Preface

A few words on the second edition. By a nice coincidence, Springer's proposal to revise the book came when I was giving a serious thought to the idea. Ten years have passed and my research shifted mostly to other topics, which discouraged my attempt, but there always seemed to be a small community interested in the book, which gave me hope that the work is not useless. The proposal tipped the balance.

Besides correcting some errors and typos, the new version has a few additions and modifications. Chapter 9, dedicated to optimization problems using the atomic norm and to the related super-resolution problem, is completely new. The Bounded Real Lemma (BRL) for trigonometric polynomial is central to the solution; it was a great reward to see that this BRL, which is the contribution that I consider the most personal and of which I was very proud at the time, has been applied in all its forms in a topic that I never foresaw. To help reading this chapter, all theory regarding the BRL is now gathered in Chap. 4. Another new topic, mentioned mostly in passing, is that of hybrid polynomials, having both real and trigonometric variables. The convex optimization software has greatly evolved, especially toward user convenience; some of the programs shown in the book are now written for CVX, which attracted immediate popularity due to its simple and versatile language; other programs use Pos3Poly, which is a package built on top of CVX, especially for optimization with positive polynomials.

On the contents of the book. Although trigonometric polynomials appear naturally in discrete-time signal processing and their positivity characterizes many design problems, it was only in the late 1990s that an exact and computationally useful parameterization of nonnegative trigonometric polynomials was found. The idea of parameterizing the coefficients of the polynomial as a linear function of the elements of a positive semidefinite matrix was already present (somewhat in disguise) in the previous literature; however, its implementation needed the emergence of semidefinite programming (SDP) methods in the early 1990s and, shortly after, of freely available SDP libraries. The following result is the foundation of this book. Any trigonometric polynomial

$$R(z) = \sum_{k=-n}^{n} r_k z^{-k}, \quad r_{-k} = r_k^*, \tag{1}$$

that is nonnegative on the unit circle (for $|z| = 1$), can be parameterized with a positive semidefinite matrix $\boldsymbol{Q}$ by

$$r_k = \mathrm{tr}[\boldsymbol{\Theta}_k \boldsymbol{Q}], \quad k = -n : n, \tag{2}$$

where $\boldsymbol{\Theta}_k$ is an elementary Toeplitz matrix, with ones on diagonal k and zeros elsewhere, and tr is the trace operator. The matrix $\boldsymbol{Q}$ is named *Gram matrix*. The parameterization (2) allows the description of a nonnegative trigonometric polynomial through a linear matrix inequality (LMI). Hence, SDP is applicable.

The book has two parts. In a simplistic classification, the first four chapters contain the theory and the other five chapters deal with applications. Here is a description of their contents that could help orient the lecture. Although the book treats also (inevitably) polynomials of real variable, we discuss here only the results pertaining to trigonometric polynomials, which have the lion's share.

Chapter 1 is written only in terms of polynomials. It starts with the spectral factorization of polynomials (1) that are nonnegative on the unit circle and which can be written as

$$R(z) = H(z)H^*(z^{-1}), \tag{3}$$

where $H(z)$ is causal and the asterisk denotes complex conjugated coefficients. It also describes polynomials (1) that are nonnegative on an interval, as a simple function of two nonnegative polynomials.

Chapter 2 is built around the Gram matrix parameterization (2) and contains examples of use and several side results linking it to the Kalman–Yakubovich–Popov lemma and spectral factorization. More importantly, it gives alternative parameterizations that are more efficient, for example, the *Gram-pair* parameterization, in which the matrix $\boldsymbol{Q}$ from (2) is replaced by two smaller positive definite matrices.

In Chap. 3, the presentation goes to multivariate polynomials. The most prominent trigonometric polynomial becomes now the *sum-of-squares*

$$R(\boldsymbol{z}) = \sum_{\ell=1}^{\nu} H_\ell(\boldsymbol{z})H_\ell^*(\boldsymbol{z}^{-1}). \tag{4}$$

(We use bold letters, like $\boldsymbol{z} = (z_1, \ldots, z_d)$, to denote multidimensional entities.) The polynomials $H_\ell(\boldsymbol{z})$ have support on the positive orthant, while the support of $R(\boldsymbol{z})$ is symmetric with respect to the origin. It turns out that all trigonometric polynomials that are strictly positive on the unit d-circle (where $|z_1| = \cdots = |z_d| = 1$) are also sum-of-squares; note that sum-of-squares are by construction nonnegative on the unit d-circle. However, the degrees of the polynomials $H_\ell(\boldsymbol{z})$ from (4) can be arbitrarily high. A parameterization like (2) holds, this time for sum-of-squares polynomials. In a practical implementation, an optimization problem with non-negative polynomials can be solved only in a relaxed way, with sum-of-squares whose factors $H_\ell(z)$ have the degrees bounded to a convenient value. Typically, a higher relaxation degree leads to a better approximation of the original problem, but with a higher complexity, due to the higher size of the Gram matrix $\boldsymbol{Q}$. Chapter 3 contains also the multivariate version of the Gram-pair parameterization and the means for reducing the size of the Gram matrix for sparse polynomials. The chapter ends with a short presentation of polynomials with matrix coefficients, for which, mutatis mutandis, all previous results hold true.

Chapter 4, dealing also with multivariate polynomials, is dedicated to the most general results, which are of three types.

Polynomials positive on domains. Let

$$\mathcal{D} = \left\{ \boldsymbol{\omega} \in [-\pi, \pi]^d \mid D_\ell(\boldsymbol{\omega}) \geq 0,\ \ell = 1 : L \right\} \tag{5}$$

be a frequency domain defined by the positivity of L given trigonometric polynomials $D_\ell(\mathbf{z})$. Then, any trigonometric polynomial $R(z)$ that is positive on $\mathcal{D}$ can be expressed as

$$R(z) = S_0(z) + \sum_{\ell=1}^{L} D_\ell(z) \cdot S_\ell(z), \tag{6}$$

where $S_\ell(z)$, $\ell = 0 : L$, are sum-of-squares. Using a Gram matrix (or a pair of matrices) to parameterize the sum-of-squares, we associate an LMI with $R(z)$.

Bounded Real Lemma. Let $H(z)$ be a polynomial with positive orthant support. Then, the inequality $|H(z)| < \gamma$, with $\gamma \in \mathbb{R}$, can be written in the form of an LMI; see Theorems 4.26 (basic general result), 4.32 (extension to matrix coefficients), and 4.35 (Gram-pair version). This LMI makes possible the formulation of some optimization problems in terms of $H(z)$; the lack of spectral factorization (3) for multivariate polynomials can be thus circumvented in some cases.

Positivstellensatz. We add equalities to the set (5), obtaining

$$\mathcal{D}_E = \left\{ \boldsymbol{\omega} \in [-\pi, \pi]^d \,\middle|\, \begin{array}{l} E_k(\boldsymbol{\omega}) = 0,\ k = 1 : K \\ D_\ell(\boldsymbol{\omega}) \geq 0,\ \ell = 1 : L \end{array} \right\}. \tag{7}$$

Sum-of-squares polynomials can be used to determine whether the set (7) is empty. This happens if and only if there exist polynomials $U_k(\mathbf{z})$ and sum-of-squares polynomials $S_\ell(\mathbf{z})$ such that

$$1 + \sum_{k=1}^{K} E_k(z) U_k(z) + S_0(z) + \sum_{\ell=1}^{L} D_\ell(z) S_\ell(z) = 0. \tag{8}$$

In all the results from Chap. 4 listed above, the degrees of the variable polynomials can be high. Hence, only relaxed versions (i.e., sufficient conditions) can be actually implemented.

In Chaps. 5–9, each basic theoretical result is applied at least once. With univariate polynomials, the typical optimization problems are obtained by replacing the unknown FIR filter $H(z)$, in which the problem is not convex, with its squared magnitude (3), in which the problem is convex (and SDP). After solving the SDP problem, the desired filter is obtained by spectral factorization, with algorithms discussed in Appendix B. The optimization problems are signal processing classics, ranging from the design of FIR and IIR filters to the design of filterbanks and wavelets.

With multivariate polynomials, the applications are the design of 2-D FIR and IIR filters, H_∞ deconvolution, and stability tests, including robust stability. One interesting conclusion is that the relaxations of minimal degree, obtained, e.g., by taking in (4) the degrees of the factors $H_\ell(z)$ equal to the degree of $R(z)$, give practically optimal solutions in almost all problems. So, the limitations of relaxations are mostly theoretical. This allows solving optimally some problems for which no other known algorithm could guarantee practical optimality.

The BRL is used in filter design, deconvolution, and especially in all optimization problems involving the atomic norm and deconvolution, such as line spectrum and direction of arrival estimation.

Each chapter ends with bibliographical notes and a number of problems, whose difficulty ranges from very simple to medium. There are no solutions given in the book, but some hints are provided for many of the "not-so-trivial" problems. The programs for solving the numerical examples are available at http://www.schur.pub.ro/postrigpol_book.htm; in case of trouble, e-mail to bogdan.dumitrescu@upb.ro.

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