Chapter 2
Formal Matrix Rings

In this chapter, we define formal matrix rings of order 2 and formal matrix rings of arbitrary order n. Their main properties are considered and examples of such rings are given.

We indicate the relationship between formal matrix rings, endomorphism rings of modules, and systems of orthogonal idempotents of rings.

For formal matrix rings, the Jacobson radical and the prime radical are described. We find when a formal matrix ring is Artinian, Noetherian, regular, unit-regular, and of stable rank 1.

In the last section, clean and k-good matrix rings are considered.

2.1 Construction of Formal Matrix Rings of Order 2

Let R, S be two rings, M an R-S-bimodule and N an S-R-bimodule. We denote by K the set of all matrices of the form

$$\begin{pmatrix} r & m \\ n & s \end{pmatrix}, \quad \text{where } r \in R,\ s \in S,\ m \in M,\ n \in N.$$

With respect to matrix addition, K is an Abelian group. To turn K into a ring, we have to be able to calculate the "product" $mn \in R$ and the "product" $nm \in S$. This is done as follows.

We assume that there are two bimodule homomorphisms $\varphi\colon M \otimes_S N \to R$ and $\psi\colon N \otimes_R M \to S$; to simplify notation, we also write $\varphi(m \otimes n) = mn$ and $\psi(n \otimes m) = nm$ for all $m \in M$ and $n \in N$. Now we can multiply matrices in K similarly to the case of ordinary matrix rings:

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$$\begin{pmatrix} r & m \\ n & s \end{pmatrix}\begin{pmatrix} r_1 & m_1 \\ n_1 & s_1 \end{pmatrix} = \begin{pmatrix} rr_1 + mn_1 & rm_1 + ms_1 \\ nr_1 + sn_1 & nm_1 + ss_1 \end{pmatrix},$$
$$r, r_1 \in R, \quad s, s_1 \in S, \quad m, m_1 \in M, \quad n, n_1 \in N.$$

In the above definition, rm_1, ms_1, nr_1, sn_1 denote the corresponding module products. We also assume that, for all $m,\ m' \in M$ and $n,\ n' \in N$, the two additional associativity relations

$$(mn)m' = m(nm'), \qquad (nm)n' = n(mn') \tag{$*$}$$

hold. The set K is a ring with the mentioned operations of addition and multiplication. When checking the ring axioms, we have to consider the main properties of the tensor product and the property that φ and ψ are bimodule homomorphisms. We also have the converse: if K is a ring, then the above associativity relations $(*)$ hold. The ring K is called the *formal matrix ring* (of order 2); it is denoted by $\begin{pmatrix} R & M \\ N & S \end{pmatrix}$. The term *ring of generalized matrices* is also used. Sometimes, we simply refers to a "matrix ring".

If $N = 0$ or $M = 0$, then K is a *ring of formal upper or lower triangular matrices*

$$\begin{pmatrix} R & M \\ 0 & S \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} R & 0 \\ N & S \end{pmatrix}, \quad \text{respectively.}$$

To define a ring of formal upper or lower triangular matrices, it is not necessary to use the homomorphisms φ and ψ.

The images I and J of the homomorphisms φ and ψ are ideals of the rings R and S, respectively. They are called the *trace ideals* of the ring K. We say that K is a *ring with zero trace ideals* or a *trivial* ring provided $\varphi = 0 = \psi$, i.e. $I = 0 = J$. Of course, a ring of formal triangular matrices is a ring with zero trace ideals.

We denote by MN (respectively, NM) the set of all finite sums of elements of the form mn (respectively, nm). The relations

$$I = MN, \quad J = NM, \quad IM = MJ, \quad NI = JN$$

hold. How to formulate the problem of studying formal matrix rings? By studying the ring $\begin{pmatrix} R & M \\ N & S \end{pmatrix}$, we mean figuring out how the properties of this ring depend on the properties of the rings R and S, the properties of the bimodules M and N, and the properties of the homomorphisms φ and ψ.

Sometimes, it is convenient to identify certain matrices with corresponding elements. For example, we can identify the matrix $\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}$ with the element $r \in R$, and so on. We make similar agreements for matrix sets. For example, the set of matrices

$\begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix}$ can be written in the form (X, Y) or simply X if $Y = 0$. We use similar rules for matrices with zero upper row.

Let T be a ring. In T, we preserve the previous addition and define a new multiplication $\circ$ by the relation $x \circ y = yx$, $x, y \in T$. As a result, we obtain a new ring T° which is called the ring *opposite* to T. One can directly verify that the opposite ring to the ring $\begin{pmatrix} R & M \\ N & S \end{pmatrix}$ is isomorphic to the formal matrix ring $\begin{pmatrix} R^\circ & N \\ M & S^\circ \end{pmatrix}$ where N is considered as an R°-S°-bimodule and M is considered as an S°-R°-bimodule. We also remark that

$$\begin{pmatrix} R & M \\ N & S \end{pmatrix} \cong \begin{pmatrix} S & N \\ M & R \end{pmatrix}.$$

If V is a right T-module, then the relation $tv = vt$, $t \in T$, $v \in V$, defines a structure of a left T°-module on V, and conversely.

If $M = 0 = N$, then the ring K can be identified with the direct product $R \times S$. Usually, we assume that the product $R \times S$ is a matrix ring.

Let K be a formal matrix ring $\begin{pmatrix} R & M \\ N & S \end{pmatrix}$. Conforming to our agreement about representations of matrices, we can represent the relation

$$K = \begin{pmatrix} eKe & eK(1-e) \\ (1-e)Ke & (1-e)K(1-e) \end{pmatrix}, \tag{2.1}$$

where $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Using this approach, the action of the corresponding homomorphisms φ and ψ coincides with the multiplication in the ring K.

In a certain sense, the converse holds. Namely, let an abstract ring T contain a non-zero idempotent e which is not equal to the identity element. We can form the formal matrix ring

$$K = \begin{pmatrix} eTe & eT(1-e) \\ (1-e)Te & (1-e)T(1-e) \end{pmatrix}.$$

The rings T and K are isomorphic. The correspondence

$$t \to \begin{pmatrix} ete & et(1-e) \\ (1-e)te & (1-e)t(1-e) \end{pmatrix}, \quad t \in T,$$

defines the corresponding isomorphism.

Let K be some formal matrix ring represented in the form (2.1). It is not difficult to specify the structure of ideals and factor rings of the ring K; see also the end of Sect. 2.4 and Propositions 4.2.3 and 4.2.4.

If L is an ideal of the ring K, then one can directly verify that L coincides with the set of matrices

$$\begin{pmatrix} eLe & eL(1-e) \\ (1-e)Le & (1-e)L(1-e) \end{pmatrix}$$

where eLe and $(1-e)L(1-e)$ are ideals of the rings R and S, respectively, and $eL(1-e)$ and $(1-e)Le$ are subbimodules of M and N, respectively. The subgroups appearing in one of four positions in L coincide with the sets of the corresponding components of elements in L.

We form the matrix group

$$\overline{K} = \begin{pmatrix} eKe/eLe & eK(1-e)/eL(1-e) \\ (1-e)Ke/(1-e)Le & (1-e)K(1-e)/(1-e)L(1-e) \end{pmatrix}.$$

In fact, we have a formal matrix ring $\overline{K}$; we consider this matrix ring in the above general sense. The multiplication of matrices in $\overline{K}$ is induced by the multiplication in K. One can directly verify that the mapping

$$K/L \to \overline{K}, \qquad \begin{pmatrix} r & m \\ n & s \end{pmatrix} + L \to \begin{pmatrix} \overline{r} & \overline{m} \\ \overline{n} & \overline{s} \end{pmatrix},$$

is a ring isomorphism where the dash denotes the corresponding residue class.

A concrete formal matrix ring is defined with the use of two bimodule homomorphisms φ and ψ. In general, the choice of another pair of homomorphisms leads to another ring. We can formulate the problem of classifying formal matrix rings depending on the corresponding pairs of bimodule homomorphisms. The following **isomorphism problem** is related to the above problem:

Let K and K_1 be two formal matrix rings with bimodule homomorphisms φ, ψ and φ_1, ψ_1, respectively. How should the homomorphisms φ, ψ and φ_1, ψ_1 be linked for an isomorphism $K \cong K_1$ to exist?

How many formal matrix rings are there? It follows from the above that the class of formal matrix rings coincides with the class of rings possessing non-trivial idempotents (if we assume that direct products of rings are matrix rings).

The class of formal matrix rings also coincides with the class of endomorphism rings of modules which are decomposable into a direct sum. Indeed, let $G = A \oplus B$ be a right module over some ring T. The endomorphism ring of G is isomorphic to the matrix ring $\begin{pmatrix} \mathrm{End}_T A & \mathrm{Hom}_T(B, A) \\ \mathrm{Hom}_T(A, B) & \mathrm{End}_T B \end{pmatrix}$ with ordinary operations of addition and multiplication of matrices (the product of homomorphisms is taken to be their composite). Conversely, for the ring $K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$, we have the decomposition $K_K = (R, M) \oplus (N, S)$ into a direct sum of right ideals such that the ring $\mathrm{End}_K(K)$ is isomorphic to the ring $\begin{pmatrix} R & M \\ N & S \end{pmatrix}$.

There are many types of rings which are not necessarily of a matrix origin but are close to formal matrix rings. In particular, various rings of triangular matrices appear. In [35], the authors study so-called *trivial extensions* of rings, which are defined as follows. If R is a ring and M is an R-R-bimodule, then we denote by T the direct sum of Abelian groups R and M, $T = \{(r, m) \mid r \in R,\ m \in M\}$. The group T is turned into a ring if the multiplication is defined by the relation $(r, m)(r_1, m_1) = (rr_1, rm_1 + mr_1)$. This ring is a trivial extension of the ring R with the use of the bimodule M.

Now we consider the ring of triangular matrices $\begin{pmatrix} R & M \\ 0 & R \end{pmatrix}$ and its subring $\Gamma = \left\{ \begin{pmatrix} r & m \\ 0 & r \end{pmatrix} \,\middle|\, r \in R,\ m \in M \right\}$. The rings T and Γ are isomorphic under the correspondence $(r, m) \to \begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$. Thus, trivial extensions consist of triangular matrices.

Every ring of formal triangular matrices $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is a trivial extension. Indeed, M can be considered as an $(R \times S)$-$(R \times S)$-bimodule if we assume that $(r, s)m = rm$ and $m(r, s) = ms$. We take the trivial extension $T = \{((r, s), m) \mid r \in R,\ s \in S,\ m \in M\}$ of the ring $R \times S$. The correspondence $\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \to ((r, s), m)$ defines an isomorphism from the ring K onto the ring T. However, there exists a class of rings of triangular matrices containing trivial extensions. Let $f\colon R \to S$ be a ring homomorphism. In the ring $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$, all matrices of the form $\begin{pmatrix} r & m \\ 0 & f(r) \end{pmatrix}$ form a subring.

Here is a more general construction of a ring extension; see [94]. Again, let M be an R-R-bimodule and $\Phi\colon M \otimes_R M \to R$ an R-R-bimodule homomorphism. We define a multiplication in $R \oplus M$ by the relation

$$(r, m)(r_1, m_1) = (rr_1 + \Phi(m \otimes m_1), rm_1 + mr_1).$$

This multiplication is associative if and only if

$$\Phi(m \otimes m_1)m_2 = m\,\Phi(m_1 \otimes m_2) \tag{2.2}$$

for all $m, m_1, m_2 \in M$. In such a case, $R \oplus M$ is a ring. This ring is denoted by $R \times_\Phi M$ and is called a *semi-trivial extension* of the ring R with the use of M and Φ.

Each formal matrix ring is a semi-trivial extension. Let $\begin{pmatrix} R & M \\ N & S \end{pmatrix}$ be a formal matrix ring with the bimodule homomorphisms φ and ψ. Set $T = R \times S$, $V = M \times N$ and consider V as a natural T-T-bimodule. We denote by Φ the T-T-bimodule homomorphism

$$(\varphi, \psi)\colon V \otimes_T V \to T.$$

It satisfies the corresponding relation (2.2). Consequently, we have a semi-trivial extension $T \times_\Phi V$. The rings $T \times_\Phi V$ and $\begin{pmatrix} R & M \\ N & S \end{pmatrix}$ are isomorphic under the correspondence

$$(r, s) + (m, n) \to \begin{pmatrix} r & m \\ n & s \end{pmatrix}.$$

On the other hand, each semi-trivial extension is embedded in a suitable formal matrix ring. Indeed, let $T \times_\Phi V$ be a semi-trivial extension. The relation corresponding to (2.2) is equivalent to the property that $\begin{pmatrix} T & V \\ V & T \end{pmatrix}$ is a formal matrix ring. The bimodule homomorphisms of this ring coincide with Φ. The mapping

$$T \times_\Phi V \to \begin{pmatrix} T & V \\ V & T \end{pmatrix}, \quad (t, v) \to \begin{pmatrix} t & v \\ v & t \end{pmatrix},$$

is a ring embedding of rings. Thus, we can identify $T \times_\Phi V$ with the matrix ring of the form $\begin{pmatrix} t & v \\ v & t \end{pmatrix}$.

Let T be a commutative ring and R, S two T-algebras. Then the ring $K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ is a T-algebra. In this case, we say that K is a *formal matrix algebra*.

2.2 Examples of Formal Matrix Rings of Order 2

Here are some examples of formal matrix rings.

(1). Let S be a ring, M a right S-module, $R = \mathrm{End}_S M$, and $M^* = \mathrm{Hom}_S(M, S)$. Then M is an R-S-bimodule and M^* is an S-R-bimodule, where

$$(s\alpha)m = s\alpha(m), \quad (\alpha r)m = \alpha(r(m)),$$
$$\alpha \in M^*, \quad s \in S, \quad r \in R, \quad m \in M.$$

There exist an R-R-bimodule homomorphism $\varphi\colon M \otimes_S M^* \to R$ and an S-S-bimodule homomorphism $\psi\colon M^* \otimes_R M \to S$, which are defined by the formulas

$$\left(\varphi\left(\sum m_i \otimes \alpha_i\right)\right)(m) = \sum m_i\alpha_i(m), \quad \psi\left(\sum \alpha_i \otimes m_i\right) = \sum \alpha_i(m_i),$$

where $m_i, m \in M$ and $\alpha_i \in M^*$. We obtain a matrix ring $\begin{pmatrix} R & M \\ M^* & S \end{pmatrix}$, since the two required associativity relations $(*)$ from Sect. 2.1 hold for φ and ψ.

(2). Let X and Y be a left and a right ideal of the ring R, respectively. Further, let S be any subring in R with $YX \subseteq S \subseteq X \cap Y$. Then $\begin{pmatrix} R & X \\ Y & S \end{pmatrix}$ is a formal matrix ring in which the actions of the mappings φ and ψ are the restrictions of multiplication in R. As a special case, we obtain the ring $\begin{pmatrix} R & Re \\ eR & eRe \end{pmatrix}$, where e is an idempotent.

(3). Let R be a ring, Y a right ideal of R, and S any subring in R containing Y as an ideal. Then S is called a *subidealizer* of the ideal Y in R, and $\begin{pmatrix} R & R \\ Y & S \end{pmatrix}$ is a formal matrix ring.

(4) Endomorphism rings of Abelian groups. If G is an Abelian group and $G = A \oplus B$, then the endomorphism ring $\operatorname{End} G$ of G is a formal matrix ring; see Sect. 2.1. Abelian groups provide many interesting useful examples of formal matrix rings. First of all, we have rings of triangular matrices. For example, the endomorphism rings of the groups $\mathbb{Q} \oplus \mathbb{Z}$, $\mathbb{Z}(p^n) \oplus \mathbb{Z}$ and $\mathbb{Z}(p^\infty) \oplus \mathbb{Q}$ are isomorphic to the rings $\begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{pmatrix}$, $\begin{pmatrix} \mathbb{Z}_{p^n} & \mathbb{Z}_{p^n} \\ 0 & \mathbb{Z} \end{pmatrix}$, and $\begin{pmatrix} \widehat{\mathbb{Z}}_p & A_p \\ 0 & \mathbb{Q} \end{pmatrix}$ respectively, where $\widehat{\mathbb{Z}}_p$ is the ring of p-adic integers and A_p is the field of p-adic numbers.

The endomorphism ring of the p-group $\mathbb{Z}(p^n) \oplus \mathbb{Z}(p^m)$, $n < m$, is an informative illustration of the notion of a formal matrix ring. We can identify this ring with the formal matrix ring $\begin{pmatrix} \mathbb{Z}_{p^n} & \mathbb{Z}_{p^n} \\ \mathbb{Z}_{p^n} & \mathbb{Z}_{p^m} \end{pmatrix}$. We denote this ring by K or $\begin{pmatrix} R & M \\ N & S \end{pmatrix}$.

How can we multiply matrices in the ring K? First of all, we remark that $\mathbb{Z}_{p^n} = \mathbb{Z}_{p^m}/(p^{m-n} \cdot 1)$. Therefore, the rings R and S act on M and N by ordinary multiplication, in a uniquely possible way. Then we pass to the homomorphisms $\varphi\colon M \otimes_S N \to R$ and $\psi\colon N \otimes_R M \to S$. If we consider K as the initial endomorphism ring, then the action of φ and ψ is reduced to the composition of the corresponding homomorphisms. Taking this into consideration, we obtain the following. If $\overline{a} \in M$ and $\overline{b} \in N$, where the dash denotes the residue class, then

$$\varphi(\overline{a} \otimes \overline{b}) = \overline{a} \circ \overline{b} = p^{m-n}\overline{a}\overline{b}.$$

Then we have

$$\psi(\overline{b} \otimes \overline{a}) = \overline{b} \circ \overline{a} = p^{m-n}\overline{b}\overline{a},$$

where the last symbols $\overline{b}$ and $\overline{a}$ denote the residue classes in $\mathbb{Z}_{p^m}$ with representatives b and a, respectively.

The trace ideals I and J of the ring K are equal to the ideal $(p^{m-n} \cdot 1)$ of the ring $\mathbb{Z}_{p^n}$ and the ideal $(p^{m-n} \cdot 1)$ of the ring $\mathbb{Z}_{p^m}$, respectively. Therefore, $I \subseteq J(R)$ and $J \subseteq J(S)$. There exists a surjective homomorphism $e\colon S \to R$, $e(\overline{y}) = \overline{y}$, $\overline{y} \in S$. In addition, $\operatorname{Ker}(e) \subseteq J(S)$ and the relation $e(\overline{b} \circ \overline{a}) = \overline{a} \circ \overline{b}$ hold.

In the paper [28], the case $n = 1$ and $m = 2$ is considered in detail. In the ring K, all invertible matrices are described. This is used to construct cryptosystems.

(5) Full matrix rings. Let R be a ring. The full matrix ring $M(n, R)$ can be represented in the form of a formal matrix ring of order 2

$$M(n, R) = \begin{pmatrix} R & M(1 \times (n-1), R) \\ M((n-1) \times 1, R) & M(n-1, R) \end{pmatrix}.$$

This ring provides an example of a ring of block matrices. A more general situation will appear in the proof of Proposition 2.3.3 and in the first paragraph after the proof of Proposition 2.3.3.

(6) see [29]. Let R be a ring, G a finite subgroup of the automorphism group of the ring R and R^G the ring of invariants of the ring R, i.e. R^G is the subring $\{x \in R \mid x^g = x \text{ for all } g \in G\}$. We consider the skew group ring $R * G$ consisting of all formal sums of the form $\sum_{g \in G} r_g g$, $r_g \in R$. The sums are added component-wise. For multiplication, we use the distributivity law and the relation $rg \cdot sh = rs^{g^{-1}} gh$ for all $r, s \in R$ and $g, h \in G$. It is clear that R is a left R^G-module and a right R^G-module. We consider R as a left and right $R * G$-module as follows:

$$x \cdot r = \sum_{g \in G} r_g r^{g^{-1}}, \quad r \cdot x = \sum_{g \in G} (r r_g)^g$$

for any elements $x = \sum_{g \in G} r_g g \in R * G$ and $r \in R$. The mappings

$$\varphi \colon R \otimes_{R*G} R \to R^G \quad \text{and} \quad \psi \colon R \otimes_{R^G} R \to R * G$$

are defined with the use of the relations

$$\varphi(x \otimes y) = \sum_{g \in G} (xy)^g \quad \text{and} \quad \psi(y \otimes x) = \sum_{g \in G} y x^{g^{-1}} g$$

respectively.

The two required associativity conditions $(*)$ from Sect. 2.1 hold, and we eventually obtain the ring $\begin{pmatrix} R^G & R \\ R & R * G \end{pmatrix}$.

2.3 Formal Matrix Rings of Order $n \geq 2$

We make several remarks about formal matrix rings of arbitrary order n. The case $n = 2$, considered in Sect. 2.1, is sufficient to understand how to define such rings.

We fix a positive integer $n \geq 2$. Let $R_1, \dots, R_n$ be rings, M_{ij} R_i-R_j-bimodules and $M_{ii} = R_i$, $i, j = 1, \dots, n$. We assume that for any $i, j, k = 1, \dots, n$ such that

$i \neq j$, $j \neq k$, there is an R_i-R_k-bimodule homomorphism

$$\varphi_{ijk}\colon M_{ij} \otimes_{R_j} M_{jk} \to M_{ik}.$$

For subscripts $i = j$ and $j = k$, we assume that φ_{iik} and φ_{ikk} are canonical isomorphisms

$$R_i \otimes_{R_i} M_{ik} \to M_{ik}, \quad M_{ij} \otimes_{R_j} R_j \to M_{ij}.$$

Instead of $\varphi_{ijk}(a \otimes b)$, we write $a \circ b$ or simply ab. Using this notation, we also assume that $(ab)c = a(bc)$ for all elements $a \in M_{ij}$, $b \in M_{jk}$, $c \in M_{k\ell}$ and subscripts i, j, k, ℓ.

We denote by K the set of all matrices (a_{ij}) of order n with values in the bimodules M_{ij}. The set K is a ring with standard matrix operations of addition and multiplication. We represent K in the form

$$\begin{pmatrix} R_1 & M_{12} & \dots & M_{1n} \\ M_{21} & R_2 & \dots & M_{2n} \\ \dots & \dots & \dots & \dots \\ M_{n1} & M_{n2} & \dots & R_n \end{pmatrix}. \tag{2.3}$$

We say that K is a *formal matrix ring of order* n. If $M_{ij} = 0$ for all i, j with $i < j$ (resp., $j < i$), then we say that K is a *ring of formal lower* (resp., *upper*) *triangular matrices*.

For every $k = 1, \dots, n$, set

$$I_k = \sum_{i \neq k} \operatorname{Im}(\varphi_{kik}), \quad \text{where} \quad \varphi_{kik}\colon M_{ki} \otimes_{R_i} M_{ik} \to R_k.$$

Equivalently, $I_k = \sum_{i \neq k} M_{ki} M_{ik}$, where $M_{ki} M_{ik}$ is the set of all finite sums of elements of the form ab, $a \in M_{ki}$, $b \in M_{ik}$. Then I_k is an ideal of the ring R_k. The ideals $I_1, \dots, I_n$ are called the *trace ideals* of the ring K.

For a better understanding of the structure of formal matrix rings, we determine their interrelations with idempotents and endomorphism rings.

Proposition 2.3.1 *A ring K is a formal matrix ring of order $n \geq 2$ if and only if K contains a complete orthogonal system consisting of n non-zero idempotents.*

Proof If K is a formal matrix ring of order n, then the matrix units $E_{11}, \dots, E_{nn}$ (see Sect. 4.1) form the required system of idempotents of the ring K.

Conversely, if $\{e_1, \dots, e_n\}$ is a complete orthogonal system of non-zero idempotents of some ring T, then T is isomorphic to the ring of formal matrices

$$\begin{pmatrix} e_1Te_1 & e_1Te_2 & \dots & e_1Te_n \\ e_2Te_1 & e_2Te_2 & \dots & e_2Te_n \\ \dots & \dots & \dots & \dots \\ e_nTe_1 & e_nTe_2 & \dots & e_nTe_n \end{pmatrix};$$

see Sect. 2.1 in connection to such a ring. □

The case of the direct sums of two modules, examined in Sect. 2.1, can be extended to the direct sums of any finite number of summands.

Proposition 2.3.2 *The class of formal matrix rings of order n coincides with the class of endomorphism rings of modules which are decomposable into a direct sum of n non-zero summands.*

Formal matrix rings of any order n appear in concrete problems. Formal matrix rings of order 2 are usually studied in the general theory; this case is considered mainly because of technical convenience, the case $n > 2$ can be reduced in some sense to the case of matrices of order 2.

Proposition 2.3.3 *A formal matrix ring of order $n > 2$ is isomorphic to some formal matrix ring of order k for every $k = 2, \dots, n-1$.*

Proof The assertion becomes quite clear if we consider the representation of matrix rings with the use of idempotents or endomorphism rings; see Proposition 2.3.1 and 2.3.2. It is sufficient to "enlarge" in some way idempotents or direct summands. Of course, there is also a direct proof. For example, take $k = 2$. We introduce the following notation for sets of matrices. Set $R = R_1$, $M = (M_{12}, \dots, M_{1n})$,

$$N = \begin{pmatrix} M_{21} \\ \vdots \\ M_{n1} \end{pmatrix}, \quad S = \begin{pmatrix} R_2 & M_{23} & \dots & M_{2n} \\ \dots & \dots & \dots & \dots \\ M_{n2} & M_{n3} & \dots & R_n \end{pmatrix}.$$

Here S is a formal matrix ring of order $n-1$, M is an R-S-bimodule, N is an S-R-bimodule, and the module multiplications are defined as products of rows and columns on matrices. The homomorphisms φ_{ijk} defining multiplication in K induce the bimodule homomorphisms $\varphi\colon M \otimes_S N \to R$ and $\psi\colon N \otimes_R M \to S$. In addition, the two required associativity relations $(*)$ from Sect. 2.1 hold. As a result, we have the formal matrix ring $\begin{pmatrix} R & M \\ N & S \end{pmatrix}$ and the isomorphism $K \cong \begin{pmatrix} R & M \\ N & S \end{pmatrix}$. The isomorphism is obtained by decomposition of each matrix into four blocks:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \to \begin{pmatrix} (a_{11}) & (a_{12} \dots a_{1n}) \\ \begin{pmatrix} a_{21} \\ \dots \\ a_{n1} \end{pmatrix} & \begin{pmatrix} a_{22} \dots a_{2n} \\ \dots\dots\dots \\ a_{n2} \dots a_{nn} \end{pmatrix} \end{pmatrix}.$$

□

In the proof of the proposition, we actually obtain that formal matrices can be decomposed into blocks, similar to ordinary matrices, i.e., we can represent formal matrices in the form of block matrices. Actions over block matrices are similar to the actions in the case where we have elements instead of blocks. Multiplication of block matrices of the same order is always possible if the factors have the same block decompositions.

Thus, any formal matrix ring can be considered as a ring of (formal) block matrices. Rings of block upper (lower) triangular matrices naturally appear. Rings of (formal) block matrices are used in the theory of finite-dimensional algebras. In particular, rings of block triangular matrices over fields naturally appear in this theory; see [11].

There are a number of constructions which allow us to construct formal matrix rings of larger order starting from formal matrix rings of smaller order. We begin by considering a first method.

Assume that we have a formal matrix ring of the form (2.3). We fix some sequence of positive integers $s_1, \dots, s_n$ and denote by $\overline{M}_{ij}$ the set of $s_i \times s_j$ matrices with elements in M_{ij} (we recall that $M_{ii} = R$). Further, let $\overline{K}$ be the set of all block matrices $(\overline{M}_{ij})$, $i, j = 1, \dots, n$. On these matrices, we define operations of addition and multiplication as usual, i.e. the addition is component-wise and $A_{ij} \cdot A_{jk} \in \overline{M}_{ik}$ for any matrices $A_{ij} \in \overline{M}_{ij}$, $A_{jk} \in \overline{M}_{jk}$. Then $\overline{K}$ is turned into a ring of formal block matrices; in addition, $\overline{K}$ is a formal matrix ring of order $s_1 + \ldots + s_n$.

Subsequently, we will use a second simple method of construction of formal matrix rings of larger order. Assume that we have a formal matrix ring of order 2, $K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$.

We show that there exists a formal matrix ring

$$K_4 = \begin{pmatrix} K & \begin{pmatrix} M \\ S \end{pmatrix} \\ \begin{pmatrix} N & S \end{pmatrix} & S \end{pmatrix}.$$

First of all, $\begin{pmatrix} M \\ S \end{pmatrix}$ is a natural K-S-bimodule and $(N \quad S)$ is an S-K-bimodule. The mapping

$$\varphi\colon \begin{pmatrix} M \\ S \end{pmatrix} \otimes_S (N \quad S) \to K, \qquad \begin{pmatrix} m \\ x \end{pmatrix} \otimes (n, y) \to \begin{pmatrix} mn & my \\ xn & xy \end{pmatrix},$$

is a K-K-bimodule homomorphism and the mapping

$$\psi\colon (N \quad S) \otimes_K \begin{pmatrix} M \\ S \end{pmatrix} \to S, \qquad (n, y) \otimes \begin{pmatrix} m \\ x \end{pmatrix} \to nm + yx,$$

is an S-S-bimodule homomorphism. For φ and ψ from Sect. 2.1, the two familiar associativity identities $(*)$ from Sect. 2.1 hold. Consequently, the specified ring K_4 exists. The same method is used to define the ring

$$K_2 = \begin{pmatrix} K & \begin{pmatrix} R \\ N \end{pmatrix} \\ (R\ M) & R \end{pmatrix}.$$

Now we remark that, in addition to the ring K, there always exists a ring $L = \begin{pmatrix} S & N \\ M & R \end{pmatrix}$. These rings are isomorphic under the correspondence

$$\begin{pmatrix} r & m \\ n & s \end{pmatrix} \to \begin{pmatrix} s & n \\ m & r \end{pmatrix}.$$

Therefore, in addition to the rings K_2 and K_4, there exist two rings K_1 and K_3 which are isomorphic to K_2 and K_4, respectively. However, we can also construct them directly.

We temporarily turn our attention to the ring of upper triangular matrices of order 3 (they will appear again in Sect. 3.1). Such a ring Γ can be represented in the form $\begin{pmatrix} R & M & L \\ 0 & S & N \\ 0 & 0 & T \end{pmatrix}$, where R, S, T are three rings, M is an R-S-bimodule, L is an R-T-bimodule, and N is an S-T-bimodule. Among the bimodule homomorphisms, only $M \otimes_S N \to L$ and natural isomorphisms of the form $R \otimes_R M \to M$ are non-zero. There are two ways to turn Γ into a ring of triangular matrices of order 2. By the first method,

$$\begin{pmatrix} r & m & \ell \\ 0 & s & n \\ 0 & 0 & t \end{pmatrix} \hookrightarrow \begin{pmatrix} \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} & \begin{pmatrix} \ell \\ n \end{pmatrix} \\ (0 \quad 0) & (t) \end{pmatrix}.$$

In this case $\begin{pmatrix} L \\ N \end{pmatrix}$ is a $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$-$T$-bimodule. By the second method, (M, L) is an R-$\begin{pmatrix} S & N \\ 0 & T \end{pmatrix}$-bimodule.

2.4 Some Ideals of Formal Matrix Rings

For a formal matrix ring of order n, we find the Jacobson radical and the prime radical. First, we consider the case $n = 2$.

Assume that we have a ring $K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$. We define four subbimodules of the bimodules M and N. Set

$$J_\ell(M) = \{m \in M \mid Nm \subseteq J(S)\}, \qquad J_r(M) = \{m \in M \mid mN \subseteq J(R)\},$$
$$J_\ell(N) = \{n \in N \mid Mn \subseteq J(R)\}, \qquad J_r(N) = \{n \in N \mid nM \subseteq J(S)\}.$$

Now we form the following sets of matrices:

$$J_\ell(K) = \begin{pmatrix} J(R) & J_\ell(M) \\ J_\ell(N) & J(S) \end{pmatrix}, \qquad J_r(K) = \begin{pmatrix} J(R) & J_r(M) \\ J_r(N) & J(S) \end{pmatrix}.$$

One can directly verify that $J_\ell(K)$ and $J_r(K)$ are a left ideal and a right ideal of the ring K, respectively.

Theorem 2.4.1 ([100]) *We have the relations*

$$J_\ell(K) = J(K) = J_r(K).$$

Proof We have $J(K) = \begin{pmatrix} X & B \\ C & Y \end{pmatrix}$, where X, Y are ideals of the rings R and S, respectively, and B and C are subbimodules in M and N, respectively; see Sect. 2.1. The relations

$$X = eJ(K)e = J(eKe) = J(R)$$

hold, where $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Similarly, we have $Y = J(S)$. Then we have

$$B \subseteq J_\ell(M) \cap J_r(M), \quad C \subseteq J_\ell(N) \cap J_r(N).$$

We have proved that $J(K) \subseteq J_\ell(K) \cap J_r(K)$.

Now we take an arbitrary matrix $\begin{pmatrix} r & m \\ n & s \end{pmatrix}$ in $J_r(K)$ and the identity matrix E. The matrices $E - \begin{pmatrix} r & m \\ 0 & 0 \end{pmatrix}$, $E - \begin{pmatrix} 0 & 0 \\ n & s \end{pmatrix}$ are right invertible in K. Their right inverse matrices are $\begin{pmatrix} x & xm \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ yn & y \end{pmatrix}$ respectively, where x and y are right inverse to $1 - r$ and $1 - s$ respectively. Consequently, the matrices $\begin{pmatrix} r & m \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ n & s \end{pmatrix}$ and $\begin{pmatrix} r & m \\ n & s \end{pmatrix}$ are contained in $J(K)$. Therefore, $J_r(K) \subseteq J(K)$. Similarly, we have $J_\ell(K) \subseteq J(K)$. $\square$

We have that $J_\ell(M) = J_r(M)$ and $J_\ell(N) = J_r(N)$. We denote these ideals by $J(M)$ and $J(N)$, respectively. Thus, we have the relation $J(K) = \begin{pmatrix} J(R) & J(M) \\ J(N) & J(S) \end{pmatrix}$.

For an arbitrary ring T, the intersection of all prime ideals of T is called the *prime radical*; it is denoted by $P(T)$.

It is well-known that the prime radical of the ring T coincides with the set of all strongly nilpotent elements of T. We recall that an element $a \in T$ is said to be *strongly nilpotent* if each sequence $a_0, a_1, a_2, \ldots$, such that

$$a_0 = a, \quad a_{n+1} \in a_n T a_n, \quad \forall n \in \mathbb{N},$$

is constantly zero from some term onwards.

We define ideals $P_\ell(M)$, $P_r(M)$, $P_\ell(N)$ and $P_r(N)$ which are similar to the ideals $J_\ell(M)$, $J_r(M)$, $J_\ell(N)$ and $J_r(N)$, respectively. We restrict ourselves to the "left-side" case. Set

$$P_\ell(M) = \{m \in M \mid Nm \subseteq P(S)\}, \quad P_r(M) = \{m \in M \mid mN \subseteq P(R)\}.$$

Then, let $P_\ell(K) = \begin{pmatrix} P(R) & P_\ell(M) \\ P_\ell(N) & P(S) \end{pmatrix}$.

Theorem 2.4.2 ([100]) $P_\ell(K) = P(K) = P_r(K)$.

The proof of Theorem 2.4.2 is a verification of the above relations using the definition of strongly nilpotent elements. □

We denote by $P(M)$ the equal ideals $P_\ell(M)$ and $P_r(M)$; we also denote the equal ideals $P_\ell(N)$ and $P_r(N)$ by $P(N)$.

Now we pass to a formal matrix ring K of any order n of the form (2.3) from Sect. 2.3. For any two subscripts i and j, we define the subbimodules

$$J_\ell(M_{ij}) = \{x \in M_{ij} \mid M_{ji}x \subseteq J(R_j)\}, \qquad J_r(M_{ij}) = \{x \in M_{ij} \mid xM_{ji} \subseteq J(R_i)\}.$$

For $i = j$, we obtain $J_\ell(R_i) = J_r(R_i) = J(R_i)$.

Theorem 2.4.3 *We have the relation*

$$J(K) = \begin{pmatrix} J(R_1) & J_\ell(M_{12}) & \ldots & J_\ell(M_{1n}) \\ J_\ell(M_{21}) & J(R_2) & \ldots & J_\ell(M_{2n}) \\ \ldots & \ldots & \ldots & \ldots \\ J_\ell(M_{n1}) & J_\ell(M_{n2}) & \ldots & J(R_n) \end{pmatrix}. \qquad (*)$$

A similar relation also holds, where the subscript ℓ is replaced by r.

Proof The case $n = 2$ is considered in Theorem 2.4.1. Let K be a formal matrix ring of order $n \geq 3$. We represent K as a ring of block matrices $\begin{pmatrix} R & M \\ N & R_n \end{pmatrix}$, where R is a formal matrix ring of order $n-1$ and M, N are the corresponding bimodules; see the proof of Proposition 2.3.3. By Theorem 2.4.1, we have $J(K) = \begin{pmatrix} J(R) & J(M) \\ J(N) & J(R_n) \end{pmatrix}$. By the induction hypothesis, the radical $J(R)$ has the required form. We have to show that the set in the right part of relations $(*)$ coincides with $\begin{pmatrix} J(R) & J(M) \\ J(N) & J(R_n) \end{pmatrix}$. It is sufficient to verify that

$$\begin{pmatrix} J_\ell(M_{1\,n}) \\ \dots \\ J_\ell(M_{n-1\,n}) \end{pmatrix} = J(M) \quad \text{and} \quad (J_\ell(M_{n1}), \dots, J_\ell(M_{n\,n-1})) = J(N),$$

where $J(M) = \{m \in M \mid Nm \subseteq J(R_n)\}$, $J(N) = \{n \in N \mid Mn \subseteq J(R)\}$.

The required assertion follows from the definition of the subbimodules $J_\ell(M_{ij})$. We just need the following point. If $x \in J_\ell(M_{nj})$, then $M_{in}x \subseteq J_\ell(M_{ij})$ for all distinct $i, j = 1, \dots, n$. Indeed, we have

$$M_{ji}M_{in}x \subseteq M_{jn}x \subseteq J(R_j).$$

The proof of the analogue of relations $(*)$ for "the subscript r" is symmetrical to the above proof. □

We have that $J_\ell(M_{ij}) = J_r(M_{ij})$ for distinct i and j. We denote this subbimodule by $J(M_{ij})$.

The prime radical $P(K)$ has a similar structure. Similar to the subbimodules $J_\ell(M_{ij})$ and $J_r(M_{ij})$, we define subbimodules $P_\ell(M_{ij})$ and $P_r(M_{ij})$. The following result holds.

Theorem 2.4.4 *We have the relation*

$$P(K) = \begin{pmatrix} P(R_(1) & P_\ell(M_{12}) & \dots & P_\ell(M_{1n}) \\ P_\ell(M_{21}) & P(R_2) & \dots & P_\ell(M_{2n}) \\ \dots & \dots & \dots & \dots \\ P_\ell(M_{n1}) & P_\ell(M_{n2}) & \dots & P(R_n) \end{pmatrix}$$

and a similar relation in which the subscript ℓ is replaced by the subscript r.

Now consider the structure of ideals of the ring K. The material of Sect. 2.1 concerning ideals and factor rings can be applied to formal matrix rings of any order n. An ideal L of the ring K is of the form

$$\begin{pmatrix} I_1 & A_{12} & \dots & A_{1n} \\ A_{21} & I_2 & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & I_n \end{pmatrix},$$

where I_i is an ideal of the ring R and A_{ij} is a subbimodule in M_{ij}. It is not difficult to determine certain interrelations between the ideals and subbimodules; in one special case, they are given in Sect. 4.2. The set of matrices

$$\begin{pmatrix} R_1/I_1 & M_{12}/A_{12} & \dots & M_{1n}/A_{1n} \\ M_{21}/A_{21} & R_2/I_2 & \dots & M_{2n}/A_{2n} \\ \dots & \dots & \dots & \dots \\ M_{n1}/A_{n1} & M_{n2}/A_{n2} & \dots & R_n/I_n \end{pmatrix}$$

naturally forms a formal matrix ring which is canonically isomorphic to the factor ring K/L.

Concluding this section, we calculate the center of a formal matrix ring. We recall that the center of some ring T is denoted by $C(T)$.

Lemma 2.4.5 *The center of the formal matrix ring K consists of all diagonal matrices* $\operatorname{diag}(r_1, r_2, \dots, r_n)$ *such that $r_i \in C(R_i)$ and $r_i m = m r_j$ for all $m \in M_{ij}$ and distinct i, j.*

Proof It is clear that diagonal matrices with the mentioned structure are contained in $C(K)$.

Now we assume that a matrix $D = (d_{ij})$ is contained in $C(K)$. It follows from the relations $DE_{kk} = E_{kk}D$ that $d_{ik} = 0 = d_{kj}$ for $i \neq k$, $k \neq j$. Therefore, $d_{ij} = 0$ for $i \neq j$ and D is a diagonal matrix.

Now we fix subscripts i, j and an element $m \in M_{ij}$. Let A_{ij} be the matrix which has m in position (i, j) and zeros in the remaining positions. It follows from the relations $DA_{ij} = A_{ij}D$ that $d_i m = m d_j$. In particular, for $i = j$, we have $d_i \in C(R_i)$. $\square$

2.5 Ring Properties

We find out when a formal matrix ring of order 2 is Artinian, Noetherian, regular, or unit-regular.

Theorem 2.5.1 ([104]) *A formal matrix ring $K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ is left Artinian if and only if R, S are left Artinian rings and ${}_RM$, ${}_SN$ are Artinian modules. Similar assertions for a right Artinian ring K and a left or right Noetherian ring K also hold.*

Proof $\Rightarrow$. Let $A_1 \supseteq A_2 \supseteq \dots$ be a descending chain of left ideals of the ring R and $C_1 \supseteq C_2 \supseteq \dots$ a descending chain of R-submodules of the module M. Then we have the following descending chains in K:

$$\begin{pmatrix} A_1 & 0 \\ NA_1 & 0 \end{pmatrix} \supseteq \begin{pmatrix} A_2 & 0 \\ NA_2 & 0 \end{pmatrix} \dots,$$
$$\begin{pmatrix} 0 & C_1 \\ 0 & NC_1 \end{pmatrix} \supseteq \begin{pmatrix} 0 & C_2 \\ 0 & NC_2 \end{pmatrix} \dots.$$

By assumption, there exists a subscript n such that $A_n = A_{n+1} = \dots$ and $C_n = C_{n+1} = \dots$. Consequently, ${}_RR$ and ${}_RM$ are Artinian modules. Similarly, ${}_SS$ and ${}_SN$ are Artinian modules.

$\Leftarrow$. The ring K, considered as a left K-module, is a pair $(R \oplus M, N \oplus S)$, where $R \oplus M$ is a left R-module and $N \oplus S$ is a left S-module; the structure of K is considered in Sect. 3.1. Let $L_1 \supseteq L_2 \supseteq \dots$ be a descending chain of left ideals of the ring K. Making use of the material of Sect. 3.1, we can write $L_k = (X_k, Y_k)$, where X_k is a submodule of the Artinian module $R \oplus M$ and Y_k is a submodule of the Artinian module $N \oplus S$. In addition, the inclusions $X_1 \supseteq X_2 \supseteq \dots$ and $Y_1 \supseteq Y_2 \supseteq \dots$ hold. Consequently, these last two chains stabilize. Therefore, the chain of left ideals $L_1 \supseteq L_2 \supseteq \dots$ also stabilizes. Consequently, K is a left Artinian ring.

The remaining assertions are similarly proved. $\square$

We say that a ring T *is of stable rank 1* if for any two elements $a, b \in T$ with $aT + bT = T$, there exists an element $z \in T$ such that $a + bz$ is an invertible element.

For any two elements $a, b \in K$, we set

$$\mathrm{diag}(a, b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad T_{12}(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad T_{21}(b) = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}.$$

Theorem 2.5.2 ([24]) *A ring* $K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ *is of stable rank* 1 *if and only if the rings* R, S *are of stable rank* 1.

Proof $\Rightarrow$. Set $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $R \cong eKe$, $S \cong (1-e)K(1-e)$, and one can directly verify that R and S are of stable rank 1.

$\Leftarrow$. We take arbitrary matrices

$$A = \begin{pmatrix} a_1 & m_1 \\ n_1 & a_2 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & m_2 \\ n_2 & b_2 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 & m \\ n & x_2 \end{pmatrix}$$

and assume that $AX + B$ is the identity matrix E of the ring K.

The matrix $G = \begin{pmatrix} A & B \\ -E & X \end{pmatrix}$ of order 2 over K is an invertible matrix with inverse $\begin{pmatrix} X & XA - E \\ E & A \end{pmatrix}$. In particular, the relation $a_1x_1 + m_1n + b_1 = 1$ holds in the ring R. Since the ring R is of stable rank 1, there exists an element $r \in R$ such that $a_1 + m_1nr + b_1r = u$, where u is an invertible element. Then there exists an element $n' \in N'$ such that

$$G \cdot \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ nr & 1 \end{pmatrix} & 0 \\ \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} & E \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} u & m_1 \\ n' & a_2 \end{pmatrix} & B \\ \begin{pmatrix} * & 0 \\ 0 & -1 \end{pmatrix} & X \end{pmatrix};$$

here and hereafter, $(*)$ denotes elements which are not important for us. We can verify that there exist elements $s_1, s_2 \in S$, $n'' \in N$, two invertible diagonal matrices and one matrix $T_{21}(*)$ in $M(2, K)$ such that

$$\mathrm{diag}(*, *) \cdot G \cdot \mathrm{diag}(*, *) \cdot T_{21}(*) =$$
$$\begin{pmatrix} \begin{pmatrix} u & m_1 \\ 0 & s_1 \end{pmatrix} & \begin{pmatrix} b_1 & m_2 \\ n'' & s_2 \end{pmatrix} \\ \begin{pmatrix} * & 0 \\ 0 & -1 \end{pmatrix} & X \end{pmatrix} = H.$$

Similarly, there exist elements $s_3, s_4 \in S$ and $m', m'' \in M$ such that $s_1s_3 + n''m'' + s_2s_4 = 1$ in S. Since S is of stable rank 1, there exist elements $s, v \in S$ such that the element v is invertible and $s_1 + n''m''s + s_2s_4s = v$.

There exists a matrix $T_{21}(*)$ such that

$$H \cdot T_{21}(*) = \begin{pmatrix} \begin{pmatrix} u & * \\ 0 & v \end{pmatrix} & \begin{pmatrix} b_1 & m_2 \\ n'' & s_2 \end{pmatrix} \\ * & X \end{pmatrix} = V.$$

Since V and $\begin{pmatrix} u & * \\ 0 & v \end{pmatrix}$ are invertible matrices, and V can be represented in the form

$$V = \mathrm{diag}(*, *) \cdot T_{21}(*) \cdot T_{12}(*),$$

where the diagonal matrix is invertible.

As a result, we obtain the relation

$$\mathrm{diag}(*, *) \cdot G \cdot \mathrm{diag}(*, *) \cdot T_{21}(*) \cdot T_{21}(*) = \mathrm{diag}(*, *) \cdot T_{21}(*) \cdot T_{12}(*)$$

or

$$G \cdot \mathrm{diag}(*, *) \cdot T_{21}(*) = \mathrm{diag}(*, *) \cdot T_{21}(*) \cdot T_{12}(*).$$

Consequently, there exist a matrix Y and invertible matrices U, V, W, P in $M(2, K)$ such that

$$G \cdot \mathrm{diag}(V, W) \cdot T_{21}(Y) = \mathrm{diag}(U, P) \cdot T_{21}(*) \cdot T_{12}(*).$$

Then we obtain the relations

$$AV + BWY = U, \quad A + B(WYV^{-1}) = UV^{-1},$$

where UV^{-1} is an invertible matrix; which is what we required. □

Here are several well-known definitions.

Up to Theorem 2.5.6, the symbol R denotes an arbitrary ring.

An element $r \in R$ is said to be *regular* if there exists an element $x \in R$ such that $r = rxr$; a ring R is said to be *regular* if every element of R is regular.

An element $r \in R$ is said to be *unit-regular* if there exists an invertible element $v \in R$ with $r = rvr$; an element r is unit-regular if and only if $r = ue_1 = e_2u$, where e_1, e_2 are idempotents and u is an invertible element. A ring R is said to be *unit-regular* if every element of R is unit-regular.

If $r = rxr$, then the elements rx and xr are idempotents of the ring R. Therefore, one can directly verify that every principal left (right) ideal of a regular ring R is generated by an idempotent. Consequently, every principal left (right) ideal of a regular ring R is a direct summand of the left (right) R-module R.

Lemma 2.5.3 *If R is a ring and y is an element in R such that $a - aya$ is a regular element, then the element a is also regular.*

Proof There exists an element $z \in R$ such that

$$(a - aya)z(a - aya) = a - aya.$$

Set

$$x = z - zay - yaz + yazay + y.$$

One can directly verify that $axa = a$; therefore, a is a regular element. □

If $x \in R$, then $r(x)$ denotes the *right annihilator* of the element x, i.e., $r(x)$ is the right ideal $\{y \in R \mid xy = 0\}$.

Proposition 2.5.4 ([41]) *For a regular ring R, the following conditions are equivalent.*

(1) *R is a unit-regular ring.*
(2) *$(1 - e)R \cong (1 - f)R$ for any idempotents $e, f \in R$ with $eR \cong fR$.*
(3) *$r(x) \cong R/xR$ for any element $x \in R$.*

Proof (1)⇒(2). Let e and f be idempotents in R with $eR \cong fR$. We have the direct decompositions

$$R_R = eR \oplus (1 - e)R, \quad R_R = fR \oplus (1 - f)R.$$

Let x be an element in R such that $x(1-e)R = 0$ and left multiplication by x is an isomorphism $eR \to fR$. There exists an invertible element $u \in R$ with $xux = x$. Since $x(ux-1) = 0$, we have

$$R \subseteq uxR + (1-e)R = ufR + (1-e)R.$$

In addition,

$$xuR = xR = fR \quad \text{and} \quad (xu)^2 = xu.$$

Therefore,

$$ufR \cap (1-e)R = 0 \quad \text{and} \quad R = ufR \oplus u(1-e)R.$$

Since u is an invertible element, we also have

$$R = ufR \oplus u(1-f)R.$$

Finally, we obtain

$$(1-e)R \cong u(1-f)R \cong (1-f)R.$$

(2) $\Rightarrow$ (3). Let $x \in R$. There exists an element $y \in R$ with $xyx = x$. Since xy and yx are idempotents, there exist direct decompositions

$$R_R = yxR \oplus r(x) = xR \oplus (1-xy)R.$$

Left multiplication of the right ideal yxR by x defines an isomorphism $yxR \to xR$. It follows from (2) that

$$r(x) \cong (1-xy)R \cong R/xR.$$

(3) $\Rightarrow$ (1). For a given element $x \in R$, there exists an element $y \in R$ with $xyx = x$. As above, we have direct decompositions

$$R_R = yxR \oplus r(x) = xR \oplus (1-xy)R.$$

It follows from (3) that

$$r(x) \cong R/xR \cong (1-xy)R.$$

Left multiplication of the right ideal yxR by x defines an isomorphism $yxR \to xR$. As a result, there exists an isomorphism $\varphi\colon R_R \to R_R$. Let u be an invertible element in R such that left multiplication by u is the inverse isomorphism to φ. Then $ux = yx$ and $xux = xyx = x$. $\square$

Proposition 2.5.5 ([41]) *A regular ring R is of stable rank* 1 *if and only if R is a unit-regular ring.*

Proof $\Rightarrow$. Let $a \in R$. Then $axa = a$ for some $x \in R$ and $R_R = aR \oplus (1 - ax)R$; see the proof of the implication (2)$\Rightarrow$(3) of Proposition 2.5.4. Since R is of stable rank 1, $a + (1 - ax)y$ is an invertible element for some element $y \in R$. Therefore, $(a + (1 - ax)y)u = 1$, where u is an invertible element. We obtain

$$a = axa = ax(a + (1 - ax)y)ua = axaua = aua.$$

Consequently, R is a unit-regular ring.

$\Leftarrow$. Assume that we have the relation $aR + bR = R$. All principal right ideals of the regular ring R are direct summands in R_R. We take some direct decomposition $R_R = aR \oplus I$ and consider the restriction of the projection $R_R \to I$ to bR. Then $bR = (aR \cap bR) \oplus J$ for some right ideal J. We have a direct decomposition $R_R = aR \oplus J$. We also have a direct decomposition $R_R = L \oplus K$, where $K = r(a)$; see the proof of the implication (2)$\Rightarrow$(3) of Proposition 2.5.4. Since $L \cong aR$, it follows from Proposition 2.5.4 that $K \cong J$. Consequently, there exists an element $c \in R$ such that $cL = 0$ and left multiplication by c induces an isomorphism from K onto J. Thus, $cR = J \subseteq bR$ and $c = by$ for some element $y \in R$. We also have that left multiplication by a induces an isomorphism from L onto aR. By considering the relations $aK = 0 = cL$, we obtain that left multiplication by $a + c$ induces an isomorphism $L \oplus K = R_R$ onto $aR \oplus J = R_R$. Consequently, $a + by = a + c$ is an invertible element of the ring R. $\square$

The paper of Brown and McCoy [21] contains an elementary proof of the property that the matrix ring $M(n, R)$ over a regular ring R is a regular ring. We extend this proof to formal matrix rings of order 2. For formal matrix rings of any order, the assertion (1) of the following Theorem 2.5.6 can be proved with the use of Lemma 1.6 in the book [41]; see also [104].

Theorem 2.5.6 *For a formal matrix ring $K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ the following assertions hold.*

(1) K is a regular ring if and only if R, S are regular rings and $x \in xNx$, $y \in yMy$ for all $x \in M$ and $y \in N$.
(2) K is a unit-regular ring if and only if R, S are unit-regular rings and $x \in xNx$, $y \in yMy$ for all $x \in M$ and $y \in N$.

Proof **(1)**. $\Rightarrow$. One can directly verify that R and S are regular rings and $x \in xNx$, $y \in yMy$ for all $x \in M$ and $y \in N$. For any $x \in M$, there exists a matrix $\begin{pmatrix} a & x' \\ y & b \end{pmatrix} \in K$ such that

$$\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & x' \\ y & b \end{pmatrix} \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}.$$

Therefore, we have $x = xyx \in xNx$. Similarly, we have that $y \in yMy$ for all $y \in N$.

$\Leftarrow$. If $a \in R$, then a' denotes an arbitrary element with $aa'a = a$. The elements b', x', y' are defined similarly, where $b \in S$, $x \in M$ and $y \in N$.

Set $X = \begin{pmatrix} 0 & 0 \\ x' & 0 \end{pmatrix}$ and $B = A - AXA$. The matrix B has the form $\begin{pmatrix} c & 0 \\ z & d \end{pmatrix}$. Then we construct the matrix $Y = \begin{pmatrix} c' & 0 \\ 0 & d' \end{pmatrix}$ and calculate

$$B - BYB = C = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}$$

for some $n \in N$. Thus, we take the matrix $Z = \begin{pmatrix} 0 & n' \\ 0 & 0 \end{pmatrix}$ and see that $C = CZC$. This means that C is a regular element in K. By Lemma 2.5.3, B and A are also regular elements.

(2). The assertion follows from (1), Theorem 2.5.2 and Proposition 2.5.5. □

2.6 Additive Problems

As earlier, the Jacobson radical of the ring R is denoted by $J(R)$.

Let R be a ring. An element $r \in R$ is said to be *clean* if it can be represented in the form $r = u + e$, where e is an idempotent and u is an invertible element. The ring R is said to be *clean* if every element of R is clean. Basic information about clean rings is contained in [93, 109, 110].

Invertible or idempotent elements are clean. If $x \in J(R)$, then $x = (x - 1) + 1$, where $x - 1$ is an invertible element. Consequently, x is a clean element. Therefore, every local ring is a clean ring.

Lemma 2.6.1 ([45]) *Let R be a ring and e an idempotent in R such that eRe and $(1 - e)R(1 - e)$ are clean rings. Then R is a clean ring.*

Proof We identify the ring R with the formal matrix ring

$$\begin{pmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{pmatrix};$$

see Sect. 2.1.

Now we take the matrix $A = \begin{pmatrix} a & x \\ y & b \end{pmatrix} \in R$. There exists a decomposition $a = u + f$, where u is an invertible element of the ring eRe and $f = f^2 \in eRe$. Then

$$b - yu^{-1}x \in (1 - e)R(1 - e).$$

Therefore, the ring $(1-e)R(1-e)$ contains an idempotent g and an invertible element v with $b - yu^{-1}x = v + g$. We have the relations

$$A = \begin{pmatrix} u+f & x \\ y & v+g+yu^{-1}x \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} + \begin{pmatrix} u & x \\ y & v+yu^{-1}x \end{pmatrix}.$$

Then $\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}$ is an idempotent of the ring R. It remains to verify that $\begin{pmatrix} u & x \\ y & v+yu^{-1}x \end{pmatrix}$ is an invertible matrix. We have the relations

$$\begin{pmatrix} e & 0 \\ -yu^{-1} & 1-e \end{pmatrix} \begin{pmatrix} u & x \\ y & v+yu^{-1}x \end{pmatrix} \begin{pmatrix} e & -u^{-1}x \\ 0 & 1-e \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix},$$

in which the first matrix and the last two matrices are invertible. □

It is easy to prove the following proposition by induction.

Proposition 2.6.2 *Let R be a ring and $e_1, e_2, \ldots, e_n$ pairwise orthogonal idempotents of the ring R such that $1 = e_1 + e_2 + \ldots + e_n$ and all e_iRe_i are clean rings. Then R is a clean ring.*

The following result can be proved with the use of Propositions 2.6.2 and 2.3.1 concerning the correspondence between formal matrix rings and systems of orthogonal idempotents.

Corollary 2.6.3 *Let K be a formal matrix ring of order n and each of the rings $R_1, R_2, \ldots, R_n$ be clean. Then K is a clean ring.*

There is another way to prove Corollary 2.6.3. First, we apply Lemma 2.6.1 to the ring $\begin{pmatrix} R & M \\ N & S \end{pmatrix}$. A formal matrix ring of order $n > 2$ is a ring of formal block matrices of order 2. Therefore, we can use induction on n.

Proposition 2.6.4 ([93]) *A ring R with Jacobson radical $J(R)$ is clean if and only if $R/J(R)$ is a clean ring and idempotents can be lifted modulo $J(R)$.*

Proof $\Rightarrow$. All homomorphic images of clean rings are clean rings.

Let x be an arbitrary element of the ring R. We have $x = u + e$, where u is an invertible element and e is an idempotent. We have the relations

$$u(x - u^{-1}(1-e)u) = ue + u^2 - u + eu = x^2 - x,$$
$$x - u^{-1}(1-e)u = u^{-1}(x^2 - x).$$

We denote by x an element $f \in R$ such that $f^2 - f \in J(R)$. It follows from the relations

$$f - u^{-1}(1-e)u = u^{-1}(f^2 - f) \in J(R)$$

that idempotents can be lifted modulo $J(R)$.

$\Leftarrow$. Let x be an arbitrary element in R. We denote by $\overline{x}$ a natural image of the element x in the factor ring $R/J(R)$. There exist elements $e, u, v \in R$ such that $\overline{e}^2 = \overline{e}$ and

$$\overline{u}\,\overline{v} = \overline{v}\,\overline{u} = \overline{1}, \quad \overline{x} = \overline{u} + \overline{e}.$$

Since idempotents can be be lifted modulo $J(R)$, we can assume that $e^2 = e$. In addition, $uv = 1 + s$ and $vu = 1 + t$ for some elements $s, t \in J(R)$. Since $1 + s$ and $1 + t$ are invertible elements, uv and vu are invertible elements. Therefore, u and v are invertible elements. Then $x = u + r + e$ for some element $r \in J(R)$, and $u + r$ is an invertible element. □

Let k be a positive integer and R a ring. An element a of R is said to be k-*good* if a is the sum of k invertible elements of the ring R. A ring is said to be k-*good* if all of its elements are k-good.

Proposition 2.6.5 *If K is a formal matrix ring of order n and all the rings $R_1, R_2, \dots, R_n$ are k-good, then K is a k-good ring.*

Proof We consider only the case $k = 2$. By the argument given after Corollary 2.6.3, we assume that $K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$.

We consider an arbitrary matrix $X = \begin{pmatrix} r & m \\ n & s \end{pmatrix}$ in the ring K. We denote by A, B and C the matrices

$$\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix},$$

respectively. Since R and S are 2-good rings,

$$B = U + V, \quad \text{where } U = \begin{pmatrix} u_1 & 0 \\ 0 & v_1 \end{pmatrix}, \quad V = \begin{pmatrix} u_2 & 0 \\ 0 & v_2 \end{pmatrix}$$

are invertible matrices. Set

$$A' = A + U = \begin{pmatrix} u_1 & m \\ 0 & v_1 \end{pmatrix}, \quad C' = C + V = \begin{pmatrix} u_2 & 0 \\ n & v_2 \end{pmatrix}.$$

Now we obtain

$$X = A + B + C = A' + C',$$

where A' and C' are invertible matrices. □

There exists an interesting interrelation between clean and 2-good rings.

An element $s \in R$ is called an *involution* if $s^2 = 1$. When we say that 2 *is invertible in* R, we mean that $2 \cdot 1_R$ is an invertible element. Instead of $r \cdot 2^{-1}$, we write $r/2$.

Proposition 2.6.6 ([23]) *Let R be a ring with $2^{-1} \in R$. The ring R is clean if and only if every element x of R is the sum of an invertible element and an involution (consequently, x is a 2-good element).*

Proof We assume that R is a clean ring and $x \in R$. Then $(x+1)/2 = u + e$, where $e = e^2$ and u is an invertible element. Then $x = 2u + (2e - 1)$, where $2u$ is an invertible element and $(2e-1)^2 = 1$.

Conversely, let us assume that every element of the ring R is the sum of an invertible element and an involution. For a given $x \in R$, the element $2x - 1$ can be represented in the form $2x - 1 = u + s$, where u is an invertible element and $s^2 = 1$. Then $x = u/2 + (s+1)/2$, where $u/2$ is an invertible element and $(s+1)/2$ is an idempotent. $\square$

We recall that I and J denote the trace ideals of the ring $K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ defined in Sect. 2.1; namely, $I = MN$ and $J = NM$. For the ring K with $I \subseteq J(R)$ and $J \subseteq J(S)$, some of the above assertions can be reversed. First, we formulate several general facts.

Lemma 2.6.7 *Let K be a formal matrix ring such that $I \subseteq J(R)$ and $J \subseteq J(S)$.*

(1) $J(K) = \begin{pmatrix} J(R) & M \\ N & J(S) \end{pmatrix}$ *and*

$$K/J(K) \cong R/J(R) \times S/J(S).$$

(2) The matrix $\begin{pmatrix} a & x \\ y & b \end{pmatrix}$ is invertible in K if and only if the elements a and b are invertible in R and S, respectively.

Proof **(1)**. The assertion directly follows from Theorem 2.4.1.

(2). The assertion follows from the property that for any invertible element u of some ring T and every $t \in J(T)$, the element $u + t$ is invertible in T. $\square$

A ring T is said to be *directly finite* if $ba = 1$ for any elements a, b with $ab = 1$.

Theorem 2.6.8 (see [104]) *Let $K = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ be a formal matrix ring such that $I \subseteq J(R)$ and $J \subseteq J(S)$.*

(1) K is a clean ring if and only if R and S are clean rings.
(2) K is a 2-good ring if and only if R and S are 2-good rings.
(3) K is a directly finite ring if and only if R and S are directly finite rings.

Proof It follows from Corollary 2.6.3 and Proposition 2.6.5 that it is sufficient to prove the necessity of the conditions in (1) and (2).

(1). $\Rightarrow$. It follows from Proposition 2.6.4 and the isomorphism from Lemma 2.6.7 that $R/J(R)$ is a clean ring. Since K is a clean ring, it follows from the proof of Proposition 2.6.4 that for each matrix X in K, there exists an idempotent matrix W with $X - W \in K(X^2 - X)$. We assume that a similar property holds for all elements of the ring R. Then it follows from the proof of Proposition 2.6.4 that idempotents can be lifted modulo $J(R)$ and R is a clean ring by Proposition 2.6.4. Similarly, it is verified that S is a clean ring.

Thus, let $x \in R$. There exists an idempotent matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K\left(\begin{pmatrix} x^2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\right) = K\begin{pmatrix} x^2 - x & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$\begin{pmatrix} x - a & -b \\ -c & -d \end{pmatrix} \in \begin{pmatrix} R(x^2 - x) & 0 \\ N(x^2 - x) & 0 \end{pmatrix}.$$

Therefore, $b = 0 = d$, whence $a^2 = a$, i.e., a is an idempotent. Then $x - a \in R(x^2 - x)$; which is what we required.

(2). $\Rightarrow$. For an element $r \in R$, we have

$$\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} u_1 & a \\ b & v_1 \end{pmatrix} + \begin{pmatrix} u_2 & c \\ d & v_2 \end{pmatrix},$$

where the last two matrices are invertible. Consequently, $r = u_1 + u_2$, where elements u_1 and u_2 are invertible by Lemma 2.6.7.

(3). The necessity of the conditions may be directly verified; the inclusions $I \subseteq J(R)$ and $J \subseteq J(S)$ are not required.

Now let $\begin{pmatrix} a & x \\ y & b \end{pmatrix}\begin{pmatrix} a' & x' \\ y' & b' \end{pmatrix} = 1$ in K. Then $aa' + xy' = 1$ in R and $yx' + bb' = 1$ in S. Since $xy' \in J(R)$ and $yx' \in J(S)$, the products aa' and bb' are invertible in R and S, respectively. Since R and S are directly finite rings, a and b are invertible elements. By Lemma 2.6.7, $\begin{pmatrix} a & x \\ y & b \end{pmatrix}$ is an invertible matrix, whence $\begin{pmatrix} a' & x' \\ y' & b' \end{pmatrix}\begin{pmatrix} a & x \\ y & b \end{pmatrix} = 1$. □

In Sect. 2.3, the trace ideals $I_1, \dots, I_n$ of the formal matrix ring K of order n were defined. For such a ring K, we can prove the analogue of Theorem 2.6.8 provided conditions $I_k \subseteq J(R_k)$, $k = 1, \dots, n$, hold. We can use induction on n by representing K as rings of formal block matrices of order 2.

We present some of Henriksen's interesting results about good ordinary matrix rings $M(n, R)$ over an arbitrary ring R.

Lemma 2.6.9 *For $n > 1$, every diagonal matrix in $M(n, R)$ is 2-good.*

Proof Let $D = \mathrm{diag}(a_1, a_2, \ldots, a_n)$ be a diagonal matrix. We consider the matrices

$$U = \begin{pmatrix} a_1 & 0 & 0 & \ldots & 0 & 0 & 1 \\ 1 & a_2 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & a_3 & \ldots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1 & a_{n-1} & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 \end{pmatrix},$$

$$V = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 & -1 \\ -1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & -1 & 0 & \ldots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & -1 & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & -1 & a_n \end{pmatrix}.$$

With the use of elementary transformations of rows or columns, the matrices U and V can be transformed to the identity matrix. Consequently, U and V are invertible matrices. In addition, $D = U + V$. $\square$

Lemma 2.6.10 (Kaplansky) *For any $n \geq 1$, every matrix in $M(n, R)$ is the sum of a diagonal matrix and an invertible matrix.*

Proof We use induction on n. If $n = 1$ and $a \in R$, then $a = (a - 1) + 1$ and the assertion holds.

We assume that the assertion holds for some $n \geq 1$. If $A' \in M(n + 1, R)$, then the matrix A' can be represented in the following block form:

$$A' = \begin{pmatrix} A & B \\ C & d \end{pmatrix},$$

where $A \in M(n, R)$, $d \in R$, B is a column vector and C is a row vector. By the induction hypothesis, $A = D + V$, where $D, V \in M(n, R)$, D is a diagonal matrix and V is an invertible matrix. We consider the matrices

$$D' = \begin{pmatrix} D & 0 \\ 0 & d - 1 - CV^{-1}B \end{pmatrix}, \; V' = \begin{pmatrix} V & B \\ C & 1 + CV^{-1}B \end{pmatrix}.$$

Then $A' = D' + V'$, where D' is a diagonal matrix. Let E be the identity matrix in $M(n, R)$ and

$$P = \begin{pmatrix} E & 0 \\ -CV^{-1} & 1 \end{pmatrix}, \; Q = \begin{pmatrix} V^{-1} & V^{-1}B \\ 0 & 1 \end{pmatrix}.$$

Then $PV'Q$ is the identity matrix in $M(n+1, R)$ and P, Q are invertible matrices in $M(n+1, R)$. Therefore, V' is an invertible matrix; which is what we required. □

By combining the last two lemmas, we obtain the following result.

Theorem 2.6.11 ([56]) *For $n > 1$, the matrix ring $M(n, R)$ is a 3-good ring for any ring R.*

We can show that $M(n, R)$ is also a 4-good ring.

A matrix ring $M(n, R)$ is not necessarily a 2-good ring. Let $n > 1$, F an arbitrary field and $R = F[x_1, \dots, x_n]$ the polynomial ring. In [56], it is shown that the ring $M(n, R)$ contains a matrix which is not 2-good.

Of course, there exists a ring R such that $M(n, R)$ is a 2-good ring for any $n > 1$. A ring R is called a *diagonalizable* ring if for any n, each matrix $A \in M(n, R)$ is equivalent to some diagonal matrix D, i.e., $UAV = D$ for some invertible matrices U and V. For example, commutative principal ideal rings and commutative valuation rings are diagonalizable rings.

The following theorem follows from Lemma 2.6.9 and the definition of a diagonalizable ring.

Theorem 2.6.12 *If R is a diagonalizable ring and $n > 1$, then $M(n, R)$ is a 2-good ring.*

Every formal matrix M is the sum of a diagonal matrix and an invertible matrix, i.e., Kaplansky's lemma 2.6.10 holds for M. The proof of Lemma 2.6.10 can be directly extended to formal matrices. For arbitrary formal matrices, the analogue of Lemma 2.6.9 is not true. This follows, for example, from Theorem 2.6.8(2).

In the literature, there are various notions close to those of a clean element and a clean ring. An element $r \in R$ is said to be *strongly clean* if r can be represented in the form $r = u + e$, where u is an invertible element, e is an idempotent, and $ue = eu$. A ring R is said to be *strongly clean* if every element of R is strongly clean.

Strongly clean matrix rings and individual strongly clean matrices over local rings have been studied in numerous papers. The paper [19] contains a complete characterization of commutative local rings R such that $M(n, R)$ is a strongly clean ring. Let R be an arbitrary local ring. In [105, 115], the authors determined when $M(2, R)$ is a strongly clean ring and $M(2, R, s)$ is a strongly clean ring, respectively; see the end of Sect. 4.1 for more about the ring $M(2, R, s)$.

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