated, because the closed loop establishes itself as linear continuous periodic (LCP) nonstationary system.

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Hence the traditional approach by ordinary (continuous or discrete) transfer functions or state-space descriptions with matrices of finite size cannot be used. The exact description is ordinarily in the focus, when the concept of sampled-data (SD) systems (in a stricter sense) is addressed. As in the LTI case, two principle approaches have been established for the description of SD systems—one works in time domain and uses state-space representations, the other one works in frequency domain and applies input–output representations. The lifting method [4, 27] in state space and two methods in frequency domain, namely the frequency response (FR) operator [5], and the parametric transfer matrix (PTM) [23, 24] are most common. The PTM completely bases on a frequency domain description, so transformation to state space is not required. This fact makes PTM interesting for the solution of design problems for SD systems with delay [10, 11].

In the monographs [21, 22], it was already shown that the PTM $W(\lambda, t)$ is useful for analysis and design of linear continuous periodic (LCP) systems. In contrast to the ordinary transfer matrix $W(\lambda)$ for continuous LTI systems, the PTM does not only depend on the complex frequency variable λ, but also on real parameter time t. In numerous works, see, e.g., [7, 9, 16–18, 20, 25], the PTM was used to solve various kinds of control problems mainly for single-input single-output (SISO) SD systems with arbitrary delay, including stability and stabilization, modal control, advanced statistical analysis, $\mathscr{H}_2$, $\mathscr{H}_\infty$ or $\mathscr{L}_2$ optimization, see [12–15]. In [24], the PTM method was generalized for multi-input multi-output (MIMO) SD systems.

In [11] the PTM is constructed for SD systems containing an LCP process and delay, under the assumption that the input signal acts to the sampler. The present paper constructs the PTM in the practically more important case, when the input signal acts directly to the LCP element. Moreover, the properties of the PTM as function of the complex variable λ are investigated. The results open possibilities to solve for this class of systems important kinds of control problems, e.g., stability and stabilization, modal control, advanced statistical analysis, $\mathscr{H}_2$, $\mathscr{H}_\infty$ or $\mathscr{L}_2$ optimization with the help of methods and tools, which are elaborated by the authors and co-workers [3, 19].

2 Mathematical Description of LCP Processes

Below, it is supposed that the LCP process is described by the state equation

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = A(t)v(t) + B(t)r(t) \tag{1}$$

and the output equation

$$y(t) = C(t)v(t). \tag{2}$$

Here, $v(t), r(t), y(t)$ are vectors of dimensions $\chi \times 1$, $m \times 1$, $n \times 1$, respectively, and $A(t) = A(t+T)$, $B(t) = B(t+T)$, $C(t) = C(t+T)$ are real periodic matrices of

appropriate size. Furthermore, we suppose that these matrices are continuous and of bounded variation in $[0, T]$.
Assume the matrix equation

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = A(t)v(t), \tag{3}$$

where $v(t)$ is a $\chi \times \chi$ matrix. Under the initial condition $v(0) = I_\chi$, where I_χ is the $\chi \times \chi$ identity matrix, we obtain the solution $H(t)$. As is known, [26], the matrix $H(t)$ allows a representation of the form

$$H(t) = L(t)\mathrm{e}^{Nt}, \tag{4}$$

where $L(t) = L(t + T)$ is a nonsingular continuously differentiable $\chi \times \chi$ matrix, and N is a constant $\chi \times \chi$ matrix. Hereby, without loss of generality, the matrices $L(t)$ and N can be assumed to be real.
With the help of the Lyapunov transformation

$$v(t) = L(t)v_L(t), \tag{5}$$

state equation (1) appears as

$$\frac{\mathrm{d}v_L(t)}{\mathrm{d}t} = Nv_L(t) + L^{-1}(t)B(t)r(t), \tag{6}$$

and the output equation as

$$y(t) = C(t)L(t)v_L(t). \tag{7}$$

Indeed, differentiating (5) yields

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = \frac{\mathrm{d}L(t)}{\mathrm{d}t}v_L(t) + L(t)\frac{\mathrm{d}v_L(t)}{\mathrm{d}t}. \tag{8}$$

Moreover, from (4) we find

$$L(t) = H(t)\mathrm{e}^{-Nt}, \tag{9}$$

and after differentiation

$$\begin{aligned}\frac{\mathrm{d}L(t)}{\mathrm{d}t} &= \frac{\mathrm{d}H(t)}{\mathrm{d}t}\mathrm{e}^{-Nt} - H(t)\mathrm{e}^{-Nt}N = A(t)H(t)\mathrm{e}^{-Nt} - L(t)N \\ &= A(t)L(t) - L(t)N.\end{aligned} \tag{10}$$

Inserting (10) into (8), we achieve

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = A(t)L(t)v_L(t) - L(t)Nv_L(t) + L(t)\frac{\mathrm{d}v_L(t)}{\mathrm{d}t} \tag{11}$$

Formula (1) with the help of (5) can be written in the form

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = A(t)L(t)v_L(t) + B(t)u(t) \tag{12}$$

Comparing (11) and (12), we achieve

$$L(t)Nv_L(t) + L(t)\frac{\mathrm{d}v_L(t)}{\mathrm{d}t} = B(t)u(t). \tag{13}$$

Since the matrix $L(t)$ for all t is nonsingular, so from (13) we directly derive (6). Moreover, inserting (5) into (2) yields (7).

Further on, the totality of equations (6), (7) is called the $\mathscr{L}$-equivalent LCP process.

The matrix

$$M = \mathrm{e}^{NT} \tag{14}$$

commonly is named the monodromy matrix of state equation (1), and the roots of equation

$$\det(I_\chi - \zeta M) = 0 \tag{15}$$

are its inverse multipliers.

3 PTM of Open SD System with LCP Process and Delay

In this section, we consider the open SD system with delay, as presented in Fig. 1.

In Fig. 1, DC stands for the digital controller, containing the analogue to digital converter (ADC), the control program (CP), and the digital to analogue converter (DAC), respectively described by the equations

$$\begin{aligned} \xi_k &= x(kT), \quad (k = 0, \pm 1, \dots) \\ \alpha(\zeta)\psi_k &= \beta(\zeta)\xi_k, \\ u(t) &= h(t - kT)\psi_k, \quad kT + 0 \le t \le (k+1)T - 0. \end{aligned} \tag{16}$$

Here $x(t), \psi_k, u(t)$ are $n \times 1$, $q \times 1$, $m \times 1$ vectors, and ζ is the shift operator for one step backward, [2]. Further, $\alpha(\zeta), \beta(\zeta)$ are polynomial matrices of size $q \times q$ and

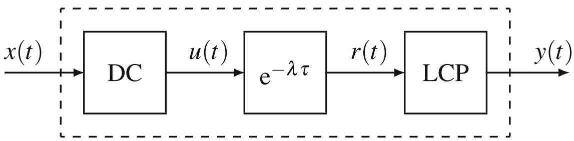

Fig. 1 Open SD system with LCP process and delay

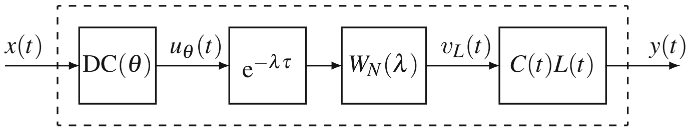

Fig. 2 Open SD system after Lyapunov transformation

$q \times n$, respectively, satisfying the causality condition det $\alpha(0) \neq 0$. Moreover, $h(t)$ is an $m \times q$ matrix, where its elements are of bounded variation at the interval $0 < t < T$. Finally, in Fig. 1 τ is a real constant, for which we use the two decompositions

$$\tau = \mu T + \theta = (\mu + 1)T - \gamma, \tag{17}$$

where μ is a nonnegative integer, and $0 \leq \theta < T, 0 < \gamma \leq T$.

Below, we will always suppose that the LCP process is given by its $\mathscr{L}$-equivalent model (6), (7). The open chain under investigation can be configured to the structure of Fig. 2, containing a dynamic time-invariant continuous element.

In Fig. 2 the block DC(θ) is the conventional digital controller, described by the equations

$$\begin{aligned} \xi_k &= x(kT), \quad (k = 0, \pm 1, ...), \\ \alpha(\zeta)\psi_k &= \beta(\zeta)\xi_k, \\ u_\theta(t) &= h_\theta(t - kT)\psi_k, \quad kT + 0 \leq t \leq (k+1)T - 0, \end{aligned} \tag{18}$$

where

$$h_\theta(t) = L^{-1}(t + \theta)B(t + \theta)h(t). \tag{19}$$

Moreover, in Fig. 2 $W_N(\lambda)$ is the rational matrix

$$W_N(\lambda) = (\lambda I_\chi - N)^{-1}. \tag{20}$$

As is seen from Fig. 2, a conventional DC and a continuous LTI block are acting between the input $x(t)$ and output $v_L(t)$. Therefore, formulae from [24] can be applied for the calculation of the PTM $W_{Lx}(\lambda, t)$ from the input $x(t)$ to the output $v_L(t)$ with the result

$$W_{Lx}(\lambda, t) = \mathrm{e}^{-\lambda t}\mathrm{e}^{-(\mu+1)\lambda T}\tilde{\mathscr{I}}_N(\lambda, t)\tilde{W}_d(\lambda). \tag{21}$$

Here

$$\begin{aligned} \tilde{\mathscr{I}}_N(\lambda, t) &\stackrel{\triangle}{=} \int_0^T \tilde{D}_N(T, \lambda, t + \gamma - \nu)h_\theta(\nu)\,\mathrm{d}\nu, \\ \tilde{W}_d(\lambda) &= \alpha^{-1}(\zeta)\beta)\zeta)\Big|_{\zeta=\mathrm{e}^{-\lambda T}}, \end{aligned} \tag{22}$$

where

$$\tilde{D}_N(T,\lambda,t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} W_N(\lambda + k\mathrm{j}\omega)\mathrm{e}^{(\lambda+k\mathrm{j}\omega)t}, \tag{23}$$

which is determined for $-\infty < t < \infty$ by the formulae [24]

$$\begin{aligned} \tilde{D}_N(T,\lambda,t) &= \tilde{\mathscr{D}}_N(T,\lambda,t) = \mathrm{e}^{Nt}(I_\chi - \mathrm{e}^{-\lambda T}M)^{-1} \\ &= (I_\chi - \mathrm{e}^{-\lambda T}M)^{-1}\mathrm{e}^{Nt}, \quad 0 < t < T, \\ \tilde{D}_N(T,\lambda,t+T) &= \tilde{D}_N(T,\lambda,t)\mathrm{e}^{\lambda T}. \end{aligned} \tag{24}$$

Using (24), (17) and (4), from (22), we obtain

$$\tilde{\mathscr{I}}_N(\lambda,t) = \mathrm{e}^{Nt}\tilde{F}(\lambda,t)\left[(I_\chi - e^{-\lambda T}M)^{-1}P(\theta) - R(t,\theta))\right], \tag{25}$$

where we denoted

$$\begin{aligned} \tilde{F}(\lambda,t) &= \begin{cases} M, & 0 < t < \theta, \\ \mathrm{e}^{\lambda T}I_\chi, & \theta < t < T, \end{cases} \\ P(\theta) &= \int_0^T H^{-1}(\nu+\theta)B(\nu+\theta)h(\nu)\,\mathrm{d}\nu, \\ R(t,\theta) &= \begin{cases} \int_{t+\gamma}^T H^{-1}(\nu+\theta)B(\nu+\theta)h(\nu)\,\mathrm{d}\nu, & 0 < t < \theta, \\ \int_{t-\theta}^T H^{-1}(\nu+\theta)B(\nu+\theta)h(\nu)\,\mathrm{d}\nu, & \theta < t < T. \end{cases} \end{aligned} \tag{26}$$

Formula (25) is extended onto the whole axis $-\infty < t < \infty$ with the help of the relation

$$\tilde{\mathscr{I}}_N(\lambda,t+T) = \tilde{\mathscr{I}}_N(\lambda,t)\mathrm{e}^{\lambda T}. \tag{27}$$

Regarding (21)–(27), it can be shown that the PTM $W_{Lx}(\lambda,t)$ is continuous with respect to t for all t.

Now, from Fig. 2, we easily derive an expression for the PTM $W_{yx}(\lambda,t)$ from the input $x(t)$ to the output $y(t)$

$$\begin{aligned} W_{yx}(\lambda,t) &= C(t)L(t)W_{Lx}(\lambda,t) \\ &= \mathrm{e}^{-\lambda t}\mathrm{e}^{-(\mu+1)\lambda T}C(t)L(t)\tilde{\mathscr{I}}_N(\lambda,t)\tilde{W}_d(\lambda). \end{aligned} \tag{28}$$

Under the taken propositions, the PTM $W_{yx}(\lambda,t)$ is continuous with respect to t.

4 PTM of Closed SD System with LCP Process and Delay

Consider the closed SD system $\mathscr{S}_\tau$, shown in Fig. 3 described by the equations

$$\begin{aligned} \frac{\mathrm{d}v(t)}{\mathrm{d}t} &= A(t)v(t) + B_1(t)g(t) + B(t)u(t-\tau), \\ y(t) &= C(t)v(t), \\ \xi_k &= y(kT), \quad (k = 0, \pm 1, \ldots), \\ \alpha(\zeta)\psi_k &= \beta(\zeta)\xi_k, \\ u(t) &= h(t-kT)\psi_k, \quad kT + 0 \le t \le (k+1)T - 0, \end{aligned} \tag{29}$$

where all propositions of the last section should still hold, and in addition $B_1(t) = B_1(t+T)$ is a $\chi \times \ell$ matrix, where its elements are continuous functions of bounded variation inside the period. When we apply in (29) Lyapunov transformation (5), then we achieve the equivalent system of equations, which is the starting point of the further investigations

$$\begin{aligned} \frac{\mathrm{d}v_L(t)}{\mathrm{d}t} &= Nv_L(t) + L^{-1}(t)B_1(t)g(t) + u_\theta(t-\tau), \\ y(t) &= C(t)L(t)v_L(t), \\ \xi_k &= C(0)v_L(kT), \quad (k = 0, \pm 1, \ldots), \\ \alpha(\zeta)\psi_k &= \beta(\zeta)\xi_k, \\ u_\theta(t) &= h_\theta(t-kT)\psi_k, \quad kT + 0 \le t \le (k+1)T - 0, \end{aligned} \tag{30}$$

where we used that $L(0) = I_\chi$.

Observing the structure of the open system in Fig. 2, the closed system $\mathscr{S}_\tau$ can be configured to the structure shown in Fig. 4.

According to the general approach, [23, 24], for the determination of the PTM for the system $\mathscr{S}_\tau$, we assume

$$g(t) = \mathrm{e}^{\lambda t} I_\ell \tag{31}$$

and find the solution of equation (30), where

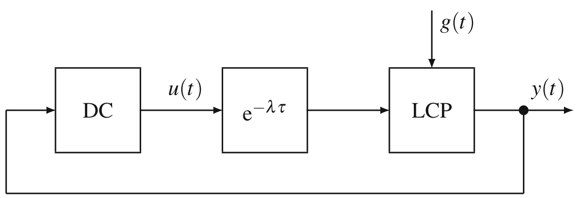

Fig. 3 Closed SD system $\mathscr{S}_\tau$

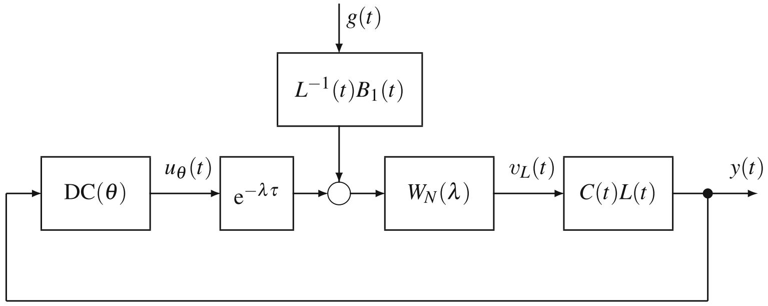

Fig. 4 Closed SD system $\mathscr{S}_\tau$ after Lyapunov transformation

$$
\begin{aligned}
v_L(\lambda,t) &= \mathrm{e}^{\lambda t} W_{Lg}(\lambda,t), \qquad & W_{Lg}(\lambda,t) &= W_{Lg}(\lambda,t+T),\\
y(\lambda,t) &= \mathrm{e}^{\lambda t} W_{yg}(\lambda,t), \qquad & W_{yg}(\lambda,t) &= W_{yg}(\lambda,t+T),
\end{aligned}
\tag{32}
$$

where $W_{Lg}(\lambda,t)$ and $W_{yg}(\lambda,t)$ are the PTM from the input $g(t)$ to the outputs $v_L(t)$ and $y(t)$, respectively.

Theorem 1 *If (17) is valid, then the PTM $W_{Lg}(\lambda,t)$ and $W_{yg}(\lambda,t)$ are determined by the formulae*

$$
\begin{aligned}
W_{Lg}(\lambda,t) &= \mathrm{e}^{-\lambda t}\mathrm{e}^{-(\mu+1)\lambda T}\tilde{\mathscr{I}}_N(\lambda,t)\tilde{R}(\lambda)C(0)g_1(\lambda,0) + g_1(\lambda,t),\\
W_{yg}(\lambda,t) &= C(t)L(t)W_{Lg}(\lambda,t),
\end{aligned}
\tag{33}
$$

where, in addition to the above notations

$$
\begin{aligned}
\tilde{R}(\lambda) &= \tilde{W}_d(\lambda)\left[I_n - \mathrm{e}^{-(\mu+1)\lambda T}C(0)\tilde{\mathscr{I}}_N(\lambda,0)\tilde{W}_d(\lambda)\right]^{-1},\\
g_1(\lambda,t) &= \mathrm{e}^{-\lambda t}\int_0^T \tilde{D}_N(T,\lambda,t-\nu)L^{-1}(\nu)B_1(\nu)\mathrm{e}^{\lambda\nu}\,\mathrm{d}\nu.
\end{aligned}
\tag{34}
$$

Proof Assume in system $\mathscr{S}_\tau$ the regime (31), (32). Then the closed system in Fig. 4, respecting the stroboscopic property of the digital controller [23], can be configured to the open system in Fig. 5.

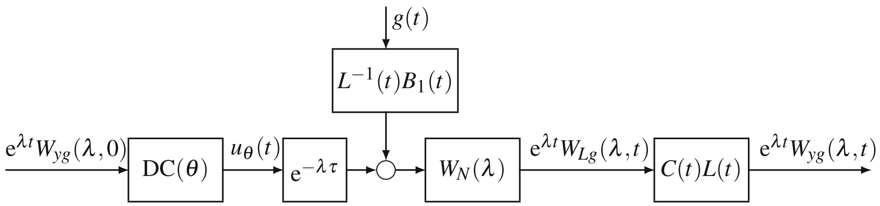

Fig. 5 Open loop equivalent of Fig. 4

Applying the formulae of Sect. 2, after reduction by $e^{\lambda t}$, we find

$$C(t)L(t)W_{Lx}(\lambda,t)W_{yg}(\lambda,0)+C(t)L(t)g_1(\lambda,t)=W_{yg}(\lambda,t), \tag{35}$$

where $g_1(\lambda,t)$ is the matrix in (34). Since the left and right side of the last equation are continuous according to t, so for $t=0$, we obtain

$$\begin{aligned} W_{yg}(\lambda,0) &= \left[I_n - C(0)W_{Lx}(\lambda,0)\right]^{-1} C(0)g_1(\lambda,0) \\ &= \left[I_n - e^{-(m+1)\lambda T}C(0)\tilde{\mathscr{I}}_N(\lambda,0)\tilde{W}_d(\lambda)\right]^{-1} C(0)g_1(\lambda,0). \end{aligned} \tag{36}$$

Substituting this expression on the left side of (35), we achieve the second formula in (33). The first formula in (33) directly follows from (36) and Fig. 5. □

5 PTM of System $\mathscr{S}_\tau$ as Function of the Argument λ

As was stated before, the PTM $W_{Lg}(\lambda,t)$ and $W_{yg}(\lambda,t)$ depend continuously on t for all λ, excluding a certain set of singular points. In this section we will show that these singular points are poles, and therefore, for all t the matrices $W_{Lg}(\lambda,t)$, $W_{yg}(\lambda,t)$, $W_{vg}(\lambda,t)$ are meromorphic functions of the argument λ.

Denote

$$\begin{aligned} \tilde{a}(\lambda) &= I_\chi - e^{-\lambda T}M, \\ \tilde{b}(\lambda) &= M\left[e^{-\lambda T}MA_3(\theta)+A_1(\theta)\right], \end{aligned} \tag{37}$$

where the matrices $A_1(\theta)$ and $A_3(\theta)$ are defined by the relations

$$\begin{aligned} A_1(\theta) &= \int_0^\gamma H^{-1}(\nu+\theta)B(\nu+\theta)h(\nu)\,d\nu, \\ A_3(\theta) &= \int_\gamma^T H^{-1}(\nu+\theta)B(\nu+\theta)h(\nu)\,d\nu. \end{aligned} \tag{38}$$

Further assume

$$\tilde{\alpha}(\lambda) = \alpha(\zeta)|_{\zeta=e^{-\lambda T}}, \quad \tilde{\beta}(\lambda) = \beta(\zeta)\big|_{\zeta=e^{-\lambda T}}. \tag{39}$$

Then, by using (37)–(39), we can construct the block matrix

$$\tilde{Q}_T(\lambda,\tilde{\alpha},\tilde{\beta}) = \begin{bmatrix} I_\chi - e^{-\lambda T}M & 0_{\chi n} & -e^{-(\mu+1)\lambda T}\tilde{b}(\lambda) \\ -C(0) & I_n & 0_{nq} \\ 0_{q\chi} & -\tilde{\beta}(\lambda) & \tilde{\alpha}(\lambda) \end{bmatrix}, \tag{40}$$

where 0_{ik} stands for the $i \times k$ zero matrix.

Theorem 2 *The PTM $W_{vg}(\lambda, t)$ and $W_{yg}(\lambda, t)$ for all t are meromorphic functions of the argument λ, and they permit representations of the form*

$$W_{vg}(\lambda, t) = \frac{P_L(\lambda, t)}{\det \tilde{Q}_T(\lambda, \tilde{\alpha}, \tilde{\beta})},$$
$$W_{yg}(\lambda, t) = \frac{P_y(\lambda, t)}{\det \tilde{Q}_T(\lambda, \tilde{\alpha}, \tilde{\beta})}, \tag{41}$$

where the matrices $P_L(\lambda, t)$ and $P_y(\lambda, t)$ for all t are integral functions of the argument λ.

Proof Enhancing the first equation in (29) to matrix input, under (31), we obtain with the notation $v(\lambda, kT) = v_k(\lambda)$

$$v(\lambda, t) = H(t)M^{-k}v_k(\lambda) + \int_{kT}^{t} H(t)H^{-1}(v)B_1(v)e^{\lambda v}\,dv$$
$$+ \int_{kT}^{t} H(t)H^{-1}(v)B(v)h(v-\tau)\,dv. \tag{42}$$

When (17) is fulfilled, we find, as in [2]

$$u(t-\tau) = \begin{cases} h(t-kT+\gamma)\psi_{k-\mu-1}(\lambda), & kT < t < kT+\theta, \\ h(t-kT-\theta)\psi_{k-\mu}(\lambda), & kT+\theta < t < (k+1)T. \end{cases} \tag{43}$$

With the help of this equation, taking in (42) $t = (k+1)T$, we obtain a discrete model of the vector state for input (31), which after substituting k by $k-1$, takes the form

$$v_k(\lambda) = Mv_{k-1}(\lambda) + M^2A_3(\theta)\psi_{k-\mu-2}(\lambda) + MA_1(\theta)\psi_{k-\mu-1} + e^{(k-1)\lambda T}G(\lambda), \tag{44}$$

where the $\chi \times \ell$ matrix

$$G(\lambda) = M\int_0^T H^{-1}(v)B_1(v)e^{\lambda v}\,dv \tag{45}$$

is an integral function of the argument λ. Combining (44) with the discrete model for the output $y(t)$ and the equations of the digital control program, we obtain a discrete model of the closed system $\mathscr{S}_\tau$ for the input (31)

$$\begin{aligned} &v_k(\lambda) = Mv_{k-1}(\lambda) + M^2A_3(\theta)\psi_{k-\mu-2}(\lambda) + MA_1(\theta)\psi_{k-\mu-1}(\lambda) + e^{(k-1)\lambda T}G(\lambda), \\ &\xi_k(\lambda) = C(0)v_k(\lambda), \\ &\alpha_0\psi_k(\lambda) + \cdots + \alpha_\rho\psi_{k-\rho}(\lambda) = \beta_0\xi_k(\lambda) + \cdots + \beta_\rho\xi_{k-\rho}(\lambda), \end{aligned} \tag{46}$$

where $L(kT) = I_\chi$ has been used. In analogy to Theorem 7.4 in [24], it can be shown that matrix (42) takes the form

$$v(\lambda, t) = \mathrm{e}^{\lambda t} W_{vg}(\lambda, t), \quad W_{vg}(\lambda, t) = W_{vg}(\lambda, t + T) \tag{47}$$

if and only if the sequences $v_k(\lambda)$ and $\psi_k(\lambda)$ are chosen as solution of system of linear matrix equations (46), satisfying the conditions

$$\begin{aligned} &v_k(\lambda) = \mathrm{e}^{\lambda T} v_{k-1}(\lambda), \quad \psi_k(\lambda) = \mathrm{e}^{\lambda T} \psi_{k-1}(\lambda), \\ &\xi_k(\lambda) = \mathrm{e}^{\lambda T} \xi_{k-1}(\lambda). \end{aligned} \tag{48}$$

Due to these conditions, from (46), we find the linear system of matrix equations according to the matrices $v_0(\lambda)$, $\psi_0(\lambda)$, $\xi_0(\lambda)$

$$\begin{aligned} &(I_\chi - \mathrm{e}^{-\lambda T} M) v_0(\lambda) - \mathrm{e}^{-(\mu+1)\lambda T} \tilde{b}(\lambda) \psi_0(\lambda) = \mathrm{e}^{-\lambda T} G(\lambda), \\ &\xi_0(\lambda) = C(0) v_0(\lambda), \\ &\tilde{\alpha}(\lambda) \psi_0(\lambda) = \tilde{\beta}(\lambda) \xi_0(\lambda). \end{aligned} \tag{49}$$

Introduce the block matrices

$$Z(\lambda) = \begin{bmatrix} v_0(\lambda) \\ \xi_0(\lambda) \\ \psi_0(\lambda) \end{bmatrix}, \qquad \bar{G}(\lambda) = \begin{bmatrix} G(\lambda) \\ 0_{n\ell} \\ 0_{q\ell} \end{bmatrix}. \tag{50}$$

Then Eq. (48) can be written in the form

$$\tilde{Q}_T(\lambda, \tilde{\alpha}, \tilde{\beta}) Z(\lambda) = \mathrm{e}^{-\lambda T} \bar{G}(\lambda), \tag{51}$$

from which we directly find

$$Z(\lambda) = \tilde{Q}^{-1}(\lambda, \tilde{\alpha}, \tilde{\beta}) \bar{G}(\lambda) \mathrm{e}^{-\lambda T}. \tag{52}$$

Since $\bar{G}(\lambda)$ is an integral function of the argument λ, from (52) we obtain the expressions

$$v_0(\lambda) = \frac{M_v(\lambda)}{\det \tilde{Q}_T(\lambda, \tilde{\alpha}, \tilde{\beta})}, \quad \xi_0(\lambda) = \frac{M_\xi(\lambda)}{\det \tilde{Q}_T(\lambda, \tilde{\alpha}, \tilde{\beta})}, \quad \psi_0(\lambda) = \frac{M_\psi(\lambda)}{\det \tilde{Q}_T(\lambda, \tilde{\alpha}, \tilde{\beta})}, \tag{53}$$

in which the numerators are integral functions (matrices) of the argument λ. Further on, like in the proof of Theorem 7.4 in [24], it can be shown that the expression

$$W_{vg}(\lambda, t) = \mathrm{e}^{-\lambda t} v_0(\lambda, t), \tag{54}$$

where the matrix $v_0(\lambda, t)$ is determined by formulae (42), (43), (52) yields the PTM $W_{vg}(\lambda, t)$ in form (41). Analogously, the representation of the PTM $W_{yg}(\lambda, t)$ takes the form

$$W_{yg}(\lambda, t) = \mathrm{e}^{-\lambda t} C(t)L(t)v_0(\lambda, t), \tag{55}$$

so the proof is complete. □

Remark 1 A direct solution of equations (49) yields the relations

$$\begin{aligned} v_0(\lambda) &= \left[I_\chi - \mathrm{e}^{-\lambda T} M - \mathrm{e}^{-(\mu+1)\lambda T} \tilde{b}(\lambda) \tilde{W}_d(\lambda) C(0)\right]^{-1} G(\lambda) e^{-\lambda T}, \\ \xi_0(\lambda) &= C(0)v_0(\lambda), \\ \psi_0(\lambda) &= \tilde{W}_d(\lambda)\xi_0(\lambda), \end{aligned} \tag{56}$$

which can be used for the practical construction of the PTM for the system $\mathscr{S}_\tau$.

6 Numerical Example

Construct the PTM $W_{yg}(\lambda, t)$ for the system $\mathscr{S}_\tau$ with the LCP process of first order

$$\begin{aligned} \frac{\mathrm{d}v(t)}{\mathrm{d}t} &= (a - \frac{\sin t}{2 - \cos t})v(t) + g(t) + u(t - \pi), \\ y(t) &= v(t), \\ \xi_k &= y(kT), \quad (k = 0, \pm 1, \ldots), \\ \alpha(\zeta)\psi_k &= \beta(\zeta)\xi_k, \\ u(t) &= \psi_k, \quad kT < t < (k+1)T, \end{aligned} \tag{57}$$

where $a \neq 0$ is a real constant. In the actual case, we have

$$\begin{aligned} &A(t) = a - \frac{\sin t}{2 - \cos t}, \quad B(t) = B_1(t) = C(t) = 1, \\ &h(t) = 1. \end{aligned} \tag{58}$$

Moreover,

$$T = 2\pi, \quad \theta = \gamma = \pi, \quad \mu = 0, \tag{59}$$

and $\alpha(\zeta), \beta(\zeta)$ are the polynomials of the digital controller. For analysis, the controller is assumed to be given, in case of controller design, these polynomials will be determined later, using expressions with them as parameters.
It is easy to verify that in the actual case

$$H(t) = \frac{\mathrm{e}^{at}}{2 - \cos t}, \quad L(t) = \frac{1}{2 - \cos t}, \quad N = a, \quad M = \mathrm{e}^{2\pi a}, \tag{60}$$

and, hence

$$H^{-1}(t) = e^{-at}(2 - \cos t), \quad L^{-1}(t) = 2 - \cos t\,. \tag{61}$$

Therefore, formulae (38) and (45) yield

$$\begin{aligned} A_1(\theta) &= e^{-\pi a} \int_0^{\pi} e^{-av}(2 + \cos v)\, dv, \\ A_3(\theta) &= e^{-\pi a} \int_{\pi}^{2\pi} e^{-av}(2 + \cos v)\, dv, \\ G(\lambda) &= e^{2\pi a} \int_0^{2\pi} e^{-av}(2 - \cos v) e^{\lambda v}\, dv \end{aligned} \tag{62}$$

and from (37), we find

$$\begin{aligned} I_\chi - e^{-2\pi\lambda} M &= 1 - e^{-2\pi\lambda} e^{2\pi a}, \\ \tilde{b}(\lambda) &= e^{2\pi a} \left[e^{-2\pi\lambda} e^{2\pi a} \tilde{A}_3(\theta) + A_1(\theta) \right]. \end{aligned} \tag{63}$$

With the help of (57)–(62), from (55) we obtain

$$v_0(\lambda) = \frac{e^{-2\pi\lambda} \tilde{\alpha}(\lambda) G(\lambda)}{\det \tilde{Q}_T(\lambda, \tilde{\alpha}, \tilde{\beta})}, \qquad \psi_0(\lambda) = \frac{e^{-2\pi\lambda} \tilde{\beta}(\lambda) G(\lambda)}{\det \tilde{Q}_T(\tilde{\alpha}, \tilde{\beta})}, \tag{64}$$

where

$$\det \tilde{Q}_T(\lambda, \tilde{\alpha}, \tilde{\beta}) = (1 - e^{-2\pi\lambda} e^{2\pi a}) \tilde{\alpha}(\lambda) - e^{-2\pi\lambda} \tilde{b}(\lambda) \tilde{\beta}(\lambda). \tag{65}$$

Using (42), (43) and (63), we find from the equations of the LCP process, that for $0 \le t \le T$, the representation of $W_{yg}(\lambda, t)$ in form (41) apppears as

$$W_{yg}(\lambda, t) = \frac{e^{(a-\lambda)t} e^{-2\pi\lambda}}{(2 - \cos t) \det \tilde{Q}_T(\lambda, \tilde{\alpha}, \tilde{\beta})} G(\lambda) \tilde{H}_{yg}(\lambda, t) + \frac{e^{(a-\lambda)t}}{2 - \cos t} \int_0^t (2 - \cos v) e^{(\lambda - a)v}\, dv \tag{66}$$

where

$$\tilde{H}_{yg}(\lambda, t) = \begin{cases} \tilde{\alpha}(\lambda) + \tilde{\beta}(\lambda) e^{-2\pi\lambda} \displaystyle\int_0^t e^{-av}(2 - \cos v)\, dv, & 0 \le t \le \pi, \\ \tilde{\alpha}(\lambda) + \tilde{\beta}(\lambda) \left[e^{-2\pi\lambda} \displaystyle\int_0^{\pi} e^{-av}(2 - \cos v)\, dv \right. \\ \qquad \left. + \displaystyle\int_{\pi}^t e^{-av}(2 - \cos v)\, dv \right] & \pi \le t \le 2\pi. \end{cases} \tag{67}$$

7 Conclusions

The contribution provides closed expressions for the parametric transfer matrices (PTM) of sampled-data (SD) systems containing a periodic continuous process and delay, where the external excitation acts on the input to the process. It is shown that the PTM $W_{yx}(\lambda, t)$ for all t is a meromorphic function of the complex variable λ, the poles of it are among the set of eigenvalues of certain matrices, depending on λ, but not depending on the parameter t. An example is given. The achieved results open possibilities to solve, in analogy to that of previous works of the authors, various control problems, including stability and stabilzation, advanced statistical analysis, $\mathscr{H}_2$, $\mathscr{L}_2$ and $\mathscr{H}_\infty$ optimization for systems of the considered class. The development of computational tools for analysis and design of digital filters and controllers interacting with RLCP processes, will be a main task in future work.

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