

Chapter 2

Renormalisation and the Conformal Anomaly

In order to study the conformal anomaly and related Green functions of the energy-momentum tensor, we need to renormalise the quantum field theory in curved spacetime including sources for all the operators of interest. In this section, which follows [7], we restrict to two dimensions and consider an interaction Lagrangian of the form $\mathcal{L} = g^I O_I$, where the O_I are a complete set of dimension two operators. The corresponding sources are simply the position-dependent couplings $g^I(x)$. In addition to the conventional renormalisation of the bare parameters, there are then further counterterms depending on the spacetime curvature and on derivatives of the couplings. The latter provide the extra counterterms required to define renormalised Green functions of the composite operators O_I , removing the contact-term divergences which arise even in flat spacetime.

2.1 Renormalisation and Local Couplings

In dimensional regularisation, the action is therefore

$$S = S_0 + \int d^n x \sqrt{-g} \left(g_B^I O_I + \frac{1}{2(n-1)} c_B R - \Lambda_B \right), \quad (2.1)$$

where the free action S_0 is invariant under Weyl transformations. The bare couplings are defined as¹

$$\begin{aligned} g_B^I &= \mu^{k_I \epsilon} (g^I + L^I(g)) \\ c_B &= \mu^{-\epsilon} (c + L_c(g)) \\ \Lambda_B &= \mu^{-\epsilon} (\Lambda + L_\Lambda(g)) , \end{aligned} \quad (2.2)$$

where

$$L_\Lambda = \frac{1}{2} A_{IJ}(g) \partial_\mu g^I \partial^\mu g^J . \quad (2.3)$$

Here, $\epsilon = 2 - n$ and L^I, L_c, A_{IJ} denote series of poles in $1/\epsilon$ as usual. The renormalised couplings c and Λ will normally be taken as zero.

Beta functions β_c and β_Λ are defined from these couplings in the same way as the standard beta function β^i . Explicitly,

$$\hat{\beta}^I \equiv \mu \frac{d}{d\mu} g^I = -k_I \epsilon g^I + \beta^I , \quad (2.4)$$

with

$$\beta^I = -k_I \epsilon L^I - \mu \frac{d}{d\mu} L^I . \quad (2.5)$$

Note that this is a mass-independent renormalisation scheme so the counterterms have no explicit μ -dependence, i.e. $\mu \partial L^I / \partial \mu = 0$. In the same way,

$$\hat{\beta}_c \equiv \mu \frac{d}{d\mu} c = \epsilon c + \beta_c , \quad (2.6)$$

with

$$\beta_c = \epsilon L_c - \mu \frac{d}{d\mu} L_c , \quad (2.7)$$

and

$$\hat{\beta}_\Lambda = \epsilon \Lambda + \beta_\Lambda , \quad (2.8)$$

¹The power $\mu^{k_I \epsilon}$ in g_B^I fixes the dimension of the bare operator O_{BI} away from $n = 2$. This allows for composite operators with different combinations of the elementary fields, whose dimension is set by the dimensionality/Weyl invariance of the free action. The renormalised couplings are dimensionless, while with the definition in (2.17) the renormalised operators O_I all have dimension n before the physical limit $n \rightarrow 2$ is taken. Note that no sum over the index on k_I is implied here or in subsequent equations.

with

$$\beta_\Lambda = \epsilon L_\Lambda - \mu \frac{d}{d\mu} L_\Lambda . \quad (2.9)$$

To evaluate the final term, we use

$$\mu \frac{d}{d\mu} = \mu \frac{\partial}{\partial \mu} + \int d^n x \left(\hat{\beta}^I \frac{\delta}{\delta g^I(x)} + \hat{\beta}_c \frac{\delta}{\delta c(x)} + \hat{\beta}_\Lambda \frac{\delta}{\delta \Lambda(x)} \right) , \quad (2.10)$$

and all couplings are ultimately set to their constant physical values. In what follows, we frequently simply write $\hat{\beta}^I \frac{\partial}{\partial g^I}$ as shorthand for $\int d^n x \hat{\beta}^I \frac{\delta}{\delta g^I(x)}$ and also abbreviate $\frac{\partial}{\partial g^I} \equiv \partial_I$. It then follows that

$$\beta_\Lambda = \frac{1}{2} \chi_{IJ} \partial_\mu g^I \partial^\mu g^J \quad (2.11)$$

with

$$\chi_{IJ} = \epsilon A_{IJ} - \mathcal{L}_{\hat{\beta}} A_{IJ} , \quad (2.12)$$

where the Lie derivative is given by

$$\mathcal{L}_{\hat{\beta}} A_{IJ} = \hat{\beta}^K \partial_K A_{IJ} + \partial_I \hat{\beta}^K A_{KJ} + \partial_J \hat{\beta}^K A_{IK} . \quad (2.13)$$

2.2 Trace Anomaly

Recall that the energy-momentum tensor is derived as the variation of the curved-spacetime action with respect to the metric:

$$T_{\mu\nu}(x) = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}(x)} . \quad (2.14)$$

Since the metric is itself a finite parameter, this defines a renormalised operator. To evaluate this, we need the variation of the curvature counterterms, especially

$$\frac{\delta}{\delta g^{\mu\nu}(x)} R(y) = \sqrt{-g} (R_{\mu\nu} + \Delta_{\mu\nu}) \delta(x, y) , \quad (2.15)$$

where we define $\Delta_{\mu\nu} = g_{\mu\nu}D^2 - D_\mu D_\nu$. After contracting with the metric, this gives the following expression in terms of bare quantities for the trace of the energy-momentum tensor:

$$T^\mu{}_\mu = \epsilon \left(-k_I g_B^I O_{BI} + \frac{1}{2(n-1)} c_B R - \Lambda_B \right) + D^2 c_B + 2\mu^{-\epsilon} \Lambda . \quad (2.16)$$

The renormalised operators O_I are defined as the functional derivatives of the action with respect to the couplings, viz.

$$O_i = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^I} . \quad (2.17)$$

The curvature and derivative counterterms therefore contribute:

$$\begin{aligned} O_i &= \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^I} \int d^n x \sqrt{-g} \left(g_B^J O_{BJ} + \frac{1}{2(n-1)} c_B R - \Lambda_B \right) \\ &= Z_I^J O_{BJ} + \mu^{-\epsilon} \left(\frac{1}{2(n-1)} \partial_i L_c R - \frac{1}{2} \partial_I A_{JK} \partial g^J \partial g^K + D_\mu (A_{IJ} \partial^\mu g^J) \right) . \end{aligned} \quad (2.18)$$

with

$$Z_{IJ} = \mu^{k_I \epsilon} (\delta_{IJ} + \partial_I L_J) . \quad (2.19)$$

This can be re-expressed, after a short but intricate calculation (see Appendix A), in terms of the beta functions:

$$T^\mu{}_\mu = \hat{\beta}^I O_I + \mu^{-\epsilon} \left(\frac{1}{2(n-1)} \hat{\beta}^I c R - \hat{\beta}^\Lambda \right) + \mu^{-\epsilon} (D^2 c + 2\Lambda) + \mu^{-\epsilon} D_\mu Z^\mu , \quad (2.20)$$

with

$$Z_\mu = \partial_\mu L_c - \hat{\beta}^I A_{IJ} \partial_\mu g^J . \quad (2.21)$$

Finally, we define the key function w_i [7] through

$$Z_\mu = w_I \partial_\mu g^I \quad (2.22)$$

and reducing to $n = 2$ and setting the couplings $c = \Lambda = 0$, we find our final result for the energy-momentum tensor trace:

$$T^\mu{}_\mu = \beta^I O_I + \frac{1}{2} \beta_c R - \beta_\Lambda + D_\mu Z^\mu . \quad (2.23)$$

The rhs is the conformal, or trace, anomaly. As well as the familiar operator and curvature contributions, this expression includes the dependence on derivatives of the position-dependent couplings. These are crucial in renormalising higher-point Green functions involving $T^\mu{}_\mu$. Evaluating with the couplings set to constants, we recover the familiar expression for the conformal anomaly:

$$\langle T^\mu{}_\mu \rangle - \langle \Theta \rangle = \frac{1}{2} \beta_c R, \quad (2.24)$$

where $\Theta \equiv \beta^I O_I$ is the usual operator anomaly and $\frac{1}{2} \beta_c R$ is the curvature contribution.

Since all the other terms in (2.23) are finite, it follows that Z^μ itself must also be finite. This gives rise to a consistency condition which plays an important role in the analysis of the c -theorem. Indeed, as we see later, this is equivalent to the Wess-Zumino consistency condition which follow from the abelian nature of the Weyl anomaly. To derive this, we start from the definition (2.21) of Z_μ and apply the operator $\epsilon - \hat{\beta}^K \partial_K$ to both sides, or equivalently act with $\mu \frac{d}{d\mu}$ on $\mu^{-\epsilon} Z_\mu$. We have

$$\left(\epsilon - \hat{\beta}^K \partial_K \right) \left(\partial_\mu L_c - \hat{\beta}^I A_{IJ} \partial_\mu g^J \right) = \left(\partial_I \beta_c - \chi_{IJ} \hat{\beta}^J \right) \partial_\mu g^I \quad (2.25)$$

and

$$\left(\epsilon - \hat{\beta}^K \partial_K \right) \left(w_I \partial_\mu g^I \right) = - \left(\epsilon g^J \partial_J w_I + \beta^J \partial_J w_I + \partial_I \beta^J w_J \right) \partial_\mu g^I. \quad (2.26)$$

Equating the terms of $O(1)$, we isolate the crucial identity

$$\partial_I \beta_c = \chi_{IJ} \beta^J - \mathcal{L}_\beta w_I. \quad (2.27)$$

If we now define

$$\tilde{\beta}_c = \beta_c + \beta^I w_I \quad (2.28)$$

we can rewrite (2.27) in the form

$$\partial_I \tilde{\beta}_c = \chi_{IJ} \beta^J + (\partial_I w_J - \partial_J w_I) \beta^J. \quad (2.29)$$

Notice the crucial role played here by the function w_I , introduced by Osborn [7] in his analysis of renormalisation in the presence of position-dependent couplings.

Finally, contracting with the beta function, we derive the Weyl consistency condition for the RG flow of the modified coefficient $\tilde{\beta}_c$ of the Euler density in

the trace anomaly²:

$$\beta^I \partial_I \tilde{\beta}_c = \chi_{IJ} \beta^I \beta^J . \quad (2.30)$$

Clearly, this condition has a close resemblance to the form of the c -theorem in two dimensions [7]. We will explore this connection further in what follows.

2.3 Renormalisation of Two-Point Green Functions

As noted above, the position-dependent couplings $g^I(x)$ act as sources for the composite operators O_I , while the counterterms proportional to $\partial_\mu g^I(x)$ provide the contact terms required to renormalise their higher-point Green functions.

We first define the renormalised two-point functions of the energy-momentum tensor in curved spacetime so as to be symmetric, i.e.³

$$i\langle T_{\mu\nu}(x) T_{\rho\sigma}(y) \rangle = \frac{2}{\sqrt{-g(x)}} \frac{2}{\sqrt{-g(y)}} \frac{\delta}{\delta g^{\mu\nu}(x)} \frac{\delta}{\delta g^{\rho\sigma}(y)} W \quad (2.31)$$

²Notice that there is an arbitrariness in the choice of RG scheme defining the various functions involved here. This can be parametrised by the following changes:

$$\delta\beta_c = \mathcal{L}_\beta b \equiv \beta^I \partial_I b , \quad \delta\chi_{IJ} = \mathcal{L}_\beta a_{IJ} ,$$

which implies

$$\delta\tilde{\beta}_c = a_{IJ} \beta^I \beta^J , \quad \delta w_I = -\partial_I b + a_{IJ} \beta^J .$$

This leaves the form of the consistency relation (2.30) invariant.

³We further define the two-point functions of the trace $T^\mu{}_\mu(x)$ so that

$$\begin{aligned} i\langle T^\mu{}_\mu(x) T^\rho{}_\rho(y) \rangle &\equiv g^{\mu\nu} g^{\rho\sigma} i\langle T_{\mu\nu}(x) T_{\rho\sigma}(y) \rangle \\ &= \frac{2}{\sqrt{-g}} g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}(x)} \left(\frac{2}{\sqrt{-g}} g^{\rho\sigma} \frac{\delta W}{\delta g^{\rho\sigma}(y)} \right) - (n+2) \langle T^\mu{}_\mu(x) \rangle \delta(x, y) , \end{aligned}$$

where the ambiguity in the definition of the renormalised Green function is reflected in the presence of the VEV of $T^\mu{}_\mu(x)$, itself defined by

$$\langle T^\mu{}_\mu(x) \rangle \equiv g^{\mu\nu} \langle T_{\mu\nu}(x) \rangle .$$

With Minkowski signature, these two-point functions are the usual time-ordered products. Note that throughout this paper $\delta(x, y)$ is the delta function density, where $\delta(x - y) = \sqrt{-g} \delta(x, y)$, and here we have left the dimension n general for future convenience.

Similarly, letting $\Theta = \beta^I O_I$ as the operator part of the trace anomaly, we define

$$\begin{aligned} i\langle \Theta(x) \Theta(y) \rangle &\equiv \beta^I \beta^J i\langle O_I(x) O_J(y) \rangle \\ &= \frac{1}{\sqrt{-g}} \beta^I \frac{\delta}{\delta g^I(x)} \left(\frac{1}{\sqrt{-g}} \beta^J \frac{\delta W}{\delta g^J(y)} \right) - \beta^I \partial_I \beta^J \langle O_J(x) \rangle \delta(x, y) . \end{aligned}$$

and similarly

$$i\langle O_I(x) O_J(y) \rangle = \frac{1}{\sqrt{-g(x)}} \frac{1}{\sqrt{-g(y)}} \frac{\delta}{\delta g^I(x)} \frac{\delta}{\delta g^J(y)} W. \quad (2.32)$$

With these definitions, we readily find the following relations between the bare and renormalised two-point functions:

$$\begin{aligned} i\langle T^\mu{}_\mu(x) T^\rho{}_\rho(y) \rangle - i\langle T^\mu{}_\mu(x) T^\rho{}_\rho(y) \rangle_B + 2\langle T^\mu{}_\mu(x) \rangle \delta(x, y) \\ = \frac{1}{\sqrt{-g(x)}} \frac{1}{\sqrt{-g(y)}} \left\langle \frac{\delta^2 S}{\delta \sigma(x) \delta \sigma(y)} \right\rangle \end{aligned} \quad (2.33)$$

and similarly

$$i\langle T^\mu{}_\mu(x) O_I(y) \rangle - i\langle T^\mu{}_\mu(x) O_I(y) \rangle_B = \frac{1}{\sqrt{-g(x)}} \frac{1}{\sqrt{-g(y)}} \left\langle \frac{\delta^2 S}{\delta \sigma(x) \delta g^I(y)} \right\rangle \quad (2.34)$$

$$i\langle O_I(x) O_J(y) \rangle - i\langle O_I(x) O_J(y) \rangle_B = \frac{1}{\sqrt{-g(x)}} \frac{1}{\sqrt{-g(y)}} \left\langle \frac{\delta^2 S}{\delta g^I(x) \delta g^J(y)} \right\rangle \quad (2.35)$$

The evaluation of the counterterms on the rhs of (2.33)–(2.35) by differentiation of the action (2.1) is again very intricate and details are given in Appendix A. For the Green functions of interest here, we eventually find:

$$\begin{aligned} i\langle T^\mu{}_\mu(x) T^\rho{}_\rho(y) \rangle - i\langle T^\mu{}_\mu(x) T^\rho{}_\rho(y) \rangle_B + (2 - \epsilon) \langle T^\mu{}_\mu(x) \rangle \delta(x, y) \\ = \epsilon \mu^{-\epsilon} L_c D^2 \delta(x, y) \end{aligned} \quad (2.36)$$

$$i\langle T^\mu{}_\mu(x) O_I(y) \rangle - i\langle T^\mu{}_\mu(x) O_I(y) \rangle_B = \mu^{-\epsilon} \partial_I L_c D^2 \delta(x, y) \quad (2.37)$$

$$\begin{aligned} i\langle \hat{\Theta}(x) O_I(y) \rangle - i\langle \hat{\Theta}(x) O_I(y) \rangle_B + \partial_I \hat{\beta}^J \langle O_J(x) \rangle \delta(x, y) \\ = -\mu^{-\epsilon} \left[\frac{1}{2(n-1)} \partial_I \beta_c R \delta(x, y) - A_{IJ} \hat{\beta}^J D^2 \delta(x, y) \right] \end{aligned} \quad (2.38)$$

where $\hat{\Theta} = \hat{\beta}^I O_I$ and as usual we have set $c = \Lambda = 0$ and omitted terms of $O(\partial_\mu g^I)$.

We can derive important identities amongst the renormalised connected Green functions by using the operator relation $T^\mu{}_\mu \sim \beta^I O_I$ inside the *bare* Green

functions. This immediately gives, for $n = 2$,

$$i\langle T^\mu{}_\mu(x) T^\rho{}_\rho(y) \rangle - i\langle T^\mu{}_\mu(x) \Theta(y) \rangle + 2\langle T^\mu{}_\mu(x) \rangle \delta(x, y) = \beta_c D^2 \delta(x, y) \quad (2.39)$$

$$\begin{aligned} i\langle T^\mu{}_\mu(x) O_I(y) \rangle - i\langle \Theta(x) O_I(y) \rangle - \partial_I \beta^J \langle O_J(x) \rangle \delta(x, y) \\ = \frac{1}{2} \partial_I \beta_c R \delta(x, y) + w_I D^2 \delta(x, y) \end{aligned} \quad (2.40)$$

where we have used the identities $\beta_c = \epsilon L_c - \hat{\beta}^I \partial_I L_c$ and $w_I = \partial_I L_c - A_{IJ} \hat{\beta}^J$. Notice the appearance here of $\gamma_I^J = \partial_I \beta^J$, which is the anomalous dimension matrix for the operators O_I .

Combining these, we find an identity which will play an important role in our analysis of the c -theorem, viz.

$$\begin{aligned} i\langle T^\mu{}_\mu(x) T^\rho{}_\rho(y) \rangle - i\langle \Theta(x) \Theta(y) \rangle + 2\langle T^\mu{}_\mu(x) \rangle \delta(x, y) - \beta^I \partial_I \beta^J \langle O_J(x) \rangle \delta(x, y) \\ = \frac{1}{2} \beta^I \partial_I \beta_c R \delta(x, y) + \tilde{\beta}_c D^2 \delta(x, y) \end{aligned} \quad (2.41)$$

where $\tilde{\beta}_c = \beta_c + \beta^I w_I$ as defined in (2.28). Notice that requiring finiteness of this identity between renormalised Green functions implies the same consistency condition previously derived in (2.29) from finiteness of the VEV $\langle T^\mu{}_\mu \rangle$ itself.

Finally, it is important for what follows to see how the $\tilde{\beta}_c$ terms have arisen here as a result of the Weyl variation of the Ricci scalar (see Appendix A). This is despite the fact that the integral of R over the whole spacetime is a topological invariant in two dimensions, which implies that its Weyl variation vanishes. To see this explicitly, note that under the Weyl rescaling $\delta g^{\mu\nu} = 2\sigma g^{\mu\nu}$, the Ricci scalar transforms in n dimensions as $\delta R = 2\sigma R + 2(n-1)D^2\sigma$. Then, using (2.15),

$$\frac{\delta}{\delta g^{\mu\nu}} \int d^n x \sqrt{-g} R = \sqrt{-g} G_{\mu\nu}, \quad (2.42)$$

where $G_{\mu\nu}$ is the Einstein tensor, and so

$$\frac{\delta}{\delta \sigma} \int d^n x \sqrt{-g} R = \left(1 - \frac{n}{2}\right) \sqrt{-g} R. \quad (2.43)$$

which indeed vanishes when $n = 2$.

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