

Chapter 2

Derivatives and Differentials of Intuitionistic Fuzzy Functions

Calculus, which is an important branch of classical mathematics, is the mathematical study of change. Like the calculus of real numbers and the complex numbers, the calculus of IFNs is very significant to the theory environment. Thus, this chapter aims to do work in the calculus in intuitionistic fuzzy environment. To begin with, we introduce the concept of intuitionistic fuzzy functions (IFFs) (Lei and Xu 2015b), which are the main research subjects of intuitionistic fuzzy calculus. Then, we study the derivatives and differentials of IFFs in detail.

Firstly, we introduce some topological knowledge of IFNs, which makes preparations for defining the limit and the continuity of functions of IFNs.

When considering the complex plane $\mathbb{C}$, let $a \in \mathbb{C}$, $r \in (0, +\infty)$, then we call the set

$$\{z : |z - a| < r, z \in \mathbb{C}\}$$

a neighborhood or r -neighborhood of a , denoted by $U(a, r)$. Moreover, the set

$$\{z : |z - a| \leq r\}$$

is called a closed neighborhood of a , denoted by $\overline{U}(a, r)$.

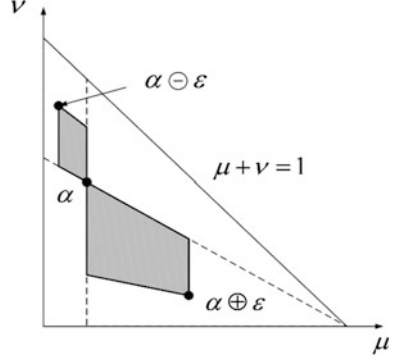
Hence, we can analogize the definition of neighborhood in the $\mathbb{C}$ to get a similar notion in $\mathbf{\Delta}$, which is described as follows:

Definition 2.1 Let $\alpha \in \mathbf{\Delta}$ and $\mathbf{0} \trianglelefteq \varepsilon \trianglelefteq \alpha$. Then we call the set

$$\{X : |X \ominus \alpha| \triangleleft \varepsilon, X \in \mathcal{S}_{\oplus}(\alpha) \cup \mathcal{S}_{\ominus}(\alpha)\}$$

a neighborhood or ε -neighborhood of α , denoted by $U(\alpha, \varepsilon)$, where

Fig. 2.1 Closed neighborhood of α



$$|X \ominus \alpha| = \begin{cases} X \ominus \alpha, & \text{if } X \in \mathcal{S}_{\oplus}(\alpha) \\ \alpha \ominus X, & \text{if } X \in \mathcal{S}_{\ominus}(\alpha) \end{cases}$$

In addition, $\{X : |X \ominus \alpha| \leq \varepsilon, X \in \mathcal{S}_{\oplus}(\alpha) \cup \mathcal{S}_{\ominus}(\alpha)\}$ is called a closed neighborhood of α , denoted by $\bar{U}(\alpha, \varepsilon)$, which is shown in Fig. 2.1.

For any given set $E \subset \mathbf{A}$, if for any ε ($0 \leq \varepsilon \leq \alpha$), there always exists an infinity of IFNs in the set $U(\alpha, \varepsilon) \cap E$, then we call α an accumulation point of E .

2.1 Intuitionistic Fuzzy Functions

In this section, we study the concept of functions related to IFNs (Lei and Xu, 2015b, c, 2016a).

Let E ($E \subset \mathbf{A}$) be a non-empty set, which consists of some IFNs. Then we call $\varphi : E \rightarrow \mathbf{A}$ an intuitionistic fuzzy function (IFF) defined in E , which is denoted by

$$Y = \varphi(X), X \in E$$

where E is the domain of φ , X is the independent variable, and Y is the dependent variable. According to the definition of IFF, we assume that an IFF φ consists of two real functions f and g , which are

$$\varphi(X) = (f(\mu, \nu), g(\mu, \nu)), X = (\mu, \nu) \in E$$

where f and g satisfy the conditions: $0 \leq f(\mu, \nu) \leq 1$, $0 \leq g(\mu, \nu) \leq 1$ and $0 \leq f(\mu, \nu) + g(\mu, \nu) \leq 1$ for any $(\mu, \nu) \in E$.

In addition, we call that the IFF $\varphi = (f, g)$ is meaningful at some points $X = (\mu, \nu)$ if only $f(\mu, \nu)$ and $g(\mu, \nu)$ of $\varphi(X) = (f(\mu, \nu), g(\mu, \nu))$ meet the above three inequalities, otherwise φ is meaningless at X . If φ is meaningful at all points

of one set, then we call that φ is meaningful in the set. Obviously, the IFF φ is meaningful in its domain E .

Reviewing the knowledge of derivative of real and complex functions, we know that its definition is the limit value of the expression $\frac{f(y)-f(x)}{y-x}$ when $y \rightarrow x$, where x and y are both real or complex numbers. However, for the IFNs, there exists a question whether $\varphi(Y) \ominus \varphi(X) = \left(\frac{\mu_{\varphi(Y)} - \mu_{\varphi(X)}}{1 - \mu_{\varphi(X)}}, \frac{\nu_{\varphi(Y)}}{\nu_{\varphi(X)}} \right)$ is still an IFN when $Y \ominus X$ is an IFN, which means that if $\varphi(Y)$ will fall into $S_{\oplus}(\varphi(X))$ when Y falls into $S_{\oplus}(X)$. Unfortunately, the answer about the question is negative. In order to solve the problem, the following definition is provided:

Definition 2.2 (Lei and Xu 2015a) Let $\varphi = (f, g)$ be an IFF in the set E , X and Y be both IFNs in E . If $X \trianglelefteq Y$, $\varphi(X) \trianglelefteq \varphi(Y)$ holds, then we call φ a monotonically increasing IFF.

Based on the concept of monotonically increasing IFF, we know that

$$\varphi(Y) \ominus \varphi(X) = \left(\frac{\mu_{\varphi(Y)} - \mu_{\varphi(X)}}{1 - \mu_{\varphi(X)}}, \frac{\nu_{\varphi(Y)}}{\nu_{\varphi(X)}} \right)$$

must be an IFN when $Y \ominus X$ is an IFN, which is just the property of IFFs we want. Considering $\varphi(X) = \lambda X (X \in \blacktriangle)$, we get that $\varphi(X) = \lambda X$ is a monotonically increasing IFF due to that $\varphi(Y) \ominus \varphi(X) = \lambda Y \ominus \lambda X = \lambda(Y \ominus X)$.

In addition, the intuitionistic fuzzy weighted aggregation (IFWA) operator was proposed by Xu (2007), which has the following form:

$$IFWA_{\omega}(\alpha_1, \alpha_2, \dots, \alpha_n) = \bigoplus_{i=1}^n \omega_i \alpha_i = \left(1 - \prod_{i=1}^n (1 - \mu_{\alpha_i})^{\omega_i}, \prod_{i=1}^n \nu_{\alpha_i}^{\omega_i} \right)$$

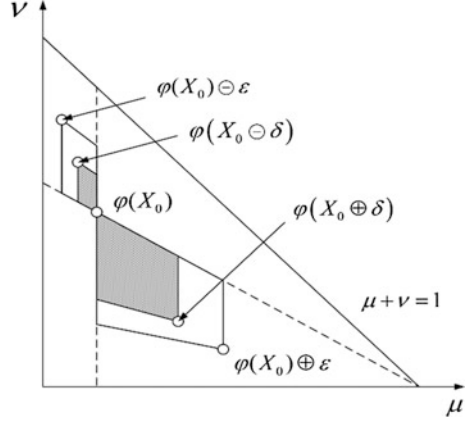
where α_i ($i = 1, 2, \dots, n$) is a collection of IFNs, and ω_i ($i = 1, 2, \dots, n$) is the weights of α_i ($i = 1, 2, \dots, n$) with $\omega_i \geq 0$ and $\sum_{i=1}^n \omega_i = 1$. Apparently, the function $IFWA_{\omega}(\alpha_1, \alpha_2, \dots, \alpha_n)$ is also a monotonically increasing IFF of any α_i .

It is worth noting that IFFs studied in this chapter are both assumed as monotonically increasing IFFs. Below we investigate the limit and continuity of IFFs (Lei and Xu 2015b):

We define $\varphi = (f, g)$ in the set E , and let X_0 be an accumulation point of E (Maybe $X_0 \notin E$) and α be an intuitionistic fuzzy constant, then for any given $\varepsilon \triangleright O$, there always exists an IFN $\delta \triangleright O$, such that if only $X \in U(X_0, \delta)$, we have $\varphi(X) \in U(\alpha, \varepsilon)$. We call $\varphi(X)$ approaches α when X approaches X_0 , which can be written by

$$\lim_{X \rightarrow X_0, X \in E} \varphi(X) = \alpha$$

According to the definition of neighborhood and the limit of IFFs, we can define the limit of IFFs from another angle:

Fig. 2.2 Continuity of $\varphi(X)$ 

For any given $\varepsilon \triangleright \mathbf{O}$, there is an IFN $\delta \triangleright \mathbf{O}$, such that when $\mathbf{O} \triangleleft |X \ominus X_0| \triangleleft \delta$, we have $|\varphi(X) \ominus \varphi(X_0)| \triangleleft \varepsilon$. Moreover, we introduce the continuity of IFFs as follows:

Assume that φ is an IFF in the set E , X_0 is an accumulation point of E and $X_0 \in E$, if $\lim_{X \rightarrow X_0, X \in E} \varphi(X) = \varphi(X_0)$, then we call $\varphi(X)$ to be continuous at X_0 . It means that for any given $\varepsilon \triangleright \mathbf{O}$, there always exists an IFN $\delta \triangleright \mathbf{O}$, such that if only $X \in U(X_0, \delta)$, then $\varphi(X) \in U(\varphi(X_0), \varepsilon)$, which can be shown in Fig. 2.2.

Because φ is a monotonically increasing IFF, we obtain that if $\varphi(X)$ is continuous at X_0 , then we have the following conclusions:

- (1) When $X \in \mathcal{S}_{\oplus}(X_0)$, $\lim_{X \rightarrow X_0, X \in E} \varphi(X) \ominus \varphi(X_0) = \mathbf{O}$, denoted as $\lim_{X \rightarrow X_0^{\oplus}, X \in E} \varphi(X) \ominus \varphi(X_0) = \mathbf{O}$.
- (2) When $X \in \mathcal{S}_{\ominus}(X_0)$, $\lim_{X \rightarrow X_0, X \in E} \varphi(X_0) \ominus \varphi(X) = \mathbf{O}$, denoted as $\lim_{X \rightarrow X_0^{\ominus}, X \in E} \varphi(X_0) \ominus \varphi(X) = \mathbf{O}$.

The above conclusions are similar to the left-continuity and the right-continuity of real functions: $\lim_{x \rightarrow x_0^-} f(x) = f(x_0)$ and $\lim_{x \rightarrow x_0^+} f(x) = f(x_0)$. Consequently, we can define the continuity of IFFs as follows:

For any given $\varepsilon \triangleright \mathbf{O}$, there is an IFN $\delta \triangleright \mathbf{O}$, such that when $\mathbf{O} \triangleleft |X \ominus X_0| \triangleleft \delta$, we have

$$\lim_{X \rightarrow X_0, X \in E} |\varphi(X) \ominus \varphi(X_0)| = \mathbf{O}$$

Obviously, the functions $\varphi(X) = \lambda X$ and $IFWA_{\omega}(\alpha_1, \alpha_2, \dots, \alpha_n)$ are both continuous.

2.2 Derivatives of Intuitionistic Fuzzy Functions

Based on the discussions about the IFFs in the last section, this section studies the derivatives of IFFs.

By analogizing the definitions of derivatives of real and complex functions, which are both defined as the limit of $\frac{f(y)-f(x)}{y-x}$ (that is $\lim_{y \rightarrow x} \frac{f(y)-f(x)}{y-x}$), we can give a definition about the derivative of IFF as follows:

Definition 2.3 (Lei and Xu 2015b) Let $\varphi = (f, g)$ be a monotonically increasing IFF defined in the set E , X be an accumulation point of E (Maybe there is $X \notin E$). If $\lim_{Y \rightarrow X^{\oplus}, Y \in E} \frac{\varphi(Y) \ominus \varphi(X)}{Y \ominus X}$ is still an IFN, then we call it the right derivative of φ at X , denoted by $\varphi'_{\oplus}(X)$. Similarly, $\lim_{Y \rightarrow X^{\ominus}, Y \in E} \frac{\varphi(X) \ominus \varphi(Y)}{X \ominus Y}$ is the left derivative if it is an IFN, which can be noted by $\varphi'_{\ominus}(X)$. In addition, if the left and the right derivatives are both IFNs and equal to each other, then we call φ is derivable at X and $\lim_{Y \rightarrow X, Y \in E} \frac{|\varphi(Y) \ominus \varphi(X)|}{|Y \ominus X|}$ is the derivative of φ at X , denoted by $\frac{d\varphi(X)}{dX}$. If φ exists the derivative at every point in a set E , then we call that φ is derivable in E .

Moreover, another form of the left and right derivatives can be expressed as:

$$\lim_{Y \rightarrow X^{\oplus}} \frac{\varphi(Y) \ominus \varphi(X)}{Y \ominus X} \Leftrightarrow \lim_{\Delta X \rightarrow 0} \frac{\varphi(X \oplus \Delta X) \ominus \varphi(X)}{\Delta X}$$

and

$$\lim_{Y \rightarrow X^{\ominus}} \frac{\varphi(X) \ominus \varphi(Y)}{X \ominus Y} \Leftrightarrow \lim_{\Delta X \rightarrow 0} \frac{\varphi(X) \ominus \varphi(X \ominus \Delta X)}{\Delta X}$$

After defining the derivative of φ , we give the following theorem:

Theorem 2.1 (Lei and Xu 2015b) Let $\varphi(X) = (f(\mu, v), g(\mu, v))$ be a monotonically increasing IFF in the set E , If φ is derivable in E , if and only if φ meets:

$$\frac{\partial f(\mu, v)}{\partial v} = \frac{\partial g(\mu, v)}{\partial \mu} = 0 \quad \text{and} \quad 0 \leq \frac{1 - \mu}{1 - f(\mu, v)} \frac{\partial f(\mu, v)}{\partial \mu} \leq \frac{v}{g(\mu, v)} \frac{\partial g(v)}{\partial v} \leq 1$$

Due to $f'_v = g'_\mu = 0$, $\varphi(X)$ can be written as $(f(\mu), g(v))$. Then, the derivative of φ has the following form:

$$\frac{d\varphi(X)}{dX} = \lim_{Y \rightarrow X} \frac{|\varphi(Y) \ominus \varphi(X)|}{|Y \ominus X|} = \left(\frac{1 - \mu}{1 - f(\mu)} \frac{df(\mu)}{d\mu}, 1 - \frac{v}{g(v)} \frac{dg(v)}{dv} \right)$$

Proof We first prove the right derivative of φ , which is $\lim_{Y \rightarrow X^{\oplus}} \frac{\varphi(Y) \ominus \varphi(X)}{Y \ominus X}$, where $Y \in \mathcal{S}_{\oplus}(X)$.

Assume that $Y = (\mu + \Delta\mu, v + \Delta v) = (\mu', v') \in \mathcal{S}_{\oplus}(X)$, which means $Y \triangleright X$, then there is $\varphi(Y) = (f(\mu + \Delta\mu, v + \Delta v), g(\mu + \Delta\mu, v + \Delta v)) = (f(\mu', v'), g(\mu', v'))$. Hence, we can get the following derivation process:

$$\begin{aligned}
 \frac{d\varphi(X)}{dX} &= \lim_{Y \rightarrow X} \frac{\varphi(Y) \ominus \varphi(X)}{Y \ominus X} \\
 &= \lim_{\substack{\mu' \rightarrow \mu \\ v' \rightarrow v}} \frac{(f(\mu', v'), g(\mu', v')) \ominus (f(\mu, v), g(\mu, v))}{(\mu', v') \ominus (\mu, v)} \\
 &= \lim_{\substack{\mu' \rightarrow \mu \\ v' \rightarrow v}} \frac{\left(\frac{f(\mu', v') - f(\mu, v)}{1 - f(\mu, v)}, \frac{g(\mu', v')}{g(\mu, v)} \right)}{\left(\frac{\mu' - \mu}{1 - \mu}, \frac{v'}{v} \right)} \\
 &= \lim_{\substack{\mu' \rightarrow \mu \\ v' \rightarrow v}} \left(\frac{f(\mu', v') - f(\mu, v)}{1 - f(\mu, v)} \frac{1 - \mu}{\mu' - \mu}, \frac{\frac{g(\mu', v')}{g(\mu, v)} - \frac{v'}{v}}{1 - \frac{v'}{v}} \right) \\
 &= \left(\lim_{\substack{\mu' \rightarrow \mu \\ v' \rightarrow v}} \frac{f(\mu', v') - f(\mu, v)}{1 - f(\mu, v)} \frac{1 - \mu}{\mu' - \mu}, \lim_{\substack{\mu' \rightarrow \mu \\ v' \rightarrow v}} \frac{\frac{g(\mu', v')}{g(\mu, v)} - \frac{v'}{v}}{1 - \frac{v'}{v}} \right)
 \end{aligned}$$

Next, we respectively discuss the membership and the non-membership of the above equation in detail. For the membership part, it can be simplified as:

$$\begin{aligned}
 &\lim_{\substack{\mu' \rightarrow \mu \\ v' \rightarrow v}} \frac{f(\mu', v') - f(\mu, v)}{1 - f(\mu, v)} \frac{1 - \mu}{\mu' - \mu} \\
 &= \lim_{\substack{\mu' \rightarrow \mu \\ v' \rightarrow v}} \frac{1 - \mu}{1 - f(\mu, v)} \frac{f(\mu', v') - f(\mu, v)}{\mu' - \mu} \\
 &= \lim_{\substack{\mu' \rightarrow \mu \\ v' \rightarrow v}} \frac{1 - \mu}{1 - f(\mu, v)} \left(\frac{f(\mu', v') - f(\mu, v)}{\mu' - \mu} + \frac{f(\mu, v') - f(\mu, v)}{v' - v} \frac{v' - v}{\mu' - \mu} \right) \\
 &= \frac{1 - \mu}{1 - f(\mu, v)} \left(\frac{\partial f(\mu, v)}{\partial \mu} + \frac{\partial f(\mu, v)}{\partial v} \cos \theta \right)
 \end{aligned}$$

and for the non-membership part, it can be calculated as:

$$\begin{aligned}
\lim_{\substack{\mu' \rightarrow \mu \\ v' \rightarrow v}} \frac{\frac{g(\mu', v') - v'}{g(\mu, v)} - \frac{v'}{v}}{1 - \frac{v'}{v}} &= \lim_{\substack{\mu' \rightarrow \mu \\ v' \rightarrow v}} \frac{v}{g(\mu, v)} \frac{g(\mu', v') - g(\mu', v)}{v - v'} + \frac{\frac{v}{g(\mu, v)} g(\mu', v) - v'}{v - v'} \\
&= \lim_{\substack{\mu' \rightarrow \mu \\ v' \rightarrow v}} \frac{v}{g(\mu, v)} \frac{g(\mu', v') - g(\mu', v)}{v - v'} \\
&\quad + \frac{\frac{v}{g(\mu, v)} g(\mu', v) - \frac{v}{g(\mu, v)} g(\mu, v) + \frac{v}{g(\mu, v)} g(\mu, v) - v'}{v - v'} \\
&= \lim_{\substack{\mu' \rightarrow \mu \\ v' \rightarrow v}} \frac{v}{g(\mu, v)} \frac{g(\mu', v') - g(\mu', v)}{v - v'} + \frac{v}{g(\mu, v)} \frac{g(\mu', v) - g(\mu, v)}{\mu - \mu'} \frac{\mu - \mu'}{v - v'} + 1 \\
&= 1 - \frac{v}{g(\mu, v)} \left(\frac{\partial g(\mu, v)}{\partial v} + \frac{\partial g(\mu, v)}{\partial \mu} \frac{1}{\cos \theta} \right)
\end{aligned}$$

Therefore, the right derivative of φ can be written as:

$$\varphi'_{\oplus} = \left(\frac{1 - \mu}{1 - f(\mu, v)} \left(\frac{\partial f(\mu, v)}{\partial \mu} + \frac{\partial f(\mu, v)}{\partial v} \cos \theta \right), 1 - \frac{v}{g(\mu, v)} \left(\frac{\partial g(\mu, v)}{\partial v} + \frac{\partial g(\mu, v)}{\partial \mu} \frac{1}{\cos \theta} \right) \right)$$

In order to make the above expression of $\varphi'_{\oplus}$ not depend on the change of $\cos \theta$, we let

$$\frac{\partial f(\mu, v)}{\partial v} = \frac{\partial g(\mu, v)}{\partial \mu} = 0$$

Based on the condition $f'_v = g'_\mu = 0$, we note $\varphi(X) = (f(\mu), g(v))$. Moreover, the expression of $\varphi'_{\oplus}$ is

$$\varphi'_{\oplus} = \lim_{Y \rightarrow X^{\oplus}} \frac{\varphi(Y) \ominus \varphi(X)}{Y \ominus X} = \left(\frac{1 - \mu}{1 - f(\mu)} \frac{df(\mu)}{d\mu}, 1 - \frac{v}{g(v)} \frac{dg(v)}{dv} \right)$$

Similarly, we can also prove $\varphi'_{\ominus}$, and it can be represented as:

$$\varphi'_{\ominus} = \lim_{Y \rightarrow X^{\ominus}} \frac{\varphi(X) \ominus \varphi(Y)}{X \ominus Y} = \left(\frac{1 - \mu}{1 - f(\mu)} \frac{df(\mu)}{d\mu}, 1 - \frac{v}{g(v)} \frac{dg(v)}{dv} \right)$$

Hence, if only f'_μ and g'_v exist, the derivative of φ exists and

$$\varphi' = \varphi'_{\oplus} = \varphi'_{\ominus} = \left(\frac{1 - \mu}{1 - f(\mu)} \frac{df(\mu)}{d\mu}, 1 - \frac{v}{g(v)} \frac{dg(v)}{dv} \right)$$

Moreover, in order to make $\left(\frac{1-\mu}{1-f(\mu)} \frac{df(\mu)}{d\mu}, 1 - \frac{v}{g(v)} \frac{dg(v)}{dv}\right)$ to be still an IFN, the inequalities $0 \leq \frac{1-\mu}{1-f(\mu)} \frac{df(\mu)}{d\mu} \leq \frac{v}{g(v)} \frac{dg(v)}{dv} \leq 1$ need hold. ■

Theorem 2.1 gives the criterion to judge whether an IFF is derivable, which is similar to the “C-R condition” in the complex number field.

In what follows, we introduce several examples (Lei and Xu 2015b):

- (1) For $\varphi(X) = \alpha_0$, where α_0 is a constant. Then, its derivative $\frac{d\varphi(X)}{dX} = \mathbf{O}$, which is similar to $\frac{df(x)}{dx} = 0$ for the real constant function $f(x) = c$.
- (2) Let $\varphi(X) = \lambda X$, then we have $\frac{d\varphi(X)}{dX} = (\lambda, 1 - \lambda)$. Specially, for the identity IFF $\varphi(X) = X$, there is $\frac{d\varphi(X)}{dX} = \mathbf{E}$. It is like the situation that $\frac{df(x)}{dx} = 1$ when $f(x) = x$.
- (3) When considering $\varphi(\alpha_i) = IFWA_{\omega}(\alpha_1, \alpha_2, \dots, \alpha_n) (i = 1, 2, \dots, n)$, there is $\frac{dIFWA_{\omega}(\alpha_1, \alpha_2, \dots, \alpha_n)}{d\alpha_i} = (\omega_i, 1 - \omega_i)$, which expresses the information of the weights of the independent variable α_i .

For the real functions $f(x) = kx + a$ and $g(x) = kx + b$, there is $f'(x) = g'(x)$. For IFFs, the similar conclusion (Lei and Xu 2015b) can be obtained by the following two illustrations:

- (4) For $\varphi(X) = \lambda X \oplus \alpha$ and $\psi(X) = \lambda X \oplus \beta$, there is $\frac{d\varphi(X)}{dX} = \frac{d\psi(X)}{dX}$.
- (5) For $\varphi(X) = K \otimes X \oplus \alpha$ and $\psi(X) = K \otimes X \oplus \beta$, there is $\frac{d\varphi(X)}{dX} = \frac{d\psi(X)}{dX}$.

In the following, the concept of elasticity coefficient, which is usually used in economics, will be introduced to analyze the derivative of IFFs:

Let $y = f(x)$ be a real function, then we call

$$E_{f(x)} = \lim_{\Delta x \rightarrow 0} \frac{\frac{\Delta y}{y}}{\frac{\Delta x}{x}} = \frac{\frac{dy}{y}}{\frac{dx}{x}} = \frac{dy}{dx} \frac{x}{y}$$

the elasticity coefficient of $f(x)$ at x .

According to the definition of elasticity coefficient, we get that the elasticity coefficient actually describes what percentage of the dependent variable y will change if 1% of x happens. It is just an indicator to depict the sensitivity of y to the argument x . Moreover, there is also

$$\frac{1-\mu}{1-f(\mu)} \frac{df(\mu)}{d\mu} = \frac{\mu}{1-f(\mu)} \frac{d(1-f(\mu))}{d\mu} \frac{d\mu}{d(1-\mu)} \frac{1-\mu}{\mu} = \frac{E_{1-f(\mu)}}{E_{1-\mu}}$$

and

$$1 - \frac{v}{g(v)} \frac{dg(v)}{dv} = 1 - \frac{E_{g(v)}}{E_v}$$

Thus, the expression of the derivative of $\varphi(X)$ can be written (Lei and Xu 2015b) as

$$\frac{d\varphi(X)}{dX} = \left(\frac{1-\mu}{1-f(\mu)} \frac{df(\mu)}{d\mu}, 1 - \frac{v}{g(v)} \frac{dg(v)}{dv} \right) = \left(\frac{E_{1-f(\mu)}}{E_{1-\mu}}, 1 - \frac{E_{g(v)}}{E_v} \right)$$

If we understand the IFF $\varphi : (\mu, v) \rightarrow (f(\mu), g(v))$ by utilizing its interval form introduced in Chap. 1, that is, $\varphi : [v, 1-\mu] \rightarrow [g(v), 1-f(\mu)]$, then we can get that the derivative of $\varphi(X)$ at X essentially depicts that the reaction extent of intervals endpoints of $[g(v), 1-f(\mu)]$ when the endpoints of $[v, 1-\mu]$ change (Lei and Xu 2015b).

Below we introduce the notion of the compound IFFs, and the chain rule of derivatives:

Definition 2.4 (Lei and Xu 2015a) Let $\varphi(X)$ be an IFF defined in a set B , and $X(t)$ be an IFF in a set A . Then, the set G is a non-empty of A that satisfies $X(t) \in B$ for any $t \in G$, which means

$$G = \{t | t \in A, X(t) \in B\} \neq \emptyset$$

For any $t \in G$, according to the corresponding relation X , there is an IFN $X(t)$ belonging to B . Then according to the corresponding relation φ , there is an IFN $\varphi(X)$. Hence, we define an IFF $\varphi \circ X$ in the set G as a compound IFF, which is

$$(\varphi \circ X)(t) = \varphi(X(t)), t \in G$$

where X is called the middle variable.

It needs to point out that if there are no special instructions, both the IFFs φ and X in a compound IFF $\varphi \circ X$ are monotonically increasing IFFs.

Theorem 2.2 (Lei and Xu 2015a) *Let $\varphi(X(t))$ be a compound IFF. If the derivatives of φ and X both exist, then the derivative of $\varphi(X(t))$ also exists, and*

$$\frac{d\varphi(X(t))}{dt} = \frac{d\varphi(X(t))}{dX(t)} \otimes \frac{dX(t)}{dt}$$

Proof We prove it in two different ways:

Method 1. Firstly, we can give a proof based on the definition of derivative of IFFs. For the right derivative, there is

$$\begin{aligned}
\frac{d\varphi(X(t))}{dt} &= \lim_{\Delta t \rightarrow 0} \frac{\varphi(X(t \oplus \Delta t)) \ominus \varphi(X(t))}{\Delta t} \\
&= \lim_{\Delta t \rightarrow 0} \left(\frac{\varphi(X(t \oplus \Delta t)) \ominus \varphi(X(t))}{X(t \oplus \Delta t) \ominus X(t)} \otimes \frac{X(t \oplus \Delta t) \ominus X(t)}{\Delta t} \right) \\
&= \lim_{\Delta t \rightarrow 0} \frac{\varphi(X(t \oplus \Delta t)) \ominus \varphi(X(t))}{X(t \oplus \Delta t) \ominus X(t)} \otimes \lim_{\Delta t \rightarrow 0} \frac{X(t \oplus \Delta t) \ominus X(t)}{\Delta t} \\
&= \frac{d\varphi(X(t))}{dX(t)} \otimes \frac{dX(t)}{dt}
\end{aligned}$$

Similarly, we can get the same conclusion for the left derivative.

Method 2. We study its membership and non-membership, respectively. The proof is shown as follows:

Assume that $\varphi(\alpha) = (f_\varphi(\mu_\alpha), g_\varphi(v_\alpha))$ and $X(t) = (f_X(\mu_t), g_X(v_t))$ are the IFFs of the independent $\alpha = (\mu_\alpha, v_\alpha)$ and $t = (\mu_t, v_t)$, respectively, and $\varphi(X(t)) = (f_\varphi(f_X(\mu_t)), g_\varphi(g_X(v_t)))$ is a compound IFF, then

$$\begin{aligned}
\frac{d\varphi(X(t))}{dt} &= \left(\frac{1 - \mu_t}{1 - f_\varphi(f_X(\mu_t))} \frac{df_\varphi(f_X(\mu_t))}{d\mu_t}, 1 - \frac{v_t}{g_\varphi(g_X(v_t))} \frac{dg_\varphi(g_X(v_t))}{dv_t} \right) \\
&= \left(\frac{1 - \mu_t}{1 - f_X(\mu_t)} \frac{df_X(\mu_t)}{d\mu_t} \frac{1 - f_X(\mu_t)}{1 - f_\varphi(f_X(\mu_t))} \frac{df_\varphi(f_X(\mu_t))}{df_X(\mu_t)}, \right. \\
&\quad \left. 1 - \left(1 - \frac{v_t}{g_X(v_t)} \frac{dg_X(v_t)}{dv_t} \right) \left(1 - \frac{g_X(v_t)}{g_\varphi(g_X(v_t))} \frac{dg_\varphi(g_X(v_t))}{dg_X(v_t)} \right) \right) \\
&= \left(\frac{1 - \mu_t}{1 - f_X(\mu_t)} \frac{df_X(\mu_t)}{d\mu_t}, 1 - \frac{v_t}{g_X(v_t)} \frac{dg_X(v_t)}{dv_t} \right) \\
&\quad \otimes \left(\frac{1 - f_X(\mu_t)}{1 - f_\varphi(f_X(\mu_t))} \frac{df_\varphi(f_X(\mu_t))}{df_X(\mu_t)}, 1 - \frac{g_X(v_t)}{g_\varphi(g_X(v_t))} \frac{dg_\varphi(g_X(v_t))}{dg_X(v_t)} \right) \\
&= \frac{d\varphi(X(t))}{dX(t)} \otimes \frac{dX(t)}{dt}
\end{aligned}$$

which completes the proof of the theorem. ■

According to Theorem 2.2, the derivative of the compound function is also equal to the product of the derivatives of the components in intuitionistic fuzzy calculus, which is the same as one of the traditional calculus. Furthermore, we provide some examples (Lei & Xu, 2015a) as follows:

- (1) Let $\varphi_1(\alpha) = \lambda\alpha \oplus C$, $\varphi_2(\alpha) = \lambda\alpha$, $\varphi_3(\alpha) = \alpha \oplus C$ and $\varphi_4(\alpha) = \alpha$ be four IFFs, where $0 \leq \lambda \leq 1$. Then, according to the derivatives of IFFs, there are $\frac{d\varphi_1(\alpha)}{d\alpha} = (\lambda, 1 - \lambda)$, $\frac{d\varphi_2(\alpha)}{d\alpha} = (\lambda, 1 - \lambda)$, $\frac{d\varphi_3(\alpha)}{d\alpha} = E$ and $\frac{d\varphi_4(\alpha)}{d\alpha} = E$. Moreover, we can conduct the following conclusions:

(a) If $\varphi_1(\alpha) = \lambda\alpha \oplus C$ is expressed as $\varphi_1(\varphi_4(\alpha)) = \lambda\varphi_4(\alpha) \oplus C$, then there is

$$\begin{aligned} \frac{d\varphi_1(\alpha)}{d\alpha} &= \frac{d\varphi_1(\varphi_4(\alpha))}{d\alpha} \\ &= \frac{d\varphi_1(\varphi_4(\alpha))}{d\varphi_4(\alpha)} \otimes \frac{d\varphi_4(\alpha)}{d\alpha} = \frac{d\varphi_1(\alpha)}{d\alpha} \otimes \frac{d\varphi_4(\alpha)}{d\alpha} \\ &= (\lambda, 1 - \lambda) \otimes E = (\lambda, 1 - \lambda) \end{aligned}$$

(b) If $\varphi_1(\alpha) = \lambda\alpha \oplus C$ is written as $\varphi_1(\varphi_3(\alpha)) = \varphi_3(\alpha) \oplus C$, then

$$\frac{d\varphi_1(\alpha)}{d\alpha} = \frac{d\varphi_1(\varphi_3(\alpha))}{d\alpha} = \frac{d\varphi_1(\varphi_3(\alpha))}{d\varphi_3(\alpha)} \otimes \frac{d\varphi_3(\alpha)}{d\alpha} = (\lambda, 1 - \lambda)$$

(2) Considering $\varphi(\alpha) = \lambda\alpha \oplus C$ and $IFWA_\omega(\alpha_1, \alpha_2, \dots, \alpha_n) = \bigoplus_{i=1}^n \omega_i \alpha_i$, then when regarding α_1 as its independent variable, the derivative of $\varphi(IFWA_\omega(\alpha_1, \alpha_2, \dots, \alpha_n))$ can be calculated:

$$\begin{aligned} &\frac{d\varphi(IFWA_\omega(\alpha_1, \alpha_2, \dots, \alpha_n))}{d\alpha_1} \\ &= \frac{d\varphi(IFWA_\omega(\alpha_1, \alpha_2, \dots, \alpha_n))}{dIFWA_\omega(\alpha_1, \alpha_2, \dots, \alpha_n)} \otimes \frac{dIFWA_\omega(\alpha_1, \alpha_2, \dots, \alpha_n)}{d\alpha_1} \\ &= (\lambda, 1 - \lambda) \otimes (\lambda_1, 1 - \lambda_1) \\ &= (\lambda\lambda_1, 1 - \lambda\lambda_1) \end{aligned}$$

What's more, we know $\varphi(IFWA_\omega(\alpha_1, \alpha_2, \dots, \alpha_n)) = \bigoplus_{i=1}^n \lambda\lambda_i \alpha_i \oplus C$, then

$$\frac{d\varphi(IFWA_\omega(\alpha_1, \alpha_2, \dots, \alpha_n))}{d\alpha_1} = (\lambda\lambda_1, 1 - \lambda\lambda_1)$$

Before introducing the properties of derivatives of IFFs, we first show several concepts as follows:

Definition 2.5 (Lei and Xu 2015b) For any given IFN $\alpha = (\mu, \nu)$, we define $U(\alpha) = \mu$, $V(\alpha) = \nu$ and $\pi(\alpha) = 1 - \mu - \nu$.

Based on Definition 2.5, we get $U(\alpha) = 0.1$, $V(\alpha) = 0.6$ and $\pi(\alpha) = 0.3$ for $\alpha = (0.1, 0.6)$. In addition, any IFN $\alpha = (\mu, \nu)$ can be denoted by $\alpha = (U(\alpha), V(\alpha))$. Similarly, any IFF $\varphi(X) = (f(\mu), g(\nu))$ can be written as $\varphi(X) = (U(\varphi(X)), V(\varphi(X)))$.

Theorem 2.3 (Lei and Xu 2015a) Let $\varphi(X)$ and $\varphi_i(X)$ ($i = 1, 2, \dots, n$) be $n + 1$ derivable IFFs, then

(1) $\frac{d}{dX} \left(\bigoplus_{i=1}^n \varphi_i(X) \right) = \left(\sum_{i=1}^n U(\varphi'_i(X)), 1 - \sum_{i=1}^n (1 - V(\varphi'_i(X))) \right)$, where $\varphi'_i(X)$ represents the derivative of $\varphi_i(X)$, which is $\frac{d\varphi_i(X)}{dX}$.

(2) If $\varphi_i(X) \trianglelefteq \varphi_j(X)$ ($1 \leq i, j \leq n$), then

$$\frac{d}{dX} (\varphi_j(X) \ominus \varphi_i(X)) = (U(\varphi'_j(X)) - U(\varphi'_i(X)), 1 - (V(\varphi'_i(X)) - V(\varphi'_j(X))))$$

(3) $\frac{d}{dX} (\lambda \varphi(X)) = (\lambda, 1 - \lambda) \otimes \frac{d\varphi(X)}{dX}$, where $0 \leq \lambda \leq 1$.

Proof We can prove the conclusion (1) based on the derivatives of IFFs. When $i = 1, 2$, there is

$$\varphi_1(X) \oplus \varphi_2(X) = (f_1(\mu) + f_2(\mu) - f_1(\mu)f_2(\mu), g_1(v)g_2(v))$$

Hence, $\frac{d}{dX} \left(\bigoplus_{i=1}^2 \varphi_i(X) \right) = \frac{d}{dX} (f_1(\mu) + f_2(\mu) - f_1(\mu)f_2(\mu), g_1(v)g_2(v))$. Then, its membership and non-membership degrees are respectively discussed as follows:

(a) Considering the membership degree of $\frac{d}{dX} (f_1(\mu) + f_2(\mu) - f_1(\mu)f_2(\mu), g_1(v)g_2(v))$, we have

$$\frac{1 - \mu}{(1 - f_1(\mu))(1 - f_2(\mu))} (f'_1(\mu)(1 - f_2(\mu)) + f'_2(\mu)(1 - f_1(\mu))) = \sum_{i=1}^2 U(\varphi'_i(X))$$

(b) For its non-membership degree, there is

$$1 - \frac{v}{g_1(v)g_2(v)} (g'_1(v)g_2(v) + g_1(v)g'_2(v)) = 1 - \sum_{i=1}^n V(\varphi'_i(X))$$

On the basis of (a) and (b), we get that the conclusion (1) holds when $i = 1, 2$. Moreover, according to the mathematical induction, it is easy to prove that it also holds when $i = 1, 2, \dots, n$.

For the conclusion (2), we also investigate its membership and non-membership degrees, respectively, whose proofs are similar to those of the conclusion (1), and are omitted here.

In what follows, we prove the conclusion (3) by utilizing the chain rule of derivatives:

$$\frac{d(\lambda \varphi(X))}{dX} = \frac{d(\lambda \varphi(X))}{d\varphi(X)} \otimes \frac{d\varphi(X)}{dX} = (\lambda, 1 - \lambda) \otimes \frac{d\varphi(X)}{dX}$$

Thus, the proof of this theorem is completed. ■

2.3 Differentials of Intuitionistic Fuzzy Functions

Let $\varphi(X)$ be an IFF, and the function value $\varphi(X_0)$ is derivable at X_0 . If we want to get the function values of some points near X_0 , such as $\varphi(X_0 \oplus \Delta X)$ at $X_0 \oplus \Delta X$, it is usual to acquire the approximation of $\varphi(X_0 \oplus \Delta X)$ instead of the precise value of $\varphi(X_0 \oplus \Delta X)$ in practical applications. Hence, we will focus on the methods to calculate the approximate of IFFs in this chapter.

For a monotonically increasing IFF $\varphi(X)$, $\Delta Y = \varphi(X_0 \oplus \Delta X) \ominus \varphi(X_0)$ or $\varphi(X_0 \oplus \Delta X) = \varphi(X_0) \oplus \Delta Y$, then we only need to handle the approximation of ΔY in order to get the expression $\varphi(X_0 \oplus \Delta X)$. Obviously, ΔY is related to ΔX . To facilitate the calculation, ΔY will be replaced by a simple IFF depending on ΔX .

Now we give a definition of differential of IFFs as follows:

Definition 2.6 (Lei and Xu 2015b) Let $\varphi(X)$ be a derivable IFF. If we note a tiny change of X as ΔX , then

$$d\varphi(X) = \frac{d\varphi(X)}{dX} \otimes \Delta X$$

is called the differential of $\varphi(X)$. In addition, $dX = E \otimes \Delta X = \Delta X$ due to $\frac{dX}{dX} = E$ for the identity IFF $\varphi(X) = X$. Therefore, the differential can be rewritten as:

$$d\varphi(X) = \frac{d\varphi(X)}{dX} \otimes dX$$

Based on the differential of $\varphi(X)$, the following theorem is provided to reveal the relationship between the increment of φ ($\Delta\varphi(X)$) and its differential ($d\varphi(X)$):

Theorem 2.4 (Lei and Xu 2015b) Let $\varphi(X) = (f(\mu), g(v))$ be a monotonically increasing IFF, and also be derivable, then

$$\varphi(Y) \ominus \varphi(X) \approx \frac{d\varphi(X)}{dX} \otimes (Y \ominus X)$$

where $X \leq Y$. If we note $\Delta X = Y \ominus X = \left(\frac{\Delta\mu}{1-\mu}, 1 - \frac{\Delta v}{v}\right)$ and $\Delta\varphi = \varphi(Y) \ominus \varphi(X)$, then the equality can be actually expressed as $\Delta\varphi \approx d\varphi$, which satisfies the following conditions:

$$\lim_{\Delta\mu \rightarrow 0} \frac{U(\Delta\varphi) - U(d\varphi)}{\Delta\mu} = 0, \quad \lim_{\Delta v \rightarrow 0} \frac{V(\Delta\varphi) - V(d\varphi)}{\Delta v} = 0$$

Proof If $X \leq Y$, X and Y are expressed as (μ, v) and (μ', v') , respectively, then

$$\Delta X = Y \ominus X = (\mu', v') \ominus (\mu, v) = \left(\frac{\mu' - \mu}{1 - \mu}, \frac{v'}{v} \right)$$

and there is

$$\frac{d\varphi(X)}{dX} = \left(\frac{1 - \mu}{1 - f(\mu)} \frac{df(\mu)}{d\mu}, 1 - \frac{v}{g(v)} \frac{dg(v)}{dv} \right)$$

Hence, we have

$$d\varphi = \frac{d\varphi(X)}{dX} \otimes \Delta X = \left(\frac{\mu' - \mu}{1 - f(\mu)} \frac{df(\mu)}{d\mu}, 1 - \frac{v - v'}{g(v)} \frac{dg(v)}{dv} \right)$$

Furthermore

$$\begin{aligned} \varphi(X) \oplus \frac{d\varphi(X)}{dX} \otimes \Delta X &= \left(f(\mu) + \frac{\mu' - \mu}{1 - f(\mu)} \frac{df(\mu)}{d\mu} - f(\mu) \frac{\mu' - \mu}{1 - f(\mu)} \frac{df(\mu)}{d\mu}, g(v) \left(1 - \frac{v - v'}{g(v)} \frac{dg(v)}{dv} \right) \right) \\ &= \left(f(\mu) + (\mu' - \mu) \frac{df(\mu)}{d\mu}, g(v) + (v' - v) \frac{dg(v)}{dv} \right) \\ &= (f(\mu) + (f(\mu') - f(\mu) + o(\mu' - \mu)), g(v) + (g(v') - g(v) + o(v' - v))) \\ &= (f(\mu') + o(\mu' - \mu), g(v') + o(v' - v)) \\ &\approx F(Y) \end{aligned}$$

Hence, $\varphi(Y) \ominus \varphi(X) \approx \frac{d\varphi(X)}{dX} \otimes (Y \ominus X)$ holds, which satisfies the conditions:
 $\lim_{\Delta\mu \rightarrow 0} \frac{U(\Delta\varphi) - U(d\varphi)}{\Delta\mu} = 0$ and $\lim_{\Delta v \rightarrow 0} \frac{V(\Delta\varphi) - V(d\varphi)}{\Delta v} = 0$. ■

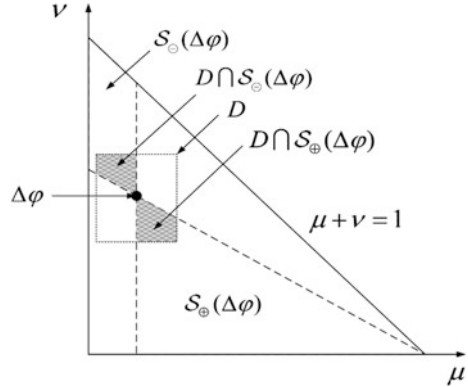
Theorem 2.4 describes the fact that $d\varphi$ closely approximates to $\Delta\varphi$ by comparing the membership degrees ($U(\Delta\varphi)$ and $U(d\varphi)$) and the non-membership degrees ($V(\Delta\varphi)$ and $V(d\varphi)$) of $d\varphi$ and $\Delta\varphi$, respectively. Moreover, this theorem manifests that the differences $U(\Delta\varphi) - U(d\varphi)$ and $V(\Delta\varphi) - V(d\varphi)$ are the infinitesimals of higher order of $\Delta\mu$ and Δv , respectively. However, we doubt whether the following hold:

$$\varphi(Y) \ominus \varphi(X) = \frac{d\varphi(X)}{dX} \otimes (Y \ominus X) \oplus \varepsilon$$

or

$$\varphi(Y) \ominus \varphi(X) = \frac{d\varphi(X)}{dX} \otimes (Y \ominus X) \ominus \varepsilon$$

Fig. 2.3 The relationship between $d\phi$ and $\Delta\phi$



which is $\Delta\phi = d\phi \oplus \varepsilon$ or $\Delta\phi = d\phi \ominus \varepsilon$. In order to investigate this problem, Fig. 2.3 is provided:

Because $U(\Delta\phi) - U(d\phi)$ and $V(\Delta\phi) - V(d\phi)$ are respectively the infinitesimals of higher order of $\Delta\mu$ and $\Delta\nu$, hence, we know that $d\phi$ must fall into a rectangular area like D in Fig. 2.2. If $d\phi$ is located in $D \cap S_{\oplus}(\Delta\phi)$, then $\Delta\phi = d\phi \ominus \varepsilon$. In addition, $\Delta\phi = d\phi \oplus \varepsilon$ if only $d\phi$ falls into $D \cap S_{\ominus}(\Delta\phi)$. Generally, there is not always $\Delta\phi = d\phi \oplus \varepsilon$ or $\Delta\phi = d\phi \ominus \varepsilon$.

In the following, we analyze the differentials of IFFs by utilizing the elasticity coefficient. The formula in Theorem 2.4 can be transformed into the interval form (Lei & Xu, 2015b) as follows:

$$\begin{aligned}
 \phi(Y) \ominus \phi(X) &\approx \frac{d\phi(X)}{dX} \otimes (Y \ominus X) \\
 &\Leftrightarrow \left(\frac{f(\mu') - f(\mu)}{1 - f(\mu)}, \frac{g(v')}{g(v)} \right) \approx \left(\frac{1 - \mu}{1 - f(\mu)} \frac{df(\mu)}{d\mu}, 1 - \frac{v}{g(v)} \frac{dg(v)}{dv} \right) \otimes \left(\frac{\mu' - \mu}{1 - \mu}, \frac{v'}{v} \right) \\
 &\Leftrightarrow \left[\frac{f(\mu') - f(\mu)}{1 - f(\mu)}, \frac{g(v) - g(v')}{g(v)} \right] \approx \left[\frac{1 - \mu}{1 - f(\mu)} \frac{df(\mu)}{d\mu}, \frac{v}{g(v)} \frac{dg(v)}{dv} \right] \times \left[\frac{\mu' - \mu}{1 - \mu}, \frac{v - v'}{v} \right]
 \end{aligned}$$

Therefore, it is easy to obtain that the formula in Theorem 2.4 reveals how to estimate the relative increment of $[f(\mu), 1 - g(v)]$ when the relative increment of $[\mu, 1 - v]$ and the range of the elasticity coefficient $\left[\frac{1 - \mu}{1 - f(\mu)} \frac{df(\mu)}{d\mu}, \frac{v}{g(v)} \frac{dg(v)}{dv} \right]$ are both known.

Two examples (Lei and Xu 2015b) are provided to show Theorem 3.4 below:

- (1) Let $\phi(X) = (f(\mu), g(v)) = \lambda X$ ($0 < \lambda \leq 1$), then $f(\mu) = 1 - (1 - \mu)^\lambda$ and $g(v) = v^\lambda$. In addition, there is

$$\frac{d\varphi(X)}{dX} = \left(\frac{1-\mu}{(1-\mu)^\lambda} \lambda (1-\mu)^{\lambda-1}, 1-\lambda \right) = (\lambda, 1-\lambda)$$

Thus, according to Theorem 3.4, there is $\varphi(X \oplus \Delta X) \ominus \varphi(X) \approx (\lambda, 1-\lambda) \otimes \Delta X$. Moreover, by the operational law of IFNs: $\lambda(\alpha_1 \oplus \alpha_2) = \lambda\alpha_1 \oplus \lambda\alpha_2$, $\varphi(X \oplus \Delta X) \ominus \varphi(X) = \lambda(X \oplus \Delta X) \ominus \lambda X = \lambda \cdot \Delta X$ holds.

Without loss of generality, we assume $\lambda = 0.2$ and $\Delta X = (0.02, 0.97)$, then

$$(\lambda, 1-\lambda) \otimes \Delta X = (0.004, 0.994) \text{ and } \lambda \cdot \Delta X = (0.0040324, 0.9939267)$$

Obviously, replacing $\lambda \cdot \Delta X$ by $(\lambda, 1-\lambda) \otimes \Delta X$ is approximate.

- (2) Assume that there are four IFNs $\alpha_1 = (0.3, 0.4)$, $\alpha_2 = (0.2, 0.5)$, $\alpha_3 = (0.1, 0.2)$ and $\alpha_4 = (0.3, 0.4)$, and their weights are respectively 0.2, 0.4, 0.1 and 0.3, then we have

$$\begin{aligned} IFWA_\omega(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= \left(1 - \prod_{j=1}^4 (1 - \mu_{\alpha_j})^{\omega_j}, \prod_{j=1}^4 v_{\alpha_j}^{\omega_j} \right) \\ &= \left(1 - (1 - 0.3)^{0.2} \times (1 - 0.2)^{0.4} \times (1 - 0.1)^{0.1} \right. \\ &\quad \times \left. (1 - 0.3)^{0.3}, 0.4^{0.2} \times 0.5^{0.4} \times 0.2^{0.1} \times 0.4^{0.3} \right) \\ &= (0.243, 0.408) \end{aligned}$$

If some decision makers think that their assessment $\alpha_1 = (0.3, 0.4)$ has an estimated error, and want to give the value again. Let the new value be $\alpha'_1 = (\mu'_1, v'_1)$, then

- (1) If assuming $\alpha_1 \trianglelefteq \alpha'_1$, which means that there exists an IFN β_1 such that $\alpha'_1 = \alpha_1 \oplus \beta_1$, and $\alpha'_1 = (0.4, 0.3)$, then $\beta_1 = \alpha'_1 \ominus \alpha_1 = (0.143, 0.75)$. Hence, there is

$$\begin{aligned} IFWA_\omega(\alpha'_1, \alpha_2, \alpha_3, \alpha_4) &\approx IFWA_\omega(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \oplus (\omega_1, 1 - \omega_1) \otimes (\alpha'_1 \ominus \alpha_1) \\ &= (0.243, 0.408) \oplus (0.2, 0.8) \otimes (0.143, 0.75) \\ &= (0.265, 0.388) \end{aligned}$$

- (2) When $\alpha'_1 \trianglelefteq \alpha_1$, which means that there is β_2 such that $\alpha'_1 = \alpha_1 \ominus \beta_2$, and $\alpha'_1 = (0.2, 0.5)$, then $\beta_2 = \alpha_1 \ominus \alpha'_1 = (0.125, 0.8)$. Hence, there is

$$\begin{aligned}
IFWA_{\omega}(\alpha'_1, \alpha_2, \alpha_3, \alpha_4) &\approx IFWA_{\omega}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \ominus (\omega_1, 1 - \omega_1) \otimes (\alpha_1 \ominus \alpha'_1) \\
&= (0.243, 0.408) \ominus (0.2, 0.8) \otimes (0.125, 0.8) \\
&= (0.224, 0.425)
\end{aligned}$$

The following theorem shows that the situation holds when $\Delta\varphi = d\varphi$:

Theorem 2.5 (Lei and Xu 2015b) *Let $\varphi(X) = (f(\mu), g(v))$ be an IFF, which satisfies*

$$\frac{d^2f(\mu)}{d\mu^2} = \frac{d^2g(v)}{dv^2} = 0$$

when $X \leq Y$, then

$$\varphi(Y) \ominus \varphi(X) = \frac{d\varphi(X)}{dX} \otimes (Y \ominus X)$$

Based on the proof of Theorem 2.4, it is easy to prove Theorem 2.5. So it is omitted here. Next, we give an example (Lei and Xu 2015b) to illustrate this theorem:

Let $\alpha_0 = (\mu_0, v_0)$, $\alpha = (\mu, v)$ and $X = (\mu_X, v_X)$ be three IFNs, and $\varphi(\alpha) = \alpha_0 \otimes \alpha$ be an IFF. Because $f(\mu) = \mu_0\mu$ and $g(v) = v + v_0 - v_0v$ satisfy

$$\frac{d^2f(\mu)}{d\mu^2} = \frac{d^2g(v)}{dv^2} = 0$$

According to Theorem 2.5, we have

$$\varphi(\alpha') = \varphi(\alpha) \oplus \frac{d\varphi(\alpha)}{d\alpha} \otimes (\alpha' \ominus \alpha)$$

In addition, by using the formula of derivative of IFFs, we have

$$\frac{d\varphi(\alpha)}{d\alpha} = \alpha_0 \otimes \left(\frac{1 - \mu}{1 - \mu_0\mu}, 1 - \frac{v}{v + v_0 - v_0v} \right)$$

Hence, there is

$$\alpha_0 \otimes (\alpha \oplus X) = \alpha_0 \otimes \alpha \oplus \alpha_0 \otimes X \otimes \left(\frac{1 - \mu}{1 - \mu_0\mu}, 1 - \frac{v}{v + v_0 - v_0v} \right)$$

Now we use the following (1)–(3) to test the above equality:

$$\begin{aligned}
 (1) \quad \alpha' &= \alpha \oplus X = (\mu + \mu_X - \mu\mu_X, v v_X). \\
 (2) \quad \frac{d\varphi(\alpha)}{d\alpha} \otimes X &= (\mu_X, v_X) \otimes (\mu_0, v_0) \otimes \left(\frac{1 - \mu}{1 - \mu_0\mu}, \quad 1 - \frac{v}{v + v_0 - v_0v} \right) \\
 &= (\mu_X, v_X) \otimes \left(\frac{\mu_0 - \mu_0\mu}{1 - \mu_0\mu}, \quad \frac{v_0}{v + v_0 - v_0v} \right) \\
 &= \left(\frac{\mu_X\mu_0 - \mu_X\mu_0\mu}{1 - \mu_0\mu}, \quad v_X + \frac{v_0}{v + v_0 - v_0v} - \frac{v_X v_0}{v + v_0 - v_0v} \right) \\
 (3) \quad \varphi(\alpha) \oplus \frac{d\varphi(\alpha)}{d\alpha} \otimes X &= \alpha_0 \otimes \alpha \oplus \frac{d\varphi(\alpha)}{d\alpha} \otimes X \\
 &= (\mu_0\mu + \mu_0\mu_X - \mu_0\mu\mu_X, \quad v_0 + v v_X - v_0 v v_X)
 \end{aligned}$$

Furthermore,

$$\varphi(\alpha') = \alpha_0 \otimes (\alpha \oplus X) = (\mu_0\mu + \mu_0\mu_X - \mu_0\mu\mu_X, \quad v_0 + v v_X - v_0 v v_X)$$

Hence, $\varphi(\alpha') = \varphi(\alpha) \oplus \frac{d\varphi(\alpha)}{d\alpha} \otimes (\alpha' \ominus \alpha)$ holds, which is

$$\alpha_0 \otimes (\alpha \oplus X) = \alpha_0 \otimes \alpha \oplus \alpha_0 \otimes X \otimes \left(\frac{1 - \mu}{1 - \mu_0\mu}, \quad 1 - \frac{v}{v + v_0 - v_0v} \right)$$

which indicates $\alpha_0 \otimes (\alpha \oplus X) \neq \alpha_0 \otimes \alpha \oplus \alpha_0 \otimes X$ holds.

In the last section, we have studied the chain rule of derivatives of the compound IFFs. It is natural to obtain the form invariance of differential in intuitionistic fuzzy calculus. Next, we show the important property (Lei and Xu 2015a) as follows:

Let $\varphi(\psi(t))$ be a compound IFF, which consists of two IFFs $Y = \varphi(X)$ and $X = \psi(t)$. If both Y'_X and X'_t exist, then $Y'_t = Y'_X \otimes X'_t$.

If considering the middle variable X as the independent variable of Y , then

$$dY = Y'_X \otimes dX$$

On the other hand, if we do not consider X as the independent variable of Y , but t , then

$$dY = Y'_t \otimes dt$$

However, if we replace Y'_t with $Y'_X \otimes X'_t$ and notice that $X'_t \otimes dt$ is actually dX , then we have the following derivational process:

$$dY = Y'_t \otimes dt = Y'_X \otimes X'_t \otimes dt = Y'_X \otimes dX$$

which is just the form invariance of differential in intuitionistic fuzzy calculus.

Based on these analyses, we know that the differential form remains unchanged when the original variable of IFFs is replaced by a new variable. The only difference is that dX will not be regarded as the increment ΔX but the differential of $X = \psi(t)$ when we consider t as the independent variable of Y .

2.4 Conclusions

In this chapter, we have mainly studied the derivatives and differentials of IFFs. Firstly, we have defined the concept of IFF, and pointed out that the IFFs are the main research object in intuitionistic fuzzy calculus. In addition, we have also given the definitions of monotonically increasing IFFs and continuous IFFs. By taking the limit value of difference quotients of IFFs, we have defined the derivatives of IFFs. After that, we have studied some important proprieties of derivatives, like the chain rule of derivatives of the compound IFFs. In addition, this chapter has also investigated the derivatives of IFFs by introducing the notion of elasticity coefficient.

Based on the derivatives of IFFs, we have defined the differentials of IFFs, and proven the relationship between the increment of φ ($\Delta\varphi(X)$) and its differential ($d\varphi(X)$). Finally, we have revealed the form invariance of differential in intuitionistic fuzzy calculus.

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