Chapter 2
Gram Matrix Representation

Abstract There are several ways of characterizing nonnegative polynomials that may be interesting for a mathematician. However, not all of them are appropriate for computational purposes, by "computational" understanding primarily optimization methods. Nonnegative polynomials have a basic property extremely useful in optimization: They form a convex set. So, an optimization problem whose variables are the coefficients of a nonnegative polynomial has a unique solution (or, in the degenerate case, multiple solutions belonging to a convex set), if the objective and the other constraints besides positivity are also convex. Convexity is not enough for obtaining efficiently a reliable solution. Efficiency and reliability are specific only to some classes of convex optimization, such as linear programming (LP), second-order cone problems (SOCP), and semidefinite programming (SDP). SDP includes LP and SOCP and is probably the most important advance in optimization in the last decade of the previous century. See some basic information on SDP in Appendix A. In this chapter, we present a parameterization of nonnegative polynomials that is intimately related to SDP. Each polynomial can be associated with a set of matrices, called *Gram* matrices (Choi et al., Proc Symp Pure Math 58:103–126, 1995, [1]); if the polynomial is nonnegative, then there is at least a positive semidefinite Gram matrix associated with it. Solving optimization problems with nonnegative polynomials may thus be reduced, in many cases, to SDP. We give several examples of such problems and of programs that solve them. Spectral factorization is important in this context, and we present several techniques for its computation. Besides the standard, or trace, parameterization, we discuss several other possibilities that may have computational advantages.

2.1 Parameterization of Trigonometric Polynomials

Let us start with some notations. The vector

$$\boldsymbol{\psi}_n(z) = [1\ z\ z^2\ \ldots\ z^n]^T \tag{2.1}$$

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contains the canonical basis for polynomials of degree n in z. Whenever the degree results from the context, we denote $\boldsymbol{\psi}(z)$ the vector from (2.1). A causal polynomial (1.10) can be written in the form $H(z) = \boldsymbol{h}^T \boldsymbol{\psi}(z^{-1})$, where $\boldsymbol{h} = [h_0\ h_1\ \dots\ h_n]^T \in \mathbb{R}^{n+1}$ (or $\mathbb{C}^{n+1}$) is the vector of its coefficients. We use the notation $\boldsymbol{\psi}(\omega)$ for $\boldsymbol{\psi}(\mathrm{e}^{j\omega})$; remark that $\boldsymbol{\psi}^T(-\omega) = \boldsymbol{\psi}^H(\omega)$. Also, we denote $n' = n + 1$.

Definition 2.1 Consider the trigonometric polynomial $R \in \mathbb{C}_n[z]$, defined as in (1.1). A Hermitian matrix $\boldsymbol{Q} \in \mathbb{C}^{n' \times n'}$ is called a *Gram* matrix associated with $R(z)$ if

$$R(z) = \boldsymbol{\psi}^T(z^{-1}) \cdot \boldsymbol{Q} \cdot \boldsymbol{\psi}(z). \tag{2.2}$$

We denote $\mathcal{G}(R)$ the set of Gram matrices associated with $R(z)$. ■

If $R \in \mathbb{R}_n[z]$, then the matrix $\boldsymbol{Q}$ obeying to (2.2) belongs to $\mathbb{R}^{n' \times n'}$ and is symmetric.

Example 2.2 Let us consider polynomials of degree two with real coefficients, $R(z) = r_2 z^{-2} + r_1 z^{-1} + r_0 + r_1 z + r_2 z^2$. A few computations show that if

$$R(z) = [1\ z^{-1}\ z^{-2}] \begin{bmatrix} q_{00} & q_{10} & q_{20} \\ q_{10} & q_{11} & q_{21} \\ q_{20} & q_{21} & q_{22} \end{bmatrix} \begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix},$$

where $\boldsymbol{Q} \in \mathbb{R}^{3 \times 3}$ is a Gram matrix associated with $R(z)$, then

$$\begin{aligned} r_0 &= q_{00} + q_{11} + q_{22}, \\ r_1 &= q_{10} + q_{21}, \\ r_2 &= q_{20}. \end{aligned} \tag{2.3}$$

Hence, any Gram matrix associated with $R(z)$ has the form

$$\begin{aligned} \boldsymbol{Q} &= \begin{bmatrix} r_0 - q_{11} - q_{22} & r_1 - q_{21} & r_2 \\ r_1 - q_{21} & q_{11} & q_{21} \\ r_2 & q_{21} & q_{22} \end{bmatrix} \\ &= \begin{bmatrix} r_0 & r_1 & r_2 \\ r_1 & 0 & 0 \\ r_2 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -q_{11} - q_{22} & -q_{21} & 0 \\ -q_{21} & q_{11} & q_{21} \\ 0 & q_{21} & q_{22} \end{bmatrix}. \end{aligned}$$

It is clear that, in general, any Hermitian matrix in $\mathbb{C}^{n' \times n'}$ produces a Hermitian polynomial through the mapping (2.2), which is many-to-one. For instance, taking

$$R(z) = 2z^{-2} - 3z^{-1} + 6 - 3z + 2z^2 = (2 - z^{-1} + z^{-2})(2 - z + z^2), \tag{2.4}$$

the following three matrices

$$\boldsymbol{Q}_0 = \begin{bmatrix} 6 & -3 & 2 \\ -3 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{Q}_1 = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix},$$
$$\boldsymbol{Q}_2 = \begin{bmatrix} 2.2 & -1.5 & 2.0 \\ -1.5 & 1.6 & -1.5 \\ 2.0 & -1.5 & 2.2 \end{bmatrix} \tag{2.5}$$

are Gram matrices associated with $R(z)$. ■

A natural (and simple) question regards the relation between the coefficients of $R(z)$ and the elements of $\boldsymbol{Q} \in \mathcal{G}(R)$. From (2.3), we may infer that r_k is the sum of elements of $\boldsymbol{Q}$ along diagonal $-k$ (the main diagonal has number 0, and the lower triangle diagonals have negative numbers, as in MATLAB). This is indeed the case.

Theorem 2.3 *If $R \in \mathbb{C}_n[z]$ and $\boldsymbol{Q} \in \mathcal{G}(R)$, then the relation*

$$r_k = \operatorname{tr}[\boldsymbol{\Theta}_k \boldsymbol{Q}] = \sum_{i=\max(0,k)}^{\min(n+k,n)} q_{i,i-k}, \quad k = -n : n, \tag{2.6}$$

holds, where $\boldsymbol{\Theta}_k$ is the elementary Toeplitz matrix with ones on the k-th diagonal and zeros elsewhere and $\operatorname{tr}\boldsymbol{X}$ *is the trace of the matrix $\boldsymbol{X}$. We name (2.6) the* trace *parameterization of the trigonometric polynomial $R(z)$.*

Proof We recall that $\operatorname{tr}[\boldsymbol{ABC}] = \operatorname{tr}[\boldsymbol{CAB}]$, where $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are matrices of appropriate sizes and also that $a = \operatorname{tr}[a]$, if a is a scalar. The relation (2.2) can be written as

$$R(z) = \boldsymbol{\psi}^T(z^{-1}) \cdot \boldsymbol{Q} \cdot \boldsymbol{\psi}(z) = \operatorname{tr}[\boldsymbol{\psi}(z) \cdot \boldsymbol{\psi}^T(z^{-1}) \cdot \boldsymbol{Q}] = \operatorname{tr}[\boldsymbol{\Psi}(z) \cdot \boldsymbol{Q}],$$

where

$$\boldsymbol{\Psi}(z) = \begin{bmatrix} 1 \\ z \\ \vdots \\ z^n \end{bmatrix} [1 \; z^{-1} \; \dots \; z^{-n}] = \begin{bmatrix} 1 & z^{-1} & \dots & z^{-n} \\ z & 1 & \ddots & z^{-n+1} \\ \vdots & \ddots & \ddots & \vdots \\ z^n & z^{n-1} & \dots & 1 \end{bmatrix} = \sum_{k=-n}^{n} \boldsymbol{\Theta}_k z^{-k}. \tag{2.7}$$

Combining the above two relations, we obtain

$$R(z) = \sum_{k=-n}^{n} \operatorname{tr}[\boldsymbol{\Theta}_k \boldsymbol{Q}] z^{-k},$$

which proves (2.6) after identification with (1.1). ■

Example 2.4 For a polynomial of degree 2, as in Example 2.2, the trace parameterization (2.6) tells that

$$r_0 = \operatorname{tr} \boldsymbol{Q}, \quad r_1 = \operatorname{tr} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{Q}, \quad r_2 = \operatorname{tr} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{Q}.$$

We are now ready to state the main result, namely that the set $\mathbb{C}_n[z]$ of nonnegative polynomials (of order n) and the set of positive semidefinite matrices of size $n' \times n'$ are connected by the trace parameterization mapping (2.6).

Theorem 2.5 *A polynomial $R \in \mathbb{C}_n[z]$ is nonnegative (positive) on the unit circle if and only if there exists a positive semidefinite (definite) matrix $\boldsymbol{Q} \in \mathbb{C}^{n' \times n'}$ such that (2.6) holds.*

Proof If $\boldsymbol{Q} \succeq 0$ exists such that (2.6) holds, then, using the definition (2.2) of a Gram matrix, we can write

$$R(\omega) = [1 \ \mathrm{e}^{-j\omega} \ \ldots \mathrm{e}^{-jn\omega}] \cdot \boldsymbol{Q} \cdot \begin{bmatrix} 1 \\ \mathrm{e}^{j\omega} \\ \vdots \\ \mathrm{e}^{jn\omega} \end{bmatrix} = \boldsymbol{\psi}^H(\omega) \cdot \boldsymbol{Q} \cdot \boldsymbol{\psi}(\omega) \geq 0,$$

for all ω. The same reasoning shows that if $\boldsymbol{Q} \succ 0$, then $R(\omega) > 0$.

Reciprocally, if $R(\omega) \geq 0$, then the spectral factorization Theorem 1.1 says that

$$R(z) = H(z)H^*(z^{-1}) = \boldsymbol{h}^T \boldsymbol{\psi}(z^{-1}) \cdot \boldsymbol{h}^H \boldsymbol{\psi}(z) = \boldsymbol{\psi}^T(z^{-1}) \cdot \boldsymbol{h}\boldsymbol{h}^H \cdot \boldsymbol{\psi}(z).$$

It results that

$$\boldsymbol{Q}_1 = \boldsymbol{h}\boldsymbol{h}^H \succeq 0 \tag{2.8}$$

is a Gram matrix associated with $R(z)$. Note that $\operatorname{rank} \boldsymbol{Q}_1 = 1$.

If $R(\omega) > 0$, since $[-\pi, \pi]$ is compact, there exists $\varepsilon > 0$ such that $R_\varepsilon(z) = R(z) - \varepsilon$ is nonnegative. Denoting $H_\varepsilon(z)$ a spectral factor of $R_\varepsilon(z)$ and noticing that $\boldsymbol{\psi}^T(z^{-1}) \cdot \boldsymbol{\psi}(z) = n'$, it results as above that

$$R(z) = \boldsymbol{\psi}^T(z^{-1}) \cdot \left(\boldsymbol{h}\boldsymbol{h}^H + (\varepsilon/n')\boldsymbol{I}\right) \cdot \boldsymbol{\psi}(z)$$

and so $\boldsymbol{h}\boldsymbol{h}^H + (\varepsilon/n')\boldsymbol{I} \succ 0$ is a Gram matrix associated with $R(z)$. ■

Example 2.6 Returning to Example 2.2 and the three Gram matrices from (2.5), we notice that $\boldsymbol{Q}_1$ is defined as in (2.8) and so is positive semidefinite of rank 1 and also $\boldsymbol{Q}_2 \succ 0$. We conclude that $R(\omega) > 0$. The fact that a Gram matrix, in our case $\boldsymbol{Q}_0$, is not definite has no consequence on the positivity of $R(z)$. ■

Remark 2.7 Theorem 2.5 establishes a linear relation between the elements of two convex sets: nonnegative polynomials and positive semidefinite matrices. On one side, we have the usual parameterization of $\overline{\mathbb{CP}}_n[z]$ using the $n + 1$ coefficients of the polynomial $R(z)$. On the other side, we have an overparameterization, using the

$n(n+1)/2$ independent elements of the Gram matrix $\boldsymbol{Q}$. The high number of parameters of the latter is compensated by the reliability and efficiency of optimization algorithms dealing with linear combinations of positive semidefinite matrices, which belong to the class of semidefinite programming. ■

Remark 2.8 If the polynomial has complex coefficients, Theorem 2.5 can be formulated in terms of real matrices, as follows. The polynomial $R(z)$ is nonnegative if and only if there exist matrices $\boldsymbol{Q}_r, \boldsymbol{Q}_i \in \mathbb{R}^{n' \times n'}$ such that

$$\mathrm{Re}\, r_k = \mathrm{tr}[\boldsymbol{\Theta}_k \boldsymbol{Q}_r], \quad \mathrm{Im}\, r_k = \mathrm{tr}[\boldsymbol{\Theta}_k \boldsymbol{Q}_i] \tag{2.9}$$

and

$$\begin{bmatrix} \boldsymbol{Q}_r & -\boldsymbol{Q}_i \\ \boldsymbol{Q}_i & \boldsymbol{Q}_r \end{bmatrix} \succeq 0. \tag{2.10}$$

Indeed, putting $\boldsymbol{Q} = \boldsymbol{Q}_r + j\boldsymbol{Q}_i$, relation (2.10) is equivalent to $\boldsymbol{Q} \succeq 0$ and relation (2.9) is equivalent to (2.6). (Such a formulation might be useful when using SDP algorithms that work only with real matrices. However, all important SDP libraries are able to deal with complex matrices.) A more efficient way to parameterize complex polynomials with real Gram matrices will be presented in Sect. 2.8.1. ■

Remark 2.9 (Sum-of-squares decomposition) Let $R \in \mathbb{C}_n[z]$ be a nonnegative trigonometric polynomial, and let $\boldsymbol{Q} \succeq 0$ be a positive semidefinite Gram matrix associated with it. A distinct sum-of-squares decomposition (1.23) of $R(z)$ can be derived from each such Gram matrix. Let

$$\boldsymbol{Q} = \sum_{\ell=1}^{\nu} \lambda_\ell^2 \boldsymbol{x}_\ell \boldsymbol{x}_\ell^H, \tag{2.11}$$

be the eigendecomposition of $\boldsymbol{Q}$, in which ν is the rank, λ_ℓ^2 the eigenvalues, and $\boldsymbol{x}_\ell$ the eigenvectors, $\ell = 1 : \nu$. Inserting (2.11) into (2.2), we obtain the sum-of-squares decomposition

$$R(z) = \sum_{\ell=1}^{\nu} [\lambda_\ell \boldsymbol{\psi}^T(z^{-1}) \boldsymbol{x}_\ell] \cdot [\lambda_\ell \boldsymbol{x}_\ell^H \boldsymbol{\psi}(z)] = \sum_{\ell=1}^{\nu} H_\ell(z) H_\ell^*(z^{-1}), \tag{2.12}$$

where

$$H_\ell(z) = \lambda_\ell \boldsymbol{\psi}^T(z^{-1}) \boldsymbol{x}_\ell. \tag{2.13}$$

So, the sum-of-squares (2.12) has a number of terms equal to the rank of the Gram matrix $\boldsymbol{Q}$. If the Gram matrix is $\boldsymbol{Q}_1$ from (2.8), then the spectral factorization (1.11) is obtained. ■

Remark 2.10 (Toeplitz Gram matrices) There is a single Toeplitz Gram matrix of size $n' \times n'$ associated with a given polynomial $R(z)$ of degree n, namely

$\boldsymbol{Q} = \mathrm{Toep}(r_0/(n+1), r_1/n, \ldots, r_n)$. If $R(\omega) \geq 0$, is this matrix positive semidefinite? For example, for the positive polynomial (2.4), this matrix is

$$\boldsymbol{Q} = \begin{bmatrix} 2.0 & -1.5 & 2.0 \\ -1.5 & 2.0 & -1.5 \\ 2.0 & -1.5 & 2.0 \end{bmatrix}$$

and is positive semidefinite (and singular). Modifying r_2 to e.g., 2.001 keeps the polynomial positive, but the Toeplitz Gram matrix is no more positive semidefinite. So, in general, there is no connection between the nonnegativity of the polynomial and the positive semidefiniteness of the Toeplitz Gram matrix.

However, we can show that, for any nonnegative $R(z)$, there is an arbitrarily close $\tilde{R}(z)$ for which the Toeplitz Gram matrix is positive semidefinite. The trick is to remove the size restrictions. We can artificially consider the degree of $R(z)$ to be $m > n$, by adding coefficients $r_k = 0$, $k = n+1 : m$. Remember now Theorem 1.8, which states that the Toeplitz matrices $\boldsymbol{R}_m$ defined in (1.28) are positive semidefinite. For any m, the polynomial $\tilde{R}(z)$, with coefficients defined by

$$\tilde{r}_k = \mathrm{tr}\left[\boldsymbol{\Theta}_k \cdot \frac{1}{m+1}\boldsymbol{R}_m\right] = \left(1 - \frac{|k|}{m+1}\right) r_k,$$

is thus nonnegative and the Gram matrix $\boldsymbol{R}_m/(m+1)$ is Toeplitz and positive semidefinite. For large enough m, the polynomial $\tilde{R}(z)$ is arbitrarily close to $R(z)$. ■

2.2 Optimization Using the Trace Parameterization

We present now some simple problems that can be solved using the trace parameterization (2.6) and SDP.

Let us notice first that, given the polynomial $R(z)$, the set $\mathcal{G}(R)$ is convex. Indeed, for any $\alpha \in [0, 1]$ and $\boldsymbol{Q}$, $\tilde{\boldsymbol{Q}} \in \mathcal{G}(R)$, it is immediate from (2.2) that $\alpha\boldsymbol{Q} + (1-\alpha)\tilde{\boldsymbol{Q}} \in \mathcal{G}(R)$. Moreover, if $R(\omega) \geq 0$, then the set of positive semidefinite Gram matrices associated with $R(z)$ is also convex, as the intersection of two convex sets.

Problem (*Most_positive_Gram_matrix*) It is clear that, given $R \in \mathbb{C}_n[z]$ with $R(\omega) > 0$, there are an infinite number of positive definite Gram matrices in $\mathcal{G}(R)$. This results, for example, by taking all possible values for the parameter ε appearing at the end of the proof of Theorem 2.5. A distinguished member of $\mathcal{G}(R)$ is the most positive one, i.e., the most nonsingular. The distance to nonsingularity is measured by the smallest singular value, or, as we deal with positive definite matrices, by the smallest eigenvalue. So, we want the matrix in $\mathcal{G}(R)$ having the largest smallest eigenvalue, namely the solution of the optimization problem

$$\begin{aligned} \lambda^\star = \max_{\lambda, \boldsymbol{Q}} \ & \lambda \\ \text{s.t.} \ & \operatorname{tr}[\boldsymbol{\Theta}_k \boldsymbol{Q}] = r_k, \quad k = 0:n \\ & \lambda \geq 0, \quad \boldsymbol{Q} \succeq \lambda \boldsymbol{I} \end{aligned} \tag{2.14}$$

The inequality $\boldsymbol{Q} \succeq \lambda \boldsymbol{I}$ ensures that maximization of λ is equivalent to maximization of the smallest eigenvalue of the Gram matrix $\boldsymbol{Q}$. (We always assume that the Gram matrices are Hermitian and will not specify it explicitly from now on.) The optimization problem (2.14) is a semidefinite program, since the variables are the positive semidefinite matrix $\boldsymbol{Q}$ and the positive scalar λ and the constraints are linear equalities in the elements of $\boldsymbol{Q}$.

Although in evolved optimization problem solvers such as CVX the problem (2.14) can be posed as it is, some popular SDP libraries need the problem in the standard equality form shown in Appendix A. The transformation to standard form can be made by denoting $\tilde{\boldsymbol{Q}} = \boldsymbol{Q} - \lambda \boldsymbol{I}$; as the matrix $\tilde{\boldsymbol{Q}}$ is positive semidefinite, it can serve as variable in the SDP problem. Since $\boldsymbol{\Theta}_0 = \boldsymbol{I}$, it follows that $\operatorname{tr}\boldsymbol{\Theta}_0 = n'$, while for $k \neq 0$ we have $\operatorname{tr}\boldsymbol{\Theta}_k = 0$. We conclude that

$$\operatorname{tr}[\boldsymbol{\Theta}_k \tilde{\boldsymbol{Q}}] = \begin{cases} r_0 - n'\lambda, & \text{if } k = 0, \\ r_k, & \text{otherwise.} \end{cases}$$

Thus, the problem (2.14) is equivalent to the standard SDP problem

$$\begin{aligned} \lambda^\star = \max_{\lambda, \tilde{\boldsymbol{Q}}} \ & \lambda \\ \text{s.t.} \ & n'\lambda + \operatorname{tr}\tilde{\boldsymbol{Q}} = r_0 \\ & \operatorname{tr}[\boldsymbol{\Theta}_k \tilde{\boldsymbol{Q}}] = r_k, \quad k = 1:n \\ & \lambda \geq 0, \quad \tilde{\boldsymbol{Q}} \succeq 0 \end{aligned} \tag{2.15}$$

We finally note that the SDP problem (2.14) or (2.15) gives no solution if the polynomial $R(z)$ is not nonnegative, as no positive semidefinite Gram matrix exists. However, the problem of finding the Gram matrix with maximum smallest eigenvalue is well defined; removing the constraint $\lambda \geq 0$ in (2.14) or (2.15) and thus leaving λ free is the single necessary modification. ■

Example 2.11 In Example 2.2, none of the three Gram matrices from (2.5) is the most positive one. Solving the SDP problem (2.14), we obtain

$$\boldsymbol{Q} = \begin{bmatrix} 2.2917 & -1.5000 & 2.0000 \\ -1.5000 & 1.4167 & -1.5000 \\ 2.0000 & -1.5000 & 2.2917 \end{bmatrix}.$$

The smallest eigenvalue of this matrix is $\lambda^\star = 0.2917$. For comparison, the smallest eigenvalue of the matrix $\boldsymbol{Q}_2$ from (2.5) is 0.2. ■

Problem (*Min_poly_value*) A problem related to that of finding the most positive Gram matrix is to compute the minimum value on the unit circle of a given trigonometric polynomial $R(z)$. Let $R \in \mathbb{C}_n[z]$ be a polynomial not necessarily nonnegative. We want to find

$$\mu^\star = \min_{\omega \in [-\pi, \pi]} R(\omega). \tag{2.16}$$

Certainly, this is a problem that can be solved using elementary tools, so the solution given below may not be the most efficient, although it is instructive in the current context. Since $\mu^\star$ is the maximum scalar for which $R(\omega) - \mu^\star$ is nonnegative, we can connect (2.16) to nonnegative polynomials by transforming it into

$$\begin{aligned} \mu^\star = \ & \max_{\mu} \ \mu \\ & \text{s.t.} \ \ R(\omega) - \mu \ge 0, \quad \forall \omega \in [-\pi, \pi] \end{aligned} \tag{2.17}$$

Denoting $\tilde{R}(z) = R(z) - \mu$, we note that $\tilde{r}_0 = r_0 - \mu$ and $\tilde{r}_k = r_k$, $k = 1 : n$. Using the trace parameterization (and reminding again that $\boldsymbol{\Theta}_0 = \boldsymbol{I}$), the problem (2.17) can brought to the following SDP form

$$\begin{aligned} \mu^\star = \ & \max_{\mu, \tilde{\boldsymbol{Q}}} \ \mu \\ & \text{s.t.} \ \ \mu + \operatorname{tr} \tilde{\boldsymbol{Q}} = r_0 \\ & \qquad \operatorname{tr}[\boldsymbol{\Theta}_k \tilde{\boldsymbol{Q}}] = r_k, \quad k = 1 : n \\ & \qquad \tilde{\boldsymbol{Q}} \succeq 0 \end{aligned} \tag{2.18}$$

If $R(\omega) \ge 0$, the SDP problems (2.14) and (2.18) are equivalent, expressing the connection between the most positive Gram matrix and the minimum value of a polynomial. The equivalence is shown by the relations

$$\mu = n'\lambda, \quad \tilde{\boldsymbol{Q}} = \boldsymbol{Q} - \lambda \boldsymbol{I} \tag{2.19}$$

between the variables of the two problems, which are obvious if we look at the (2.15), which is equivalent to (2.14) and becomes identical to (2.18) by taking $\mu = n'\lambda$. So, the optimal values of (2.14) and (2.18) are related by $\mu^\star = n'\lambda^\star$, and solving one problem leads immediately to the solution of the other through (2.19).

SeDuMi, CVX, and Pos3Poly programs for solving the SDP problems (2.14) and (2.18) are presented and commented in Sect. 2.12.1. ■

Example 2.12 The minimum value on the unit circle of the polynomial (2.4) (of degree $n = 2$) considered in Example 2.2 is $\mu^\star = 3\lambda^\star = 3 \cdot 0.2917 = 0.8750$. ■

Problem (*Nearest_autocorrelation*) We return to a problem discussed in the previous chapter: Given a symmetric (or Hermitian) sequence $\hat{r}_k$, $k = -n : n$, find the nonnegative sequence r_k that is nearest from $\hat{r}_k$. The optimization problem to be solved is (1.22); remind that $\boldsymbol{r} = [r_0 \ r_1 \ \ldots \ r_n]^T$. Expressing the nonnegativity condition with the trace parameterization, we obtain the problem

$$\begin{aligned} \min_{\boldsymbol{r},\boldsymbol{Q}} \ & (\boldsymbol{r}-\hat{\boldsymbol{r}})^H \boldsymbol{\Gamma}(\boldsymbol{r}-\hat{\boldsymbol{r}}) \\ \text{s.t.} \ & \operatorname{tr}[\boldsymbol{\Theta}_k \boldsymbol{Q}] = r_k, \quad k = 0:n \\ & \boldsymbol{Q} \succeq 0 \end{aligned} \tag{2.20}$$

where $\boldsymbol{\Gamma} \succ 0$. To bring (2.20) to a standard form, notice that

$$(\boldsymbol{r}-\hat{\boldsymbol{r}})^H \boldsymbol{\Gamma}(\boldsymbol{r}-\hat{\boldsymbol{r}}) = \|\boldsymbol{\Gamma}^{1/2}(\boldsymbol{r}-\hat{\boldsymbol{r}})\|^2, \tag{2.21}$$

where $\boldsymbol{\Gamma}^{1/2}$ is the square root of $\boldsymbol{\Gamma}$, i.e., the positive definite matrix $\boldsymbol{X}$ such that $\boldsymbol{X}^H\boldsymbol{X} = \boldsymbol{\Gamma}$. An alternative possibility in (2.21) is to use the Cholesky factor of $\boldsymbol{\Gamma}$ instead of $\boldsymbol{\Gamma}^{1/2}$. Using the same trick as in passing from (2.16) to (2.17), we obtain

$$\begin{aligned} \min_{\alpha,\boldsymbol{r},\boldsymbol{Q}} \ & \alpha \\ \text{s.t.} \ & \|\boldsymbol{\Gamma}^{1/2}(\boldsymbol{r}-\hat{\boldsymbol{r}})\| \le \alpha \\ & \operatorname{tr}[\boldsymbol{\Theta}_k \boldsymbol{Q}] = r_k, \quad k = 0:n \\ & \boldsymbol{Q} \succeq 0 \end{aligned} \tag{2.22}$$

The first constraint has a second-order cone form, so (2.22) is a semidefinite-quadratic-linear programming (SQLP) problem. We have only to bring it to one of the standard forms shown in Appendix A. To this purpose, denote

$$\boldsymbol{y} = \boldsymbol{\Gamma}^{1/2}(\boldsymbol{r}-\hat{\boldsymbol{r}})$$

and, in $\boldsymbol{r} - \boldsymbol{\Gamma}^{-1/2}\boldsymbol{y} = \hat{\boldsymbol{r}}$, replace $\boldsymbol{r}$ by its trace parameterization. So, the problem (2.22) is equivalent to

$$\begin{aligned} \min_{\alpha,\boldsymbol{y},\boldsymbol{Q}} \ & \alpha \\ \text{s.t.} \ & \begin{bmatrix} \vdots \\ \operatorname{tr}[\boldsymbol{\Theta}_k \boldsymbol{Q}] \\ \vdots \end{bmatrix} - \boldsymbol{\Gamma}^{-1/2}\boldsymbol{y} = \hat{\boldsymbol{r}} \\ & \boldsymbol{Q} \succeq 0, \quad \|\boldsymbol{y}\| \le \alpha \end{aligned} \tag{2.23}$$

This is a standard SQLP problem in equality form. ■

Remark 2.13 (Complexity issues) As discussed in Appendix A, the complexity of an SDP problem in equality form is $O(n^2m^2)$, where $n \times n$ is the size of the variable positive semidefinite matrix and m is the number of equality constraints. The scalar or SOC variables (from (2.14) and (2.23), respectively) do not change significantly the complexity. Since the size of the Gram matrix $\boldsymbol{Q}$ is $(n+1) \times (n+1)$ and the number of equality constraints is $n+1$, we can appreciate that the complexity of the three problems—*Most_positive_Gram_matrix*, *Min_poly_value* and *Nearest_autocorrelation*—formulated in SDP form in this section is $O(n^4)$. ■

2.3 Toeplitz Quadratic Optimization

In the previous section, we have presented several optimization problems in which the variable was genuinely a nonnegative polynomial. Here, we discuss a problem that can be transformed—in a general way—into one with nonnegative polynomials. The idea is to replace the variable causal polynomial $H(z)$ with $R(z) = H(z)H^*(z^{-1})$ (i.e., with its squared magnitude on the unit circle), solve the presumably easier problem with $R(z)$ as variable, and finally recover $H(z)$ by spectral factorization. We have already met a somewhat similar problem, namely *MA_Estimation* in Sect. 1.2.

Consider the quadratic optimization problem

$$\begin{aligned} &\min_{\boldsymbol{h}} \ \boldsymbol{h}^H \boldsymbol{A}_0 \boldsymbol{h} \\ &\text{s.t.} \ \ \boldsymbol{h}^H \boldsymbol{A}_\ell \boldsymbol{h} = b_\ell, \ \ \ell = 1 : L \end{aligned} \tag{2.24}$$

where the matrices $\boldsymbol{A}_\ell$, $\ell = 0 : L$, and the scalars b_ℓ, $\ell = 1 : L$, are given. The matrix $\boldsymbol{A}_0$ is positive semidefinite. The variable is the vector $\boldsymbol{h} \in \mathbb{C}^{n+1}$; we can interpret its elements as the coefficients of the causal filter (1.10). Although the objective function is convex, the problem (2.24) is not convex, in general; a notorious exception occurs when $L = 1$, $\boldsymbol{A}_1 = \boldsymbol{I}$, for which the solution is an eigenvector of $\boldsymbol{A}_0$ corresponding to the minimal eigenvalue. We treat here only the case where all the matrices $\boldsymbol{A}_\ell$ are *Toeplitz* and Hermitian; that the matrices are Hermitian is not a particularization, due to the quadratic form of the objective and constraints; for an anti-Hermitian matrix $\boldsymbol{A}$ (with $\boldsymbol{A}^H = -\boldsymbol{A}$), the quadratic function is $\boldsymbol{h}^H \boldsymbol{A} \boldsymbol{h} = 0$; so, if the matrices $\boldsymbol{A}_\ell$ were not Hermitian, they could be replaced with their Hermitian part $(\boldsymbol{A} + \boldsymbol{A}^H)/2$. We note also that if the matrices $\boldsymbol{A}_\ell$, $\ell = 1 : L$, are real, the equality constraints from (2.24) can be changed into inequalities without changing the character of the solution presented below.

We denote

$$\boldsymbol{A}_\ell = \text{Toep}(a_{\ell 0}, \ldots, a_{\ell n}) \tag{2.25}$$

and notice that

$$\boldsymbol{A}_\ell = a_{\ell 0} \boldsymbol{\Theta}_0 + \sum_{k=1}^{n} (a_{\ell k} \boldsymbol{\Theta}_k + a_{\ell k}^* \boldsymbol{\Theta}_{-k}). \tag{2.26}$$

If we consider $H(z)$ as the spectral factor of a nonnegative polynomial $R(z)$, i.e., relation (1.11) holds, then the coefficients of $H(z)$ and $R(z)$ are related through (1.17), which is equivalent to

$$r_k = \boldsymbol{h}^H \boldsymbol{\Theta}_k \boldsymbol{h}. \tag{2.27}$$

From (2.26) and (2.27), it results that

$$\boldsymbol{h}^H \boldsymbol{A}_\ell \boldsymbol{h} = r_0 + \sum_{k=1}^{n} (a_{\ell k} r_k + a_{\ell k}^* r_k^*) = r_0 + 2 \sum_{k=1}^{n} \text{Re}(a_{\ell k} r_k). \tag{2.28}$$

So, the problem (2.24) can be transformed into

$$\begin{aligned} \min_{\boldsymbol{r}} \ & r_0 + 2\textstyle\sum_{k=1}^{n} \mathrm{Re}(a_{0k} r_k) \\ \text{s.t.} \ & r_0 + 2\textstyle\sum_{k=1}^{n} \mathrm{Re}(a_{\ell k} r_k) = b_\ell, \ \ell = 1 : L \\ & R(\omega) \ge 0, \ \forall \omega \in [-\pi, \pi] \end{aligned} \tag{2.29}$$

This is a convex optimization problem! The variables are the coefficients of a nonnegative polynomial, and the quadratic objective and constraints from (2.24) are now linear. We can use the trace parameterization (2.6) to transform (2.29) into an SDP problem. Inserting $r_k = \mathrm{tr}[\boldsymbol{\Theta}_k \boldsymbol{Q}]$ into (2.26), we obtain

$$\boldsymbol{h}^H \boldsymbol{A}_\ell \boldsymbol{h} = \mathrm{tr}\left[a_{\ell 0} \boldsymbol{\Theta}_0 \boldsymbol{Q} + \sum_{k=1}^{n} (a_{\ell k} \boldsymbol{\Theta}_k + a_{\ell k}^* \boldsymbol{\Theta}_{-k}) \boldsymbol{Q} \right] = \mathrm{tr}[\boldsymbol{A}_\ell \boldsymbol{Q}]. \tag{2.30}$$

Using this equality, the problem (2.29) is equivalent to the SDP problem

$$\begin{aligned} \min_{\boldsymbol{Q}} \ & \mathrm{tr}[\boldsymbol{A}_0 \boldsymbol{Q}] \\ \text{s.t.} \ & \mathrm{tr}[\boldsymbol{A}_\ell \boldsymbol{Q}] = b_\ell, \ \ell = 1 : L \\ & \boldsymbol{Q} \succeq 0 \end{aligned} \tag{2.31}$$

We conclude that the solution of the Toeplitz quadratic optimization problem (2.24) can be obtained as follows:

1. Solve the SDP problem (2.31) for the positive semidefinite matrix $\boldsymbol{Q}$.
2. Compute $R(z)$ with (2.6): $r_k = \mathrm{tr}[\boldsymbol{\Theta}_k \boldsymbol{Q}]$.
3. Obtain $\boldsymbol{h}$ from the spectral factorization of $R(z)$.

It is clear from the above method that any spectral factor of $R(z)$ is a solution to (2.24). Spectral factorization algorithms compute usually (and reliably) only the minimum-phase (or maximum-phase) factor. This might be the only drawback of the method; however, in signal processing applications, the minimum-phase spectral factor is often the desired one.

Examples of problems of the form (2.24) and interpretations of their solutions will be given in Chap. 6.

2.4 Duality

As mentioned in Remark 1.5, the set $\overline{\mathbb{RP}}_n[z]$ of nonnegative trigonometric polynomials of degree n is a cone. Due to the interest in optimization problems, we naturally look at the *dual* cone, defined by

$$\overline{\mathbb{RP}}_n^\star[z] = \{\boldsymbol{y} \in \mathbb{R}^{n+1} \mid \boldsymbol{y}^T \boldsymbol{r} \ge 0, \ \forall R \in \overline{\mathbb{RP}}_n[z]\}. \tag{2.32}$$

Theorem 2.14 *The dual cone (2.32) is the space of sequences* $\boldsymbol{y} \in \mathbb{R}^{n+1}$ *for which* $Toep(2y_0, y_1, \ldots, y_n) \succeq 0$. *(In other words, the dual cone can be identified with the space of positive semidefinite Toeplitz matrices.)*

Proof Since the polynomial $R(z)$ is nonnegative, it admits a spectral factorization (1.11), relation (1.17) holds and we can write

$$\boldsymbol{y}^T\boldsymbol{r} = \sum_{k=0}^{n} y_k \sum_{i=k}^{n} h_i h_{i-k} = \frac{1}{2}\boldsymbol{h}^T \begin{bmatrix} 2y_0 & y_1 & \ldots & y_n \\ y_1 & 2y_0 & \ddots & y_{n-1} \\ \vdots & \ddots & \ddots & \vdots \\ y_n & y_{n-1} & \ldots & 2y_0 \end{bmatrix} \boldsymbol{h}.$$

Since the above quadratic form is nonnegative for all $\boldsymbol{h} \in \mathbb{R}^{n+1}$, it follows that the matrix $\text{Toep}(2y_0, y_1, \ldots, y_n)$ is positive semidefinite. ■

Knowing the form of the dual cone, we can build easier the duals of optimization problems with nonnegative trigonometric polynomials. Let us consider the problem (1.22), where, for simplicity, we take $\boldsymbol{\Gamma} = \boldsymbol{I}$. The function dual to

$$f(\boldsymbol{r}) \stackrel{\Delta}{=} (\boldsymbol{r} - \hat{\boldsymbol{r}})^T(\boldsymbol{r} - \hat{\boldsymbol{r}}) \tag{2.33}$$

is

$$g(\boldsymbol{y}) = \inf_{\boldsymbol{r}} \left[f(\boldsymbol{r}) - \boldsymbol{y}^T\boldsymbol{r} \right],$$

where the Lagrangean multiplier $\boldsymbol{y}$ belongs to the dual cone. The minimum is obtained trivially for $\boldsymbol{y} = 2(\boldsymbol{r} - \hat{\boldsymbol{r}})$ and so

$$g(\boldsymbol{y}) = -\frac{1}{4}\boldsymbol{y}^T\boldsymbol{y} - \boldsymbol{y}^T\hat{\boldsymbol{r}}.$$

The optimization problem dual to (1.22) is

$$\begin{array}{llcll} \max\limits_{\boldsymbol{y}} & g(\boldsymbol{y}) & \iff & \min\limits_{\boldsymbol{y}} & \frac{1}{4}\boldsymbol{y}^T\boldsymbol{y} + \boldsymbol{y}^T\hat{\boldsymbol{r}} \\ \text{s.t.} & \boldsymbol{y} \in \overline{\mathbb{RP}}_n^\star[z] & & \text{s.t.} & \text{Toep}(2y_0, y_1, \ldots, y_n) \succeq 0 \end{array} \tag{2.34}$$

and is (as all duals are) a convex problem. Since (1.22) is convex and the Slater condition holds (which translates to the mere existence of strictly positive polynomials), it follows that the problems (1.22) and (2.34) have the same optimal value.

Moreover, we see immediately that (2.34) is an SDP problem (more precisely, it can be written as an SQLP one) since its constraint is the positive semidefinite matrix

$$\boldsymbol{Y} = \text{Toep}(2y_0, y_1, \ldots, y_n) = 2y_0\boldsymbol{I} + \sum_{k=1}^{n} y_k(\boldsymbol{\Theta}_k + \boldsymbol{\Theta}_k^T) \tag{2.35}$$

that depends linearly on the variables y_k.

We can now derive the dual of (2.34), using the scalar product specific to the space of positive semidefinite matrices, when building the Lagrangean function. The new primal (i.e., dual of the dual) function is

$$\tilde{f}(\boldsymbol{Q}) = \inf_{\boldsymbol{y}} \left(-g(\boldsymbol{y}) - \mathrm{tr}[\boldsymbol{Q}\boldsymbol{Y}]\right), \tag{2.36}$$

where the Lagrangean multiplier is $\boldsymbol{Q} \succeq 0$. We note that, due to (2.35), we have

$$\frac{\partial \mathrm{tr}[\boldsymbol{Q}\boldsymbol{Y}]}{\partial y_k} = \mathrm{tr}[(\boldsymbol{\Theta}_k + \boldsymbol{\Theta}_k^T)\boldsymbol{Q}] = 2\mathrm{tr}[\boldsymbol{\Theta}_k \boldsymbol{Q}].$$

Since the function to be minimized in (2.36) is quadratic, the minimum is obtained by equating its derivative with zero, giving

$$\frac{1}{2}\boldsymbol{y} = 2\begin{bmatrix} \vdots \\ \mathrm{tr}[\boldsymbol{\Theta}_k \boldsymbol{Q}] \\ \vdots \end{bmatrix} - \hat{\boldsymbol{r}}.$$

The dual of (2.34) is identical to (1.22), for $\boldsymbol{r} \in \mathbb{R}^{n+1}$ given by

$$r_k = 2\mathrm{tr}[\boldsymbol{\Theta}_k \boldsymbol{Q}].$$

Barring an insignificant factor of 2 that can be included in $\boldsymbol{Q}$, we have obtained again the trace parameterization of nonnegative polynomials, stated by Theorem 2.5. Although this is not a complete proof, it is an instructive result on how the Lagrangean duality mechanism can be used.

2.5 Kalman–Yakubovich–Popov Lemma

We show here that the trace parameterization (2.6) can be derived from the Kalman–Yakubovich–Popov (KYP) lemma. Consider a discrete-time system with transfer function $G(z) = \boldsymbol{C}(z\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}$, where $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ is a state-space model. Assume that the state-space representation is minimal (or $(\boldsymbol{A}, \boldsymbol{B})$ is controllable, $(\boldsymbol{C}, \boldsymbol{A})$ is observable). The KYP lemma states that the system is positive real, i.e.,

$$\mathrm{Re}[G(\omega)] \geq 0, \ \forall \omega \in [-\pi, \pi],$$

if and only if there exists a matrix $\boldsymbol{P} \succeq 0$ such that

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{P} - \boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A} & \text{sym} \\ \boldsymbol{C} - \boldsymbol{B}^T\boldsymbol{P}\boldsymbol{A} & (\boldsymbol{D} + \boldsymbol{D}^T) - \boldsymbol{B}^T\boldsymbol{P}\boldsymbol{B} \end{bmatrix} \succeq 0. \tag{2.37}$$

The causal part $R_+(z)$ of a nonnegative trigonometric polynomial, defined in (1.2), is positive real. Its controllable state-space realization is

$$\boldsymbol{A} = \boldsymbol{\Theta}_1 = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 \end{bmatrix}, \ \boldsymbol{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{2.38}$$
$$\boldsymbol{C} = \begin{bmatrix} r_n & \dots & r_2 & r_1 \end{bmatrix}, \quad \boldsymbol{D} = r_0/2.$$

Replacing these matrices in (2.37), we obtain

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{P} & \boldsymbol{C}^T \\ \boldsymbol{C} & \boldsymbol{D} + \boldsymbol{D}^T \end{bmatrix} - \begin{bmatrix} \boldsymbol{A}^T \\ \boldsymbol{B}^T \end{bmatrix} \boldsymbol{P} \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \end{bmatrix}$$
$$= \left[\begin{array}{c|c} & r_n \\ \boldsymbol{P} & \vdots \\ & r_1 \\ \hline r_n \ \dots \ r_1 & r_0 \end{array}\right] - \left[\begin{array}{c|c} 0 & 0 \ \dots \ 0 \\ \hline 0 & \\ \vdots & \boldsymbol{P} \\ 0 & \end{array}\right] \succeq 0. \tag{2.39}$$

Remarking that

$$\operatorname{tr} \boldsymbol{\Theta}_k \left(\begin{bmatrix} \boldsymbol{P} & \boldsymbol{0} \\ \boldsymbol{0} & 0 \end{bmatrix} - \begin{bmatrix} 0 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P} \end{bmatrix} \right) = 0, \tag{2.40}$$

for all $k = 0 : n$, the relation (2.39) is equivalent to $\operatorname{tr}[\boldsymbol{\Theta}_k \boldsymbol{Q}] = r_k$, i.e., the trace parameterization (2.6).

Despite the equivalence, it is not efficient to solve problems with nonnegative polynomials, as those presented in Sect. 2.2, by using the constraint (2.39), $\boldsymbol{P} \succeq 0$, instead of the trace parameterization. The LMI (2.39) has size $(n + 1) \times (n + 1)$ (as in the trace parameterization), but it contains $O(n^2)$ scalar variables in the matrix $\boldsymbol{P}$ (compared to only $n + 1$ equalities for the trace parameterization, which is an LMI in equality form). Consequently, the use of the KYP lemma leads to a complexity of $O(n^6)$, much higher than the $O(n^4)$ needed by the trace parameterization.

2.6 Spectral Factorization from a Gram Matrix

Let $R(z)$ be a nonnegative trigonometric polynomial and $H(z)$ a spectral factor respecting (1.11). As we have seen, we can express certain optimization problems involving $R(z)$ in terms of its associated Gram matrices. In this section, we explore how the spectral factor $H(z)$ can be computed directly from a Gram matrix $\boldsymbol{Q}$, and not by using (2.6) to get $R(z)$ and then obtaining $H(z)$ with one of the spectral factorization algorithms described in Appendix B.

2.6.1 SDP Computation of a Rank-1 Gram Matrix

We have remarked that the positive semidefinite matrix $\boldsymbol{Q}_1 = \boldsymbol{h}\boldsymbol{h}^H$ is a Gram matrix associated with $R(z)$ (see the lines preceding (2.8)). So, if

$$\boldsymbol{Q} = \begin{bmatrix} q_{00} & \boldsymbol{q}^H \\ \boldsymbol{q} & \hat{\boldsymbol{Q}} \end{bmatrix} \succeq 0 \tag{2.41}$$

is a rank-1 Gram matrix, then the spectral factor can be readily obtained from its first column as

$$\boldsymbol{h} = \frac{1}{\sqrt{q_{00}}} \begin{bmatrix} q_{00} \\ \boldsymbol{q} \end{bmatrix}. \tag{2.42}$$

The following theorem gives the conditions to obtain such a Gram matrix, more precisely the one giving the minimum-phase spectral factor.

Theorem 2.15 *Let $R \in \mathbb{C}_n[z]$ be a nonnegative trigonometric polynomial. Let $\boldsymbol{Q} \in \mathbb{C}^{n' \times n'}$ be the positive semidefinite Gram matrix (2.41) associated with $R(z)$ which has the largest element q_{00}. Then, the rank of the matrix $\boldsymbol{Q}$ is equal to 1 and $\boldsymbol{Q} = \boldsymbol{h}\boldsymbol{h}^H$, where the vector $\boldsymbol{h}$ contains the coefficients of the minimum-phase spectral factor of $R(z)$.*

Proof The set $\mathcal{G}(R)$ of positive semidefinite Gram matrices $\boldsymbol{Q}$ associated with $R(z)$ is convex and closed and the function $f(\boldsymbol{Q}) = q_{00}$ is linear and so is convex. It results that the maximum of $f(\boldsymbol{Q})$ is attained for some $\boldsymbol{Q} \in \mathcal{G}(R)$, $\boldsymbol{Q} \succeq 0$, and thus, the Gram matrix asserted in the theorem indeed exists. Let us assume that its rank is not one. Then, writing $\boldsymbol{Q}$ as in (2.41), there exists a nonzero matrix $\boldsymbol{P} \succeq 0$ such that

$$\begin{bmatrix} q_{00} & \boldsymbol{q}^H \\ \boldsymbol{q} & \hat{\boldsymbol{Q}} - \boldsymbol{P} \end{bmatrix} \succeq 0. \tag{2.43}$$

For example, we can take $\boldsymbol{P} = \hat{\boldsymbol{Q}} - \boldsymbol{q}\boldsymbol{q}^H / q_{00}$, i.e., the Schur complement of q_{00} in $\boldsymbol{Q}$. For this $\boldsymbol{P}$, it is clear that $\boldsymbol{P} = \boldsymbol{0}$ only if the rank $\boldsymbol{Q} = 1$, which we have assumed not true. Let us write

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \hat{\boldsymbol{P}} \end{bmatrix},$$

where $\hat{p}_{00} > 0$. So, we put in evidence the first nonzero (and positive, since $\boldsymbol{P} \succeq 0$ and $\boldsymbol{P}$ is nonzero) diagonal element of $\boldsymbol{P}$ as upper-left element of the block $\hat{\boldsymbol{P}}$. Define the matrix

$$\boldsymbol{X} = \boldsymbol{Q} - \begin{bmatrix} 0 & 0 \\ 0 & \hat{\boldsymbol{P}} \end{bmatrix} + \begin{bmatrix} \hat{\boldsymbol{P}} & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.44}$$

Since $\boldsymbol{X}$ is obtained by adding a positive semidefinite matrix to (2.43), it follows that $\boldsymbol{X} \succeq 0$. Moreover, taking (2.40) into account, it results that $\text{tr}[\boldsymbol{\Theta}_k \boldsymbol{X}] = \text{tr}[\boldsymbol{\Theta}_k \boldsymbol{Q}]$, for any $k = 0 : n$, and so $\boldsymbol{X} \in \mathcal{G}(R)$. Finally, we note that $x_{00} = q_{00} + \hat{p}_{00} > q_{00}$. We have thus built a Gram matrix associated with $R(z)$, whose upper-left element is greater than q_{00}, which is impossible. We conclude that the rank of $\boldsymbol{Q}$ is one and so $\boldsymbol{Q} = \boldsymbol{h}\boldsymbol{h}^H$, with $\boldsymbol{h}$ defined by (2.42).

That $\boldsymbol{h}$ is minimum-phase follows from the well-known Robinson's energy delay property, stating that the minimum-phase filter has the most energy concentrated in its first coefficients. Let $\boldsymbol{g}$ be a spectral factor of $R(z)$ having at least one zero outside the unit circle; if $\boldsymbol{h}$ is minimum-phase, then

$$\sum_{i=0}^{k} |h_i|^2 \geq \sum_{i=0}^{k} |g_i|^2, \quad \forall k = 0 : n-1,$$

and reciprocally. Moreover, for $k = 0$, the inequality is strict, i.e., $|h_0|^2 > |g_0|^2$. Since for the vector (2.42) we have $q_{00} = |h_0|^2 > |g_0|^2$ and q_{00} is maximum, it results that $\boldsymbol{h}$ is minimum-phase. ■

We conclude that the spectral factorization of a polynomial $R(z)$ can be computed with the following algorithm.

1. Solve the SDP problem

$$\begin{aligned} &\max_{\boldsymbol{Q}} \; q_{00} \\ &\text{s.t.} \;\; \text{tr}[\boldsymbol{\Theta}_k \boldsymbol{Q}] = r_k, \quad k = 0 : n \\ &\qquad \boldsymbol{Q} \succeq 0 \end{aligned} \tag{2.45}$$

2. Writing the solution $\boldsymbol{Q}$ as in (2.41), compute the minimum-phase spectral factor $\boldsymbol{h}$ with (2.42).

This spectral factorization algorithm has generally a higher complexity than many of those presented in Appendix B. However, it has two advantages. With appropriate modifications described in Sect. B.5, it can be used for polynomials with matrix coefficients (a topic discussed later in Sect. 3.10). Also, it can be combined with certain optimization problems in order to avoid spectral factorization as a separate operation. Consider for example the Toeplitz quadratic optimization problem (2.24). Instead of solving (2.31), computing $R(z)$, and then its spectral factor, we can solve

$$\begin{aligned} &\min_{\boldsymbol{Q}} \; \text{tr}[\boldsymbol{A}_0 \boldsymbol{Q}] - \alpha q_{00} \\ &\text{s.t.} \;\; \text{tr}[\boldsymbol{A}_\ell \boldsymbol{Q}] = b_\ell, \; \ell = 1 : L \\ &\qquad \boldsymbol{Q} \succeq 0 \end{aligned} \tag{2.46}$$

where α is a constant. This constant should be small enough such that (2.46) gives (approximately) the same $R(z)$ as the original problem (2.31). However, the Gram matrix $\boldsymbol{Q}$ given by (2.46) will have rank equal to 1. (That there is a rank-1 solution to (2.31) is ensured by its equivalence to (2.24)!)

2.6.2 Spectral Factorization Using a Riccati Equation

Let $(\boldsymbol{\Theta}_1^T, \tilde{\boldsymbol{h}}, \boldsymbol{c}^T, h_0)$ be the observable state-space realization of $H(z)$, where

$$\tilde{\boldsymbol{h}} = [h_n \ \dots \ h_1]^T, \quad \boldsymbol{c} = [0 \ \dots \ 0 \ 1]^T. \tag{2.47}$$

Given $R(z)$, the state-space formalism can be used to obtain a spectral factorization algorithm, based on solving a Riccati equation, as follows. Note also that in the spectral factorization relation (1.11), we can always take $H(z)$ such that h_0 is real.

Theorem 2.16 *Let $R \in \mathbb{C}_n[z]$ be a nonnegative polynomial and denote*

$$\tilde{\boldsymbol{r}} = [r_n \ \dots \ r_1]^T.$$

Let $\boldsymbol{\Xi}$ be the (positive semidefinite) solution of the discrete-time matrix Riccati equation

$$\boldsymbol{\Xi} = \boldsymbol{\Theta}_1^T \boldsymbol{\Xi} \boldsymbol{\Theta}_1 + (\tilde{\boldsymbol{r}} - \boldsymbol{\Theta}_1^T \boldsymbol{\Xi} \boldsymbol{c})(r_0 - \boldsymbol{c}^T \boldsymbol{\Xi} \boldsymbol{c})^{-1} (\tilde{\boldsymbol{r}} - \boldsymbol{\Theta}_1^T \boldsymbol{\Xi} \boldsymbol{c})^H. \tag{2.48}$$

The minimum-phase spectral factor of $R(z)$ is given by

$$\begin{aligned} h_0 &= (r_0 - \boldsymbol{c}^T \boldsymbol{\Xi} \boldsymbol{c})^{1/2}, \\ \tilde{\boldsymbol{h}} &= (\tilde{\boldsymbol{r}} - \boldsymbol{\Theta}_1^T \boldsymbol{\Xi} \boldsymbol{c}) / h_0, \end{aligned} \tag{2.49}$$

where $\tilde{\boldsymbol{h}}$ and $\boldsymbol{c}$ are like in (2.47).

This is a relatively well-known result; for completeness, the proof is given in Sect. 2.12.2. Note that the matrix $\boldsymbol{\Theta}_1$ is stable (has all eigenvalues inside the unit circle) and thus makes possible the existence of a positive semidefinite solution $\boldsymbol{\Xi}$ of the Riccati equation (2.48); this ensures the minimum-phase property of the spectral factor.

So, the minimum-phase spectral factor is computed simply with (2.49), after solving the Riccati equation (2.48); due to the special form of $\boldsymbol{\Theta}_1$ and $\boldsymbol{c}$, some computations are trivial; for example, $\boldsymbol{c}^T \boldsymbol{\Xi} \boldsymbol{c}$ is the element of $\boldsymbol{\Xi}$ from the lower-right corner, etc. Note that relations (2.48), (2.49) can be written in the equivalent form

$$\begin{aligned} h_0^2 &= r_0 - \boldsymbol{c}^T \boldsymbol{\Xi} \boldsymbol{c}, \\ h_0 \tilde{\boldsymbol{h}} &= \tilde{\boldsymbol{r}} - \boldsymbol{\Theta}_1^T \boldsymbol{\Xi} \boldsymbol{c}, \\ \tilde{\boldsymbol{h}} \tilde{\boldsymbol{h}}^H &= \boldsymbol{\Xi} - \boldsymbol{\Theta}_1^T \boldsymbol{\Xi} \boldsymbol{\Theta}_1. \end{aligned} \tag{2.50}$$

Now, let assume that we have a positive semidefinite Gram matrix $\boldsymbol{Q}$ associated with $R(z)$, split as follows:

$$\boldsymbol{Q} = \begin{bmatrix} \tilde{\boldsymbol{Q}} & \boldsymbol{s} \\ \boldsymbol{s}^H & \rho \end{bmatrix}, \tag{2.51}$$

where ρ is a scalar. Writing $\boldsymbol{Q}$ as in (2.39) and identifying the blocks with (2.51), we obtain

$$\begin{aligned} \rho &= r_0 - \boldsymbol{c}^T \boldsymbol{P} \boldsymbol{c}, \\ \boldsymbol{s} &= \tilde{\boldsymbol{r}} - \boldsymbol{\Theta}_1^T \boldsymbol{P} \boldsymbol{c}, \\ \tilde{\boldsymbol{Q}} &= \boldsymbol{P} - \boldsymbol{\Theta}_1^T \boldsymbol{P} \boldsymbol{\Theta}_1. \end{aligned} \tag{2.52}$$

Subtracting (2.52) from (2.50) and denoting $\boldsymbol{\Pi} = \boldsymbol{P} - \boldsymbol{\Xi}$, we obtain

$$\begin{aligned} h_0^2 &= \rho + \boldsymbol{c}^T \boldsymbol{\Pi} \boldsymbol{c}, \\ h_0 \tilde{\boldsymbol{h}} &= \boldsymbol{s} + \boldsymbol{\Theta}_1^T \boldsymbol{\Pi} \boldsymbol{c}, \\ \tilde{\boldsymbol{h}} \tilde{\boldsymbol{h}}^H &= \tilde{\boldsymbol{Q}} - \boldsymbol{\Pi} + \boldsymbol{\Theta}_1^T \boldsymbol{\Pi} \boldsymbol{\Theta}_1. \end{aligned} \tag{2.53}$$

These relations give the spectral factorization algorithm working directly with the Gram matrix $\boldsymbol{Q}$, split as in (2.51):

1. Compute the matrix $\boldsymbol{\Pi}$ by solving the Riccati equation

$$\boldsymbol{\Pi} = \tilde{\boldsymbol{Q}} + \boldsymbol{\Theta}_1^T \boldsymbol{\Pi} \boldsymbol{\Theta}_1 - (\boldsymbol{s} + \boldsymbol{\Theta}_1^T \boldsymbol{\Pi} \boldsymbol{c})(\rho + \boldsymbol{c}^T \boldsymbol{\Pi} \boldsymbol{c})^{-1} (\boldsymbol{s} + \boldsymbol{\Theta}_1^T \boldsymbol{\Pi} \boldsymbol{c})^H. \tag{2.54}$$

2. Compute the minimum-phase spectral factor $H(z)$ with

$$\begin{aligned} h_0 &= (\rho + \boldsymbol{c}^T \boldsymbol{\Pi} \boldsymbol{c})^{1/2}, \\ \tilde{\boldsymbol{h}} &= (\boldsymbol{s} + \boldsymbol{\Theta}_1^T \boldsymbol{\Pi} \boldsymbol{c}) / h_0. \end{aligned} \tag{2.55}$$

In principle, the algorithm for solving the Riccati equation may fail if the polynomial $R(z)$ has zeros on the unit circle (and so a symplectic matrix built with the parameters of the equation has eigenvalues on the unit circle). However, in practice, the algorithm based on solving (2.54) works very well; this appears to be due to the presence of small numerical errors in the Gram matrix, and so in $\tilde{\boldsymbol{Q}}$. (On the contrary, the algorithm based on solving (2.48) was observed to fail!) Although this algorithm is rather slow and can be used only for degrees up to 200–300, the author's experience recommends it as very safe.

2.7 Parameterization of Real Polynomials

The presentation of the Gram matrix concept given in Sect. 2.1 can be followed with few modifications for the case of polynomials of real variable. Since we are interested by positive polynomials, we consider only even degrees. Most of the proofs are given at the end of the chapter.

Definition 2.17 Consider the polynomial $P \in \mathbb{R}_{2n}[t]$. A symmetric matrix $\boldsymbol{Q} \in \mathbb{R}^{n' \times n'}$, where $n' = n + 1$, is called a *Gram* matrix associated with $P(t)$ if

$$P(t) = \boldsymbol{\psi}_n^T(t) \cdot \boldsymbol{Q} \cdot \boldsymbol{\psi}_n(t). \tag{2.56}$$

We denote $\mathcal{G}(P)$ the set of Gram matrices associated with $P(t)$. ■

Example 2.18 Consider polynomials of degree four, $P(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3 + p_4 t^4$. A Gram matrix $\boldsymbol{Q} \in \mathbb{R}^{3\times3}$ satisfies the relation

$$P(t) = [1 \;\; t \;\; t^2] \begin{bmatrix} q_{00} & q_{10} & q_{20} \\ q_{10} & q_{11} & q_{21} \\ q_{20} & q_{21} & q_{22} \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix}.$$

It results that

$$\begin{aligned} p_0 &= q_{00}, \\ p_1 &= 2q_{10}, \\ p_2 &= q_{11} + 2q_{20}, \\ p_3 &= 2q_{21}, \\ p_4 &= q_{22}. \end{aligned} \tag{2.57}$$

Unlike the 3×3 Gram matrix in Example 2.2, here there is only one degree of liberty left to the Gram matrices associated with $P(t)$, which have the form

$$\boldsymbol{Q} = \begin{bmatrix} p_0 & \frac{p_1}{2} & \frac{p_2}{2} \\ \frac{p_1}{2} & 0 & \frac{p_3}{2} \\ \frac{p_2}{2} & \frac{p_3}{2} & p_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -\frac{q_{11}}{2} \\ 0 & q_{11} & 0 \\ -\frac{q_{11}}{2} & 0 & 0 \end{bmatrix}.$$

In general, the mapping (2.56) that associates symmetric matrices in $\mathbb{R}^{n' \times n'}$ to real polynomials is many-to-one. For instance, taking

$$P(t) = 2 + 2t + 7t^2 - 2t^3 + t^4, \tag{2.58}$$

the following two matrices

$$\boldsymbol{Q}_0 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 7 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad \boldsymbol{Q}_1 = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & -1 \\ 2 & -1 & 1 \end{bmatrix} \tag{2.59}$$

are Gram matrices associated with $P(t)$. ■

Relations (2.57) suggest that the coefficients of $P(t)$ are obtained as sums along the antidiagonals of the Gram matrix.

Theorem 2.19 *If $P \in \mathbb{R}_{2n}[t]$ and $\boldsymbol{Q} \in \mathcal{G}(P)$, then the relation*

$$p_k = \mathrm{tr}[\boldsymbol{\Upsilon}_k \boldsymbol{Q}] = \sum_{i=\max(0,k-n)}^{\min(k,n)} q_{i,k-i}, \quad k = 0 : 2n, \tag{2.60}$$

holds, where $\boldsymbol{\Upsilon}_k$ is the elementary Hankel matrix with ones on the k-th antidiagonal and zeros elsewhere (antidiagonals are numbered from zero, starting with the upper left corner of the matrix).

Example 2.20 For a polynomial of degree 4, as in Example 2.18, the first three coefficients are given through (2.60) by

$$p_0 = \mathrm{tr} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{Q}, \quad p_1 = \mathrm{tr} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{Q}, \quad p_2 = \mathrm{tr} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \boldsymbol{Q}.$$

Theorem 2.21 *A polynomial $P \in \mathbb{R}_{2n}[t]$ is nonnegative (positive) on the real axis if and only if there exists a positive semidefinite (definite) matrix $\boldsymbol{Q} \in \mathbb{R}^{n' \times n'}$ such that (2.60) holds.*

Example 2.22 In Example 2.18, the Gram matrix $\boldsymbol{Q}_0$ from (2.59) is positive definite, which shows that the polynomial (2.58) is positive. However, the Gram matrix $\boldsymbol{Q}_1$ is not positive semidefinite. ■

Remark 2.23 (Sum-of-squares decomposition) Let $P \in \mathbb{R}_{2n}[t]$ be a nonnegative polynomial. Let $\boldsymbol{Q}$ be an associated positive semidefinite Gram matrix, whose eigenvalue decomposition is (2.11); as $\boldsymbol{Q}$ is real, the eigenvectors $\boldsymbol{x}_\ell$ are also real. Inserting (2.11) into (2.56), we obtain the sum-of-squares decomposition

$$P(t) = \sum_{\ell=1}^{\nu} \lambda_\ell^2 \cdot [\boldsymbol{\psi}^T(t)\boldsymbol{x}_\ell] \cdot [\boldsymbol{x}_\ell^T \boldsymbol{\psi}(t)] = \sum_{\ell=1}^{\nu} F_\ell(t)^2, \tag{2.61}$$

where $F_\ell(t) = \lambda_\ell \boldsymbol{\psi}^T(t)\boldsymbol{x}_\ell$. ■

As examples of SDP programs solving some simple problems involving nonnegative polynomials of real variable, we will give below short descriptions of the problems *Min_poly_value* and *Most_positive_Gram_matrix* for real polynomials. In contrast with the trigonometric polynomials case, it will result that these problems are not equivalent.

Problem (*Min_poly_value*) Let $P \in \mathbb{R}_{2n}[t]$, with $p_{2n} > 0$. We compute $\mu^\star = \min_{t\in\mathbb{R}} P(t)$ by finding the maximum $\mu \in \mathbb{R}$ for which $\tilde{P}(t) = P(t) - \mu$ is a nonnegative polynomial. Using the parameterization (2.60), the following SDP problem,

$$\begin{aligned} \mu^\star = \max_{\mu, \tilde{\boldsymbol{Q}}} \ & \mu \\ \text{s.t.} \ & \mu + \mathrm{tr}[\boldsymbol{\Upsilon}_0 \tilde{\boldsymbol{Q}}] = p_0 \\ & \mathrm{tr}[\boldsymbol{\Upsilon}_k \tilde{\boldsymbol{Q}}] = p_k, \quad k = 1:2n \\ & \tilde{\boldsymbol{Q}} \succeq 0 \end{aligned} \tag{2.62}$$

similar to (2.18), provides the solution.

Problem (*Most_positive_Gram_matrix*) Let $P \in \mathbb{R}_{2n}[t]$ be a positive polynomial. To compute the most positive Gram matrix associated with $P(t)$, we have to solve an SDP problem similar to (2.14), namely

$$\begin{aligned} \lambda^\star = \max_{\lambda, \boldsymbol{Q}} \ & \lambda \\ \text{s.t.} \ & \mathrm{tr}[\boldsymbol{\Upsilon}_k \boldsymbol{Q}] = p_k, \quad k = 0:2n \\ & \lambda \geq 0, \quad \boldsymbol{Q} \succeq \lambda \boldsymbol{I} \end{aligned} \tag{2.63}$$

To bring the above problem to standard form, we use, as in Sect. 2.2, the positive definite matrix $\tilde{\boldsymbol{Q}} = \boldsymbol{Q} - \lambda \boldsymbol{I}$. The difference is that now we have

$$\mathrm{tr}\boldsymbol{\Upsilon}_k = \begin{cases} 1, & \text{if } k \text{ is even} \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

Thus, the standard form of (2.63) is

$$\begin{aligned} \lambda^\star = \max_{\lambda, \tilde{\boldsymbol{Q}}} \ & \lambda \\ \text{s.t.} \ & \lambda + \mathrm{tr}[\boldsymbol{\Upsilon}_k \tilde{\boldsymbol{Q}}] = p_k, \quad k = 0:2:2n, \\ & \mathrm{tr}[\boldsymbol{\Upsilon}_k \tilde{\boldsymbol{Q}}] = p_k, \quad k = 1:2:2n \\ & \lambda \geq 0, \quad \tilde{\boldsymbol{Q}} \succeq 0 \end{aligned} \tag{2.64}$$

It is obvious that the constraints of (2.64) and (2.62) are different. ■

Example 2.24 Consider again the polynomial (2.58). Its minimum value for $t \in \mathbb{R}$ is 1.8628. The most positive Gram matrix associated with $P(t)$ is

$$\boldsymbol{Q} = \begin{bmatrix} 2.0000 & 1.0000 & -0.1763 \\ 1.0000 & 7.3525 & -1.0000 \\ -0.1763 & -1.0000 & 1.0000 \end{bmatrix}.$$

The smallest eigenvalue of $\boldsymbol{Q}$ is 0.8458. ■

2.8 Choosing the Right Basis

In defining the Gram matrices for trigonometric polynomials, we have used the natural basis (2.1). However, there are other possibilities. The technically simplest way is to replace the vector $\boldsymbol{\psi}(z)$ with

$$\boldsymbol{\phi}(z) = \boldsymbol{C}\boldsymbol{\psi}(z), \tag{2.65}$$

where $\boldsymbol{C} \in \mathbb{C}^{(n+1)\times(n+1)}$ is a nonsingular matrix. The relation (2.2) becomes

$$R(z) = \boldsymbol{\phi}^H(z^{-1}) \cdot \boldsymbol{C}^{-H}\boldsymbol{Q}\boldsymbol{C}^{-1} \cdot \boldsymbol{\phi}(z). \tag{2.66}$$

From Theorem 2.5, we immediately conclude that $R(z)$ is nonnegative if and only if there exist $\widehat{\boldsymbol{Q}} \succeq 0$ such that

$$R(z) = \boldsymbol{\phi}^H(z^{-1}) \cdot \widehat{\boldsymbol{Q}} \cdot \boldsymbol{\phi}(z). \tag{2.67}$$

(Since $\boldsymbol{C}$ is nonsingular, any $\widehat{\boldsymbol{Q}} \succeq 0$ can be written as $\widehat{\boldsymbol{Q}} = \boldsymbol{C}^{-H}\boldsymbol{Q}\boldsymbol{C}^{-1}$, for some $\boldsymbol{Q} \succeq 0$.) We can name $\widehat{\boldsymbol{Q}}$ a Gram matrix associated with $R(z)$, for the basis $\boldsymbol{\phi}(z)$. The parameterization (2.6) takes the form

$$r_k = \mathrm{tr}[\boldsymbol{\Theta}_k \boldsymbol{C}^H \widehat{\boldsymbol{Q}} \boldsymbol{C}] = \mathrm{tr}[\boldsymbol{C}\boldsymbol{\Theta}_k \boldsymbol{C}^H \cdot \widehat{\boldsymbol{Q}}]. \tag{2.68}$$

This general approach may be not so useful, especially as it may produce complex Gram matrices $\widehat{\boldsymbol{Q}}$ even for polynomials with real coefficients. Since $R(\omega)$ has real values, we would be more interested in associating real Gram matrices even with polynomials with complex coefficients. We introduce in the sequel several new parameterizations.

2.8.1 Basis of Trigonometric Polynomials

Let us consider a nonnegative trigonometric polynomial $R(z)$ of degree $n = 2\tilde{n}$. The spectral factorization Theorem 1.1 says that $R(\omega) = |H(\omega)|^2$, where $H(z)$ is a causal polynomial; since $H(z)$ may be multiplied with any unit-norm constant and is still a spectral factor, we can take $h_{\tilde{n}}$ real. We can also write $R(\omega) = |\tilde{H}(\omega)|^2$, with

$$\tilde{H}(z) = z^{\tilde{n}} H(z) = \sum_{k=-\tilde{n}}^{\tilde{n}} h_{k+\tilde{n}} z^{-k}. \tag{2.69}$$

It results that

$$\tilde{H}(\omega) = A(\omega) + jB(\omega), \tag{2.70}$$

where

$$\begin{aligned} A(\omega) &= h_{\tilde{n}} + \textstyle\sum_{k=1}^{\tilde{n}}[\operatorname{Re}(h_{\tilde{n}-k} + h_{\tilde{n}+k})\cos k\omega + \operatorname{Im}(-h_{\tilde{n}-k} + h_{\tilde{n}+k})\sin k\omega], \\ B(\omega) &= \textstyle\sum_{k=1}^{\tilde{n}}[\operatorname{Im}(h_{\tilde{n}-k} + h_{\tilde{n}+k})\cos k\omega + \operatorname{Re}(h_{\tilde{n}-k} - h_{\tilde{n}+k})\sin k\omega] \end{aligned}$$

are trigonometric polynomials of degree $\tilde{n}$, with real coefficients. Introducing the basis vector (of length $n+1$)

$$\boldsymbol{\chi}(\omega) = [1 \;\; \cos\omega \;\; \sin\omega \;\; \ldots \;\; \cos\tilde{n}\omega \;\; \sin\tilde{n}\omega]^T, \tag{2.71}$$

we can write

$$A(\omega) = \boldsymbol{a}^T\boldsymbol{\chi}(\omega), \quad B(\omega) = \boldsymbol{b}^T\boldsymbol{\chi}(\omega),$$

with $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^{n+1}$, and so we obtain

$$|\tilde{H}(\omega)|^2 = A(\omega)^2 + B(\omega)^2 = \boldsymbol{\chi}^T(\omega)(\boldsymbol{a}\boldsymbol{a}^T + \boldsymbol{b}\boldsymbol{b}^T)\boldsymbol{\chi}(\omega). \tag{2.72}$$

This expression leads to a result similar with Theorems 2.5 and 2.21.

Theorem 2.25 *A polynomial $R \in \mathbb{C}_{2\tilde{n}}[z]$ is nonnegative on the unit circle if and only if there exists a positive semidefinite matrix $\boldsymbol{Q} \in \mathbb{R}^{(2\tilde{n}+1)\times(2\tilde{n}+1)}$ such that*

$$R(\omega) = \boldsymbol{\chi}^T(\omega) \cdot \boldsymbol{Q} \cdot \boldsymbol{\chi}(\omega), \tag{2.73}$$

where $\boldsymbol{\chi}(\omega)$ is defined in (2.71).

Proof If there exists $\boldsymbol{Q} \succeq 0$ such that (2.73) holds, then it is clear that $R(\omega) \geq 0$ for all ω. Reciprocally, if $R(\omega) \geq 0$, then the matrix $\boldsymbol{Q} \stackrel{\Delta}{=} \boldsymbol{a}\boldsymbol{a}^T + \boldsymbol{b}\boldsymbol{b}^T \succeq 0$ from (2.72) satisfies (2.73). ■

Example 2.26 Let us take a polynomial of degree $n = 2$ (i.e., $\tilde{n} = 1$), with complex coefficients. On the unit circle, according to (2.73), we have

$$R(\omega) = [1 \;\; \cos\omega \;\; \sin\omega]\begin{bmatrix} q_{00} & q_{10} & q_{20} \\ q_{10} & q_{11} & q_{21} \\ q_{20} & q_{21} & q_{22} \end{bmatrix}\begin{bmatrix} 1 \\ \cos\omega \\ \sin\omega \end{bmatrix}.$$

By using simple trigonometric expressions such as $(\cos\omega)^2 = (1+\cos 2\omega)/2$, we obtain

$$R(\omega) = (q_{00} + \tfrac{q_{11}}{2} + \tfrac{q_{22}}{2}) + 2q_{10}\cos\omega + 2q_{20}\sin\omega + (\tfrac{q_{11}}{2} - \tfrac{q_{22}}{2})\cos 2\omega + q_{21}\sin 2\omega. \tag{2.74}$$

In the particular case where the polynomial is

$$R(z) = (2+j)z^{-2} + (3-j)z^{-1} + 9 + (3+j)z + (2-j)z^2,$$

on the unit circle we have

$$R(\omega) = 9 + 6\cos\omega - 2\sin\omega + 4\cos 2\omega + 2\sin 2\omega. \tag{2.75}$$

Identifying with (2.74), we obtain the general form of the Gram matrix

$$\boldsymbol{Q} = \begin{bmatrix} q_{00} & 3 & -1 \\ 3 & 13 - q_{00} & 2 \\ -1 & 2 & 5 - q_{00} \end{bmatrix}. \tag{2.76}$$

Taking $q_{00} = 2$, we get the matrix

$$\boldsymbol{Q} = \begin{bmatrix} 2 & 3 & -1 \\ 3 & 11 & 2 \\ -1 & 2 & 3 \end{bmatrix} \succ 0.$$

Its positivity ensures that the polynomial $R(z)$ is positive on the unit circle. Indeed, the minimum value of the polynomial is 0.5224, as obtained by solving (2.18) with the programs shown in Sect. 2.12.1. ■

Using the standard trigonometric identities

$$\begin{aligned} \cos i\omega \cos \ell\omega &= \tfrac{1}{2}[\cos(i+\ell)\omega + \cos(i-\ell)\omega], \\ \sin i\omega \sin \ell\omega &= \tfrac{1}{2}[-\cos(i+\ell)\omega + \cos(i-\ell)\omega], \end{aligned} \tag{2.77}$$

and

$$\sin i\omega \cos \ell\omega = \frac{1}{2}[\sin(i+\ell)\omega + \sin(i-\ell)\omega], \tag{2.78}$$

it can be easily shown that the relation (2.73) can be written as a linear dependence between the coefficients of $R(z)$ and the elements of the matrix $\boldsymbol{Q}$. This proves the following.

Theorem 2.27 *A polynomial $R \in \mathbb{C}_{2\tilde{n}}[z]$ is nonnegative on the unit circle if and only if there exists a positive semidefinite matrix $\boldsymbol{Q} \in \mathbb{R}^{(2\tilde{n}+1)\times(2\tilde{n}+1)}$ such that*

$$r_k = tr[\boldsymbol{\Gamma}_k \boldsymbol{Q}], \quad k = 0 : 2\tilde{n}, \tag{2.79}$$

where $\boldsymbol{\Gamma}_k$ are constant matrices.

The expressions of the matrices $\boldsymbol{\Gamma}_k$ are not derived here. Note that these matrices are in general complex; in this sense, the parameterization (2.79) is opposed to the trace parameterization (2.6), where the constant matrices $\boldsymbol{\Theta}_k$ are real, while the parameter matrix $\boldsymbol{Q}$ is complex. However, the relation (2.79) can be immediately split into $\mathrm{Re} r_k = \mathrm{tr}[(\mathrm{Re}\boldsymbol{\Gamma}_k)\boldsymbol{Q}]$, $\mathrm{Im} r_k = \mathrm{tr}[(\mathrm{Im}\boldsymbol{\Gamma}_k)\boldsymbol{Q}]$; the total number of real equalities is $2n+1$ (remind that $n = 2\tilde{n}$). In contrast, the trace parameterization (2.6) has $n+1$ complex equalities (amounting also to $2n+1$ real equalities).

For example, we derive from (2.73) that the (always real) free term is

$$r_0 = q_{00} + \frac{1}{2}\sum_{i=1}^{n} q_{ii}$$

and so $\boldsymbol{\Gamma}_0 = \text{diag}(1, 1/2, \ldots, 1/2)$. We leave the formulas for the other matrices $\boldsymbol{\Gamma}_k$ as a problem for the interested reader. A simpler case, when the polynomial has real coefficients, will be detailed in Sect. 2.8.3.

Problem (*Min_poly_value*) The parameterization (2.79) can be used to solve optimization problems in the same way as the trace parameterization. For example, the minimum value of a given polynomial $R(\omega)$ can be computed by solving

$$\begin{aligned} \mu^\star = \max_{\mu} \ & \mu \\ \text{s.t.} \ & R(\omega) - \mu = \text{tr}[\boldsymbol{\Gamma}_k \tilde{\boldsymbol{Q}}], \ k = 0 : n \\ & \tilde{\boldsymbol{Q}} \succeq 0 \end{aligned} \tag{2.80}$$

This is obviously an SDP problem. The main difference with respect to (2.18) is that the Gram matrix is now real, even though $R(z)$ has complex coefficients. The size of the matrix is the same in both problems. The number of equality constraints in (2.80) is $2n + 1$, i.e., the number of real coefficients in (1.8); that is why, for example, the matrix (2.76) depends only on a single variable, q_{00}; the 6 distinct elements of a 3×3 symmetric matrix must satisfy 5 linear equalities. In (2.18), the number of equality constraints is only $n + 1$, but these are complex equalities. Generally, we expect that (2.80) is solved faster than (2.18). We note also that the problem (2.80) is equivalent to finding the most positive matrix $\boldsymbol{Q}$ for which (2.73) holds; see **P** 2.10. ■

Example 2.28 (continued) The most positive matrix (2.76) is obtained for $q_{00} = 2.55$. Its smallest eigenvalue is 0.2612. This leads to a minimum value of $R(\omega)$ equal to 0.5224. (See again problem **P** 2.10.) ■

Let us now look at the case where the degree of the polynomial is odd, $n = 2\tilde{n} + 1$. We have now $R(\omega) = |\tilde{H}(\omega)|^2$, with

$$\tilde{H}(z) = z^{\tilde{n}+\frac{1}{2}} H(z) = \sum_{k=-\tilde{n}}^{\tilde{n}+1} h_{k+\tilde{n}} z^{-k+\frac{1}{2}}. \tag{2.81}$$

With the basis vector

$$\tilde{\boldsymbol{\chi}}(\omega) = [\cos \tfrac{\omega}{2} \ \ \sin \tfrac{\omega}{2} \ \ \ldots \ \ \cos(\tilde{n} + \tfrac{1}{2})\omega \ \ \sin(\tilde{n} + \tfrac{1}{2})\omega]^T \tag{2.82}$$

of length $n+1 = 2(\tilde{n}+1)$, it results that (2.70) holds with $A(\omega) = \boldsymbol{a}^T \tilde{\boldsymbol{\chi}}(\omega)$, $B(\omega) = \boldsymbol{b}^T \tilde{\boldsymbol{\chi}}(\omega)$. Hence, Theorem 2.25 holds also for odd-order polynomials if we replace $\boldsymbol{\chi}(\omega)$ with $\tilde{\boldsymbol{\chi}}(\omega)$ in (2.73). However, in this case, although the polynomial $R(\omega)$ is expressed as a sum-of-squares, the terms of the sum-of-squares are trigonometric polynomials in $\omega/2$. This aspect has no consequence on optimization applications that can be carried on as for even-order polynomials. A parameterization like (2.79) also holds, but with different constant matrices $\boldsymbol{\Gamma}_k$.

2.8.2 Transformation to Real Polynomials

We consider now trigonometric polynomials with *real* coefficients, having thus the form (1.4). (Polynomials in which all terms are sine functions can be treated similarly.) As already written in (1.6), using the simple substitution $t = \cos\omega$, a polynomial $R \in \mathbb{R}_n[z]$ can be expressed on the unit circle as

$$R(\omega) \equiv P(t) = \sum_{k=0}^{n} p_k t^k, \quad t \in [-1, 1]. \tag{2.83}$$

We can parameterize nonnegative trigonometric polynomials with real coefficients by using results valid for real polynomials nonnegative on an interval, specifically Theorem 1.11. For simplicity, we consider only the case $n = 2\tilde{n}$. According to (1.30), a polynomial (2.83) which is nonnegative for $t \in [-1, 1]$, can always be written as

$$P(t) = F(t)^2 + (1 - t^2)G(t)^2, \tag{2.84}$$

where $F(t)$ and $G(t)$ are polynomials of degree $\tilde{n}$ and $\tilde{n} - 1$, respectively. Since $F(t)^2$ and $G(t)^2$ are globally nonnegative polynomials, they can be characterized via (2.56), using positive semidefinite Gram matrices $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$, as follows:

$$\begin{aligned} F(t)^2 &= \boldsymbol{\psi}_{\tilde{n}}^T(t)\,\boldsymbol{Q}_1\boldsymbol{\psi}_{\tilde{n}}(t), \\ G(t)^2 &= \boldsymbol{\psi}_{\tilde{n}-1}^T(t)\,\boldsymbol{Q}_2\boldsymbol{\psi}_{\tilde{n}-1}(t). \end{aligned} \tag{2.85}$$

Replacing these equalities with their counterparts similar to (2.60), the coefficients of the polynomial $P(t)$, as resulting from the identity (2.84), are given by

$$p_k = \begin{cases} \mathrm{tr}[\boldsymbol{\Upsilon}_k \boldsymbol{Q}_1] + \mathrm{tr}[\boldsymbol{\Upsilon}_k \boldsymbol{Q}_2] - \mathrm{tr}[\boldsymbol{\Upsilon}_{k-2} \boldsymbol{Q}_2], & \text{if } k \geq 2, \\ \mathrm{tr}[\boldsymbol{\Upsilon}_k \boldsymbol{Q}_1] + \mathrm{tr}[\boldsymbol{\Upsilon}_k \boldsymbol{Q}_2], & \text{if } k < 2. \end{cases} \tag{2.86}$$

Hence, we can parameterize a nonnegative trigonometric polynomials with two positive semidefinite matrices, of sizes $(\tilde{n} + 1) \times (\tilde{n} + 1)$ and $\tilde{n} \times \tilde{n}$ ($\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$, respectively). (Note the ambiguity of notation in (2.86), where the size of a matrix $\boldsymbol{\Upsilon}_k$ is dictated by the size of the matrix multiplying it.) This is in contrast with the trace parameterization (2.6), where a single matrix of size $(n + 1) \times (n + 1)$ appears. In terms of complexity, the parameterization (2.86) seems more convenient, as the size of the matrices is twice smaller; we can hope that, at least asymptotically, the problems using (2.86) may be solved faster than when using (2.6). In practice, the speedup is not visible, since typically the former problems need significantly more iterations. Moreover, the use of (2.86) is hampered by numerical stability considerations. The transformation from $R(\omega)$ to $P(t)$, using a Chebyshev basis (see Sect. 1.5.1), is made using coefficients that have a broad range of values (practically, from 1 to 2^n); also, the Chebyshev transformation matrix from (1.44) has a large condition number. For these reason, the use of (2.86) is limited to, say, $n \leq 30$; even so, the solutions obtained using the trace parameterization (2.6) are more accurate.

We conclude that the transformation (2.83) is a bad idea, although it may seem attractive from a complexity viewpoint. However, since the Chebyshev transformation is the main troublemaker, we can try to use bases of trigonometric functions, as shown in the sequel.

2.8.3 Gram-Pair Matrix Parameterization

We consider again a nonnegative trigonometric polynomial $R(z)$ with *real* coefficients, whose degree is $n = 2\tilde{n}$. The polynomial (2.69) has the form (2.70), where

$$\begin{aligned} A(\omega) &= h_{\tilde{n}} + \sum_{k=1}^{\tilde{n}} (h_{\tilde{n}-k} + h_{\tilde{n}+k}) \cos k\omega, \\ B(\omega) &= \sum_{k=1}^{\tilde{n}} (h_{\tilde{n}-k} - h_{\tilde{n}+k}) \sin k\omega. \end{aligned} \tag{2.87}$$

As in (2.72), we obtain

$$R(\omega) = |\tilde{H}(\omega)|^2 = A(\omega)^2 + B(\omega)^2, \tag{2.88}$$

where now $A(\omega)$ is a polynomial with cosine terms, while $B(\omega)$ is a polynomial with sine terms. Let us denote the bases of such $\tilde{n}$th order polynomials with

$$\boldsymbol{\chi}_c(\omega) = [1 \ \ \cos\omega \ \ \ldots \ \ \cos\tilde{n}\omega]^T, \tag{2.89}$$

and

$$\boldsymbol{\chi}_s(\omega) = [\sin\omega \ \ \ldots \ \ \sin\tilde{n}\omega]^T. \tag{2.90}$$

With these bases, a Gram parameterization of $R(\omega)$ is possible, using two Gram matrices.

Theorem 2.29 *Let $R \in \mathbb{R}_n[z]$ be a trigonometric polynomial of order $n = 2\tilde{n}$. The polynomial is nonnegative if and only if there exist positive semidefinite matrices $\boldsymbol{Q} \in \mathbb{R}^{(\tilde{n}+1)\times(\tilde{n}+1)}$ and $\boldsymbol{S} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ such that*

$$R(\omega) = \boldsymbol{\chi}_c^T(\omega)\boldsymbol{Q}\boldsymbol{\chi}_c(\omega) + \boldsymbol{\chi}_s^T(\omega)\boldsymbol{S}\boldsymbol{\chi}_s(\omega). \tag{2.91}$$

We name $(\boldsymbol{Q}, \boldsymbol{S})$ a Gram pair associated with $R(\omega)$.

Proof If there exist $\boldsymbol{Q} \succeq 0$, $\boldsymbol{S} \succeq 0$ such that (2.91) holds, it results that $R(\omega) \geq 0$. Reciprocally, if $R(\omega) \geq 0$, then the matrices $\boldsymbol{Q} \stackrel{\Delta}{=} \boldsymbol{a}\boldsymbol{a}^T \succeq 0$ and $\boldsymbol{S} \stackrel{\Delta}{=} \boldsymbol{b}\boldsymbol{b}^T \succeq 0$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are the vectors of coefficients of the polynomials $A(\omega)$ and $B(\omega)$ from (2.87), satisfy (2.91). ∎

To be able to formulate optimization problems, we need the expressions of the coefficients of $R(\omega)$ that result from (2.91). We start by expanding the quadratic forms, thus obtaining

$$R(\omega) = \sum_{i,\ell=0}^{\tilde{n}} q_{i\ell} \cos i\omega \cos \ell\omega + \sum_{i,\ell=0}^{\tilde{n}-1} s_{i\ell} \sin(i+1)\omega \sin(\ell+1)\omega. \tag{2.92}$$

Using the trigonometric identities (2.77) in (2.92) and taking (1.4) into account, the coefficients of $R(\omega)$ are given by

$$\begin{aligned} r_0 &= q_{00} + \frac{1}{2}\sum_{i=1}^{\tilde{n}} q_{ii} + \frac{1}{2}\sum_{i=0}^{\tilde{n}-1} s_{ii}, \\ r_k &= \frac{1}{4}\left(\sum_{i+\ell=k} q_{i\ell} + \sum_{|i-\ell|=k} q_{i\ell} - \sum_{i+\ell+2=k} s_{i\ell} + \sum_{|i-\ell|=k} s_{i\ell}\right), \quad k \geq 1. \end{aligned} \tag{2.93}$$

Thus, we can formulate the following theorem, expressing the coefficients in the style of (2.6).

Theorem 2.30 *The relation (2.91) defining a Gram pair associated with the even order trigonometric polynomial $R(\omega)$ is equivalent to*

$$r_k = \operatorname{tr}[\boldsymbol{\Phi}_k \boldsymbol{Q}] + \operatorname{tr}[\boldsymbol{\Lambda}_k \boldsymbol{S}], \tag{2.94}$$

where the matrices $\boldsymbol{\Phi}_k \in \mathbb{R}^{(\tilde{n}+1)\times(\tilde{n}+1)}$ and $\boldsymbol{\Lambda}_k \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ are

$$\begin{aligned} \boldsymbol{\Phi}_0 &= \tfrac{1}{2}(\boldsymbol{\Upsilon}_0 + \boldsymbol{I}), \\ \boldsymbol{\Phi}_k &= \tfrac{1}{4}(\boldsymbol{\Upsilon}_k + \boldsymbol{\Theta}_k + \boldsymbol{\Theta}_{-k}), \quad k \geq 1, \end{aligned} \tag{2.95}$$

and, respectively

$$\begin{aligned} \boldsymbol{\Lambda}_0 &= \tfrac{1}{2}\boldsymbol{I}, \\ \boldsymbol{\Lambda}_k &= \tfrac{1}{4}(-\boldsymbol{\Upsilon}_{k-2} + \boldsymbol{\Theta}_k + \boldsymbol{\Theta}_{-k}), \quad k \geq 1. \end{aligned} \tag{2.96}$$

In the above relations, we assume that the matrices $\boldsymbol{\Upsilon}_k$ or $\boldsymbol{\Theta}_k$ are zero whenever k is out of range (i.e., negative or larger than the number of diagonals). (For example, in (2.96), $\boldsymbol{\Upsilon}_{k-2} = 0$ if $k = 1$ and $\boldsymbol{\Theta}_k = 0$ if $k \geq \tilde{n}$.)

We note that the matrices $\boldsymbol{\Phi}_k$ and $\boldsymbol{\Lambda}_k$ are symmetric. We can replace $\boldsymbol{\Theta}_k + \boldsymbol{\Theta}_{-k}$ with $2\boldsymbol{\Theta}_k$, in their expressions, and the parameterization (2.94) remains valid.

Example 2.31 If $n = 4$, and so $\tilde{n} = 2$, the first three pairs of constant matrices from (2.94) are

$$\boldsymbol{\Phi}_0 = \frac{1}{2}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right), \quad \boldsymbol{\Lambda}_0 = \frac{1}{2}\left(-\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right),$$

$$\boldsymbol{\Phi}_1 = \frac{1}{4}\left(\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\right), \quad \boldsymbol{\Lambda}_1 = \frac{1}{4}\left(-\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right),$$

$$\boldsymbol{\Phi}_2 = \frac{1}{4}\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\right), \quad \boldsymbol{\Lambda}_2 = \frac{1}{4}\left(-\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right).$$

In general, for $k = 0 : \tilde{n}$, the matrices $\boldsymbol{\Phi}_k$ have a Toeplitz+Hankel structure, while $\boldsymbol{\Phi}_k = \boldsymbol{\Upsilon}_k$ for $k > \tilde{n}$. The matrices $\boldsymbol{\Lambda}_k$ are Toeplitz for $k = 0, 1$, Toeplitz+Hankel for $k = 2 : \tilde{n} - 1$ and Hankel for $k = \tilde{n} : n$. ■

Problem (*Min_poly_value*) Using the parameterization (2.91), the minimum value of a given polynomial $R(\omega)$ can be computed by solving

$$\begin{aligned} \mu^\star = \ & \max_{\mu} \ \mu \\ & \text{s.t.} \ \ R(\omega) - \mu = \boldsymbol{\chi}_c^T(\omega)\tilde{\boldsymbol{Q}}\boldsymbol{\chi}_c(\omega) + \boldsymbol{\chi}_s^T(\omega)\tilde{\boldsymbol{S}}\boldsymbol{\chi}_s(\omega) \\ & \qquad \tilde{\boldsymbol{Q}} \succeq 0, \ \ \tilde{\boldsymbol{S}} \succeq 0 \end{aligned} \tag{2.97}$$

The form (2.94) confirms that this is an SDP problem. As in Sect. 2.8.2, the two matrices from (2.97) are twice smaller than the single matrix from the corresponding problem (2.18), where the trace parameterization (2.6) is used. As discussed in Remark 2.13, the complexity of such an SDP problem depends on the square of the size of the matrices (the number of equality constraints is the same in the two problems). So, we can expect that (2.97) is solved up to four times faster than its counterpart (2.18); however, since there are two matrices in (2.97), the speedup factor could be actually twice smaller. These are only qualitative considerations; the fact that the constant matrices from (2.94) and (2.6) are sparse makes the complexity analysis more difficult.

We also note that solving (2.97) is equivalent to finding the most positive matrices $\boldsymbol{Q}$, $\boldsymbol{S}$ for which (2.91) holds. By this, we understand that $\min(\lambda_{min}(\boldsymbol{Q}), \lambda_{min}(\boldsymbol{S}))$ is maximum. See problem **P** 2.11 for details. ■

Example 2.32 We give in Table 2.1 the times needed for finding the minimum value of a trigonometric polynomial with random coefficients by solving two SDP problems; the first is (2.18), based on the trace parameterization (2.6); the second is (2.97), based on the Gram-pair parameterization (2.91). The first two rows contain data from 2006, when the first edition of this book was written. The last two rows were obtained

Table 2.1 Times, in seconds, for finding the minimum value of a trigonometric polynomial via two SDP problems

Year	SDP problem	Parameterization	Order n				
			20	50	100	200	300
2006	(2.18)	Trace	0.26	1.00	7.0	120	800
	(2.97)	Gram pair	0.21	0.51	2.7	22	99
2016	(2.18)	Trace	0.10	0.50	1.2	5.5	18
	(2.97)	Gram pair	0.04	0.20	0.7	2.6	7.5

in 2016, when preparing the second edition. In both cases, the computers were rather average for the period. We see that the Gram-pair parameterization leads to a roughly twice faster solution for almost all sizes in 2016, but only for small sizes in 2006 (excepting very small problems, where overhead due to preparing data and other operations may be significant). The most likely reason for the much better behavior of the Gram-pair parameterization on the old computer is the lower memory requirement of this parameterization. We also note that, no matter the parameterization, we can solve in the same time problems that are twice larger than 10 years ago. We conclude that the Gram-pair parameterization is clearly faster and should be preferred to the trace parameterization. ■

If the order of the polynomial $R(\omega)$ is odd, $n = 2\tilde{n} + 1$, the pseudopolynomial (2.81) leads to an expression (2.88) where $A(\omega)$ and $B(\omega)$ depend linearly on the elements of the basis vectors

$$\tilde{\boldsymbol{\chi}}_c(\omega) = [\cos\tfrac{\omega}{2}\ \cos\tfrac{3\omega}{2}\ \ldots\ \cos(\tilde{n}+\tfrac{1}{2})\omega]^T \tag{2.98}$$

and

$$\tilde{\boldsymbol{\chi}}_s(\omega) = [\sin\tfrac{\omega}{2}\ \sin\tfrac{3\omega}{2}\ \ldots\ \sin(\tilde{n}+\tfrac{1}{2})\omega]^T \tag{2.99}$$

respectively. It is easy to see that Theorem 2.29 holds also for odd n, with $\tilde{\boldsymbol{\chi}}_c(\omega)$ and $\tilde{\boldsymbol{\chi}}_s(\omega)$ replacing $\boldsymbol{\chi}_c(\omega)$ and $\boldsymbol{\chi}_s(\omega)$, respectively, in (2.91). Also, we note that the matrices $\boldsymbol{Q}$ and $\boldsymbol{S}$ have the same size, namely $(\tilde{n}+1)\times(\tilde{n}+1)$, since the basis vectors (2.98) and (2.99) have the same length. The relations between the coefficients of $R(\omega)$ and the elements of the two Gram matrices are simpler than in the even case (2.93). They are

$$\begin{aligned} r_0 &= \frac{1}{2}\sum_{i=0}^{\tilde{n}}(q_{ii}+s_{ii}),\\ r_k &= \frac{1}{4}\left(\sum_{i+\ell+1=k}(q_{i\ell}-s_{i\ell}) + \sum_{|i-\ell|=k}(q_{i\ell}+s_{i\ell})\right), \quad k \ge 1. \end{aligned} \tag{2.100}$$

The proof of the above formulas, based on relations in the style of (2.77), is left to the reader. From (2.100) it results that, for odd order, the constant matrices $\boldsymbol{\Phi}_k$ and $\boldsymbol{\Lambda}_k$ appearing in (2.94) should be replaced with

$$\begin{aligned}\tilde{\boldsymbol{\Phi}}_k &= \tfrac{1}{4}(\boldsymbol{\Upsilon}_{k-1} + \boldsymbol{\Theta}_k + \boldsymbol{\Theta}_{-k}),\\ \tilde{\boldsymbol{\Lambda}}_k &= \tfrac{1}{4}(-\boldsymbol{\Upsilon}_{k-1} + \boldsymbol{\Theta}_k + \boldsymbol{\Theta}_{-k}),\end{aligned} \tag{2.101}$$

respectively, for $k = 0 : n$.

2.9 Interpolation Representations

Until now, in all parameterizations, we have defined the polynomials by their coefficients. Alternatively, we can use as parameters the values of the polynomial on a specified set of points. Let $\Omega = \{\omega_i\}_{i=1:2n+1} \subset (-\pi, \pi]$ be a set of $2n+1$ frequency points. We return to the general case of trigonometric polynomials with complex coefficients. If the values

$$\rho_i = R(\omega_i), \quad i = 1 : 2n+1, \tag{2.102}$$

of the nth order trigonometric polynomial $R(z)$ are known, then the polynomial is completely determined.

For simplicity, let us consider only the case where $n = 2\tilde{n}$. We work with the basis of trigonometric polynomials

$$\boldsymbol{\varphi}(\omega) = \boldsymbol{C}\boldsymbol{\chi}(\omega), \tag{2.103}$$

where $\boldsymbol{C}$ is a nonsingular matrix; so, we consider all bases that are similar to (2.71). We can now state a characterization of nonnegative polynomials defined in terms of the values (2.102).

Theorem 2.33 *The trigonometric polynomial $R \in \mathbb{C}_n[z]$ satisfying (2.102) is nonnegative if and only if there exists a positive semidefinite matrix $\widehat{\boldsymbol{Q}} \in \mathbb{R}^{(n+1)\times(n+1)}$ such that*

$$\rho_i = \boldsymbol{\varphi}^T(\omega_i) \cdot \widehat{\boldsymbol{Q}} \cdot \boldsymbol{\varphi}(\omega_i), \tag{2.104}$$

for all the $2n+1$ points $\omega_i \in \Omega$.

Proof The nth order polynomial $\boldsymbol{\varphi}^T(\omega)\widehat{\boldsymbol{Q}}\boldsymbol{\varphi}(\omega)$ has the values ρ_i for the $2n+1$ frequencies $\omega_i \in \Omega$ and so is identical to $R(\omega)$ (which satisfies the relations (2.102)). Due to (2.103), it results that $R(\omega) = \boldsymbol{\chi}^T(\omega)\boldsymbol{C}^T\widehat{\boldsymbol{Q}}\boldsymbol{C}\boldsymbol{\varphi}(\omega)$, which is nonnegative if and only if $\widehat{\boldsymbol{Q}} \succeq 0$. ■

In principle, any basis $\boldsymbol{\varphi}(\omega)$ and any set of points Ω may be used. However, some choices are more appealing by offering a simple interpretation of some elements of the matrix $\widehat{\boldsymbol{Q}}$. One interesting basis is given by the Dirichlet kernel

$$D_{\tilde{n}}(\omega) = \frac{1}{2\tilde{n}+1} \sum_{k=-\tilde{n}}^{\tilde{n}} \mathrm{e}^{-jk\omega} = \frac{1}{2\tilde{n}+1} \frac{\sin \frac{(2\tilde{n}+1)\omega}{2}}{\sin \frac{\omega}{2}}. \tag{2.105}$$

Denote $\tau = 2\pi/(2\tilde{n}+1)$. We note that

$$D_{\tilde{n}}(\ell\tau) = \begin{cases} 1, & \text{if } \ell = 0, \\ 0, & \text{if } \ell \in \mathbb{Z} \setminus \{0\}. \end{cases} \tag{2.106}$$

This means that the $2\tilde{n}+1$ polynomials from the vector

$$\boldsymbol{\varphi}(\omega) = [D_{\tilde{n}}(\omega + \tilde{n}\tau) \ \ldots \ D_{\tilde{n}}(\omega) \ \ldots \ D_{\tilde{n}}(\omega - \tilde{n}\tau)]^T \tag{2.107}$$

form of basis for the space of $\tilde{n}$th order trigonometric polynomials. Moreover, any $\tilde{n}$th order polynomial $S(\omega)$ can be expressed as

$$S(\omega) = \sum_{\ell=-\tilde{n}}^{\tilde{n}} S(\ell\tau) D_{\tilde{n}}(\omega - \ell\tau). \tag{2.108}$$

(It is clear that the above equality holds for the points $\ell\tau$; a dimensionality argument shows that it holds everywhere.)

We can use the basis (2.107) in the representation (2.104). The points $\ell\tau$, with $\ell = -\tilde{n} : \tilde{n}$, are very good candidates for the set Ω (note that other $2\tilde{n}$ points are needed to complete the set). Since $\boldsymbol{\varphi}(\ell\tau)$ is a unit vector, it results immediately that

$$R(\ell\tau) = \widehat{q}_{\ell+\tilde{n},\ell+\tilde{n}}, \quad \ell = -\tilde{n} : \tilde{n}, \tag{2.109}$$

i.e., the diagonal elements of the Gram matrix $\widehat{\boldsymbol{Q}}$ are equal to the values of the polynomial in the given points.

Example 2.34 Let us take $\tilde{n} = 1$. Since $\tau = 2\pi/3$, the vector (2.107) is

$$\boldsymbol{\varphi}(\omega) = [D_1(\omega + \tfrac{2\pi}{3}) \ D_1(\omega) \ D_1(\omega - \tfrac{2\pi}{3})]^T,$$

with

$$\begin{aligned} D_1(\omega) &= \tfrac{1}{3}(1 + 2\cos\omega), \\ D_1(\omega + \tfrac{2\pi}{3}) &= \tfrac{1}{3}(1 - \cos\omega - \sqrt{3}\sin\omega), \\ D_1(\omega - \tfrac{2\pi}{3}) &= \tfrac{1}{3}(1 - \cos\omega + \sqrt{3}\sin\omega). \end{aligned}$$

It results that the relation (2.103) becomes

$$\begin{bmatrix} D_1(\omega + \frac{2\pi}{3}) \\ D_1(\omega) \\ D_1(\omega - \frac{2\pi}{3}) \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & -1 & -\sqrt{3} \\ 1 & 2 & 0 \\ 1 & -1 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 1 \\ \cos\omega \\ \sin\omega \end{bmatrix}.$$

We now represent the polynomial (2.75) using the basis (2.107), i.e., in the form $R(\omega) = \boldsymbol{\varphi}^T(\omega)\widehat{\boldsymbol{Q}}\boldsymbol{\varphi}(\omega)$. Using (2.76), the Gram matrices satisfying this relation have the form

$$\widehat{\boldsymbol{Q}} = \boldsymbol{C}^{-T}\boldsymbol{Q}\boldsymbol{C} = \begin{bmatrix} 4+2\sqrt{3} & -5-\frac{\sqrt{3}}{2}+\alpha & -\frac{7}{2}+\alpha \\ -5-\frac{\sqrt{3}}{2}+\alpha & 19 & -5+\frac{\sqrt{3}}{2}+\alpha \\ -\frac{7}{2}+\alpha & -5+\frac{\sqrt{3}}{2}+\alpha & 4-2\sqrt{3} \end{bmatrix},$$

where α is a free parameter (equal to $3q_{00}/2$, where q_{00} is the parameter from (2.76)). It is easy to see that the diagonal entries of the matrix $\widehat{\boldsymbol{Q}}$ are, in order, the values $R(-2\pi/3)$, $R(0)$ and $R(2\pi/3)$. ■

Problem (*Min_poly_value*) Using the parameterization (2.104), the minimum value of the polynomial $R(\omega)$ can be found by solving the SDP problem

$$\begin{aligned} \mu^\star = {} & \max_{\mu} \ \mu \\ & \text{s.t.} \ \ R(\omega_i) - \mu = \boldsymbol{\varphi}^T(\omega_i)\widehat{\boldsymbol{Q}}\boldsymbol{\varphi}(\omega_i), \ \ i = 1:2n+1 \\ & \qquad \widehat{\boldsymbol{Q}} \succeq 0 \end{aligned} \tag{2.110}$$

The difference with respect to (2.80) is that the equality constraints are defined using polynomial values (and not coefficients). We note that the constraints can be written in the equivalent form

$$R(\omega_i) - \mu = \operatorname{tr}[\boldsymbol{A}_i\widehat{\boldsymbol{Q}}], \ \ \text{with } \boldsymbol{A}_i = \boldsymbol{\varphi}(\omega_i)\boldsymbol{\varphi}^T(\omega_i). \tag{2.111}$$

The rank of the matrices $\boldsymbol{A}_i$ is 1. (Moreover, if the Dirichlet kernel is used for generating a basis (2.107) as discussed above, some of these matrices have only one diagonal element equal to 1, the others being zero.) ■

Remark 2.35 If the polynomial $R(z)$ has real coefficients, then $n+1$ values (2.102) are sufficient for describing it uniquely. Moreover, we can use the Gram-pair representation (2.91) to say that $R(z)$ is nonnegative if and only if there exist $\boldsymbol{Q} \succeq 0$ and $\boldsymbol{S} \succeq 0$ such that

$$\rho_i = \boldsymbol{\chi}_c^T(\omega_i)\boldsymbol{Q}\boldsymbol{\chi}_c(\omega_i) + \boldsymbol{\chi}_s^T(\omega_i)\boldsymbol{S}\boldsymbol{\chi}_s(\omega_i), \ \ i = 1:n+1. \tag{2.112}$$

Moreover, the bases $\boldsymbol{\chi}_c(\omega)$ and $\boldsymbol{\chi}_s(\omega)$ can be replaced by (different) linear combinations of themselves. ■

2.10 Mixed Representations

This section presents a new parameterization of the coefficients of a nonnegative polynomial, using ideas from interpolation representations to make a connection with discrete transforms. We start with a general presentation, going then to particular cases. Let $R \in \mathbb{C}_n[z]$ be a nonnegative trigonometric polynomial. Then, there exists $\boldsymbol{Q} \in \mathbb{C}^{(n+1)\times(n+1)}$, $\boldsymbol{Q} \succeq 0$, such that

$$R(\omega) = \boldsymbol{\varphi}_Q^H(\omega)\boldsymbol{Q}\boldsymbol{\varphi}_Q(\omega), \tag{2.113}$$

where $\boldsymbol{\varphi}_Q(\omega)$ is a basis vector, e.g., like in (2.73). Consider a set of N frequency points ω_i, with sufficiently large N. Let $\boldsymbol{x} \in \mathbb{C}^M$ be a vector representing the polynomial, such that $R(\omega_i) = \boldsymbol{\varphi}_R^T(\omega_i)\boldsymbol{x}$, where $\boldsymbol{\varphi}_R(\omega)$ is a vector of trigonometric functions (another basis vector). The equality

$$\boldsymbol{\varphi}_R^T(\omega_i)\boldsymbol{x} = \boldsymbol{\varphi}_Q^H(\omega_i)\boldsymbol{Q}\boldsymbol{\varphi}_Q(\omega_i), \; i = 1 : N, \tag{2.114}$$

can be written as

$$\boldsymbol{A}\boldsymbol{x} = \operatorname{diag}(\boldsymbol{B}^H\boldsymbol{Q}\boldsymbol{B}), \tag{2.115}$$

where $\boldsymbol{A} \in \mathbb{C}^{N\times M}$ and $\boldsymbol{B} \in \mathbb{C}^{(n+1)\times N}$ are given by

$$\boldsymbol{A} = \begin{bmatrix} \vdots \\ \boldsymbol{\varphi}_R^T(\omega_i) \\ \vdots \end{bmatrix}, \quad \boldsymbol{B} = [\ldots \; \boldsymbol{\varphi}_Q(\omega_i) \; \ldots]. \tag{2.116}$$

We assume that the matrix $\boldsymbol{A}$ has full column rank, which happens when N is large enough ($N \geq M$ anyway). Denoting $\boldsymbol{A}^\#$ the pseudoinverse of $\boldsymbol{A}$, the relation (2.115) becomes

$$\boldsymbol{x} = \boldsymbol{A}^\#\operatorname{diag}(\boldsymbol{B}^H\boldsymbol{Q}\boldsymbol{B}), \tag{2.117}$$

which is the desired parameterization. As the relation (2.117) is fairly abstract, we will see immediately its (probably) simplest particular case. In any case, it is apparent that this parameterization is useful if the vector $\boldsymbol{x}$ contains the coefficients of the polynomial and the matrices $\boldsymbol{A}^\#$, $\boldsymbol{B}$ have "nice" properties.

2.10.1 *Complex Polynomials and the DFT*

We take the points $\omega_i = 2\pi i/N$, $i = 0 : N-1$, with $N \geq 2n+1$. Let $M = N$ and the vector of parameters be

$$\boldsymbol{x} = [r_0\, r_1\, \ldots\, r_n\, 0\, \ldots\, 0\, r_{-n}\, \ldots\, r_{-1}]^T. \tag{2.118}$$

Since the polynomial is Hermitian, we are interested only in the first $n+1$ elements of $\boldsymbol{x}$, which form the vector $\boldsymbol{r}$. Taking

$$\boldsymbol{\varphi}_R^T(\omega) = [1\ \mathrm{e}^{-j\omega}\ \ldots\ \mathrm{e}^{-j(N-1)\omega}],$$

the matrix

$$\boldsymbol{A} = \left[\mathrm{e}^{-j\frac{2\pi\ell i}{N}}\right]_{\ell,i=0:N-1} \tag{2.119}$$

is the length-N DFT matrix. We split $\boldsymbol{A} = [\boldsymbol{W}\ \boldsymbol{A}_2]$, where $\boldsymbol{W}$ contains the first $n+1$ columns of $\boldsymbol{A}$. We take $\boldsymbol{\varphi}_Q(\omega) = \boldsymbol{\psi}_n(\mathrm{e}^{j\omega})$, i.e., the standard basis from (2.1). It can be seen immediately that $\boldsymbol{B} = \boldsymbol{W}^H$. Since $\boldsymbol{A}$ (and in particular $\boldsymbol{W}$) has orthogonal columns, it follows that

$$\boldsymbol{W}^H\boldsymbol{A} = \frac{1}{N}[\boldsymbol{I}\ \ \boldsymbol{0}].$$

By multiplying (2.115) with $\boldsymbol{W}^H$, we obtain the parameterization

$$\boldsymbol{r} = \frac{1}{N}\boldsymbol{W}^H\mathrm{diag}(\boldsymbol{W}\boldsymbol{Q}\boldsymbol{W}^H) \tag{2.120}$$

of the coefficients of a nonnegative polynomial in terms of a positive semidefinite matrix.

Remark 2.36 Since we have used the same basis for the Gram matrix expression of $R(\omega)$, i.e., $\boldsymbol{\varphi}_Q(\omega) = \boldsymbol{\psi}_n(\mathrm{e}^{j\omega})$, it results that (2.120) is identical with the trace parameterization (2.6). (The mapping between the elements of the Gram matrix and the coefficients of the polynomial is linear.) See **P** 2.14 for an explicit proof. However, the parameterization (2.120) can be used directly in fast algorithms, as discussed in Sect. 2.11. ∎

2.10.2 Cosine Polynomials and the DCT

We consider now polynomials $R(z)$ with *real* coefficients; for simplicity, we look only at the even degree case, $n = 2\tilde{n}$. Using the Gram-pair parameterization (2.112), it results similarly to (2.115) that $R(z)$ is nonnegative if there exist $\boldsymbol{Q} \succeq 0$ and $\boldsymbol{S} \succeq 0$ such that

$$\boldsymbol{A}\boldsymbol{x} = \mathrm{diag}(\boldsymbol{B}_1^T\boldsymbol{Q}\boldsymbol{B}_1) + \mathrm{diag}(\boldsymbol{B}_2^T\boldsymbol{S}\boldsymbol{B}_2), \tag{2.121}$$

where $\boldsymbol{A}$ is defined as in (2.116) and

$$\boldsymbol{B}_1 = [\ldots\ \boldsymbol{\chi}_c(\omega_i)\ \ldots],\quad \boldsymbol{B}_2 = [\ldots\ \boldsymbol{\chi}_s(\omega_i)\ \ldots].$$

We take the points $\omega_i = \pi i/(N-1)$, $i = 0 : N-1$, with $N \geq n+1$. Let $M = N$ and the vector of parameters be

$$\boldsymbol{x} = [r_0\ 2r_1\ \ldots\ 2r_n\ 0\ \ldots\ 0]^T \in \mathbb{R}^N. \tag{2.122}$$

With

$$\boldsymbol{\varphi}_R^T(\omega) = [1\ \cos\omega\ \ldots\ \cos(N-1)\omega],$$

the matrix $\boldsymbol{A}$ is

$$\boldsymbol{A} = \left[\cos\frac{\pi\ell i}{N-1}\right]_{\ell,i=0:N-1}. \tag{2.123}$$

We remark that, denoting $\boldsymbol{D} = \mathrm{diag}(1/2, 1, \ldots, 1, 1/2)$, the matrix $\boldsymbol{AD}$ is the DCT-I transform. Moreover, the inverse of $\boldsymbol{A}$ is

$$\boldsymbol{A}^{-1} = \frac{2}{N-1}\boldsymbol{DAD}.$$

We denote $\boldsymbol{W} \in \mathbb{R}^{N\times(n+1)}$ the first $n+1$ columns of $\boldsymbol{A}^{-1}$ (which are also its first rows, as the matrix is symmetric), and so $\boldsymbol{W}^T\boldsymbol{A} = [\boldsymbol{I}\ \boldsymbol{0}]$. By the choice of frequency points, the other constant matrices from (2.121) are

$$\boldsymbol{B}_1 = \left[\cos\frac{\pi\ell i}{N-1}\right]_{\ell=0:n,i=0:N-1}, \quad \boldsymbol{B}_2 = \left[\sin\frac{\pi\ell(i+1)}{N-1}\right]_{\ell=0:n-1,i=0:N-1}.$$

With these notations, by multiplication with $\boldsymbol{W}^T$ in (2.121), we obtain the parameterization

$$\begin{bmatrix} r_0 \\ 2r_1 \\ \vdots \\ 2r_n \end{bmatrix} = \boldsymbol{W}^T\left(\mathrm{diag}(\boldsymbol{B}_1^T\boldsymbol{Q}\boldsymbol{B}_1) + \mathrm{diag}(\boldsymbol{B}_2^T\boldsymbol{S}\boldsymbol{B}_2)\right). \tag{2.124}$$

Remark 2.37 For reasons similar to those exposed in Remark 2.36, i.e., identity of bases and linearity, the parameterization (2.124) is identical with the Gram-pair parameterization (2.94). ■

2.11 Fast Algorithms

In this book, the presentation is focused on parameterizations of positive polynomials suited to the use of off-the-shelf SDP libraries. This approach is not only very convenient, as the implementation effort is minimal, but also efficient for polynomials of low or medium order (going to more than 100). Alternatively, SDP algorithms can be tailored to the specific of positive polynomials, obtaining fast methods. This section aims to open the path for the reader interested in such methods.

As mentioned before, a typical SDP problem involving a nonnegative trigonometric polynomial of order n has an $O(n^4)$ complexity, if either the trace (2.6) or Gram-pair (2.94) parameterizations are used. However, these parameterizations are expressed with sparse matrices, which allows a complexity reduction by simply informing the SDP library of the sparseness (actually, the current version of SeDuMi assumes that all matrices are sparse). So, in this case, the constant hidden by the $O(\cdot)$ notation is relatively small.

Fast methods have an $O(n^3)$ complexity. The identity between the dual cone (2.32) and the space of positive semidefinite Toeplitz matrices, presented in Sect. 2.4, allows the fast computation of the Hessian and gradient of the barrier function, required by interior point methods for solving SDP problems. The method from [2] uses displacement rank techniques. The method from [3] uses the Levinson–Durbin algorithm and the DFT. Unfortunately, it appears that the numerical stability of these algorithms limits their use to polynomials of relatively small degrees (less than, e.g., 50, for some applications). So, they may have no significant practical advantage over the algorithms based on the trace or Gram-pair parameterization, which are robust and for which the only limitation on the degree of the polynomials appears to be due mostly to the time necessary to obtain the solution and possibly also to memory requirements.

Another fast method is based on the interpolation representation presented in the previous section. The fact that the constant matrices appearing in constraints such as (2.111) have rank equal to one can be used for building fast algorithms, using a dual solver [4]; again, the Hessian of the barrier function can be evaluated with low complexity, precisely $O(n^3)$ operations.

Finally, the method from [5] is based on representations such as (2.120) and (2.124), that can be exploited when solving the Newton equations appearing in interior point methods. Not only the special form of the representations helps in reducing the number of operations, but also the fact that matrix multiplication can be sped up via the FFT, due to the connection of the constant matrices from (2.120) and (2.124) with discrete transforms such as the DFT and the DCT. Moreover, it seems that this method does not suffer from numerical stability problems, like the others above.

Typically, for small orders, the fast $O(n^3)$ methods are not faster than the standard $O(n^4)$ methods (especially if sparseness is used). The order n_o for which the fast methods become indeed faster depends on the implementation, the SDP algorithm, the programming and running environments, and the problem solved. From the data available in the literature and the author's experiments, it seems that n_o may be around 100.

2.12 Details and Other Facts

2.12.1 Writing Programs with Positive Trigonometric Polynomials

Solving an optimization problem with positive trigonometric polynomials requires an SDP library. We give here three programs for finding the minimum value of a trigonometric polynomial by solving (2.18). The first uses directly the SDP library SeDuMi [6]. The second works at a higher level, calling the convex optimization software CVX [7], which includes SDP but also other types of convex optimization. CVX calls SeDuMi or SDPT3 [8], depending on user's choice. CVX has the great advantage of expressing the optimization problems in a form very close to the mathematical one. The third program uses Pos3Poly [9], which is a library dedicated to positive polynomials, covering all types and situations described in this book. Pos3Poly is built on top of CVX, taking advantage of the possibility to build convex sets offered by CVX. The parameterization is hidden for the user, who simply works directly with the coefficients of the polynomial.

Table 2.2 contains the SeDuMi program for solving (2.18). The problem needs to be expressed in a standard form, here the equality form, see Appendix A. The polynomial has the form (1.1) and is given through its vector of coefficients $\boldsymbol{r} = [r_0 \ \dots \ r_n]^T$. The variable `K` contains a description of the optimization variables from (2.18): μ is a free scalar (it may have any real value), while $\tilde{\boldsymbol{Q}}$ is a matrix of size $n' \times n'$ (where $n' = n + 1$ is the number of distinct coefficients of the polynomial). Denoting the variables vector with $\boldsymbol{x} = [\mu \ \text{vec}(\tilde{\boldsymbol{Q}})^T]^T$, the constraints of (2.18) are expressed in SeDuMi as a linear system `Ax=b`; so, the first column of A contains a single nonzero value, i.e., `A(1,1)`, which represent the coefficient of μ in the first constraint from (2.18). The rows of `A` contain, starting with the second column, the vectorized elementary Toeplitz matrices; this is due to the equality $\text{tr}[\boldsymbol{\Theta}_k \tilde{\boldsymbol{Q}}] = \text{vec}(\boldsymbol{\Theta}_k)^T \text{vec}(\tilde{\boldsymbol{Q}})$. Finally, the vector `b` is the right hand side of the constraints of (2.18) and so it is equal to $\boldsymbol{r}$. In SeDuMi, the objective is to minimize $\texttt{c}^T\texttt{x}$ and so only the first component of `c` is nonzero and equal to -1, in order to maximize μ; the matrix variable $\tilde{\boldsymbol{Q}}$ does not appear in the objective. If the polynomial is complex, the variable `K.scomplex` specifies the positive semidefinite matrices that are complex, by their indices in the variable `K.s`; in our case, there is only one such variable, the Gram matrix. However, this is not enough; we should specify that the equality constraints should be regarded as equalities of complex numbers; this is done by specifying that the dual variables are complex, with `K.ycomplex`.

The program gives also the solution to problem (2.14), i.e., the most positive Gram matrix associated with the polynomial $R(z)$. The Gram matrix `Q` is computed using relation (2.19), from the solution of (2.18).

Running this program for `r = [6 -3 2]`, which represents the polynomial (2.4), gives the Gram matrix $\boldsymbol{Q}$ and the minimal value μ shown in Examples 2.11 and 2.12, respectively.

Table 2.2 SeDuMi program for solving the SDP problem (2.18)

```
function [m,Q] = minpol1(r)

n = length(r);  % length of the polynomial

K.f = 1;        % one free variable (m)
K.s = [n];      % one pos. semidef. matrix of size nxn (Q)
nrA = n;        % number of equality constraints
                % (one for each coefficient of r)
ncA = 1+n*n;    % number of scalar variables (in m and Q)

if ~isreal(r)   % specify complex data, if this is the case
  K.scomplex = 1;
  K.ycomplex = 1:n;
end

A = sparse(nrA,ncA);
b = r(:);       % right hand term of equality constraints
c = zeros(ncA,1);
c(1) = -1;      % the objective is to maximize m

e = ones(n,1);  % generate Toeplitz elementary matrices
for k = 1:n
  A(k,2:end) = vec( spdiags(e,k-1,n,n) )';
end
A(1,1) = 1;     % coefficient of m in first constraint

[x,y,info] = sedumi(A,b,c,K); % call SeDuMi

m = x(1);      % recover the variables
Q = mat(x(2:end)) + (m/n)*eye(n);
```

The CVX program is shown in Table 2.3 and is practically self-explanatory. The optimization variables are declared explicitly and the constraints have a very natural expression. In general, it is not needed to put the problem in a standard form, which was very easy for (2.18), but sometimes may be cumbersome.

Finally, the Pos3Poly program is shown in Table 2.4. The function `sos_pol` can be used to declare all kinds of positive polynomials; it has two arguments: The first is a vector containing the degree of the polynomial and the size of the coefficients (here

Table 2.3 CVX program for solving (2.18)

```
function [m,Q] = minpol1_cvx(r)

n = length(r);       % length of the polynomial
cvx_begin
  variable m;
  if ~isreal(r)      % complex data
    variable Q(n,n) complex semidefinite;
  else               % real data
    variable Q(n,n) semidefinite;
  end
  maximize( m )      % variable for the minimum
  subject to         % equality constraints
    m + trace(Q) == r(1);
    e = ones(n,1);
    for k = 2:n
       vec( spdiags(e,k-1,n,n) )' * vec(Q) == r(k);
    end
cvx_end
Q = Q + (m/n)*eye(n);
```

they are scalars, but we will talk later about polynomials with matrix coefficients), and the second is a structure describing the type of the polynomial; in our case, it is useful only when declaring that the polynomial is complex. The output of `sos_pol` is a variable vector representing the positive trigonometric polynomial and containing its coefficients (like `r` contains the coefficients of $R(z)$). Hence, the constraint can be written as a single vector equality. Since Pos3Poly hides the parameterization, we do not have access to the Gram matrix.

It is clear that the programming effort decreases as we go from SeDuMi to CVX and then to Pos3Poly. If the reader intends to solve only a small number of simple problems involving positive polynomials, then CVX might be the easiest way. For those who need to solve many or more difficult problems, or for those who do not want to read this book in detail, Pos3Poly is probably the best choice. SeDuMi (note that Pos3Poly can also work directly on top of SeDuMi, without CVX; read the manual if interested) is hard to recommend for others than those already very experienced with it.

Table 2.4 Pos3Poly program for solving (2.18)

```
function m = minpol1_pos3poly(r)

n = length(r);   % length of the polynomial
r = r(:);        % force column vector
p = [n-1 1];     % degree and coefficients size (scalars)
ptype = [];      % for real data this variable is not necessary
if ~isreal(r)    % complex data
  ptype.complex_coef = 1;
end
cvx_begin
  variable m;    % variable for the minimum
  maximize( m )
  subject to     % equality constraints
    m*eye(n,1) + sos_pol(p, ptype) == r;
cvx_end
```

2.12.2 *Proof of Theorem 2.16*

Consider that the FIR filter $H(z)$ has a white noise $e(\ell)$ (of unit variance) at its input and the output is $y(\ell)$. The state-space model of $H(z)$ is

$$\begin{cases} \boldsymbol{\xi}(\ell+1) = \boldsymbol{\Theta}_1^T \boldsymbol{\xi}(\ell) + \tilde{\boldsymbol{h}} e(\ell), \\ y(\ell) = \boldsymbol{c}^T \boldsymbol{\xi}(\ell) + h_0 e(\ell), \end{cases} \tag{2.125}$$

where $\boldsymbol{\xi} \in \mathbb{C}^n$ is the vector of states and $\tilde{\boldsymbol{h}}$ and $\boldsymbol{c}$ are defined in (2.47). Denote

$$\boldsymbol{\Xi} = E\{\boldsymbol{\xi}(\ell)\boldsymbol{\xi}^H(\ell)\}$$

the state autocorrelation matrix. Multiplying both sides of the first equation from (2.125) with their Hermitians and taking the average, it results that

$$\boldsymbol{\Xi} = \boldsymbol{\Theta}_1^T \boldsymbol{\Xi} \boldsymbol{\Theta}_1 + \tilde{\boldsymbol{h}}\tilde{\boldsymbol{h}}^H.$$

This is the last relation from (2.50).

Using now the second equation from (2.125), we obtain $r_0 = E\{y(\ell)y^*(\ell)\} = \boldsymbol{c}^T \boldsymbol{\Xi} \boldsymbol{c} + h_0^2$, which is the first relation from (2.50) (remind that we have assumed h_0 to be real).

Finally, combining both equations from (2.125), we get

$$E\{\boldsymbol{\xi}(\ell+1)y^*(\ell)\} = \boldsymbol{\Theta}_1^T \boldsymbol{\Xi} \boldsymbol{c} + h_0\tilde{\boldsymbol{h}}. \tag{2.126}$$

Rewriting (2.125) for each scalar component of the state vector, we obtain

$$\begin{aligned}
y(\ell) &= \boldsymbol{\xi}_{n-1}(\ell) + h_0 e(\ell),\\
\boldsymbol{\xi}_{n-1}(\ell+1) &= \boldsymbol{\xi}_{n-2}(\ell) + h_1 e(\ell),\\
&\vdots\\
\boldsymbol{\xi}_{1}(\ell+1) &= \boldsymbol{\xi}_{0}(\ell) + h_{n-1} e(\ell),\\
\boldsymbol{\xi}_{0}(\ell+1) &= h_n e(\ell)
\end{aligned}$$

By substituting successively the expressions of states, these relations are equivalent to

$$\boldsymbol{\xi}_{n-k}(\ell+1) = y(\ell+k) - \sum_{i=0}^{k-1} h_i e(\ell+k-i), \quad k = 1:n.$$

With this, we obtain

$$E\{\boldsymbol{\xi}(\ell+1)y^*(\ell)\} = \tilde{\boldsymbol{r}},$$

which makes (2.126) identical with the second relation from (2.50). Thus, we have shown that all three relations from (2.50) hold. Since they are equivalent to (2.48), (2.49), the proof is ready.

2.12.3 Proof of Theorem 2.19

The relation (2.56) can be written as

$$P(t) = \operatorname{tr}[\boldsymbol{\psi}(t)\cdot\boldsymbol{\psi}^T(t)\cdot\boldsymbol{Q}] = \operatorname{tr}[\boldsymbol{\Psi}(t)\cdot\boldsymbol{Q}],$$

where

$$\boldsymbol{\Psi}(t) = \begin{bmatrix} 1\\ t\\ \vdots\\ t^n \end{bmatrix} [1\ t\ \ldots\ t^n] = \begin{bmatrix} 1 & t & \ldots & t^n\\ t & t^2 & \iddots & t^{n+1}\\ \vdots & \iddots & \iddots & \vdots\\ t^n & t^{n+1} & \ldots & t^{2n} \end{bmatrix} = \sum_{k=0}^{2n} \boldsymbol{\Upsilon}_k t^k.$$

Combining the last two relations, we obtain

$$P(t) = \sum_{k=0}^{2n} \operatorname{tr}[\boldsymbol{\Upsilon}_k \boldsymbol{Q}] t^k,$$

which proves (2.60).

2.12.4 Proof of Theorem 2.21

If $\boldsymbol{Q} \succeq 0$ (or $\boldsymbol{Q} \succ 0$) exists such that (2.60) holds, then it results directly from (2.56) that $P(t) \geq 0$ (or $P(t) > 0$), $\forall t \in \mathbb{R}$.

Reciprocally, if $P(t) \geq 0$, then, as stated by Theorem 1.7, the polynomial can be written as

$$P(t) = F^2(t) + G^2(t) = \psi^T(t) \left(\boldsymbol{f}\boldsymbol{f}^T + \boldsymbol{g}\boldsymbol{g}^T \right) \psi(t),$$

which shows that $\boldsymbol{Q} = \boldsymbol{f}\boldsymbol{f}^T + \boldsymbol{g}\boldsymbol{g}^T$ is a (rank-2) Gram matrix associated with $P(t)$.

If $P(t) > 0$, then there exists $\varepsilon > 0$ such that $P_\varepsilon(t) = P(t) - \varepsilon(1 + t^2 + \ldots + t^{2n})$ is nonnegative. Let $\boldsymbol{Q}_\varepsilon \succeq 0$ be a Gram matrix associated with $P_\varepsilon(t)$ (obtained, e.g., as above). Since $\boldsymbol{I}$ is the Gram matrix of the polynomial $1 + t^2 + \ldots + t^{2n}$, it results that $\boldsymbol{Q} = \boldsymbol{Q}_\varepsilon + \varepsilon \boldsymbol{I} \succ 0$ is a Gram matrix associated with $P(t)$.

2.13 Bibliographical and Historical Notes

The trace parameterization of trigonometric polynomials (Theorem 2.5) and its use as optimization tool have been proposed independently by several researchers [2, 3, 10–12], starting from different applications. Many of these works were motivated by the high complexity of the KYP lemma parameterization (see Sect. 2.5) of positive polynomials proposed in [13] (for FIR filter design), [14] (for MA estimation), [15] (for compaction filter design), and [16] (for the design of orthogonal pulse shapes for communication). The jump from an $O(n^6)$ complexity to $O(n^4)$ allowed a much higher range of problems to be solved.

The Toeplitz quadratic optimization problem discussed in Sect. 2.3 has been analyzed in SDP terms in [17]. Some extensions can be found in [18]. The dual-cone formulation is a simple dualization exercise; the presentation from Sect. 2.4 is taken from [3]; an excellent lecture on convex optimization and, among many others, dual cones is [19]. For the discrete-time version of the KYP lemma and other results regarding positive real systems, we recommend [20, 21].

Spectral factorization using SDP has been proposed by several authors. The proof of Theorem 2.15 using the Schur complement was presented in [22]. For Robinson's energy delay property see [23], problem 5.66. The spectral factorization method based on the Riccati equation appeared in [24]; useful information can be found in [25] (including connections with Kalman filtering). Other spectral factorization algorithms are presented in Appendix B.

The Gram-pair factorization from Sect. 2.8.3 and the real Gram representation of polynomials with complex coefficients from Theorem 2.25 have appeared in [5], in their equivalent forms from Sect. 2.10. The explicit representations from Theorems 2.27 and 2.30 have been derived here in the style of other Gram parameterizations. The idea of using interpolation representations, as presented in Sect. 2.9, appeared first in [4].

Problems

P 2.1 Are there nonnegative polynomials $R(z)$ for which the set $\mathcal{G}(R)$ of associated Gram matrices contains a single positive semidefinite matrix?

P 2.2 (problems VI.50, VI.51 [26]) The polynomial $R \in \mathbb{C}_n[z]$ is nonnegative and has the free coefficient $r_0 = 1$.

(a) Show that $R(\omega) \le n + 1$.

(b) Show that $|r_n| \le 1/2$.

Hint: use the Gram matrix representation (2.6).

(a) A Gram matrix $\boldsymbol{Q} \succeq 0$ has nonnegative eigenvalues λ_i, $i = 1 : n + 1$. Since $\sum \lambda_i = \operatorname{tr} \boldsymbol{Q} = r_0 = 1$, it results that $\max \lambda_i \le 1$. Note also that $\|\boldsymbol{\psi}(\omega)\|^2 = n + 1$. It results that $R(\omega) = \boldsymbol{\psi}^H(\omega)\boldsymbol{Q}\boldsymbol{\psi}(\omega) \le \|\boldsymbol{\psi}(\omega)\|^2 \max \lambda_i = n + 1$.

(b) The determinant of the 2×2 matrix containing the corner elements of $\boldsymbol{Q}$ is nonnegative. The sum of the diagonal elements of this 2×2 matrix is less than 1 and the other two elements are r_n and r_n^*.

P 2.3 (problems VI.57, VI.58 [26]) The polynomial $R \in \mathbb{C}_n[z]$ has the free coefficient $r_0 = 0$.

(a) Show that $R(\omega)$ cannot have the same sign for all values of ω, unless it is identically zero.

(b) Let $-m$ and M be the minimum and maximum, respectively, of the values $R(\omega)$ (note that $m \ge 0$, $M \ge 0$). Show that $M \le nm$, $m \le nM$.

Hint: any Gram matrix has $\operatorname{tr} \boldsymbol{Q} = 0$ and hence both positive and negative eigenvalues, unless it is the null matrix.

P 2.4 Are there polynomials $P \in \mathbb{R}_{2n}[t]$ with a single associated Gram matrix? What is their degree?

P 2.5 Let $P \in \mathbb{R}_{2n}[t]$ be a positive polynomial. Let $\mu^\star$ be the minimum value of $P(t)$, i.e., the optimal value of the SDP problem (2.62). Let $\lambda^\star$ be the smallest eigenvalue of the most positive Gram matrix associated with $P(t)$, i.e., the optimum of (2.64). Show that $\mu^\star \ge \lambda^\star$.

Show that there exist polynomials for which $\mu^\star = \lambda^\star$. Hint: think at polynomials with nonzero coefficients only for even powers of t.

P 2.6 Let $R \in \mathbb{R}_n[z]$ be the polynomial whose coefficients are $r_k = n + 1 - k$. (This is the triangular, or Bartlett, window.) Show that $R(\omega) \ge 0$ by finding a positive semidefinite Gram matrix associated with $R(z)$.

P 2.7 Consider the quadratic optimization problem (2.24), in which the matrices $\boldsymbol{A}_\ell$, $\ell = 0 : L$, are Hankel (and not Toeplitz). Why cannot the problem (2.24) be solved from the solution of an SDP problem (2.31) with Hankel matrices (and so there is no analogous for real polynomials of the algorithm presented in Sect. 2.3 for trigonometric polynomials)?

P 2.8 (LMI form of Theorem 1.15) The polynomial $R \in \mathbb{C}_n[z]$ is nonnegative on the interval $[\alpha, \beta] \subset (-\pi, \pi)$ if and only if there exist positive semidefinite matrices $\boldsymbol{Q}_1 \in \mathbb{C}^{(n+1)\times(n+1)}$ and $\boldsymbol{Q}_2 \in \mathbb{C}^{n\times n}$ such that

$$r_k = \mathrm{tr}[\boldsymbol{\Theta}_k \boldsymbol{Q}_1] + \mathrm{tr}[(d_1\boldsymbol{\Theta}_{k-1} + d_0\boldsymbol{\Theta}_k + d_1^*\boldsymbol{\Theta}_{k+1})\boldsymbol{Q}_2].$$

Here, d_0 and d_1 are the coefficients of the polynomial (1.34). Also, in the argument of the second trace operator, we use the notation convention that $\boldsymbol{\Theta}_k = 0$ if $k > n-1$.

P 2.9 Let $R \in \mathbb{C}_{2\tilde{n}}[z]$ be a trigonometric polynomial. Show that the parameterization (2.73) is equivalent to

$$R(z) = \boldsymbol{\phi}^T(z) \cdot \boldsymbol{Q} \cdot \boldsymbol{\phi}(z),$$

with the basis vector

$$\boldsymbol{\phi}(z) = \begin{bmatrix} 1 \\ z + z^{-1} \\ \vdots \\ z^{\tilde{n}} + z^{-\tilde{n}} \\ j(z - z^{-1}) \\ \vdots \\ j(z^{\tilde{n}} - z^{-\tilde{n}}) \end{bmatrix}.$$

Notice the resemblance with the definition (2.56) of Gram matrices for real polynomials.

P 2.10 Show that the problem (2.80) (for computing the minimum value of a complex trigonometric polynomial) is equivalent to finding the most positive matrix $\boldsymbol{Q}$ for which (2.73) holds. Hint: notice that $\|\boldsymbol{\chi}(\omega)\|^2 = \tilde{n} + 1$.

P 2.11 The SDP problem (2.97) computes the minimum value of a trigonometric polynomial $R(z)$ with real coefficients. Denote $\lambda(\boldsymbol{Q})$, $\lambda(\boldsymbol{S})$ the sets of eigenvalues of the matrices $\boldsymbol{Q}$ and $\boldsymbol{S}$, respectively, from the Gram-pair parameterization (2.91) of $R(z)$. Show that (2.97) is equivalent to finding the matrices $\boldsymbol{Q}$ and $\boldsymbol{S}$ for which the smallest eigenvalue from $\lambda(\boldsymbol{Q}) \cup \lambda(\boldsymbol{S})$ is maximum.

Hint: notice that (2.91) is equivalent to

$$R(\omega) = [\boldsymbol{\chi}_c^T(\omega)\ \boldsymbol{\chi}_s^T(\omega)] \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{S} \end{bmatrix} \begin{bmatrix} \boldsymbol{\chi}_c(\omega) \\ \boldsymbol{\chi}_s(\omega) \end{bmatrix}$$

and that the vector $[\boldsymbol{\chi}_c^T(\omega)\ \boldsymbol{\chi}_s^T(\omega)]$ has constant norm.

P 2.12 (LMI form of Theorem 1.18) The polynomial $R \in \mathbb{R}_n[z]$, with $n = 2\tilde{n}$, is nonnegative on $[\alpha, \beta] \subset [0, \pi]$ if and only if there exist positive semidefinite matrices $\boldsymbol{Q}_1 \in \mathbb{R}^{(\tilde{n}+1)\times(\tilde{n}+1)}$ and $\boldsymbol{Q}_2 \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ such that (for brevity, we denote $\cos\alpha = a$, $\cos\beta = b$)

$$r_k = \operatorname{tr}[\boldsymbol{\Phi}_k \boldsymbol{Q}_1] + \operatorname{tr}\Big[\big((-ab - \tfrac{1}{2})\boldsymbol{\Phi}_k + \tfrac{a+b}{2}(\boldsymbol{\Phi}_{k-1} + \boldsymbol{\Phi}_{k+1}) \\ - \tfrac{1}{4}(\boldsymbol{\Phi}_{k-2} + \boldsymbol{\Phi}_{k+2})\big)\ \boldsymbol{Q}_2\Big],$$

where the matrices $\boldsymbol{\Phi}_k$ are defined in (2.95).

Derive a similar result for the case of odd degree n.

P 2.13 Let $R_1(z)$, $R_2(z)$ be two trigonometric polynomials of the same degree. Prove the following:

(a) $R_1(\omega) \ge R_2(\omega)$, $\forall \omega \in [-\pi, \pi]$, if and only if there exist Gram matrices $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$, associated with $R_1(z)$ and $R_2(z)$, respectively (i.e., defined as in the trace parameterization (2.6)) such that $\boldsymbol{Q}_1 \succeq \boldsymbol{Q}_2$.

(b) $R_1(\omega) \ge R_2(\omega)$, $\forall \omega \in [-\pi, \pi]$, if and only if there exist Gram pairs $(\boldsymbol{Q}_1, \boldsymbol{S}_1)$ and $(\boldsymbol{Q}_2, \boldsymbol{S}_2)$, associated with $R_1(z)$ and $R_2(z)$, respectively (i.e., defined as in the Gram-pair parameterization (2.94)) such that $\boldsymbol{Q}_1 \succeq \boldsymbol{Q}_2$ and $\boldsymbol{S}_1 \succeq \boldsymbol{S}_2$.

Generalize this kind of results to polynomials that are positive on an interval.

P 2.14 Show that the trace parameterization (2.6) and the DFT parameterization (2.120) of a nonnegative polynomial are identical.

Hint [5]: Denote $\boldsymbol{w}_k$ the k-th column of the DFT matrix (2.119), which is also the k-th column of $\boldsymbol{W}$ from (2.120). It results from (2.120) that

$$\begin{aligned} r_k &= \tfrac{1}{N}\boldsymbol{w}_k^H \operatorname{diag}(\boldsymbol{W}\boldsymbol{Q}\boldsymbol{W}^H) = \tfrac{1}{N}\operatorname{tr}[\operatorname{diag}(\boldsymbol{w}_k^H)\boldsymbol{W}\boldsymbol{Q}\boldsymbol{W}^H] \\ &= \tfrac{1}{N}\operatorname{tr}[\boldsymbol{W}^H \operatorname{diag}(\boldsymbol{w}_k^H)\boldsymbol{W}\boldsymbol{Q}]. \end{aligned}$$

It remains to show that $\boldsymbol{W}^H \operatorname{diag}(\boldsymbol{w}_k^H)\boldsymbol{W} = N\boldsymbol{\Theta}_k$.

P 2.15 It is clear that if any polynomial $R \in \mathbb{C}[z]$ can be written as $R(\omega) = \operatorname{tr}[\boldsymbol{Q}\boldsymbol{P}(\omega)]$, with positive semidefinite matrices $\boldsymbol{Q}$ (which depends on R) and $\boldsymbol{P}(\omega)$ (the same for all polynomials), then it follows that $R(\omega) \ge 0$. Investigate what are the conditions for the reverse implication to hold. Describe the parameterizations from this chapter as particular cases of these conditions.

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