Chapter 2
Algebraic Identities, Equations and Systems

The rest of this volume, up to Chap. 5, approaches and develops several tools necessary for an adequate presentation of the material in volumes 2 and 3. We start by studying, in this chapter, some important algebraic identities, equations and systems of equations.

2.1 Algebraic Identities

Through the rest of these notes, we refer to a varying real number as a **real variable**.[1] In general, real variables will be denoted by lower case Latin letters, for example a, b, c, x, y, z etc (an important exception to this usage is mentioned in the next paragraph).

An **algebraic expression**, or simply an *expression*, is a real number formed from a finite number of real variables, possibly with the aid of one or more **algebraic operations**, i.e., additions, subtractions, multiplications, divisions, power computations and root extractions (whenever the results of these operations make sense in $\mathbb{R}$). In particular, every real variable can be seen as an algebraic expression. For another example,

$$\frac{x + \sqrt{y} - x^2 z}{yz} + 3\sqrt[5]{x^2 y z^3 - x^4}$$

is an algebraic expression which makes sense for all reals x, y, z, such that $y > 0$ and $z \neq 0$ (recall that Problem 1, page 14, assures that we can extract roots of odd index of any real number). We shall denote algebraic expressions by upper case latin letters, as E, F etc.

[1]In [5], we shall have the opportunity to consider *complex* variables.

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We say that an algebraic expression E is a **monomial** if E is a product of a given nonzero real number by powers of its variables, each of which having nonnegative integer exponents. Thus, the monomials in the real variables x and y are the expressions of the form ax^ky^l, where $a \neq 0$ is a given real number and $k, l \geq 0$ are nonnegative integers (here, we adopt the convention that $x^k = 1$ whenever $k = 0$, and $y^l = 1$ whenever $l = 0$—see Problem 4, page 11). For an arbitrary monomial, the given nonzero real number that plays the role of a in ax^ky^l is called its **coefficient**. Hence, the monomials in x, y with coefficient 2 are those of one of the forms

$$2,\ 2x\ 2y,\ 2x^2,\ 2xy,\ 2y^2,\ 2x^3,\ 2x^2y,\ 2xy^2,\ 2y^3 \text{ etc.}$$

A **polynomial expression** or simply a **polynomial** is (an expression that is) a finite sum of monomials, as, for instance,

$$2 + 3xy - \sqrt{5}x^2yz.$$

The **coefficients** of a polynomial are the coefficients of its monomials.

Let E and F be algebraic expressions. We say that equality $E = F$ is an **algebraic identity** provided it is true for all possible values of the involved real variables. In order to give a relevant example, let us consider the algebraic expression $E = (x + y)^2$. The elementary properties of the operations of addition and multiplication of real numbers (i.e., commutativity and associativity of addition and multiplication, as well as distributivity of multiplication with respect to addition) give

$$\begin{aligned} E &= (x + y)(x + y) = x(x + y) + y(x + y) \\ &= (x^2 + xy) + (yx + y^2) \\ &= x^2 + 2xy + y^2, \end{aligned}$$

for all values of the real variables x and y. Therefore, by setting $F = x^2 + 2xy + y^2$, we obtain the algebraic identity $E = F$, i.e.,

$$(x + y)^2 = x^2 + 2xy + y^2, \tag{2.1}$$

to which we refer, from now on, as the *formula* for the square of a sum of two real numbers.

The following proposition collects some important algebraic identities, which the reader must keep for future use.

Proposition 2.1 *For all* $x, y, z \in \mathbb{R}$, *we have:*

(a) $x^2 - y^2 = (x - y)(x + y)$.
(b) $(x \pm y)^2 = x^2 \pm 2xy + y^2$.
(c) $x^3 \pm y^3 = (x \pm y)(x^2 \mp xy + y^2)$.

(d) $(x \pm y)^3 = x^3 \pm y^3 \pm 3xy(x \pm y)$.
(e) $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$.

Proof We let the proofs of items (a)–(c) as exercises (see Problem 1), observing that the identity of item (b), with the $+$ sign, was established in (2.1). In item (d), let us prove the identity for $(x+y)^3$; that for $(x-y)^3$ is totally analogous: by invoking the distributivity of the multiplication with respect to addition, as well as identity (2.1), we get

$$\begin{aligned}(x + y)^3 &= (x + y)(x + y)^2 = x(x + y)^2 + y(x + y)^2 \\ &= x(x^2 + 2xy + y^2) + y(x^2 + 2xy + y^2) \\ &= (x^3 + 2x^2y + xy^2) + (x^2y + 2xy^2 + y^3) \\ &= x^3 + y^3 + 3x^2y + 3xy^2 \\ &= x^3 + y^3 + 3xy(x + y).\end{aligned}$$

In order to get the result of item (e), we apply that of item (b), with $x + y$ in the place of x and z in the place of y:

$$\begin{aligned}(x + y + z)^2 &= [(x + y) + z]^2 \\ &= (x + y)^2 + 2(x + y)z + z^2 \\ &= (x^2 + 2xy + y^2) + 2(xz + yz) + z^2 \\ &= x^2 + y^2 + z^2 + 2xy + 2xz + 2yz.\end{aligned}$$

□

The reader has certainly noticed that, in the previous proposition, one either has: (i) an identity of the form $E = F$, where E is a product of (at least two) polynomials and F is the sum of monomials we get from expanding the products in E (this is the case of the identities of items (b), (d) and (e)); or else (ii) an identity of the form $E = F$, where E is a polynomial and F is a product of (at least two) polynomials (as in items (a) and (c) of the previous proposition). In case (ii), we shall sometimes say that F is a **factorisation** of E, or that it is obtained by *factoring out* expression E.

The coming examples will give us an idea on how to apply the identities collected in the previous proposition to solve several interesting problems.

Example 2.2 Let x, y, z be real numbers, not all zero, such that $x+y+z = 0$. Explain why $xy + xz + yz \neq 0$ and, then, compute all possible values of the expression

$$\frac{x^2 + y^2 + z^2}{xy + yz + zx}.$$

Solution Squaring both sides of $x + y + z = 0$, it follows from item (e) of Proposition 2.1 that $x^2 + y^2 + z^2 + 2(xy + xz + yz) = 0$. If $xy + xz + yz = 0$, we would have $x^2+y^2+z^2 = 0$, and a simple extension of Corollary 1.5 (cf. Problem 2, page 10) would give us $x = 0$, $y = 0$ and $z = 0$, contradicting our hypotheses. Therefore, $xy+xz+yz \neq 0$, and it follows from $x^2+y^2+z^2 = -2(xy+xz+yz)$ that

$$\frac{x^2 + y^2 + z^2}{xy + yz + zx} = -2.$$

□

Our next example shows how to use the algebraic identities we know so far to prove *inequalities*.[2]

Example 2.3 (Poland) For given positive real numbers a and b, prove that $4(a^3 + b^3) \geq (a + b)^3$.

Proof By expanding the right hand side with the aid of item (d) of Proposition 2.1, it is immediate to see that the inequality we want to prove is equivalent to $a^3 + b^3 \geq a^2b + ab^2$. It now suffices to see that

$$\begin{aligned} a^3 + b^3 - a^2b - ab^2 &= a^3 - a^2b + b^3 - ab^2 = a^2(a-b) - b^2(a-b) \\ &= (a^2 - b^2)(a-b) = (a+b)(a-b)(a-b) \\ &= (a+b)(a-b)^2 \geq 0, \end{aligned}$$

for $a + b > 0$ and $(a - b)^2 \geq 0$. □

We now generalize Example 1.8.

Example 2.4 (Austria) Let a and b be positive rationals, such that $\sqrt{ab}$ is irrational. Prove that $\sqrt{a} + \sqrt{b}$ is also irrational.

Proof By contraposition, suppose that $r = \sqrt{a}+\sqrt{b}$ were a rational number. Then, $r^2 = a+b+2\sqrt{ab}$ would also be rational. However, in such a case, we would have

$$\sqrt{ab} = \frac{r^2 - a - b}{2},$$

which would be a rational number too, for, in the right hand side of the above equality, both the numerator and the denominator are rational numbers. □

Example 2.5 (Canada) For each natural number n, prove that

$$n(n+1)(n+2)(n+3)$$

is never a perfect square.

[2]We will undertake a thorough discussion of inequalities in Chap. 5 and Sects. 9.7 and 10.8.

Proof Letting $p = n(n+1)(n+2)(n+3)$, we have

$$\begin{aligned} p &= [n(n+3)][(n+1)(n+2)] \\ &= (n^2+3n)[(n^2+3n)+2] \\ &= (n^2+3n)^2 + 2(n^2+3n) \\ &= [(n^2+3n)^2 + 2(n^2+3n)+1] - 1 \\ &= [(n^2+3n)+1]^2 - 1. \end{aligned}$$

If we set $m = n^2 + 3n + 1$, we have $m > 1$ and, hence,

$$p = m^2 - 1 > m^2 - 2m + 1 = (m-1)^2.$$

Therefore, p is situated between the consecutive perfect squares $(m-1)^2$ and m^2, so that it cannot be, itself, a perfect square. □

Apart from the algebraic identities collected in Proposition 2.1, another frequently useful one is that given by the equality

$$(x-y)(x-z) = x^2 - (y+z)x + yz. \tag{2.2}$$

Observe that, at the right hand side of the above expression, both the sum $S = y + z$ and the product $P = yz$ of y and z do appear. An expression of the form $x^2 - Sx + P$, where S and P represent the sum and the product of two numbers or expressions, is called a **second degree trinomial** in x. Hence, writing (2.2) backwards, we can also see it as giving a factorisation for the second degree trinomial $x^2 - Sx + P$, where $S = y + z$ and $P = yz$:

$$x^2 - Sx + P = (x-y)(x-z). \tag{2.3}$$

The above factorisation is sometimes called **Viète's formula**, in honor of the French mathematician François Viète.[3]

The following example shows us how to apply Viète's formula.

Example 2.6 (Soviet Union) Let a, b and c be pairwise distinct real numbers. Show that the number

$$a^2(c-b) + b^2(a-c) + c^2(b-a)$$

is always different from zero.

[3]François Viète, French mathematician of the XVI century. By his pioneerism in the usage of letters to represent variables, Viète is sometimes called the father of modern Algebra.

Proof Letting S denote the given number, we have

$$\begin{aligned} S &= a^2(c-b) + b^2a - b^2c + c^2b - c^2a \\ &= a^2(c-b) + (b^2a - c^2a) + (c^2b - b^2c) \\ &= a^2(c-b) + a(b+c)(b-c) + bc(c-b) \\ &= (c-b)[a^2 - a(b+c) + bc] \\ &= (c-b)(a-b)(a-c), \end{aligned}$$

where we used (2.3) in the last equality. Now, it follows from $a \neq b$, $b \neq c$ and $c \neq a$ that $a-b, c-b, a-c \neq 0$, so that $S \neq 0$. □

A useful variant of Viète's formula is the factorisation for the expression $x^2 + Sx + P$, where, as before, $S = y + z$ and $P = yz$:

$$x^2 + Sx + P = (x+y)(x+z). \tag{2.4}$$

If we change S, y and z in (2.3) respectively by $-S$, $-y$ and $-z$, we immediately see that (2.4) is indeed equivalent to that factorisation.

The next example uses (2.4) to get yet another algebraic identity, which will be further applied in a number of places, both in this volume as well as in [4] and [5].

Example 2.7 For all $x, y, z \in \mathbb{R}$, we have

$$(x+y+z)^3 = x^3 + y^3 + z^3 + 3(x+y)(x+z)(y+z). \tag{2.5}$$

Proof Applying item (d) of Proposition 2.1 twice, first with $x+y$ in place of x and z in place of y, we successively get

$$\begin{aligned} (x+y+z)^3 &= [(x+y)+z]^3 \\ &= (x+y)^3 + z^3 + 3(x+y)z[(x+y)+z] \\ &= x^3 + y^3 + 3xy(x+y) + z^3 + 3(x+y)[(x+y)z + z^2) \\ &= x^3 + y^3 + z^3 + 3(x+y)[xy + (x+y)z + z^2] \\ &= x^3 + y^3 + z^3 + 3(x+y)(y+z)(x+z), \end{aligned}$$

where, in the last equality, we have used the variant (2.4) of Viète's formula. □

Problems: Section 2.1

1. * Prove the other items of Proposition 2.1.

2. If $m+n+p=6$, $mnp=2$ and $mn+mp+np=11$, compute all possible values of $\frac{m}{np}+\frac{n}{mp}+\frac{p}{mn}$.
3. Let a and b be nonzero real numbers, such that $a \neq b, 1$. If $\left(\frac{b}{a}\right)^2=\left(\frac{1-b}{1-a}\right)^2$, compute all possible values of $\frac{1}{a}+\frac{1}{b}$.
4. Given positive real numbers x and y, simplify the expression

$$\frac{1-\left(\frac{x}{y}\right)^{-2}}{(\sqrt{x}-\sqrt{y})^2+2\sqrt{xy}}.$$

5. For $x, y, z \neq 0$, such that $y+z \neq 0$, simplify the expression

$$\frac{(x^3+y^3+z^3)^2-(x^3-y^3-z^3)^2}{y+z}.$$

6. Let a and b be real numbers such that $ab=1$ and $a \neq b$. Simplify the expression

$$\frac{\left(a-\frac{1}{a}\right)\left(b+\frac{1}{b}\right)}{a^2-b^2}.$$

7. Let x and y be natural numbers such that $x^2+361=y^2$. Find all possible values of x.
8. Real numbers a and b are such that $a+b=m$ and $ab=n$. Compute the value of a^4+b^4 in terms of m and n.
9. If $a^2+b^2=1$, find all possible values of $\frac{1-3(ab)^2}{a^6+b^6}$.
10. (EKMC) Let a, b, c and d be real numbers such that $a^2+b^2=1$ and $c^2+d^2=1$. If $ac+bd=\frac{\sqrt{3}}{2}$, compute the value of $ad-bc$, provided it is a positive number.
11. (Brazil) Find all natural numbers x and y such that $x+y+xy=120$.
12. * Given positive distinct real numbers x and y, prove that the following *rationalisations*[4] are valid:

    (a) $\frac{1}{\sqrt{x}\pm\sqrt{y}}=\frac{\sqrt{x}\mp\sqrt{y}}{x-y}$.

    (b) $\frac{1}{\sqrt[3]{x}\pm\sqrt[3]{y}}=\frac{\sqrt[3]{x^2}\mp\sqrt[3]{xy}+\sqrt[3]{y^2}}{x\pm y}$.

    (c) $\frac{1}{\sqrt[3]{x^2}\mp\sqrt[3]{xy}+\sqrt[3]{y^2}}=\frac{\sqrt[3]{x}\pm\sqrt[3]{y}}{x\pm y}$.

[4] In an informal way, one can think of a *rationalisation* as a way of *clearing roots* from denominators.

13. * For a natural number $n > 1$, show that

$$2\left(\sqrt{n+1}-\sqrt{n}\right) < \frac{1}{\sqrt{n}} < 2\left(\sqrt{n}-\sqrt{n-1}\right).$$

14. Rationalise $\frac{1}{2+\sqrt{2}+\sqrt{3}}$.
15. Rationalise $\frac{1}{\sqrt{2}+\sqrt[3]{3}}$. More precisely, obtain integers a, b, c, d, e, f and g such that

$$\frac{1}{\sqrt{2}+\sqrt[3]{3}} = \frac{1}{g}[(a\sqrt{2}+b)+(c\sqrt{2}+d)\sqrt[3]{3}+(e\sqrt{2}+f)\sqrt[3]{9}].$$

16. Let x, y and z be nonzero real numbers, such that $x + y + z = 0$. Explain why the sum of any two of them is also nonzero, and compute all possible values of each of the following expressions:

(a) $\frac{x^2}{(y+z)^2} + \frac{y^2}{(x+z)^2} + \frac{z^2}{(x+y)^2}$.
(b) $\frac{x^3}{(y+z)^3} + \frac{y^3}{(x+z)^3} + \frac{z^3}{(x+y)^3}$.

17. Let a and b be distinct integers. Find, in terms of a and b, the quotient of the division of $a^{64} - b^{64}$ by $(a+b)(a^2+b^2)(a^4+b^4)(a^8+b^8)(a^{16}+b^{16})$.
18. * Given $n > 1$ integer and $a, b \in \mathbb{R}$, prove that the following factorizations are valid:

(a) $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + b^{n-1})$.
(b) $a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \cdots + b^{n-1})$, provided n is odd.

19. Write $x^4 + 4y^4$ as a product of two non constant polynomials in x and y, both having integer coefficients.
20. (Canada) Let $a, b, c \in \mathbb{Z}$. Prove that 6 divides $a+b+c$ if and only if 6 divides $a^3 + b^3 + c^3$.
21. (Canada) If a, b and c are real numbers for which $a + b + c = 0$, show that $a^3 + b^3 + c^3 = 3abc$.
22. Prove the *double radical formula*, also known as *Bhaskara's formula*[5]: for all positive real numbers a and b, such that $a^2 \geq b$, one has

$$\sqrt{a \pm \sqrt{b}} = \sqrt{\frac{a+\sqrt{a^2-b}}{2}} \pm \sqrt{\frac{a-\sqrt{a^2-b}}{2}}.$$

[5]In honor of the Indian mathematician of the XII century Bhaskara II, also known as Bhaskaracharya (Bhaskara, the professor). The idea behind Bhaskara's formula is that, if a and b are naturals for which $a^2 - b$ is a perfect square, then his formula provides a simpler expression for $\sqrt{a \pm \sqrt{b}}$.

23. Show that there do not exist nonzero real numbers x, y and z such that $x+y+z \neq 0$ and

$$\frac{1}{x+y+z} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

24. (Soviet Union) Let a, b and c be pairwise distinct rationals. Prove that

$$\frac{1}{(b-c)^2} + \frac{1}{(c-a)^2} + \frac{1}{(a-b)^2}$$

is the square of a rational.

25. (TT) Let a, b and c be distinct rationals. If $\sqrt[3]{a} + \sqrt[3]{b} \in \mathbb{Q}$, prove that $\sqrt[3]{a}$, $\sqrt[3]{b} \in \mathbb{Q}$.
26. (TT) Let a, b, c, d, e and f be real numbers such that $a+b+c+d+e+f=0$ and $a^3+b^3+c^3+d^3+e^3+f^3=0$. If no two of them are opposite to each other, prove that

$$(a+c)(a+d)(a+e)(a+f) = (b+c)(b+d)(b+e)(b+f).$$

27. (Poland) For positive integers $a \leq b$, do the following items:
   (a) Show that $b^3 < b^3 + 6ab + 1 < (b+2)^3$.
   (b) Find all such a and b for which both $a^3 + 6ab + 1$ and $b^3 + 6ab + 1$ are perfect cubes.

2.2 The Modulus of a Real Number

We start this section by recalling the definition of modulus of a real number, a concept which will be important in a number of places hereafter.

Definition 2.8 For $x \in \mathbb{R}$, the **modulus** of x, denoted $|x|$, is defined as

$$|x| = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}.$$

As an example, since $-5 < 0$, we have $|-5| = -(-5) = 5$; analogously, $|-\sqrt{3}| = -(-\sqrt{3}) = \sqrt{3}$ etc. More generally, an immediate consequence of the definition is that $|x| \geq 0$ for all $x \in \mathbb{R}$, with equality if and only if $x = 0$. Moreover, one always has

$$x \leq |x| = |-x|,$$

with equality if and only if $x \geq 0$. Note also that

$$|x| = \sqrt{x^2} = \max\{x, -x\}. \tag{2.6}$$

The simplest **modular equation** is the equation

$$|x - a| = b,$$

where a and b are given real numbers. Since $|x - a| \geq 0$, such an equation does not admit roots when $b < 0$. On the other hand, when $b \geq 0$, it follows from the definition of modulus that one must have either $x - a = b$ or $x - a = -b$, from where we get the roots

$$x = a + b, a - b.$$

The coming example shows how to solve a more elaborate equation in a single variable, involving the concept of modulus of a real number.

Example 2.9 Solve equation $|x + 1| + |x - 2| + |x - 5| = 7$.

Solution First of all, note that

$$|x + 1| = \begin{cases} x + 1, \text{ if } x \geq -1 \\ -x - 1, \text{ if } x < -1 \end{cases},$$

$$|x - 2| = \begin{cases} x - 2, \text{ if } x \geq 2 \\ -x + 2, \text{ if } x < 2 \end{cases}$$

and

$$|x - 5| = \begin{cases} x - 5, \text{ if } x \geq 5 \\ -x + 5, \text{ if } x < 5 \end{cases}.$$

Now, the conjunction of the conditions $x < -1$ or $x \geq -1$, $x < 2$ or $x \geq 2$, $x < 5$ or $x \geq 5$ partitions the real line into the intervals $(-\infty, -1)$, $[-1, 2)$, $[2, 5)$ e $[5, +\infty)$. Hence, in order to simplify the left hand side of the given equation, we separately consider x as varying in each one of these intervals. We thus obtain

$$|x + 1| + |x - 2| + |x - 5| = \begin{cases} -3x + 6, \text{ if } x < -1 \\ -x + 8, \text{ if } -1 \leq x < 2 \\ x + 4, \text{ if } 2 \leq x < 5 \\ 3x - 6, \text{ if } x \geq 5 \end{cases}.$$

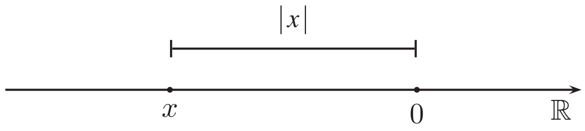

Fig. 2.1 Modulus of a real number

Finally, note that

- $-3x + 6 = 7 \Leftrightarrow x = -\frac{1}{3}$; however, since the condition $-\frac{1}{3} < -1$ is not satisfied, there are no roots in this case.
- $-x + 8 = 7 \Leftrightarrow x = 1$; since the condition $-1 \leq 1 < 2$ is satisfied, $x = 1$ is a root of the equation.
- $x + 4 = 7 \Leftrightarrow x = 3$; since the condition $2 \leq 3 < 5$ is satisfied, $x = 3$ is also a root of the equation.
- $3x - 6 = 7 \Leftrightarrow x = \frac{13}{3}$; since the condition $\frac{13}{3} \geq 5$ is not satisfied, there are no roots in this case.

Therefore, the solution set of the given equation is $S = \{1, 3\}$. □

Back to the study of the properties of modulus, let us represent the real numbers as points in the real line. It is easy to see that $|x|$ is simply the distance from (the point that represents) x to (the one representing) 0 (cf. Fig. 2.1). More generally, given $x, y \in \mathbb{R}$, we can look at $|x - y|$ as the distance from the points x and y in the real line. In fact, since $|x - y| = |y - x|$, we can suppose that $x \leq y$. Then,

$$|x - y| = y - x = \text{ distance from } x \text{ to } y \text{ in the real line.}$$

In the above reasoning, if we do not wish to consider which of x and y is the greatest one, we can write

$$|x - y| = \max\{x, y\} - \min\{x, y\}, \tag{2.7}$$

for

$$\{x, y\} = \{\max\{x, y\}, \min\{x, y\}\}. \tag{2.8}$$

There simple remarks suffice to consider the following hard example.

Example 2.10 (Yugoslavia) Let $n \in \mathbb{N}$ and $M = \{1, 2, 3, \ldots, 2n\}$. Also, let $M_1 = \{a_1, a_2, \ldots, a_n\}$ and $M_2 = \{b_1, b_2, \ldots, b_n\}$ be subsets of M such that $a_1 < a_2 < \cdots < a_n$ and $b_1 > b_2 > \cdots > b_n$. If $M_1 \cup M_2 = M$ and $M_1 \cap M_2 = \emptyset$, prove that

$$|a_1 - b_1| + |a_2 - b_2| + \cdots + |a_n - b_n| = n^2.$$

Proof It follows from (2.8) that

$$\bigcup_{i=1}^{n} \{\max\{a_i, b_i\}, \min\{a_i, b_i\}\} = \bigcup_{i=1}^{n} \{a_i, b_i\} = \{1, 2, 3, \ldots, 2n\}.$$

Also, (2.7) gives

$$\begin{aligned}|a_1 - b_1| + |a_2 - b_2| + \cdots + |a_n - b_n| &= \\ &= (\max\{a_1, b_1\} + \max\{a_2, b_2\} + \cdots + \max\{a_n, b_n\}) \\ &\quad - (\min\{a_1, b_1\} + \min\{a_2, b_2\} + \cdots + \min\{a_n, b_n\}).\end{aligned}$$

On the other hand, given integers $1 \le k, l \le n$, with $k \ne l$, we have

$$k > l \Rightarrow \max\{a_k, b_k\} \ge a_k > a_l \ge \min\{a_l, b_l\}$$

and

$$k < l \Rightarrow \max\{a_k, b_k\} \ge b_k > b_l \ge \min\{a_l, b_l\}.$$

Therefore,

$$\{\max\{a_1, b_1\}, \max\{a_2, b_2\}, \ldots, \max\{a_n, b_n\}\} = \{n + 1, n + 2, \ldots, 2n\}$$

and

$$\{\min\{a_1, b_1\}, \min\{a_2, b_2\}, \ldots, \min\{a_n, b_n\}\} = \{1, 2, \ldots, n\}.$$

Finally, the above relations give

$$\begin{aligned}|a_1 - b_1| + |a_2 - b_2| + \cdots + |a_n - b_n| &= \\ &= ((n + 1) + (n + 2) + \cdots + 2n) - (1 + 2 + \cdots + n) \\ &= n^2 + (1 + 2 + \cdots + n) - (1 + 2 + \cdots + n) = n^2.\end{aligned}$$

□

We continue our study of the concept of modulus with the following important result, which is known in mathematical literature as the **triangle inequality**.[6]

Proposition 2.11 *For all real numbers a and b, we have*

$$|a + b| \le |a| + |b|. \tag{2.9}$$

Moreover, if $a, b \ne 0$, then equality holds if and only if a and b have the same sign.

[6]At the end of Sect. 5.2, we will give an explanation of why (2.9), as well as the coming inequality (4.6), are called *triangle inequality*.

Proof Since $|a+b|$ and $|a|+|b|$ are both nonnegative, we have

$$\begin{aligned} |a+b| \le |a|+|b| &\Leftrightarrow |a+b|^2 \le (|a|+|b|)^2 \\ &\Leftrightarrow (a+b)^2 \le |a|^2+|b|^2+2|ab| \\ &\Leftrightarrow 2ab \le 2|ab|, \end{aligned}$$

which is clearly true. From the computations above it also follows that $|a+b| = |a|+|b|$ if and only if $ab = |ab|$; in turn, this happens if and only if $ab \ge 0$. Finally, if $a, b \ne 0$, then we have the equality if and only if $ab > 0$. □

Corollary 2.12 *For all real numbers a and b, we have*

$$||a|-|b|| \le |a-b|.$$

Moreover, if $a, b \ne 0$, then the equality holds if and only if a and b have the same sign.

Proof Applying the triangle inequality to $a-b$ in place of a, we get

$$|a| = |(a-b)+b| \le |a-b|+|b|$$

and, hence, $|a|-|b| \le |a-b|$. Repeating the above argument with the roles of a and b interchanged, it follows that $|b|-|a| \le |a-b|$.

Now, since $|a-b| \ge |a|-|b|, |b|-|a|$, we get

$$|a-b| \ge \max\{|a|-|b|, |b|-|a|\} = ||a|-|b||,$$

where we used (2.6) in the last equality.

Equality happens if and only if we have equality in at least one of the triangular inequalities $|a| \le |a-b|+|b|$ or $|b| \le |b-a|+|a|$. If $a, b \ne 0$ and, say, equality holds in the first one of theses inequalities, i.e., if $|a| = |a-b|+|b|$, then the condition for equality in Proposition 2.11 assures that we must have $(a-b)b \ge 0$, or, which is the same $ab \ge b^2$. In particular, we must have $ab > 0$.

Conversely, suppose that $ab > 0$, and let us show that equality holds. There are two possibilities: $a, b > 0$ or $a, b < 0$. Suppose that $a, b < 0$ (the other case can be treated in a similar way). Then, $|a| = -a$ and $|b| = -b$, so that $||a|-|b|| = |(-a)-(-b)| = |b-a| = |a-b|$. □

Given real numbers a, b and c and applying triangle inequality twice, we get

$$|a+b+c| \le |a+b|+|c| \le |a|+|b|+|c|, \tag{2.10}$$

Hence, we have the inequality

$$|a+b+c| \le |a|+|b|+|c|, \tag{2.11}$$

which is the analogous of (2.9) for three real numbers, instead of two. Therefore, we shall also refer to this last inequality as the *triangle inequality*.

If $a, b, c \neq 0$ and we have equality in (2.11), then we should also have equality in all inequalities in (2.10). In particular, we have $|a+b| \leq |a| + |b|$, and it follows from Proposition 2.11 that a and b have equal signs. Since we can also reach (2.11) by writing

$$|a+b+c| \leq |a| + |b+c| \leq |a| + |b| + |c|,$$

we conclude, analogously, that b and c should also have equal signs.

Conversely, if a, b and c all have equal signs, say $a, b, c < 0$ (the case $a, b, c > 0$ is completely analogous), then $a + b + c < 0$, so that

$$|a+b+c| = -(a+b+c) = (-a) + (-b) + (-c) = |a| + |b| + |c|.$$

Hence, we have just shown that equality holds in (2.11) if and only if a, b and c have equal signs.

As we shall see in Sect. 4.1 (cf. Problem 7, page 96), inequalities (2.9) and (2.10) can be easily generalized for n real numbers. For the time being, we end this section with the following

Example 2.13 Prove that, for every $x \in \mathbb{R}$, we have

$$|x-1| + |x-2| + |x-3| + \cdots + |x-10| \geq 25.$$

Proof It follows from the triangle inequality that

$$|x-a| + |x-b| = |x-a| + |b-x| \geq |(x-a) + (b-x)| = |b-a|.$$

Hence, grouping the summands at the left hand side in pairs, we get,

$$\begin{aligned}
|x-1| + |x-10| &\geq |10-1| = 9;\\
|x-2| + |x-9| &\geq |9-2| = 7;\\
|x-3| + |x-8| &\geq |8-3| = 5;\\
|x-4| + |x-7| &\geq |7-4| = 3;\\
|x-5| + |x-6| &\geq |6-5| = 1.
\end{aligned}$$

Adding these inequalities, we obtain that of the statement. □

Problems: Section 2.2

1. * Given real numbers a and b, show that

$$\{x \in \mathbb{R};\ |x-a| < b\} = \begin{cases} \emptyset, \text{ if } b < 0 \\ \{a\}, \text{ if } b = 0 \\ (a-b, a+b), \text{ if } b > 0 \end{cases}.$$

   Do the same for $|x-a| \le b$, $|x-a| > b$ and $|x-a| \ge b$.
2. * Prove that, for all $x, y \in \mathbb{R}$, one has $|xy| = |x| \cdot |y|$.
3. Solve, for $x \in \mathbb{R}$, the following equations:

   (a) $|x| = x - 6$.
   (b) $|x+1| + |x-2| + |x-5| = 4$.

4. Solve, for $x \in \mathbb{R} \setminus \{0, 1\}$, equation $\frac{|x|}{x} = \frac{|x-1|}{x-1}$.
5. Let a, b and c be given real numbers, with $a < b$. Discuss, in terms of a, b and c, the number of solutions of the equation

$$|x-a| + |x-b| = c.$$

6. (Mexico) Let r be a nonnegative rational number. Prove that

$$\left|\frac{r+2}{r+1} - \sqrt{2}\right| < \frac{1}{2}|r - \sqrt{2}|.$$

   (This inequality shows that the rational number $\frac{r+2}{r+1}$ approximates $\sqrt{2}$ twice as better as r does it.)
7. Prove that:

   (a) If $0 \le x \le y$, then $\frac{x}{1+x} \le \frac{y}{1+y}$.
   (b) If $a, b \in \mathbb{R}$, then $\frac{|a|}{1+|a|} + \frac{|b|}{1+|b|} \ge \frac{|a+b|}{1+|a+b|}$.

8. Let $n > 1$ be an integer. Prove that

$$|x-1| + |x-2| + \cdots + |x-2n| \ge n^2$$

   for every real x, with equality for infinitely many values of x.

2.3 A First Look at Polynomial Equations

In this section we study some particular types of *polynomial equations*, postponing a much deeper look to [5].

In general, a **polynomial equation of degree n** is an equation of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0, \tag{2.12}$$

where $n \geq 1$ is an integer and $a_0, a_1, \ldots, a_n$ are given real numbers,[7] with $a_n \neq 0$.

The simplest kind of such an equation is the **first degree equation** $ax + b = 0$, where a and b are given real numbers and $a \neq 0$. As the reader certainly knows, we have

$$ax + b = 0 \Leftrightarrow ax = -b \Leftrightarrow x = -\frac{b}{a},$$

so that $-\frac{b}{a}$ is its only root.

The second simplest kind of polynomial equation is the **second degree equation**

$$ax^2 + bx + c = 0, \tag{2.13}$$

where a, b and c are given real numbers, with $a \neq 0$. For reasons that will soon be clear, the left hand side of (2.13) is also known as the **second degree trinomial** associated to Eq. (2.13).

In order to solve (2.13), we let Δ (one reads *delta*) denote the real number

$$\Delta = b^2 - 4ac,$$

and call it the **discriminant** of the equation (or of the associated trinomial). As we shall see in a moment, the sign of Δ *discriminates* whether or not the equation has real roots. To this end, we need the following auxiliary result.

Lemma 2.14 *Given $a, b, c \in \mathbb{R}$, with $a \neq 0$, one has*

$$ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right]. \tag{2.14}$$

*This algebraic identity is called the **canonical form** of the second degree trinomial $ax^2 + bx + c$.*

[7] Here, we are using the concept of a *sequence* of real numbers. For further details in this respect, we refer the reader to Chap. 3.

Proof It suffices to see that

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right) \\ &= a\left[\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a^2} + \frac{4ac}{4a^2}\right] \\ &= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right]. \end{aligned}$$

□

Remark 2.15 The idea of adding and subtracting a certain summand out of a given algebraic expression in order to *complete a square*, as was done right after the second equality in the proof of the previous lemma, is very important and should be learned as a kind of *algebraic trick* that will be useful in a number of places, here as well as in [4] and [5]. Later in this chapter, we shall see other interesting applications of such a technique.

Proposition 2.16 *Let a, b and c be given real numbers, with* $a \neq 0$.

(a) The equation $ax^2 + bx + c = 0$ *has real roots if and only if* $\Delta \geq 0$. *Moreover, if this is so, then its roots are* $\frac{-b-\sqrt{\Delta}}{2a}$ *and* $\frac{-b+\sqrt{\Delta}}{2a}$.
(b) If $\Delta \geq 0$, *then the sum S and the product P of the roots of* $ax^2 + bx + c = 0$ *are given by* $S = -\frac{b}{a}$ *and* $P = \frac{c}{a}$.

Proof

(a) It follows from (2.14) that

$$ax^2 + bx + c = 0 \Leftrightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{\Delta}{4a^2}. \tag{2.15}$$

Since $\left(x + \frac{b}{2a}\right)^2 \geq 0$ for all $x \in \mathbb{R}$, if the equation has real roots, then one must have $\Delta \geq 0$. In this case, it transpires from (2.15) that $x + \frac{b}{2a} = \pm\frac{\sqrt{\Delta}}{2a}$, and item (a) follows.

(b) It suffices to compute

$$\frac{-b - \sqrt{\Delta}}{2a} + \frac{-b + \sqrt{\Delta}}{2a} = -\frac{b}{a}$$

and

$$\Big(\frac{-b-\sqrt{\Delta}}{2a}\Big)\Big(\frac{-b+\sqrt{\Delta}}{2a}\Big)=\frac{(-b)^2-\Delta}{4a^2}=\frac{c}{a}.$$

□

Remarks 2.17

i. When $\Delta \geq 0$, formulae $\frac{-b\pm\sqrt{\Delta}}{2a}$ for the roots of $ax^2+bx+c=0$ are known as **Bhaskara's formulae**.
ii. The formulas of item (b) are also known as **Viète's formulae**.
iii. In the notations of item (a), if $\Delta = 0$ we say that $ax^2+bx+c=0$ has *two equal roots*.

The coming example shows how one can reduce a seemingly complicated equation to a simpler one by means of a suitable **substitution of variable**.

Example 2.18 (Brazil) Find all real numbers x such that $x^2+x+1=\frac{156}{x^2+x}$.

Solution By performing the substitution $y=x^2+x$, we get the equation $y+1=\frac{156}{y}$ or, which is the same, $y^2+y-156=0$. For this last equation, since $\Delta = 1^2-4(-156)=625=25^2$, it follows that $y=\frac{-1\pm 25}{2}=-13$ or 12. Thus, we have reduced the original equation to the second degree equations $x^2+x=-13$ and $x^2+x=12$. For the first one, we have $\Delta=-51<0$, so that there are no real roots. For the second, $\Delta=49$ and, hence, $x=\frac{-1\pm 7}{2}=-4$ or 3. □

Our next example shows that it is sometimes more useful to *algebraically manipulate* a second degree equation than to solve it explicitly.

Example 2.19 Find the numerical value of $(3+\sqrt{2})^5+(3-\sqrt{2})^5$ without expanding the powers involved.

Proof Letting $u=3+\sqrt{2}$ and $v=3-\sqrt{2}$, we have $u+v=6$ and $uv=7$, so that u and v are the roots of the equation $x^2-6x+7=0$. Therefore, making $x=u$ and $x=v$ in this equation gives us $u^2-6u+7=0$ and $v^2-6v+7=0$, or, equivalently, $u^2=6u-7$ and $v^2=6v-7$. Multiplying the first equality by u^k and the second one by v^k, where $k\geq 0$ is an integer, we get

$$u^{k+2}=6u^{k+1}-7u^k \text{ and } v^{k+2}=6v^{k+1}-7v^k;$$

adding both of these, we finally arrive at

$$u^{k+2}+v^{k+2}=6(u^{k+1}+v^{k+1})-7(u^k+v^k).$$

Writing the previous relation for k respectively equal to 0, 1, 2 and 3, we successively get

$$\begin{aligned}
u^2+v^2 &= 6(u+v)-7\cdot 2 = 6\cdot 6-14 = 22;\\
u^3+v^3 &= 6(u^2+v^2)-7(u+v) = 6\cdot 22-7\cdot 6 = 90;\\
u^4+v^4 &= 6(u^3+v^3)-7(u^2+v^2) = 6\cdot 90-7\cdot 22 = 386;\\
u^5+v^5 &= 6(u^4+v^4)-7(u^3+v^3) = 6\cdot 386-7\cdot 90 = 1686.
\end{aligned}$$

□

For the next example, recall that if the sum and the product of two real numbers are positive, then both numbers are also positive.

Example 2.20 Let p and q be given real numbers. If the equation $x^2+px+q=0$ has real, positive and distinct real roots, show that the same is true for the equation $qx^2+(p-2q)x+(1-p)=0$.

Proof Note initially that $q \neq 0$, for otherwise the first equation would reduce to $x^2+px=0$, which has 0 as one of its roots, thus contradicting our hypotheses.

Now, let Δ and Δ' be, respectively, the discriminants of $x^2+px+q=0$ and $qx^2+(p-2q)x+(1-p)$. Let us first show that $\Delta'>0$, which will guarantee that the second equation has distinct real roots. Since $x^2+px+q=0$ has distinct real roots, we have $\Delta=p^2-4q>0$. Therefore,

$$\begin{aligned}
\Delta' &= (p-2q)^2-4q(1-p)\\
&= p^2-4q+4q^2\\
&= \Delta+4q^2>0.
\end{aligned}$$

Finally, according to the paragraph that immediately precedes this example, in order to show that the roots of $qx^2+(p-2q)x+(1-p)=0$ are positive, it suffices to show that the sum S' and the product P' of them are both positive. In order to do this, we recall that the roots of $x^2+px+q=0$ are known to be positive, so that (by Viète's formulae) $-p>0$ and $q>0$. Hence, again by Viète's formulae, we have

$$S' = \frac{2q-p}{q} = 2+\frac{-p}{q} > 0 \text{ and } P' = \frac{1-p}{q} = \frac{1}{q}+\frac{-p}{q} > 0,$$

as we wished to show. □

We finish our discussion of second degree equations with the following important remark: if $a\neq 0$ and $ax^2+bx+c=0$ has real roots α and β (not necessarily $\alpha\neq\beta$), then we have the factorisation

$$ax^2+bx+c = a(x-\alpha)(x-\beta). \tag{2.16}$$

In fact, it follows from item (b) of Proposition 2.16 that, for every real x,

$$\begin{aligned} a(x-\alpha)(x-\beta) &= a[x^2-(\alpha+\beta)x+\alpha\beta] \\ &= a\left[x^2-\left(-\frac{b}{a}\right)x+\frac{c}{a}\right] \\ &= ax^2+bx+c. \end{aligned}$$

It is instructive to compare (2.16) to (2.2). The right hand side of (2.16) is called the **factorised form** of the second degree trinomial ax^2+bx+c.

In turning to more general polynomial equations, it is natural to try to look at those of degrees $n=3$ and $n=4$. In these cases, formulas have been built, in terms of the coefficients of the equations, to compute their real roots, if any.[8] As professor I. Stewart teaches us in Chap. 4 of his very interesting book [25], such formulas derive from the works of the Italian mathematicians Scipione del Ferro, Girolamo Cardano, Niccolò Fontana (conhecido como Tartaglia) and Lodovico Ferrari. However, they are too complicated to be useful, and for this reason we shall not discuss them here. In order to help convincing the reader, let us just mention that Cardano's formula for the roots of the third degree polynomial equation $ax^3+bx^2+cx+d=0$ is the following[9]:

$$\sqrt[3]{q+\sqrt{q^2+(r-p^2)^3}}+\sqrt[3]{q-\sqrt{q^2+(r-p^2)^3}}+p,$$

where $p=-\frac{b}{3a}$, $q=p^3+\frac{bc-3ad}{6a^2}$ and $r=\frac{c}{3a}$ (however, see Problems 16, 21 and 22).

For polynomial Eq. (2.12) of degree $n\geq 5$, the Norwegian mathematician Niels H. Abel[10] and the French mathematician Évariste Galois,[11] both of the XIX century, proved that there exists no similar formula, built on the coefficients of the given polynomial equation, that gives its real (or even complex!) roots. Well understood, it doesn't matter how smart someone is; they proved that *it is impossible to find* such a formula, simply because it doesn't exist! For a beautiful and elementary account of the ideas involved, we refer the reader to [14].

Some particular kinds of polynomial equations of degrees 4 and 6, however, are sufficiently simple to deserve some attention, specially because appropriate

[8]As it happens, these formulas also give the *complex* roots of the corresponding equations. However, we shall postpone any considerations involving complex numbers to [5].

[9]Cf. http://www.math.vanderbilt.edu/~schectex/courses/cubic.

[10]In his 27 years of life, Abel made several deep contributions to Mathematics, among which the most famous one is perhaps the impossibility of solving general polynomial equations of degree 5.

[11]In spite of his premature death, when he was only 21 years old, Galois is considered to be one of the greatest mathematicians the world has ever seen. His work on the connection between the solvability of polynomial equations of degrees $n\geq 5$ and Group Theory constitute the foundations of what is known today as Galois' Theory, a branch of modern Algebra with applications to several distinct areas of Mathematics.

variable substitutions immediately reduce them to second degree equations. Let us first examine **biquadratic**, i.e., equations of the form

$$ax^4 + bx^2 + c = 0, \tag{2.17}$$

with $a, b, c \in \mathbb{R}$ and $a \neq 0$. The variable substitution $y = x^2$ transforms it into the second degree equation $ay^2 + by + c = 0$, which we already know how to solve in $\mathbb{R}$. Therefore, for each nonnegative root $y = \alpha$ of this last equation, solving equation $x^2 = \alpha$ gives us the pair of real roots $x = \pm\sqrt{\alpha}$ of the original biquadratic equation. Conversely, if $x = \beta$ is a real root of the given biquadratic equation, it is immediate to see that $y = \beta^2$ is a nonnegative root of the second degree equation $ay^2 + by + c = 0$. We have, thus, proved the following result.

Proposition 2.21 *Given real numbers a, b and c, with $a \neq 0$, the real roots of the biquadratic equation $ax^4 + bx^2 + c = 0$ are the reals of the form $\pm\sqrt{\alpha}$, where α is a nonnegative root of the second degree equation $ay^2 + by + c = 0$.*

In order to actually compute the real roots of a specific biquadratic equation, instead of invoking the last proposition, it is usually much better to recall our previous discussion, remembering that the variable substitution $y = x^2$ does the job of reducing it to a second degree equation.

Example 2.22 Find the real roots of the biquadratic equation $x^4 + 5x^2 - 7 = 0$.

Solution The variable substitution $y = x^2$ leads us to the second degree equation $y^2 + 5y - 7 = 0$, for which $\Delta = 53$. Hence, the roots of this last equation are $y = \frac{-5\pm\sqrt{53}}{2}$. Since $\frac{-5-\sqrt{53}}{2} < 0$, the real roots of the original biquadratic equation are the solutions of $x^2 = \frac{-5+\sqrt{53}}{2}$, i.e., are the real numbers $\pm\sqrt{\frac{-5+\sqrt{53}}{2}}$. □

Given $n \in \mathbb{N}$, we point out that we can discuss the problem of finding the roots of a polynomial equation of the form

$$ax^{2n} + bx^n + c = 0 \tag{2.18}$$

in a way quite similar to that used to study biquadratic equations. We refer the reader to Problem 16 for the corresponding details.

Let us now examine **reciprocal polynomial equations** of degree 4, i.e., polynomial equations of degree 4 having the form

$$ax^4 + bx^3 + cx^2 + bx + a = 0,$$

where a, b and c are given real numbers, with $a \neq 0$. As Problem 28 shows, the name *reciprocal* applies to a larger class of polynomial equations, and comes from the fact that $x \in \mathbb{R} \setminus \{0\}$ is a real root of it if and only if $\frac{1}{x}$ also is.

Initially, note that 0 is not a root of the equation above, for $a \neq 0$. Therefore, a real number x is a root of it if and only if it is a root of the equation

$$ax^2 + bx + c + \frac{b}{x} + \frac{a}{x^2} = 0, \tag{2.19}$$

which is obtained from the original equation by dividing both sides of the equality by x^2. Rewrite the left hand side of the last equation above as

$$a\left(x^2 + \frac{1}{x^2}\right) + b\left(x + \frac{1}{x}\right) + c = 0.$$

Now, the idea is to perform the variable substitution $y = x + \frac{1}{x}$. In order to implement it, let us first of all note that, according to (2.1), one has

$$y^2 = \left(x + \frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2}.$$

Hence, $x^2 + \frac{1}{x^2} = y^2 - 2$, so that solving (2.19) amounts to solving the second degree equation

$$a(y^2 - 2) + by + c = 0. \tag{2.20}$$

However, the above discussion hides a subtlety: it is sure that every real root $x = \alpha$ of (2.19) generates the real root $\beta = \alpha + \frac{1}{\alpha}$ of $a(y^2 - 2) + by + c = 0$. Nevertheless, the converse statement is not true: not every real root $y = \beta$ of this last equation does generate real roots $x = \alpha$ of the initial reciprocal equation. In fact, once we get a real root $y = \beta$ of $a(y^2 - 2) + by + c = 0$, in order to obtain the possible real roots of the reciprocal equation corresponding to β, we have to solve in $\mathbb{R}$ the equation

$$x + \frac{1}{x} = \beta,$$

or, which is the same, $x^2 - \beta x + 1 = 0$. Since the discriminant of this last equation is

$$\Delta = \beta^2 - 4,$$

it will have real root only if $\beta^2 - 4 \geq 0$, i.e., only if $|\beta| \geq 2$.

As was the case with biquadratic equations, in order to actually find the real roots of a given reciprocal equation of degree 4, instead of memorizing the result of the variable substitution $y = x + \frac{1}{x}$, it is more profitable to follow the steps that led us from (2.19) to (2.20). Let us see an example in this respect.

Example 2.23 Find all real roots of the reciprocal equation

$$2x^4 + 5x^3 + 6x^2 + 5x + 2 = 0.$$

Solution Dividing both sides by x^2 and grouping summands, we obtain

$$2\left(x^2+\frac{1}{x^2}\right)+5\left(x+\frac{1}{x}\right)+6=0.$$

Making the variable substitution $y=x+\frac{1}{x}$, it follows that $y^2=x^2+\frac{1}{x^2}+2$, so that the given equation is equivalent to

$$2(y^2-2)+5y+6=0.$$

Since this second degree equation has real roots $y=-2$ and $y=-\frac{1}{2}$, it follows that the real roots of the original equation are the real roots of the equations $x+\frac{1}{x}=-2$ and $x+\frac{1}{x}=-\frac{1}{2}$. The first of these equations is equivalent to $x^2+2x+1=0$ and, hence, has two real roots, both equal to -1. The second is equivalent to $2x^2+x+2=0$, which has discriminant $\Delta=-15<0$; therefore, it has no real roots. □

Problems: Section 2.3

1. Let b and c be given real numbers, such that the equation $x^2+b|x|+c=0$ has real roots. Prove that the sum of these roots is always equal to 0.
2. Solve, for $x\in\mathbb{R}$, the following equations:
   (a) $x+\sqrt{x+2}=10$.
   (b) $\sqrt{x+10}-\sqrt{2x+3}=\sqrt{1-3x}$.
   (c) $x^2+18x+30=2\sqrt{x^2+18x+45}$.
3. (IMO) In each of the cases (a) $A=\sqrt{2}$, (b) $A=1$ and (c) $A=2$, find all $x\in\mathbb{R}$ for which we have

$$\sqrt{x+\sqrt{2x-1}}+\sqrt{x-\sqrt{2x-1}}=A.$$

4. A math teacher composed three different quizzes. In the first one, he put a second degree equation. In the second, he put almost the same equation, changing just the coefficient of the monomial of second degree. Finally, in the third quiz, once more he put almost the same equation of the first, this time changing just the constant coefficient. It is known that the roots of the equation of the second quiz are 2 and 3, and that those of the third one are 2 and -7. Decide whether the second degree equation of the first quiz has real roots and, if this is so, compute them.
5. Let a and b be two distinct, nonzero real numbers. If a and b are the roots of the equation $x^2+ax+b=0$, find all possible values of $a-b$.

6. Let α and β be the roots of $x^2 - 13x + 9 = 0$, and a and b be real numbers such that the equation $x^2 + ax + b = 0$ has roots α^2 and β^2. Compute the value of $a + b$.
7. Equation $x^2 + x - 1 = 0$ has roots u and v. Find a second degree equation whose roots are u^3 and v^3.
8. The roots of the equation $x^2 - Sx + P = 0$ are the real numbers α and β. Find a second degree trinomial whose coefficients are expressions built on S and P and whose roots are the real numbers $\alpha S + P$ and $\beta S + P$.
9. Use the theory of second degree equations developed in this section to compute the value of the sum $(7 + 4\sqrt{3})^5 + (7 - 4\sqrt{3})^5$.
10. If α is a root of $x^2 - x - 1 = 0$, find all possible values of $\alpha^5 - 5\alpha$.
11. For which integer values of m does the equation $x^2 + mx + 5 = 0$ have integer roots?
12. Show that, for every $a, b, c \in \mathbb{R}$, with $a \neq 0$, the equation
$$\frac{1}{x-b} + \frac{1}{x-c} = \frac{1}{a^2}$$
has exactly two distinct real roots.
13. Solve equation $x = \sqrt{x - \frac{1}{x}} + \sqrt{1 - \frac{1}{x}}$ in the set of real numbers.
14. Show that, for every real number $a \neq 0$, the equations
$$ax^3 - x^2 - x - (a+1) = 0 \text{ and } ax^2 - x - (a+1) = 0$$
have a common root.
15. (Soviet Union) Do the following items:
    (a) For real x, write the number $x^3 - 3x^2 + 5x$ in the form
    $$a(x-1)^3 + b(x-1)^2 + c(x-1) + d,$$
    with $a, b, c, d \in \mathbb{Z}$.
    (b) If x and y are real numbers such that $x^3 - 3x^2 + 5x = 1$ and $y^3 - 3y^2 + 5y = 5$, compute all possible values of $x + y$.
16. * Let $n \in \mathbb{N}$ and $a, b, c \in \mathbb{R}$, with $a \neq 0$. Elaborate, for the equation $ax^{2n} + bx^n + c = 0$, a discussion analogous to that made on the text for biquadratic equations, and which led us to Proposition 2.21.
17. * Consider the polynomial equation of third degree $x^3 + ax^2 + bx + c = 0$, where a, b and c are given real numbers, with $c \neq 0$. If α is a (nonzero) real root of it, prove that there exist real numbers b' and c' such that we have the factorisation
$$x^3 + ax^2 + bx + c = (x - \alpha)(x^2 + b'x + c').$$

Then, conclude that the original polynomial equation has at most three (not necessarily distinct) real roots. Moreover, if this is the case, and α, β and γ are its three real roots, show that

$$x^3 + ax^2 + bx + c = (x - \alpha)(x - \beta)(x - \gamma).$$

This result generalizes the factorised form (2.16) of a second degree trinomial and is a particular case of the *division algorithm.*[12].

18. (a) Show that the real number $\alpha = \sqrt[3]{2 + \sqrt{5}} + \sqrt[3]{2 - \sqrt{5}}$ is a root of the equation $x^3 + 3x - 4 = 0$.
    (b) Conclude that α is a rational number.
19. * Establish the following version of **Girard's relations**[13] between roots and coefficients of a polynomial equation of third degree: if the real numbers a_0, a_1, a_2 and a_3 ($a_3 \neq 0$) are such that the equation

$$a_3x^3 + a_2x^2 + a_1x + a_0 = 0$$

has (not necessarily distinct) real roots x_1, x_2 and x_3, then

$$\begin{cases} x_1 + x_2 + x_3 = -\frac{a_2}{a_3} \\ x_1x_2 + x_1x_3 + x_2x_3 = \frac{a_1}{a_3} \\ x_1x_2x_3 = -\frac{a_0}{a_3} \end{cases}. \tag{2.21}$$

20. Assume that the equation $x^3 - 3x + 1 = 0$ has three real roots, say α, β and γ. Compute the values of $\alpha^2 + \beta^2 + \gamma^2$, $\alpha^3 + \beta^3 + \gamma^3$ and $\alpha^4 + \beta^4 + \gamma^4$.
21. Still concerning the third degree polynomial equation $x^3 + ax^2 + bx + c = 0$, with $a, b, c \in \mathbb{R}$, prove that there exists a real number d such that the variable substitution $y = x - d$ transforms the given equation into one of the form $y^3 + py + q = 0$, for certain real numbers p and q.
22. Concerning the equation $x^3 - 11x + 16 = 0$, do the following items:

    (a) Substitute $x = u + v$ and get an equivalent equation in the two real variables u and v.
    (b) Impose that $uv = \frac{11}{3}$ (i.e., make uv equals $-\frac{1}{3}$ times the coefficient of x in the original equation) and conclude that the equation in u and v of item (a) is equivalent to $u^6 + 16u^3 + \left(\frac{11}{3}\right)^3 = 0$.
    (c) Find u and v, and conclude that one of the roots of the given equation is

$$\sqrt[3]{-8 + \frac{\sqrt{1191}}{9}} + \sqrt[3]{-8 - \frac{\sqrt{1191}}{9}}.$$

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[12]For more details concerning this point, as well as for the generalization of the result of Problem 19, we refer the reader to [5].

[13]After Albert Girard, French mathematician of the XVII century.

The items above describe, by means of a specific example, the ideas behind Cardano's formula for the roots of a polynomial equation of third degree.

23. * Let x be a nonzero real number, such that $\left(x + \frac{1}{x}\right)^2 = 3$. Compute all possible values of $x^3 + \frac{1}{x^3}$.
24. If x is a nonzero real number such that $x + \frac{1}{x} = 4$, calculate $x^4 + \frac{1}{x^4}$.
25. If $x^2 - x - 1 = 0$, compute all possible values of $\left(x - \frac{1}{x}\right)^2 + \left(x^3 - \frac{1}{x^3}\right)^2$.
26. (Singapore) If $x^2 - 4x + 1 = 0$, compute all possible values of

$$\frac{x^6 + \frac{1}{x^6} - \left(x + \frac{1}{x}\right)^6 + 2}{x^3 + \frac{1}{x^3} - \left(x + \frac{1}{x}\right)^3}.$$

27. Solve the reciprocal equation $x^4 - 7x^3 + 14x^2 - 7x + 1 = 0$.
28. Given real numbers $a_0, a_1, \ldots, a_n$, with $a_n \neq 0$, the polynomial equation

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

is said to be **reciprocal** if $a_k = a_{n-k}$, for $0 \leq k \leq n$. If α is a real root of such an equation, show that $\alpha \neq 0$ and that $\frac{1}{\alpha}$ is also a root of it. (Hence, the name *reciprocal* justifies itself by the fact that reciprocal polynomial equations have real roots which are pairwise reciprocal.[14] In this case, one also says that $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a **reciprocal polynomial**.).

29. Do the following items:
    (a) * Given a real number $x \neq 0$, write $x^3 + \frac{1}{x^3}$ in terms of $y = x + \frac{1}{x}$.
    (b) Use the result of (a) to reduce the reciprocal equation $ax^6 + bx^5 + cx^4 + dx^3 + cx^2 + bx + a = 0$, of degree 6, to a polynomial equation of degree 3.
30. (Brazil—adapted) We are given nonzero integers $a \geq b$, such that the quadratic equation $x^2 + ax + b = 0$ has nonzero integer roots $c \geq d$. Then we form the quadratic equation $x^2+cx+d = 0$ and check if it also has nonzero integer roots. If this is so, we let $e \geq f$ be those roots and form the quadratic equation $x^2 + ex+f = 0$. We proceed in a likewise manner until we reach a quadratic equation with nonzero integer coefficients but with no integer roots. The purpose of this problem is to find all a and b for which this process continues indefinitely. To this end, do the following items:
    (a) Show that if the process is to continue indefinitely, then we can assume that $a > 0 > b$ and that every subsequent equation $x^2 + \alpha x + \beta = 0$ is also such that $\alpha > 0 > \beta$.
    (b) Under the choices of (a), let $x^2 + mx + n = 0$ and $x^2 + px + q = 0$ be two consecutive equations (i.e., such that $p > 0 > q$ are the roots of the first),

[14] We shall see in [5] that the same holds for the *complex* roots of this equation.

and let Δ and Δ' be their discriminants, respectively. Show that $\Delta' < \Delta$, unless $n = -1$ or $m = 1, n = -2$.

(c) Conclude that $a = 1$ and $b = -2$ is the only possible choice to begin with.

2.4 Linear Systems and Elimination

Let E and F be algebraic expressions in the real variables $x_1, \ldots, x_n$. By the **equation** $E = F$ in the real variables or **unknowns** $x_1, \ldots, x_n$, we mean the problem of finding all sequences[15] $(a_1, \ldots, a_n)$ of real numbers, such that E and F make sense and the equality $E = F$ holds when $x_1 = a_1, \ldots, x_n = a_n$. In this case, each such sequence $(a_1, \ldots, a_n)$ is said to be a **solution** of the equation $E = F$.

Now, let $E_1, \ldots, E_m, F_1, \ldots, F_m$ be algebraic expressions in the real variables $x_1, \ldots, x_n$. The **system of equations**

$$\begin{cases} E_1 = F_1 \\ E_2 = F_2 \\ \quad\vdots \\ E_m = F_m \end{cases} \tag{2.22}$$

is the problem of finding all sequences $(a_1, \ldots, a_n)$ of real numbers, such that the substitutions $x_1 = a_1, \ldots, x_n = a_n$ solve all of the equations $E_j = F_j$. As above, each such sequence $(a_1, \ldots, a_n)$ is said to be a **solution** of the system (2.22). To *solve* a system of equations as (2.22) means to find all of its solutions.

In this section and the next one, we shall learn how to solve some simple (though useful) systems of equations. We are also going to see that, under certain conditions, an equation $E = F$ (in several real variables) is equivalent to a system of equations like (2.22), and this will be a source of a number of interesting examples.

The simplest—and, for our purposes, also the most useful—systems of equations are the **linear systems** with two (resp. three) equations in two (resp. three) real unknowns, i.e., systems of equations of one of the forms

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases} \quad \text{or} \quad \begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}, \tag{2.23}$$

respectively. Here, the real numbers a_i, b_i, c_i, d_i are given and not all zero, being called the **coefficients** of the linear system.

[15] Although the reader is probably acquainted with the concept of sequence, we refer to Sect. 6.1 for a rigorous definition.

The most efficient method for solving linear systems like those above is the **elimination** algorithm,[16] also known as **gaussian elimination**,[17] which is based in the following result.

Lemma 2.24 *Let E and F be given algebraic expressions in the real variables x and y (or x, y and z). For reals a, b and c, the systems of equations*

$$\begin{cases} E = a \\ F = b \end{cases} \quad \textit{and} \quad \begin{cases} E + cF = a + cb \\ F = b \end{cases}$$

have the same solutions.

Proof We shall prove the lemma in the case in which E and F are algebraic expressions in the real variables x and y; the other case is completely analogous. Suppose that $x = x_0$ and $y = y_0$ solve the system on the left, so that, when we substitute x by x_0 and y by y_0 into E and F, both equalities $E = a$ and $F = b$ hold; we shall denote such a situation by writing $E(x_0, y_0) = a$ and $F(x_0, y_0) = b$. Then, substituting x by x_0 and y by y_0 into $E + cF$ gives us

$$(E + cF)(x_0, y_0) = E(x_0, y_0) + cF(x_0, y_0) = a + cb,$$

so that $x = x_0$ and $y = y_0$ also solve the system on the right. Conversely, if $x = x_0$ and $y = y_0$ solve the system on the right, then, since $E = (E+cF)-cF$, an argument entirely analogous to the above shows that $x = x_0$ and $y = y_0$ do solve the system on the left. Therefore, both systems have the same solutions. □

Back to the analysis of the linear systems (2.23), let us show that a number of careful applications of the previous lemma lead us to quick solutions of them (the *elimination algorithm* consists exactly of this).

We start by the system on the left, which, for simplicity, we write as

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}.$$

Since at least one of the coefficients a, b, c, d is nonzero, we can suppose, without loss of generality, that $a \neq 0$ (otherwise, it suffices to rewrite the system in one of the forms

[16] An **algorithm** is a finite sequence of precise procedures that, once followed, give an expected result, also known as the **output** of the algorithm. Although algorithms will play a very modest role in this volume, [5] brings a number of very interesting ones.

[17] After Joanne Carl Friedrich Gauss, German mathematician of the XVIII and XIX centuries. Gauss is generally considered to be the greatest mathematician of all times. In the several different areas of Mathematics and Physics in which he worked, like Number Theory, Analysis, Differential Geometry and Electromagnetism, there are always very important and deep results or methods that bear his name. We refer the reader to [26] for an interesting biography of Gauss.

$$\begin{cases} cx + dy = f \\ ax + by = e \end{cases}, \quad \begin{cases} by + ax = e \\ dy + cx = f \end{cases} \quad \text{or} \quad \begin{cases} dy + cx = f \\ by + ax = e \end{cases},$$

according to whether c, b or d is nonzero, and change a by this number in the following discussion).

Then, let $a \neq 0$ and $E = ax + by$, $F = cx + dy$. Changing the equation $F = f$ by the equation

$$F - \frac{c}{a}E = f - \frac{c}{a}e,$$

we get the system

$$\begin{cases} E \qquad\quad = e \\ F - \frac{c}{a}E = f - \frac{c}{a}e \end{cases},$$

and Lemma 2.24 immediately assures that the solutions of this new system coincide with those of the original one. Hence, in order to solve that system, it suffices to solve this last one. On the other hand, since

$$F - \frac{c}{a}E = (cx + dy) - \frac{c}{a}(ax + by) = \left(d - \frac{bc}{a}\right)y, \tag{2.24}$$

the last system reduces to

$$\begin{cases} ax + by = e \\ \qquad d'y = f' \end{cases},$$

where $d' = d - \frac{bc}{a}$ and $f' = f - \frac{c}{a}e$. Now, we shall consider three different cases:

- If $d' \neq 0$ (or, equivalently, $ad - bc \neq 0$), then the second equation above gives $y = \frac{f'}{d'}$, and the substitution of this value into the first equation gives $x = \frac{1}{a}(e - by) = \frac{1}{a}\left(e - \frac{bf'}{d'}\right)$. In this case, the system is said to be **determined**, for it has a unique solution.
- If $d' = 0$ (or, equivalently, $ad - bc = 0$) and $f' \neq 0$, the system is **impossible**, for the second equation reduces to $0y = f'$, which has no roots at all.
- If $d' = f' = 0$, then the second equation reduces to $0y = 0$, an equality which is true for all real values of y. Therefore, the system as a whole consists only of the equation $ax + by = 0$, which has infinitely many solutions (making $y = \alpha$, with $\alpha \in \mathbb{R}$, we get $x = -\frac{b\alpha}{a}$). For this reason, the system is said to be **undetermined**.

As was the case in the previous chapter, we would like to stress that, in specific examples, rather than memorizing the formulas obtained through the above discussion, one should just execute the elimination process. The previous discussion (more precisely, (2.24)) makes it clear that this process consists of subtracting an appropriate multiple of the first equation from the second, in order to eliminate the variable x from it (and that's where the name *elimination* comes from).

Example 2.25 Use Gaussian elimination to find all real values of a for which the system equations

$$\begin{cases} 2x + ay = 3 \\ ax + 2y = \frac{3}{2} \end{cases}$$

is impossible, undetermined or determined.

Solution Subtracting $-\frac{a}{2}$ times the first equation from the second, we get the equivalent system

$$\begin{cases} 2x + ay = 3 \\ \left(2 - \frac{a^2}{2}\right) y = \frac{3}{2}(1 - a) \end{cases}.$$

If $2 - \frac{a^2}{2} \neq 0$, which is the same as $a \neq \pm 2$, then the second equation above gives us $y = \frac{3(1-a)}{4-a^2}$, so that the first equation furnishes a single value for x, namely,

$$x = \frac{1}{2}(3 - ay) = \frac{3(4-a)}{2(4-a^2)}.$$

Therefore, the system is determined whenever $a \neq \pm 2$.

If $a = \pm 2$, then the second equation reduces to $0y = \frac{3}{2}(1 \mp 2)$, which represents an impossible equality. Therefore, the system is impossible. □

For a *geometric* interpretation of Gaussian elimination for linear systems of two equations in two real variables, see the problems of Sect. 6.2 of [4].

Let us now turn our attention to the linear system on the right, in (2.23). In order to analyse it, let $E_i = a_i x + b_i y + c_i z$, for $1 \leq i \leq 3$, so that it reduces to

$$\begin{cases} E_1 = d_1 \\ E_2 = d_2 \\ E_3 = d_3 \end{cases}.$$

As was done for linear systems in two variables, we can suppose that $a_1 \neq 0$ (the other cases being totally analogous). Changing equations $E_2 = d_2$ and $E_3 = d_3$ respectively by

$$E_2 - \frac{a_2}{a_1}E_1 = d_2 - \frac{a_2}{a_1}d_1 \text{ and } E_3 - \frac{a_3}{a_1}E_1 = d_3 - \frac{a_3}{a_1}d_1$$

(in order to eliminate the variable x from the second and third equations), we get the equivalent system

$$\begin{cases} E_1 &= d_1 \\ E_2 - \frac{a_2}{a_1}E_1 &= d_2 - \frac{a_2}{a_1}d_1 \\ E_3 - \frac{a_3}{a_1}E_1 &= d_3 - \frac{a_3}{a_1}d_1 \end{cases}.$$

Since

$$\begin{aligned} E_2 - \frac{a_2}{a_1}E_1 &= (a_2x + b_2y + c_2z) - \frac{a_2}{a_1}(a_1x + b_1y + c_1z) \\ &= \left(b_2 - \frac{a_2b_1}{a_1}\right)y + \left(c_2 - \frac{a_2c_1}{a_1}\right)z \\ &= b_2'y + c_2'z, \end{aligned}$$

and, analogously, $E_3 - \frac{a_3}{a_1}E_1 = b_3'y + c_3'z$, it suffices to solve the system

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ \qquad\quad b_2'y + c_2'z = d_2' \\ \qquad\quad b_3'y + c_3'z = d_3' \end{cases},$$

where $d_2' = d_2 - \frac{a_2}{a_1}d_1$ and $d_3' = d_3 - \frac{a_3}{a_1}d_1$.

In case all of the numbers b_2', c_2', b_3', c_3' are equal to 0, the last system above reduces to

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ \qquad\qquad\qquad 0 = d_2' \\ \qquad\qquad\qquad 0 = d_3' \end{cases}.$$

If $d_2' \neq 0$ or $d_3' \neq 0$, the system is *impossible*; if $d_2' = d_3' = 0$, the system is *undetermined*, for it is equivalent to the single equation $a_1x + b_1y + c_1z = 0$, which obviously has infinitely many solutions.

Suppose, then, that at least one of the numbers b_2', c_2', b_3' or c_3' is nonzero, say $b_2' \neq 0$ (as before, the other cases can be treated quite analogously). Then, applying Gaussian elimination to the system

$$\begin{cases} b_2'y + c_2'z = d_2' \\ b_3'y + c_3'z = d_3' \end{cases},$$

we obtain a system of the form

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ \qquad\quad b_2'y + c_2'z = d_2' \\ \qquad\qquad\quad c_3''z = d_3'' \end{cases},$$

which is also equivalent to the original system. The discussion, then, goes on as before, and we concentrate our attention in the third equation, $c_3''z = d_3''$. Also as before, we have to distinguish three distinct cases:

- If $c_3'' \neq 0$, then the third equation above gives a single value for z, say $z = \gamma$; since $b_2' \neq 0$, the substitution of this value into the second equation furnishes $y = \frac{1}{b_2'}(d_2' - c_2'z) = \frac{1}{b_2'}(d_2' - c_2'\gamma)$. Finally, since $a_1 \neq 0$, the substitution of the values thus obtained for y and z into the first equation give us a single value for x. Then, we conclude that the system is *determined*.
- If $c_3'' = 0$ and $d_3'' \neq 0$, the system is *impossible*, for the third equation reduces to $0z = d_3''$.
- If $c_3'' = d_3'' = 0$, then the third equation reduces to the equality $0z = 0$, and the system as a whole reduces to

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ \qquad\quad b_2'y + c_2'z = d_2' \end{cases}$$

or, which is the same,

$$\begin{cases} a_1x + b_1y = d_1 - c_1z \\ \qquad\quad b_2'y = d_2' - c_2'z \end{cases}.$$

Since a_1 and b_2' are both nonzero, for each real value $z = \gamma$ the system corresponding to

$$\begin{cases} a_1x + b_1y = d_1 - c_1\gamma \\ \qquad\quad b_2'y = d_2' - c_2'\gamma \end{cases}$$

is determined; therefore, the original system in x, y, z is *undetermined*.

We can summarize the above discussion by saying that the elimination algorithm for linear systems of the form

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

consists in performing, one by one, the following three steps:

1st. *Eliminate* the variable x from the second and third equations, by adding to these equations appropriate multiples of the first one, thus obtaining a system of the form

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ \qquad\quad b_2'y + c_2'z = d_2' \\ \qquad\quad b_3'y + c_3'z = d_3' \end{cases}.$$

2nd. *Eliminate* the variable y (in case $b_2' \neq 0$ in the last system above) from the third equation, by adding to it an adequate multiple of the second equation, thus obtaining a system of the form

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ \qquad\quad b_2'y + c_2'z = d_2' \\ \qquad\qquad\quad c_3''z = d_3'' \end{cases}.$$

3rd. Analyse equation $c_3''z = d_3''$ and, after this, the other two equations in succession, in order to decide whether the original system is *determined*, *undetermined* or *impossible*.

As before, it is much more useful to keep the general steps above in mind than to try to memorize any of the expressions obtained through the calculations in the previous discussions. Let us see one more example to illustrate this point.

Example 2.26 Find all real values of a for which the system of equations

$$\begin{cases} x + y - az = -1 \\ 2x + ay + z = 1 \\ ax + y - z = 2 \end{cases}$$

is impossible.

Solution Multiplying the first equation respectively by 2 and by a, and subtracting (also respectively) the results from the second and third equations, we get the equivalent system

$$\begin{cases} x + y - az = -1 \\ (a-2)y + (1+2a)z = 3 \\ (1-a)y - (1-a^2)z = 2 + a \end{cases}.$$

If $a = 1$, the last equation reduces to the impossible equality $0 = 3$, and the original system is impossible. If $a \neq 1$, add to the second equation $-\frac{a-2}{1-a}$ times the third one, thus obtaining the equivalent system

$$\begin{cases} x + y - az = -1 \\ (a^2 + a - 1)z = \frac{-a^2-3a+7}{1-a} \\ (1-a)y - (1-a^2)z = 2 + a \end{cases}.$$

(Observe that we have slightly changed the second step, so that the roles of the second and third equations of the last system are interchanged—it is the second equation that, now, has just one variable. Obviously, this is perfectly right, and shows that the elimination algorithm is quite a flexible one.)

Now, if $a^2 + a - 1 = 0$, i.e., if $a = \frac{-1\pm\sqrt{5}}{2}$, then the system is impossible, for $-a^2 - 3a + 7 \neq 0$ for these values of a and, hence, the second equation reduces to an impossible equality. If $a \neq \frac{-1\pm\sqrt{5}}{2}$, then the second equation gives us a single possible value for z; in turn, upon substitution of this value of z into the third equation, we find a single possible value for y; finally, putting these values for y and z into the first equation, we find a single possible value for x, so that the system is determined.

We conclude $a = 1$ and $a = \frac{-1\pm\sqrt{5}}{2}$ are the values of a for which the original system is impossible. □

We finish our discussion of linear systems by observing that Gaussian elimination algorithm can easily be put to work to the analysis of the general linear system in m equations and n unknowns

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \qquad\qquad \vdots & \ \ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{cases}; \tag{2.25}$$

here, the a_{ij} and b_i are given real numbers, such that at least one of the a_{ij} is nonzero.

Apart from an important particular case of general linear system, which will make its appearance in Sect. 18.1 of [5] (and will be analysed there, by other methods), we shall not make a systematic use of such systems along these notes; hence, we shall not develop the general analysis of the application of the elimination algorithm to them. If it is the case we have to solve such a linear system (as in the example below), some clever reasoning, perhaps with the aid of Lemma 2.24, will suffice.

Example 2.27 (OCM) Find all real solutions $x_1, x_2, \ldots, x_{100}$ of the linear system

$$\begin{cases} x_1 + x_2 + x_3 &= 0 \\ x_2 + x_3 + x_4 &= 0 \\ \qquad \vdots & \ \ \vdots \\ x_{98} + x_{99} + x_{100} &= 0 \\ x_{99} + x_{100} + x_1 &= 0 \\ x_{100} + x_1 + x_2 &= 0 \end{cases}.$$

Solution Observe that each x_j appears in exactly three of the given equations. Therefore, adding all of them and dividing both sides by 3, we get

$$x_1 + x_2 + x_3 + \cdots + x_{100} = 0.$$

In order to find $x_1 = 0$, just note that

$$\begin{aligned} 0 &= x_1 + (x_2 + x_3 + x_4) + \cdots + (x_{98} + x_{99} + x_{100}) \\ &= x_1 + 0 + 0 + \cdots + 0 \;=\; x_1. \end{aligned}$$

For $x_2 = 0$, write

$$\begin{aligned} 0 &= x_2 + (x_3 + x_4 + x_5) + \cdots + (x_{99} + x_{100} + x_1) \\ &= x_2 + 0 + 0 + \cdots + 0 \;=\; x_2. \end{aligned}$$

Now, $x_1 + x_2 + x_3 = 0$ implies $x_3 = 0$. Then, $x_2 + x_3 + x_4 = 0$ implies $x_4 = 0$, and so on, so that all of the x_i's are equal to 0. □

We refer the interested reader to Chap. 1 of [18] for quite a detailed exposition of the elimination algorithm for general linear systems.

Problems: Section 2.4

1. Assume that, in the linear system $\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$, we have a, b, c, d, $e, f \neq 0$. Prove that:
   (a) $\frac{a}{c} \neq \frac{b}{d} \Leftrightarrow$ the system is determined.
   (b) $\frac{a}{c} = \frac{b}{d} = \frac{e}{f} \Leftrightarrow$ the system is undetermined.
   (c) $\frac{a}{c} = \frac{b}{d} \neq \frac{e}{f} \Leftrightarrow$ the system is impossible.
2. Find all real values of a for which the system of equations

$$\begin{cases} x + 2y - 3z = 4 \\ 3x - y + 5z = 2 \\ 4x + y + (a^2 - 14)z = a + 2 \end{cases}$$

   is impossible.
3. Solve the system of equations

$$\begin{cases} \frac{2}{x} + \frac{3}{y} - \frac{1}{z} = 8 \\ \frac{3}{x} - \frac{5}{y} + \frac{2}{z} = -1 \\ \frac{7}{x} - \frac{6}{y} + \frac{3}{z} = 5 \end{cases}.$$

4. For $1 \leq i, j \leq 3$, let a_{ij} be given real numbers, such that $a_{i1}^2 + a_{i2}^2 + a_{i3}^2 = 1$ for $1 \leq i \leq 3$ and $a_{i1}a_{j1} + a_{i2}a_{j2} + a_{i3}a_{j3} = 0$, for $1 \leq i, j \leq 3$ with $i \neq j$. Given real numbers b_1, b_2, b_3, solve the linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases},$$

writing x_1, x_2 and x_3 in terms of the a_{ij} and b_i.

5. With respect to the linear system (2.25), do the following items:
   (a) If $b_1 = b_2 = \cdots = b_m = 0$, then the system always has at least one solution.
   (b) If the system has at least two different solutions, then it has infinitely many solutions.
   (c) If the system has only one solution when $b_1 = b_2 = \cdots = b_m = 0$, then, for general values of $b_1, b_2, \ldots, b_m$, it has at most one solution.

2.5 Miscellaneous

Let us now turn our attention to the next simplest type of system of equations, namely, **second degree systems**. Ultimately, such a system is just a rephrasing of a second degree equation in terms of a system of two equations in two unknowns. Nevertheless, the reader will be amazed on how they provide a number of interesting applications.

Proposition 2.28 *Let S and P be given real numbers. The system of equations*

$$\begin{cases} x + y = S \\ \quad xy = P \end{cases} \tag{2.26}$$

has real solutions if and only if $S^2 \geq 4P$. Moreover, in this case, the solutions are given by $x = \alpha$, $y = \beta$ or vice-versa, where α and β are the roots of the second degree equation $u^2 - Su + P = 0$.

Proof First of all, if α and β are the roots of $u^2 - Su + P = 0$, then we know from Viète's formulas (see item (b) of Proposition 2.16) that $\alpha + \beta = S$ and $\alpha\beta = P$. Therefore, the pairs $x = \alpha$, $y = \beta$ and $x = \beta$, $y = \alpha$ do satisfy the system of Eq. (2.26).

Conversely, let $x = x_0$, $y = y_0$ be any solution of that system. Then, the first equation gives $y_0 = S - x_0$, and the second equation then gives $P = x_0y_0 = x_0(S - x_0)$, or, which is the same, $x_0^2 - Sx_0 + P = 0$. Therefore, x_0 is a root of $u^2 - Su + P = 0$, from where we get $x_0 = \alpha$ or $x_0 = \beta$. Since $\alpha + \beta = S$, we have two possibilities:

- If $x_0 = \alpha$, then $y_0 = S - x_0 = S - \alpha = \beta$.
- If $x_0 = \beta$, then $y_0 = S - x_0 = S - \beta = \alpha$.

$\square$

The following examples show how we can sometimes reduce the search for solutions of more complex systems of equations to that of simpler systems, of one of the types we met so far.

Example 2.29 Find all real solutions of the system of equations

$$\begin{cases} (x^2+1)(y^2+1) = 10 \\ (x+y)(xy-1) = 3 \end{cases}.$$

Solution First of all, rewriting the left hand side of the first equation as

$$\begin{aligned} (x^2+1)(y^2+1) &= x^2y^2 + x^2 + y^2 + 1 \\ &= (xy)^2 + [(x+y)^2 - 2xy] + 1, \end{aligned}$$

and letting $x + y = a$ and $xy = b$, we get the system

$$\begin{cases} b^2 + a^2 - 2b = 9 \\ a(b-1) = 3 \end{cases}.$$

Now, writing

$$\begin{aligned} b^2 + a^2 - 2b &= a^2 + (b^2 - 2b + 1) - 1 \\ &= a^2 + (b-1)^2 - 1 \end{aligned}$$

and making the substitution $b - 1 = c$, we reach the system

$$\begin{cases} a^2 + c^2 = 10 \\ ac = 3 \end{cases}.$$

By squaring the second equation, we transform this last system into one of the form (2.26), whose unknowns are a^2 and c^2, and such that $S = 10$ and $P = 9$. Since the roots of the second degree equation $u^2 - 10u + 9 = 0$ are 1 and 9, it follows that $a^2 = 1$ or 9 and, hence, $a = \pm 1$ or ± 3. Then, we have the possibilities

$$(a, c) = (1, 3), (-1, -3), (3, 1) \text{ or } (-3, -1),$$

from where

$$(a, b) = (1, 4), (-1, -2), (3, 2) \text{ or } (-3, 0).$$

Finally, each of these pairs (a, b) give us another system of the form (2.26), with unknowns x and y. Solving the four systems thus obtained, we arrive at the solutions of the original system:

$$(x, y) = (1, -2), (-2, 1), (1, 2), (2, 1), (0, -3) \text{ or } (-3, 0).$$

□

Some seemingly complicated equations can be easily solved, provided we find a way to transform them into systems of equations. Since there is no general procedure that tells us when or how this can be done, each equation should be analysed separately. In this respect, the following example shows that a frequently useful algebraic trick is the introduction of new variables.

Example 2.30 (Israel) Find all real solutions of the equation

$$\sqrt[4]{13 + x} + \sqrt[4]{4 - x} = 3.$$

Solution Letting $a = \sqrt[4]{13 + x}$ and $b = \sqrt[4]{4 - x}$, we get $a + b = 3$ and $a^4 + b^4 = (13 + x) + (4 - x) = 17$. Hence, we have reduced the problem of solving the giving equation to that of solving the system of equations

$$\begin{cases} a + b = 3 \\ a^4 + b^4 = 17 \end{cases}.$$

Applying formula (2.1) for the square of a sum twice, we get

$$\begin{aligned} 17 &= a^4 + b^4 = (a^2 + b^2)^2 - 2a^2b^2 \\ &= [(a + b)^2 - 2ab]^2 - 2(ab)^2 \\ &= (9 - 2ab)^2 - 2(ab)^2 \\ &= 81 - 36ab + 2(ab)^2, \end{aligned}$$

so that

$$(ab)^2 - 18(ab) + 32 = 0.$$

Solving for ab the second degree equation above, we find $ab = 2$ or $ab = 16$. Therefore, there are two distinct possibilities:

$$(i) \begin{cases} a + b = 3 \\ ab = 2 \end{cases} \text{ or } (ii) \begin{cases} a + b = 3 \\ ab = 16 \end{cases}.$$

Possibility (i) gives $a = 1$ and $b = 2$, or vice-versa. If $a = 1$, then $13 + x = 1$ and, hence, $x = -12$. If $a = 2$, then $13 + x = 16$ and, hence, $x = 3$. Possibility (ii) does not give any real solutions, for, in this case, a and b would be real roots of the second degree equation $u^2 - 3u + 16 = 0$, which has none of them. □

The following lemma states a relatively easy sufficient condition to transform the search for the roots of an equation in one unknown into that of a system of equations.

Lemma 2.31 *If $E_1, E_2, \ldots, E_n$ are expressions involving one or more real variables, then*

$$E_1^2 + E_2^2 + \cdots + E_n^2 = 0 \Leftrightarrow \begin{cases} E_1 = 0 \\ E_2 = 0 \\ \quad\vdots \\ E_n = 0 \end{cases}. \tag{2.27}$$

Proof This is an easy generalization of Corollary 1.5 and of Problem 2, page 10. □

Example 2.32 Find all real roots of the equation

$$x^4y^2 + y^2 + 2 = 2y + 2x^2y.$$

Solution We can write the given equation as

$$(x^4y^2 - 2x^2y + 1) + (y^2 - 2y + 1) = 0,$$

or, which is the same,

$$(x^2y - 1)^2 + (y - 1)^2 = 0.$$

Therefore, by the previous lemma, the equation is equivalent to the system of equations

$$\begin{cases} x^2y - 1 = 0 \\ y - 1 \quad\; = 0 \end{cases},$$

whose solutions $y = 1, x = \pm 1$ can be obtained without difficulty. □

Problems: Section 2.5

1. (IMO) Find all real numbers x, y and z, such that the sum of any of them with the product of the other two is always equal to 2.
2. Let a be a nonzero real constant. Find, in terms of a, all real numbers x and y that solve the system of equations

$$\begin{cases} \frac{1}{x+y} + x = a + 1 \\ \frac{x}{x+y} \qquad = a \end{cases}.$$

3. Solve, for $x, y \in \mathbb{R}$, the system of equations

$$\begin{cases} \frac{x+y}{xy} + \frac{x-y}{xy} = 5 \\ \frac{xy}{x+y} + \frac{xy}{x-y} = \frac{5}{6} \end{cases}.$$

4. (IMO—adapted) Consider the system of equations

$$\begin{cases} ax_1^2 + bx_1 + c = x_2 \\ ax_2^2 + bx_2 + c = x_3 \\ ax_3^2 + bx_3 + c = x_1 \end{cases},$$

whose unknowns are x_1, x_2, x_3, where a, b and c are given real numbers, with $a \neq 0$. If $\Delta = (b-1)^2 - 4ac$, do the following items:

(a) If $\Delta < 0$, then there is no real solution.
(b) If $\Delta = 0$, then there is exactly one real solution.

5. (TT) Find all real solutions of the system of equations

$$\begin{cases} x^3 = 2y - 1 \\ y^3 = 2z - 1 \\ z^3 = 2x - 1 \end{cases}.$$

6. Solve, for $x, y \in \mathbb{R}$, the equation $x^2 + 2xy + 3y^2 + 2x + 6y + 3 = 0$.
7. (IMO) Find all real values of a for which the system of equations

$$\begin{cases} x^2 + y^2 = 4z \\ 3x + 4y + z = a \end{cases},$$

with unknowns x, y and z, has a single solution.
8. (NMC) Find all real numbers x, y, z greater than 1, such that

$$x + y + z + \frac{3}{x-1} + \frac{3}{y-1} + \frac{3}{z-1} = 2\left(\sqrt{x+2} + \sqrt{y+2} + \sqrt{z+2}\right).$$

9. Find all real roots of the equation $\sqrt[3]{x+5} + \sqrt[3]{11-x} = 6$.
10. Find all real roots of the equation $\sqrt{5 - \sqrt{5-x}} = x$.
11. (Canada) Find all real roots of the equation $x^2 + \frac{x^2}{(x+1)^2} = 3$.
12. (Canada) Solve the system of equations

$$\begin{cases} \frac{4x^2}{1+4x^2} = y \\ \frac{4y^2}{1+4y^2} = z \\ \frac{4z^2}{1+4z^2} = x \end{cases}$$

13. (Romania) Let a, b, c and d be given real numbers, such that

$$\begin{cases} a+b+c \le 3d \\ b+c+d \le 3a \\ c+d+a \le 3b \\ d+a+b \le 3c \end{cases}.$$

Prove that $a=b=c=d$.

14. * Let $E_1, E_2, \ldots, E_n$ and $F_1, F_2, \ldots, F_n$ be given expressions in one or more real variables, such that $E_1 \le F_1$, $E_2 \le F_2$, …, $E_n \le F_n$. Prove that the equation

$$E_1 + \cdots + E_n = F_1 + \cdots + F_n$$

is equivalent to the system of equations

$$\begin{cases} E_1 = F_1 \\ E_2 = F_2 \\ \quad\vdots \\ E_n = F_n \end{cases}.$$

15. (Romania) Find all real roots of the equation

$$\sqrt{4x^2 - x^4 - 3} + \sqrt{4y^2 - y^4} + \sqrt{4z^2 - z^4 + 5} = 6.$$

16. Solve, in the set of positive reals, the system of equations

$$\begin{cases} x + \frac{4}{y} = \frac{5y}{4} \\ y + \frac{4}{z} = \frac{5z}{4} \\ z + \frac{4}{x} = \frac{5x}{4} \end{cases}.$$

17. (Soviet Union—adapted)

(a) For $x > 0$, prove that $x + \frac{2}{x} \ge 2\sqrt{2}$, with equality if and only if $x = \sqrt{2}$.
(b) Solve the system of equations

$$\begin{cases} 2y = x + \frac{2}{x} \\ 2z = y + \frac{2}{y} \\ 2x = z + \frac{2}{z} \end{cases}.$$

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