Chapter 2
Generalized Type-2 Fuzzy Logic

This Chapter describes the basic concepts about generalized type-2 fuzzy sets theory. We explain the generalized type-2 fuzzy system approximation based on α-planes including the fuzzifier process, fuzzy rules, inference engine, type reducer and defuzzification process; all these definitions are used to develop the fuzzy edge detection methods.

2.1 Definition of Generalized Type-2 Fuzzy Sets

A generalized type-2 fuzzy set (T2 FS), denoted by $\tilde{A}$, is characterized by a type-2 membership function $\mu_{\tilde{A}}(x, u)$, where $x \in X$, $u \in J_x \subseteq [0, 1]$ and $0 \leq \mu_{\tilde{A}}(x, u) \leq 1$, and can be represented by Eq. (2.1) [36, 38, 47, 45].

$$\tilde{A} = \left\{ \left((x, u), \mu_{\tilde{A}}(x, u) \right) \middle| \, \forall x \in X, \, \forall u \in J_x \subseteq [0, 1] \right\} \tag{2.1}$$

If $\tilde{A}$ is continuous it can be denoted by Eq. (2.2), where $\int\int$ denotes the union for x and u;

$$\tilde{A} = \int_{x \in X} \int_{u \in J^u_{x \subseteq [0,1]}} \mu_{\tilde{A}}(x, u)/(x, u) \tag{2.2}$$

In Eq. (2.2), J_x is called the primary membership of x in $\tilde{A}$. At each value of x say $x = x'$, the two-dimensional (2-D) plane, whose axes are u and $\mu_{\tilde{A}}(x', u)$, is called a vertical slice of $\tilde{A}$ [45]. A secondary membership function is a vertical slice of $\mu_{\tilde{A}}(x, u)$. It is $\mu_{\tilde{A}}(x = x', u)$, for $x' \in X$ and $\forall_u \in J'_x \subseteq [0, 1]$, and it is described in Eq. (2.3), in which $0 \leq f_{x'}(u) \leq 1$.

C.I. Gonzalez et al., *Edge Detection Methods Based on Generalized Type-2 Fuzzy Logic*, SpringerBriefs in Computational Intelligence,
DOI 10.1007/978-3-319-53994-2_2

$$\mu_{\tilde{A}}(x = x', u) = \int\limits_{u \in J_{x'}} f_{x'}(u)/u \quad J_{x'} \subseteq [0, 1] \tag{2.3}$$

Uncertainty in the primary membership of a GT2 fuzzy set $\tilde{A}$ is represented by a bounded region; therefore, the two-dimensional support of $\mu_{\tilde{A}}(x, u)$ is called footprint of uncertainty (FOU) of $\tilde{A}$ and is denoted by Eq. (2.4).

$$FOU(\tilde{A}) = \{(x, u) \in X \times [0, 1] \mu_{\tilde{A}}(x, u) > 0\} \tag{2.4}$$

$\mathrm{FOU}(\tilde{A})$ can also be expressed as the union of all primary memberships, i.e.

$$FOU(\tilde{A}) = \bigcup_{x \in X} J_x^u \tag{2.5}$$

In Fig. 2.1 we can find a representation of a generalized type-2 membership function, and in Fig. 2.2, the footprint of uncertainty (FOU) is illustrated, which is associated with the third dimension and allows a better modeling of real world uncertainty.

2.2 α-Planes Representation

In order to limit the complexity of generalized type-2 fuzzy logic several efforts have been proposed; for example, Wagner and Hagras [36, 37] have introduced the zSlices representation and Mendel and Liu [38, 39], have put forward a representation based on α-planes. These approximation techniques decompose the three-dimensional GT2 membership function by using different kinds of cuts to obtain a collection of IT2 fuzzy sets.

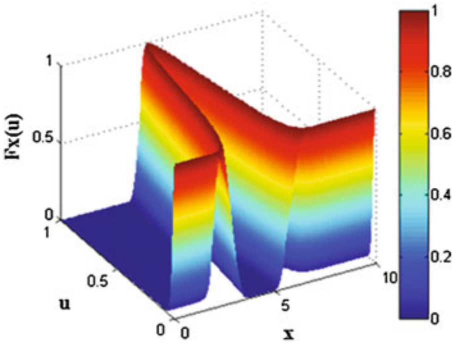

Fig. 2.1 Generalized type-2 membership function

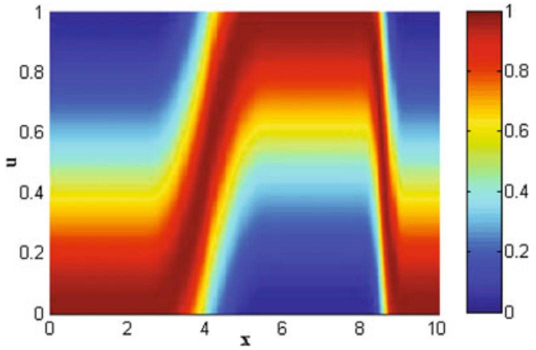

Fig. 2.2 FOU of the generalized type-2 membership function

An α-plane for the GT2 FS $\tilde{A}$, denoted by $\tilde{A}_\alpha$, is the union of all primary memberships functions of $\tilde{A}$ whose secondary grades are greater than or equal to $\alpha (0 \leq \alpha \leq 1)$ [45]. The equation of the α-plane is represented by Eq. (2.6).

$$\begin{aligned} \tilde{A}_\alpha &= \left\{ (x,u), \mu_{\tilde{A}}(x,u) \geq \alpha | \quad \forall x \in X, \forall u \in \forall u \subseteq [0,1] \right\} \\ &= \int_{\forall x \in X} \int_{\forall u \in [0,1]} \{(x,u) | f_x(u) \geq \alpha\} \end{aligned} \tag{2.6}$$

The union of all α-planes is expressed in Eq. (2.7); where $R_{\tilde{A}_\alpha}$ is one horizontal slice at level α.

$$\tilde{A} = \bigcup_{\alpha \in [0,1]} R_{\tilde{A}_\alpha} \tag{2.7}$$

2.3 Generalized Type-2 Fuzzy Systems Based on α-Planes

In Fig. 2.3, the block diagram of the generalized type-2 fuzzy inference system is presented. The generalized type-2 Mamdani fuzzy systems contain five main components: fuzzifier, fuzzy rules, inference engine, type-reduction and defuzzifier. In the following, a general description about these components is presented.

2.3.1 Fuzzifier Process

The fuzzifier maps crisp inputs into generalized type-2 fuzzy sets to process within the FSs. In this book, we will focus on the type-2 singleton fuzzifier as it is fast to compute and, thus, suitable for the generalized type-2 FSs real-time operation.

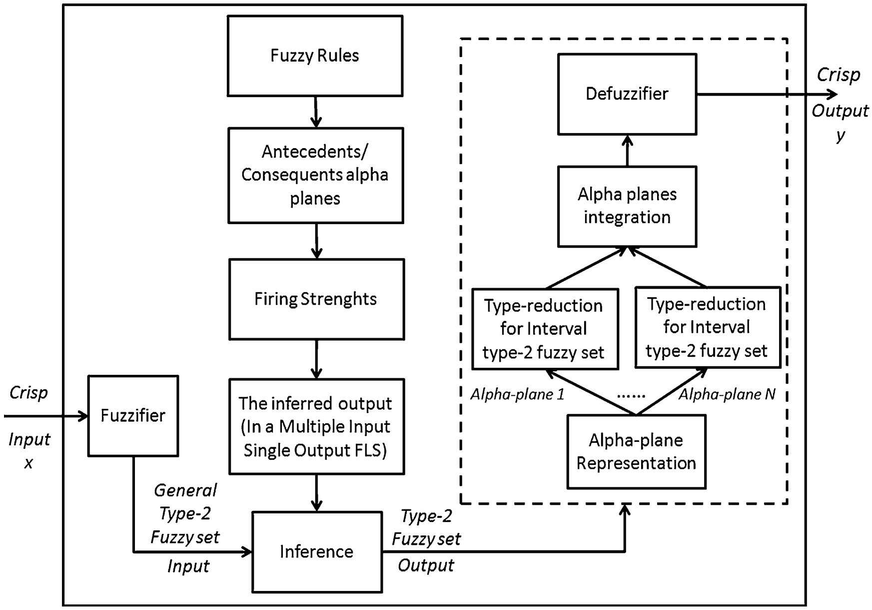

Fig. 2.3 Generalized type-2 fuzzy system based on the α-planes representation

Singleton fuzzification maps the crisp input into a fuzzy set, which has a single point of nonzero membership. Hence, the singleton fuzzifier maps the crisp input x'_p into a type-2 fuzzy singleton, whose membership function is $\mu_{\tilde{A}_p}(x_p) = 1/1$ for $x_p = x'_p$ and $\mu_{\tilde{A}_p}(x_p) = 0$ for all $x_p \neq x'_p$ for all $p = 1, 2, \ldots, P$, where P is the number of FSs inputs [34, 36].

2.3.2 *Fuzzy Rules*

The structure of the rules in the generalized type-2 FSs is the standard Mamdani-type FSs rule structure used in the type-1 FSs and in interval type-2 FSs, but in this book, we assume that the antecedents and the consequents sets are represented by generalized type-2 fuzzy sets. So for a type-2 FSs with p inputs $x_1 \in X_1, \ldots, x_p \in X_P$ and one output $y \in Y$, Multiple Input Single Output (MISO), if we assume there are M rules, the kth rule in the generalized type-2 FSs can be written as follows [32].

$$R^k : \; IF\, x_1\, is\, \tilde{F}_1^k \; and \; \ldots x_p\, is\, \tilde{F}_p^k, \;\; THEN \;\; y\, is\, \tilde{G}^k \tag{2.8}$$

2.3.3 Inference Engine

For performing the inference in the GT2 fuzzy system, the α-planes representation was used. The equation of the α-plane is represented in Eq. (2.6).

The inference engine for a generalized type-2 fuzzy system based on α-planes, can be viewed as a combination of a series of interval type-2 fuzzy sets. During this inference process, a series of operations are computed as follows.

1. Obtain the α-planes for the antecedents and consequents. In this case, the α-planes are obtained in the secondary membership functions of the antecedents $\tilde{F}_i^k$ and consequents $\tilde{G}_i^k$ of the *i*th input, *k*th rule. The α-planes of the $\tilde{F}_i^k$, create an interval type-2 fuzzy set [38, 40, 44], which is defined by Eq. (2.9).

$$\left(\tilde{F}_i^k\right)_\propto = \left\{ \int_{x_i' \in X_i} \left[\int_{\mu^\propto_{\tilde{F}_i^k}(x_i') \in \left[\underline{\mu}^\propto_{\tilde{F}_i^k}(x_i'), \overline{\mu}^\propto_{\tilde{F}_i^k}(x_i')\right]} 1/\mu^\propto_{\tilde{F}_i^k}(x_i') \right] /x \right\} \tag{2.9}$$

where $\left(\tilde{\mathrm{F}}_{\mathrm{i}}^{\mathrm{k}}\right)_\propto$ can be written as

$$\left(\tilde{F}_i^k\right)_\propto = \left\{ \int_{x_i' \in X_i} \left[\underline{\mu}^\propto_{\tilde{F}_i^k}(x_i'), \overline{\mu}^\propto_{\tilde{F}_i^k}(x_i')\right] /x_i' \right\} \tag{2.10}$$

The α-planes of the consequents $\tilde{G}_i^k$, are defined by (2.11).

$$\left(\tilde{G}_i^k\right)_\propto = \left\{ \int_{y_i \in Y_i} \left[\int_{\mu^\propto_{\tilde{G}_i^k}(y_i) \in \left[\underline{\mu}^\propto_{\tilde{G}_i^k}(y_j), \overline{\mu}^\propto_{\tilde{G}_i^k}(y_j)\right]} 1/\mu^\propto_{\tilde{G}_j^k}(y_i) \right] /y_j \right\} \tag{2.11}$$

Another expression for $\left(\tilde{G}_i^k\right)_\propto$ is

$$\left(\tilde{G}_i^k\right)_\propto = \left\{ \int_{x_i' \in X_i} \left[\underline{\mu}^\propto_{\tilde{G}_i^k}(y_j), \overline{\mu}^\propto_{\tilde{G}_i^k}(y_j)\right] /y_j \right\} \tag{2.12}$$

2. Calculate the firing strengths. The firing strengths of the rules are calculated, where the firing sets $\mu^{\alpha}_{\tilde{F}^k_i}(x'_i)$ for each α-plane (α), of the *i*th input and *k*th rule of a singleton type-2 fuzzy system are represented by Eq. (2.13).

$$\begin{aligned} \underline{\Omega}^k_{\propto}(x') &= \sqcap^n_{i=1}\left\{\underline{\mu}^{\alpha}_{\tilde{F}^k_i}(x'_i)\right\} \\ \underline{\Omega}^k_{\propto}(x') &= \sqcap^n_{i=1}\left\{\overline{\mu}^{\alpha}_{\tilde{F}^k_i}(x'_i)\right\} \end{aligned} \tag{2.13}$$

3. In a Multiple Input Single Output (MISO) fuzzy system, the inferred output $\underline{\mu}^{\alpha}_{\tilde{B}_j}(y_j)$ and $\overline{\mu}^{\alpha}_{\tilde{B}_j}(y_j)$ of each rule *k* are represented by (2.14), where $\mu^{\alpha}_{\tilde{G}^k_j}$ is the type-2 fuzzy MF that represents the αth α-plane, *k*th rule, *j*th input of the consequents.

$$\begin{aligned} \underline{\mu}^{\alpha}_{\tilde{B}_j}(y_j) &= \underline{\Omega}^k_{\propto}(x') \quad \sqcap \underline{\mu}^{\propto}_{\tilde{G}^k_j}(y_j) \\ \overline{\mu}^{\alpha}_{\tilde{B}_j}(y_j) &= \overline{\Omega}^k_{\propto}(x') \sqcap \overline{\mu}^{\propto}_{\tilde{G}^k_j}(y_j) \end{aligned} \tag{2.14}$$

4. The outputs of the fired rules (M) are combined using the join operation to produce the overall output set, which can be expressed as follows.

$$\begin{aligned} \underline{\mu}^{\alpha}_{\tilde{B}_j}(y_j) &= \sqcup^r_{k=1}\left\{\underline{\mu}^{\propto}_{\tilde{B}^k_j}(y_j)\right\} \\ \overline{\mu}^{\alpha}_{\tilde{B}_j}(y_j) &= \sqcup^r_{k=1}\left\{\overline{\mu}^{\propto}_{\tilde{B}^k_j}(y_j)\right\} \end{aligned} \tag{2.15}$$

2.3.4 Type Reducer

To perform the defuzzification process, the centroid method is used. The centroid of a generalized type-2 fuzzy set $\tilde{A}$, can be obtained by taking the union of the centroids of all the α-planes of $\tilde{A}$, and then the Karnik–Mendel algorithm can be used for computing the centroid of each α-plane. The centroid of a generalized type-2 fuzzy system, introduced by Karnik and Mendel [40, 46, 47], which is expressed in Eq. (2.16).

$$Y_C(\propto) = Centroid\left(\tilde{A}(\propto)\right) = \int\limits_{u_1 \in {}^{\propto}J_{x_1}} \cdots \int\limits_{u_N \in {}^{\propto}J_{x_N}} \propto / \frac{\sum_{i=1}^{N} x_i \mu_A(x_i)}{\sum_{i=1}^{N} \mu_A(x_i)} = \propto / [{}^{\propto}y_l, {}^{\propto}y_r] \tag{2.16}$$

where $[^{\propto}y_l, {}^{\propto}y_r]$ is the domain of the centroid. Obviously, computing ${}^{\propto}y_l$ and ${}^{\propto}y_r$ is the same as computing y_l and y_r for an interval type-2 fuzzy set; therefore, centroid type-reduction is performed by the Karnik-Mendel algorithms to compute ${}^{\propto}y_l$ and ${}^{\propto}y_r$ [47, 48].

2.3.5 Defuzzification Process

In the defuzzification phase a type-1 fuzzy output set is produced during the type-reduction process. In this book the GT2 fuzzy inference system is approximated using α-planes; so, for each α-planes the centroid type-reducer is applied with Eq. (2.16); after that the results of the α-planes are integrated by Eq. (2.17) and Eq. (2.18) [32]. Finally the defuzzifier output is obtained by using the average of y^l and y^r, this is expressed in Eq. (2.19) [39, 40].

$$\hat{y}_j^l(x') = \frac{\sum_{i=1}^{N} \propto_i^{\propto_i} y_j^l(x')}{\sum_{i=1}^{N} \propto_i} \tag{2.17}$$

$$\hat{y}_j^r(x') = \frac{\sum_{i=1}^{N} \propto_i^{\propto_i} y_j^r(x')}{\sum_{i=1}^{N} \propto_i} \tag{2.18}$$

$$\hat{y}_j(x') = \frac{\hat{y}_j^l(x') + \hat{y}_j^r(x')}{2} \tag{2.19}$$

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Edge Detection Methods Based on Generalized Type-2 Fuzzy Logic
Gonzalez, C.I.; Melin, P.; Castro, J.R.; Castillo, O.
2017, X, 89 p. 34 illus., 21 illus. in color., Softcover
ISBN: 978-3-319-53993-5