Chapter 2
The Linearized Monge-Ampère Equation

2.1 The Linearized Monge-Ampère Equation and Interior Regularity of Its Solution

2.1.1 The Linearized Monge-Ampère Equation

The linearized Monge-Ampère equation associated with a C^2 and locally uniformly convex potential u defined on some subset of $\mathbb{R}^n$ is of the form

$$L_u v := \sum_{i,j=1}^{n} U^{ij} v_{ij} \equiv \mathrm{trace}(UD^2 v) = g. \tag{2.1}$$

Here and throughout,

$$U = (U^{ij}) = (\det D^2 u)(D^2 u)^{-1}$$

is the matrix of cofactors of the Hessian matrix $D^2 u = (u_{ij})$. The coefficient matrix U of L_u arises from the linearization of the Monge-Ampère operator $\det D^2 u$ because

$$U = \frac{\partial(\det D^2 u)}{\partial(D^2 u)}.$$

One can also note that $L_u v$ is the coefficient of t in the expansion

$$\det D^2(u + tv) = \det D^2 u + t\,\mathrm{trace}(UD^2 v) + \cdots + t^n \det D^2 v.$$

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Typically, one assumes that u solves the Monge-Ampère equation

$$\det D^2 u = f \text{ for some function f satisfying the bounds } 0 < \lambda \leq f \leq \Lambda \tag{2.2}$$

where λ and Λ are positive constants. Given these bounds, U is a positive semi-definite matrix. Hence, L_u is a linear elliptic partial differential operator, possibly degenerate.

The linearized Monge-Ampère operator L_u captures two of the most important second order equations in PDEs from the simplest linear equation to one of the most important nonlinear equations. In fact, in the special case where u is a quadratic polynomial, say $u(x) = \frac{1}{2}|x|^2$, L_u becomes the Laplace operator: $L_u = \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$. On the other hand, since $L_u u = n \det D^2 u$, the Monge-Ampère equation is a special case of the linearized Monge-Ampère equation. As $U = (U^{ij})$ is divergence-free (see Lemma 3.61), that is,

$$\sum_{i=1}^{n} \partial_i U^{ij} = 0$$

for all $j = 1, \cdots, n$, the linearized Monge-Ampère equation can be written in both divergence and double divergence form:

$$L_u v = \sum_{i,j=1}^{n} \partial_i (U^{ij} v_j) = \sum_{i,j=1}^{n} \partial_{ij} (U^{ij} v).$$

2.1.2 Linearized Monge-Ampère Equations in Contexts

L_u appears in many contexts:

(1) Affine maximal surface equation in affine geometry (Chern [12], Trudinger-Wang [37–39])

$$U^{ij} w_{ij} = 0, \; w = (\det D^2 u)^{-\frac{n+1}{n+2}}$$

(2) Abreu's equation (Abreu [1], Donaldson [15–18]) in the context of existence of Kähler metrics of constant scalar curvature in complex geometry

$$U^{ij} w_{ij} = -1, \; w = (\det D^2 u)^{-1}$$

A more familiar form of the Abreu's equation is

$$\sum_{i,j=1}^{n} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j} = -1$$

where $(u^{ij}) = (D^2u)^{-1}$ is the inverse matrix of D^2u.

(3) Semigeostrophic equations in fluid mechanics (Brenier [4], Cullen-Norbury-Purser [13], Loeper [27]).

(4) Regularity of the polar factorization for time dependent maps (Loeper [26]).

2.1.3 Difficulties and Expected Regularity

The classical regularity theory for uniformly elliptic equations with measurable coefficients deals with **divergence** form operators

$$L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a^{ij} \frac{\partial}{\partial x_j} \right)$$

or **nondivergence** form operators

$$L = \sum_{i,j=1}^{n} a^{ij} \partial_{ij}$$

with positive ellipticity constants λ and Λ, that is, the eigenvalues of the coefficient matrix $A = (a^{ij})$ are bounded between λ and Λ. The important Harnack and Hölder estimates for **divergence** form equations $Lu = 0$ were established in the late 50s by De Giorgi-Nash-Moser [14, 31, 30]. The regularity theory in this case is connected with isoperimetric inequality, Sobolev embedding, Moser iteration, heat kernel, BMO (the space of functions of bounded mean oscillation). On the other hand, the Harnack and Hölder estimates for **nondivergence** form equations $Lu = 0$ were established only in the late 70s by Krylov-Safonov [22, 23]. The regularity theory is connected with the Aleksandrov-Bakelman-Pucci (ABP) maximum principle coming from the Monge-Ampère equation.

The linearized Monge-Ampère theory investigates operators of the form

$$L_u = \sum_{i,j=1}^{n} U^{ij} \partial_{ij}$$

where it is only known that the *product of the eigenvalues* of the coefficient matrix U is bounded between two constants. This comes from (2.2) because

$$\lambda^{n-1} \le \det U = (\det D^2 u)^{n-1} \le \Lambda^{n-1}.$$

Therefore, the linearized Monge-Ampère operator L_u is in general not uniformly elliptic, i.e., the eigenvalues of $U = (U^{ij})$ are not necessarily bounded away from 0 and ∞. Moreover, when considered in a bounded convex domain Ω, L_u can be possibly singular near the boundary. In other words, the linearized Monge-Ampère equation can be both degenerate and singular. The degeneracy and singularity of L_u are the main difficulties in establishing regularity results for its solutions.

A natural question is what regularity we can hope for solutions of the linearized Monge-Ampère equation $L_u v = 0$ under the structural assumption (2.2). At least on a heuristic level, they can be expected to be Hölder continuous. Indeed, strictly convex solutions of (2.2), interpreted in the sense of Aleksandrov for u not C^2 as in Definition 3.6, are $C^{1,\alpha}$ for some $\alpha \in (0,1)$ depending only on n, λ and Λ. This follows from the regularity theory of the Monge-Ampère equation; see Theorems 3.53 and 3.58. By differentiating (2.2), we see that each partial derivative $u_k = \frac{\partial u}{\partial x_k}$ $(k = 1, \cdots, n)$ is a solution of the inhomogeneous linearized Monge-Ampère equation

$$L_u u_k = f_k.$$

We can expect that the regularity for v is that of u_k, which is C^α, and hence it should be Hölder continuous. The theory of Caffarelli-Gutiérrez confirms this expectation.

2.1.4 Affine Invariance Property

The second order operator $L_u := U^{ij}\partial_{ij}$ is affine invariant, i.e., invariant with respect to linear transformations of the independent variable x of the form $x \mapsto Tx$ with $\det T = 1$. Indeed, for such T, the rescaled functions

$$\tilde{u}(x) = u(Tx) \text{ and } \tilde{v}(x) = v(Tx)$$

satisfy the same structural conditions as in (2.1) and (2.2) because

$$\det D^2\tilde{u}(x) = \det D^2 u(Tx) = f(Tx) \text{ and } L_{\tilde{u}}\tilde{v}(x) = L_u v(Tx) = g(Tx).$$

More generally, under the transformations

$$\tilde{u}(x) = u(Tx), \ \tilde{v}(x) = v(Tx),$$

the Eq. (2.1) becomes

$$L_{\tilde{u}}\tilde{v}(x) := \tilde{U}^{ij}\tilde{v}_{ij}(x) = (\det T)^2 g(Tx).$$

The last equation follows from standard computation. We have

$$D\tilde{u} = T^t Du;\ D^2\tilde{u} = T^t(D^2u)T;\ D^2\tilde{v} = T^t(D^2v)T$$

and

$$\tilde{U} = (\det D^2\tilde{u})(D^2\tilde{u})^{-1} = (\det T)^2(\det D^2u)T^{-1}(D^2u)^{-1}(T^{-1})^t = (\det T)^2 T^{-1}U(T^{-1})^t.$$

Therefore,

$$L_{\tilde{u}}\tilde{v}(x) = \operatorname{trace}(\tilde{U}D^2\tilde{v}) = (\det T)^2\operatorname{trace}(UD^2v(Tx)) = (\det T)^2 L_u v(Tx) = (\det T)^2 g(Tx).$$

The rest of the section will be devoted to interior regularity for solutions to the linearized Monge-Ampère equation. We start by recalling Krylov-Safonov's Harnack inequality for linear, uniformly elliptic equations in non-divergence form.

2.1.5 Krylov-Safonov's Harnack Inequality

In 1979, Krylov-Safonov [22, 23] established the Harnack inequality and Hölder estimates for solutions of linear elliptic equations in **non-divergence** form

$$Lv := \sum_{i,j=1}^{n} a^{ij}\frac{\partial^2 v}{\partial x_i \partial x_j} = 0 \tag{2.3}$$

where the eigenvalues of the coefficient matrix $A = (a^{ij})$ are bounded between two positive constants λ and Λ, that is

$$\lambda I_n \leq (a^{ij}) \leq \Lambda I_n. \tag{2.4}$$

The following theorem is the celebrated result of Krylov-Safonov.

Theorem 2.1 (Krylov-Safonov's Harnack Inequality, [22, 23]) *Assume (a^{ij}) satisfies (2.4). Let v be a nonnegative solution of (2.3) in Ω. Then v satisfies the Harnack inequality on Euclidean balls. More precisely, for all $B_{2r}(x_0) \subset\subset \Omega$, we have*

$$\sup_{B_r(x_0)} v \leq C(n,\lambda,\Lambda) \inf_{B_r(x_0)} v. \tag{2.5}$$

From the Harnack inequality (2.5), we obtain a Hölder estimate

$$|v(x) - v(y)| \leq C\,|x-y|^{\alpha} \sup_{B_{2r}(x_0)} |v|$$

for $x, y \in B_r(x_0)$ where α and C are positive constants depending only on n, λ, Λ.

Remark 2.2

(i) The uniform ellipticity of $A(x)$ is invariant under rigid transformation of the domain, i.e., for any orthogonal matrix O, the matrix $A(Ox)$ is also uniformly elliptic with the same ellipticity constants as $A(x)$.
(ii) Balls are invariant under orthogonal transformations.
(iii) One important fact, but hidden, in the regularity theory of uniformly elliptic equations is that the quadratic polynomials

$$P(x) = a + b \cdot x + \frac{1}{2}\,|x|^2\,,\ b \in \mathbb{R}^n,$$

are "potentials" for L, that is

$$L(P) \approx 1$$

and level surfaces of $P(x)$ are all possible balls of $\mathbb{R}^n$. Moreover,

$$|\nabla P(x) - b| \approx 1$$

for x in the ring $B_2(b) \backslash B_1(b)$.

Krylov-Safonov theory makes crucial use of the ABP estimate which bounds solution of $Lv = f$ using the boundary values of v and L^n norm of the right hand side. In general form, it states as follows; see [2, 3, 32] and also [19, Theorem 9.1].

Theorem 2.3 (ABP Maximum Principle) *Let (a^{ij}) be a measurable, positive definite matrix. For $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, we have*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + \frac{\operatorname{diam}(\Omega)}{n\omega_n^{1/n}} \left\| \frac{a^{ij}u_{ij}}{[\det(a_{ij})]^{1/n}} \right\|_{L^n(\Gamma^+)}$$

where Γ^+ is the upper contact set

$$\Gamma^+ = \{y \in \Omega | u(x) \leq u(y) + p \cdot (x-y) \text{ for all } x \in \Omega\,, \text{ for some } p = p(y) \in \mathbb{R}^n\}.$$

2.1.6 Harnack Inequality for the Linearized Monge-Ampère Equation

The regularity theory for the linearized Monge-Ampère equation was initiated in the fundamental paper [10] by Caffarelli and Gutiérrez. They developed an interior Harnack inequality theory for nonnegative solutions of the homogeneous equations

$$L_u v = 0,$$

where L_u is defined as in (2.1), in terms of the pinching of the Hessian determinant

$$\lambda \leq \det D^2 u \leq \Lambda. \tag{2.6}$$

Their approach is based on that of Krylov and Safonov [22, 23] on the Harnack inequality and Hölder estimates for linear, uniformly elliptic equations in general form, with sections replacing Euclidean balls. Before stating precisely the Harnack inequality theory of Caffarelli-Gutiérrez, we would like to see, at least heuristically, what objects are prominent in this theory.

Remark 2.4

(i) By the affine invariance property of the linearized Monge-Ampère equations (see Sect. 2.1.4), it is not hard to imagine that good estimates for the linearized Monge-Ampère equations must be formulated on domains that are invariant under affine transformations. Balls are not affine invariant.
(ii) Clearly, after an affine transformation, an ellipsoid becomes another ellipsoid.
(iii) A very important class of ellipsoid-like objects in the context of the Monge-Ampère equation and the linearized Monge-Ampère equation are sections.

The notion of sections (or cross sections) of convex solutions to the Monge-Ampère equation was first introduced and studied by Caffarelli [5–8], and plays an important role in his fundamental interior $W^{2,p}$ estimates [6]. Sections are defined as sublevel sets of convex solutions after subtracting their supporting hyperplanes. They have the same role as Euclidean balls have in the classical theory. The section of a convex function u defined on $\overline{\Omega}$ with center x_0 in $\overline{\Omega}$ and height t is defined by

$$S_u(x_0, t) = \{x \in \overline{\Omega} : u(x) < u(x_0) + \nabla u(x_0) \cdot (x - x_0) + t\}.$$

After affine transformations, the sections of u become sections of another convex function.

Example 2.5 A Euclidean ball of radius r is a section with height $r^2/2$ of the quadratic function $|x|^2/2$ whose Hessian determinant is 1. For $u(x) = |x|^2/2$, we have

$$S_u(x, h) = B_{\sqrt{2h}}(x) \cap \overline{\Omega}.$$

An important fact is the convexity of sections. They can be normalized to look like balls (John's lemma, Lemma 3.23). Illustrating (i) and (iii) in Remark 2.4, we can consider the following example.

Example 2.6 Consider the functions $u(x_1, x_2) = \frac{x_1^2}{2\varepsilon} + \frac{\varepsilon}{2}x_2^2$ and $v(x_1, x_2) = \frac{x_1^2}{2\varepsilon} - \frac{\varepsilon}{2}x_2^2 + 1$ in $\mathbb{R}^2$ where $\varepsilon \in (0, 1)$. Then $\det D^2 u = 1$ and

$$U^{ij} v_{ij} = 0.$$

We can compute for $\frac{1}{4} \le r \le \frac{1}{2}$ and $\frac{1}{4} \le t \le \frac{1}{2}$

(i)

$$\sup_{B_r(0)} v = \frac{r^2}{2\varepsilon} + 1; \inf_{B_r(0)} v = 1 - \frac{\varepsilon}{2} r^2; \sup_{B_r(0)} v \ge \frac{1}{32\varepsilon} \inf_{B_r(0)} v.$$

(ii)

$$\sup_{S_u(0,t)} v = t + 1; \inf_{S_u(0,t)} v = 1 - t.$$

The ratio $\sup v / \inf v$ does not depend on the eccentricity of the section $S_u(0, t)$ for the given range of t. This ratio becomes unbounded on balls around 0 when $\varepsilon \to 0$.

Now, if v is a nonnegative solution of the linearized Monge-Ampère equation $L_u v = 0$ in a section $S_u(x_0, 2h) \subset\subset \Omega$ then Caffarelli and Gutiérrez's theorem on the Harnack inequality says that the values of v in the concentric section of half height are comparable with each other. More precisely, we have the following:

Theorem 2.7 (Caffarelli-Gutiérrez's Harnack Inequality, [10]) *Assume that the C^2 convex function u satisfies the Monge-Ampère equation*

$$\lambda \le \det D^2 u \le \Lambda \textit{ in } \Omega.$$

Let $v \in W^{2,n}_{loc}(\Omega)$ be a nonnegative solution of

$$L_u v := U^{ij} v_{ij} = 0$$

in a section $S_u(x_0, 2h) \subset\subset \Omega$. Then

$$\sup_{S_u(x_0,h)} v \le C(n, \lambda, \Lambda) \inf_{S_u(x_0,h)} v. \tag{2.7}$$

This theory of Caffarelli and Gutiérrez is an affine invariant version of the classical Harnack inequality for uniformly elliptic equations with measurable coefficients. In fact, since the linearized Monge-Ampère operator L_u can be written in both divergence form and non-divergence form, Caffarelli-Gutiérrez's theorem is the affine invariant analogue of De Giorgi-Nash-Moser's theorem [14, 31, 30] and also Krylov-Safonov's theorem [22, 23] on Hölder continuity of solutions to uniformly elliptic equations in divergence and nondivergence form, respectively.

Remark 2.8 The Harnack estimate (2.7) also holds for nonnegative solutions to equations of the form

$$\operatorname{trace}(A(x)UD^2v) = 0$$

with A uniformly elliptic

$$C^{-1}I_n \leq A(x) \leq CI_n.$$

Thus, when $u(x) = \frac{1}{2}|x|^2$, we obtain the Krylov-Safonov's Harnack inequality for uniformly elliptic equations. Therefore, Harnack inequality also works for

$$a^{ij}v_{ij} = 0$$

with

$$\tilde{\lambda}(D^2u)^{-1} \leq (a^{ij}) \leq \tilde{\Lambda}(D^2u)^{-1}.$$

In this case, we have a Hessian^{-1}-like elliptic equation.

The Harnack inequality (2.7) implies the geometric decay of the oscillation of the solution on sections with smaller height and gives the C^α estimate for solution. Quantitatively, this says that if v solves $L_u v = 0$ in $S_u(x_0, 2) \subset\subset \Omega$ then v is C^α in $S_u(x_0, 1)$ and

$$\|v\|_{C^\alpha(S_u(x_0,1))} \leq C(n, \lambda, \Lambda, S_u(x_0, 2))\|v\|_{L^\infty(S_u(x_0,2))}.$$

The important point to be emphasized here is that α depends only on n, λ, Λ and the dependence of C on $S_u(x_0, 2)$ can be actually removed in applications if we use affine transformations to transform the convex set $S_u(x_0, 2)$ into a convex set comparable to the unit Euclidean ball. The latter point follows from John's lemma (see Lemma 3.23) on inscribing ellipsoid of maximal volume of a convex set [21]. In fact, we can obtain interior Hölder estimate for inhomogeneous equations.

Theorem 2.9 (Interior Hölder Estimate) *Assume that* $\lambda \leq \det D^2u \leq \Lambda$ *in a convex domain* $\Omega \subset \mathbb{R}^n$ *with* $u = 0$ *on* $\partial\Omega$ *where* $B_1(0) \subset \Omega \subset B_n(0)$. *Let* $f \in L^n(B_1(0))$ *and* $v \in W^{2,n}_{loc}(B_1(0))$ *be a solution of* $U^{ij}v_{ij} = f$ *in* $B_1(0)$. *Then there exist constants* $\beta \in (0, 1)$ *and* $C > 0$ *depending only on n,* λ*, and* Λ *such that*

$$|v(x) - v(y)| \leq C|x-y|^\beta\left(\|v\|_{L^\infty(B_1(0))} + \|f\|_{L^n(B_1(0))}\right) \quad \text{for all } x, y \in B_{\frac{1}{2}}(0).$$

The Harnack inequality (2.7) is also true for more general hypotheses on the Monge-Ampère measure $\mu = \det D^2 u$ such as a suitable doubling property. We say that the Borel measure μ is *doubling with respect to the center of mass* on the sections of u if there exist constants $\beta > 1$ and $0 < \alpha < 1$ such that for all sections $S_u(x_0, t)$,

$$\mu(S_u(x_0, t)) \leq \beta \mu(\alpha S_u(x_0, t)). \tag{2.8}$$

Here $\alpha S_u(x_0, t)$ denotes the α-dilation of $S_u(x_0, t)$ with respect to its center of mass x^* (computed with respect to the Lebesgue measure):

$$\alpha S_u(x_0, t) = \{x^* + \alpha(x - x^*) : x \in S_u(x_0, t)\}.$$

Maldonado [29], extending the work of Caffarelli-Gutiérrez, proved the following Harnack inequality for the linearized Monge-Ampère equation under minimal geometric condition, namely, the doubling condition (2.8).

Theorem 2.10 ([29]) *Assume that* $\det D^2 u = \mu$ *satisfies (2.8). For each compactly supported section* $S_u(x, t) \subset\subset \Omega$*, and any nonnegative solution* v *of* $L_u v = 0$ *in* $S_u(x, t)$*, we have*

$$\sup_{S_u(x, \tau t)} v \leq C \inf_{S_u(x, \tau t)} v$$

for universal τ, C *depending only on* n, β *and* α*.*

For example, the Harnack inequality holds for μ positive polynomials. If $u(x_1, x_2) = x_1^4 + x_2^2$ then $\mu = \det D^2 u = C x_1^2$ is an admissible measure. The Harnack inequality applies to equation of the Grushin-type

$$x_1^{-2} v_{11} + v_{22} = 0. \tag{2.9}$$

Remark 2.11 Equation of the type (2.9) is relevant in non-local equations such as fractional Laplace equation. By Caffarelli-Silvestre [11], we can relate the fractional Laplacian

$$(-\Delta)^s f(x) = C_{n,s} \int_{\mathbb{R}^n} \frac{f(x) - f(\xi)}{|x - \xi|^{n+2s}} d\xi,$$

where the parameter s is a real number between 0 and 1, and $C_{n,s}$ is some normalization constant, with solutions of the following extension problem. For a function $f : \mathbb{R}^n \to \mathbb{R}$, we consider the extension $v : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ that satisfies the equations

$$v(x, 0) = f(x), \ \Delta_x v + \frac{a}{y} v_y + v_{yy} = 0.$$

The last equation can also be written as

$$\operatorname{div}(y^a Dv) = 0$$

which is clearly the Euler-Lagrange equation for the functional

$$J(v) = \int_{y>0} |Dv|^2 y^a dX, \; X = (x, y).$$

We can show that

$$C(-\Delta)^s f = \lim_{y \to 0+} -y^a v_y = \frac{1}{1-a} \lim_{y \to 0} \frac{v(x, y) - v(x, 0)}{y^{1-a}}$$

for $s = \frac{1-a}{2}$ and some constant C depending on n and s, which reduces to the regular normal derivative in the case $a = 0$.

If we make the change of variables $z = \left(\frac{y}{1-a}\right)^{1-a}$, we obtain a nondivergence form equation of the type (2.9)

$$\Delta_x v + z^\alpha v_{zz} = 0$$

for $\alpha = \frac{-2a}{1-a}$. Moreover, $y^a v_y = v_z$. Thus, we can show that the following equality holds up to a multiplicative constant

$$(-\Delta)^s f(x) = -\lim_{y \to 0+} y^a v_y(x, y) = -v_z(x, 0).$$

Remark 2.12 The Harnack inequality in Theorem 2.7 has been recently extended to the boundary in [24].

2.2 Interior Harnack and Hölder Estimates for the Linearized Monge-Ampère Equation

In this section, we prove Theorems 2.7 and 2.9.

2.2.1 Proof of Caffarelli-Gutiérrez's Harnack Inequality

In this section, we prove Theorem 2.7 concerning Caffarelli-Gutiérrez's Harnack inequality for the linearized Monge-Ampère equation.

We first briefly outline the proof of the Harnack inequality (2.7) in Theorem 2.7. Our proof adapts the general scheme in proving Harnack inequality in Krylov-Safonov [22, 23], Caffarelli-Cabré [9], Caffarelli-Gutiérrez [10], Savin [33] and most recently Imbert-Silvestre [20].

By using the affine invariant property of the linearized Monge-Ampère equation as explained in Sect. 2.1.4, we can rescale the domain, and the functions u and v. Furthermore, by changing coordinates and subtracting a supporting hyperplane to the graph of u at $(x_0, u(x_0))$, we can assume that $x_0 = 0$, $u(0) = 0$, $Du(0) = 0$, $h = 2$ and that the section $S_4 = S_u(0, 4) \subset\subset \Omega$ is normalized, that is

$$B_1(0) \subset S_4 \subset B_n(0).$$

For simplicity, we denote $S_t = S_u(0, t)$.

A constant depending only on λ, Λ and n is called *universal*. We denote universal constants by $c, C, C_1, C_2, K, M, \delta, \cdots$, etc. Their values may change from line to line.

From the engulfing property of sections in Theorem 3.54, we find that if $y \in S_u(x, t)$ then

$$S_u(x, t) \subset S_u(y, \theta_0 t) \subset S_u(x, \theta_0^2 t),$$

it suffices to show that if $v \geq 0$ in S_2 then $v \leq C(n, \lambda, \Lambda)v(0)$ in S_1.

The idea of the proof is the following. We show that the distribution function of v, $|\{v > t\} \cap S_1|$ decays like $t^{-\varepsilon}$ (L^ε estimate). Thus, $v \approx v(0)$ in S_1 except a set of very small measure. If $v(x_0) \gg v(0)$ at some point x_0, then by the same method (now applying to $C_1 - C_2 v$), we find $v \gg v(0)$ in a set of positive measure which contradicts the above estimate. To study the distribution function of v, we slide generalized paraboloids associated with u of constant opening, $P(x) = -a[u(x) - u(y) - Du(y) \cdot (x - y)]$, from below till they touch the graph of v for the first time. These are the points where we use the equation and obtain the lower bound for the measure of the touching points. By increasing the opening of the sliding paraboloids, the set of touching points almost covers S_1 in measure.

There are three main steps in the proof of the L^ε estimate.

Step 1: Measure (ABP type) estimate. The rough idea is that

$$\text{Measure of contact points } \geq c \text{ Measure of vertices.}$$

This step is not difficult. The reason why it works is the following. In the ABP estimate, we need the lower bound on the determinant of the coefficient matrix which is the case here.

Step 2: Doubling estimate. This step is based on construction of subsolutions.

Step 3: This step proves the geometric decay of $|\{v > t\} \cap S_1|$. It is based on a covering lemma which is a consequence of geometric properties of sections.

Our measure estimate in Step 1 states as follows.

Lemma 2.13 (Measure Estimate) *Suppose that $v \geq 0$ is a solution of $L_u v = 0$ in a normalized section S_4. There are small, universal constants $\delta > 0, \alpha > 0$ and a large constant $M_1 > 1$ with the following properties. If $\inf_{S_\alpha} v \leq 1$ then*

$$|\{v > M_1\} \cap S_1| \leq (1 - \delta)|S_1|.$$

The key doubling estimate for Step 2 is the following lemma.

Lemma 2.14 (Doubling Estimate) *Suppose that $v \geq 0$ is a solution of $L_u v = 0$ in a normalized section S_4. Let α be the small constant in Lemma 2.13. If $v \geq 1$ in S_α then $v \geq c(n, \lambda, \Lambda)$ in S_1.*

Combining Lemmas 2.13 and 2.14, and letting $M := M_1 c(n, \lambda, \Lambda)^{-1}$, we obtain the following result:

Proposition 2.15 (Critical Density Estimate) *Suppose that $v \geq 0$ is a solution of $L_u v = 0$ in a normalized section S_4. There is a small, universal constant $\delta > 0$ and a large constant $M > 1$ with the following properties. If*

$$|\{v > M\} \cap S_1| > (1 - \delta)|S_1|$$

then $v > 1$ in S_1.

From the critical density estimate and the growing ink-spots lemma stated in Lemma 2.19, we obtain the L^ε estimate and completing the proof of Step 3.

Theorem 2.16 (Decay Estimate of the Distribution Function) *Suppose that $v \geq 0$ is a solution of $L_u v = 0$ in a normalized section S_4 with*

$$\inf_{S_u(0,1)} v \leq 1.$$

Then there are universal constants $C_1 > 1$ and $\varepsilon \in (0, 1)$ such that for all $t > 0$, we have

$$|\{v > t\} \cap S_1| \leq C_1 t^{-\varepsilon}.$$

Proof of Theorem 2.16 Let $\delta \in (0, 1)$ and $M > 1$ be the constants in Proposition 2.15. The conclusion of the theorem follows from the following decay estimate for $A_k := \{v > M^k\} \cap S_1$:

$$|A_k| \leq C_2 M^{-\varepsilon k}.$$

Note that A_k's are open sets and $A_k \subset A_1$ for all $k \geq 1$. Recalling $\inf_{S_1} v \leq 1$, by Proposition 2.15, we have

$$|A_k| \leq |A_1| \leq (1 - \delta)|S_1| \text{ for all } k.$$

From Proposition 2.15, we find that if a section $S \subset S_1$ satisfies $|S \cap A_{k+1}| > (1-\delta)|S|$, then $S \subset A_k$. Using Lemma 2.19, we obtain

$$|A_{k+1}| \leq (1-c\delta)|A_k|,$$

and therefore, by induction,

$$|A_k| \leq (1-c\delta)^{k-1}(1-\delta)|S_1| = C_2 M^{-\varepsilon k},$$

where $\varepsilon = -\log(1-c\delta)/\log M$ and $C_2 = (1-c\delta)^{-1}(1-\delta)|S_1|$. This finishes the proof. □

Proof of Theorem 2.7 Let $\delta \in (0,1)$ and $M > 1$ be the constants in Proposition 2.15 and $\varepsilon \in (0,1)$ be the constant in Theorem 2.16. By a covering argument, our theorem follows from the following claim.

Claim 2.17

$$\sup_{S_u(0,1/2)} v \leq C \inf_{S_u(0,1/2)} v.$$

This in turns follows from the following claim.

Claim 2.18 If $\inf_{S_u(0,1/2)} v \leq 1$ then for some universal constant C, we have $\sup_{S_u(0,1/2)} v \leq C$.

Indeed, for each $\tau > 0$, the function

$$v^\tau = \frac{v}{\inf_{S_u(0,1/2)} v + \tau}$$

satisfies $a^{ij} v^\tau_{ij} = 0$. We apply **Claim 2.18** to v^τ to obtain

$$\sup_{S_u(0,1/2)} v \leq C\left(\inf_{S_u(0,1/2)} v + \tau\right).$$

Sending $\tau \to 0$, we get the conclusion of **Claim 2.17.**

It remains to prove **Claim 2.18.** Let $\beta > 0$ be a universal constant to be determined later and let $h_t(x) = t(1-u(x))^{-\beta}$ be defined in $S_u(0,1)$. We consider the minimum value of t such that $h_t \geq v$ in $S_u(0,1)$. It suffices to show that t is universally bounded by a constant C because we have then

$$\sup_{S_u(0,1/2)} v \leq C \sup_{S_u(0,1/2)} (1-u(x))^{-\beta} \leq 2^\beta C.$$

If $t \leq 1$, we are done. Hence, we further assume that $t \geq 1$.

Since t is chosen to be the *minimum* value such that $h_t \geq v$, then there must exist some $x_0 \in S_u(0,1)$ such that $h_t(x_0) = v(x_0)$. Let $r = (1-u(x_0))/2$. Let $H_0 := h_t(x_0) = t(2r)^{-\beta} \geq 1$. By Theorem 3.57, there is a small constant c and large constant $p_1 = \mu^{-1}$ such that $S_u(x_0, 2cr^{p_1}) \subset S_u(0,1)$. We bound t by estimating the measure of the set $\{v \geq H_0/2\} \cap S_u(x_0, cr^{p_1})$ from above and below.

The estimate from above can be done using Theorem 2.16 which then says that

$$|\{v > H_0/2\} \cap S_u(x_0, cr^{p_1})| \leq |\{v > H_0/2\} \cap S_u(0,1)| \leq CH_0^{-\varepsilon} = Ct^{-\varepsilon}(2r)^{\beta\varepsilon}. \tag{2.10}$$

To estimate the measure of $\{v \geq H_0/2\} \cap S_u(x_0, cr^{p_1})$ from below, we apply Theorem 2.16 to $C_1 - C_2 v$ on a small but definite fraction of this section. Let ρ be the small universal constant and β be a large universal constant such that

$$M\left((1-\rho)^{-\beta} - 1\right) \leq \frac{1}{2},\ \beta \geq \frac{n}{2\mu\varepsilon}. \tag{2.11}$$

Consider the section $S_u(x_0, c_1 r^{p_1})$ where $c_1 \leq c$ is small. We claim that $1-u(x) \geq 2r - 2\rho r$ in this section. Indeed, if $x \in S_u(x_0, c_1 r^{p_1})$ then by Lemma 3.52, we have $|x - x_0| \leq C(c_1 r^{p_1})^{\mu} \leq c\rho r$ for small c_1 and hence, by the gradient estimate in Lemma 3.11

$$1 - u(x) = 2r + u(x_0) - u(x) \geq 2r - (\sup_{S_u(0,1)} |Du|)|x - x_0| \geq 2r - 2\rho r.$$

The maximum of v in the section $S_u(x_0, c_1 r^{\gamma})$ is at most the maximum of h_t which is not greater than $t(2r - 2\rho r)^{-\beta} = (1-\rho)^{-\beta} H_0$. Define the following function for $x \in S_u(x_0, c_1 r^{p_1})$

$$w(x) = \frac{(1-\rho)^{-\beta} H_0 - v(x)}{\left((1-\rho)^{-\beta} - 1\right) H_0}.$$

Note that $w(x_0) = 1$, and w is a non-negative solution of $L_u w = 0$ in $S_u(x_0, c_1 r^{p_1})$. Using Proposition 2.15, we obtain

$$|\{w \leq M\} \cap S_u(x_0, 1/4c_1 r^{p_1})| \geq \delta |S_u(x_0, 1/4c_1 r^{p_1})|.$$

In terms of the original function v, this is an estimate of a set where v is larger than

$$H_0\left((1-\rho)^{-\beta} - M\left((1-\rho)^{-\beta} - 1\right)\right) \geq \frac{H_0}{2},$$

because of the choice of ρ and β. Thus, we obtain the estimate

$$|\{v \geq H_0/2\} \cap S_u(x_0, c_1 r^{p_1})| \geq \delta |S_u(x_0, c_1 r^{p_1})|.$$

In view of (2.10), and the volume estimate on sections in Theorem 3.42, we find

$$Ct^{-\varepsilon}(2r)^{\beta\varepsilon} \geq \delta |S_u(x_0, c_1 r^{p_1})| \geq c(n, \lambda, \Lambda) r^{np_1/2} = c(n, \lambda, \Lambda) r^{\frac{n}{2\mu}}.$$

By the choice of β in (2.11), we find that t is universally bounded. $\square$

In the proof of Theorem 2.16, we use the following consequence of Vitali's covering lemma. It is often referred to as the growing ink-spots lemma which was first introduced by Krylov-Safonov [23]. The term "growing ink-spots lemma" was coined by E. M. Landis.

Lemma 2.19 (Growing Ink-Spots Lemma) *Suppose that u is a strictly convex solution to the Monge-Ampère equation $\lambda \leq \det D^2 u \leq \Lambda$ in a bounded and convex set $\Omega \subset \mathbb{R}^n$. Assume that for some $h > 0$, $S_u(0, 2h) \subset\subset \Omega$.*

Let $E \subset F \subset S_u(0, h)$ be two open sets. Assume that for some constant $\delta \in (0, 1)$, the following two assumptions are satisfied.

- *If any section $S_u(x, t) \subset S_u(0, h)$ satisfies $|S_u(x, t) \cap E| > (1 - \delta)|S_u(x, t)|$, then $S_u(x, t) \subset F$.*
- *$|E| \leq (1 - \delta)|S_u(0, h)|$.*

Then $|E| \leq (1 - c\delta)|F|$ for some constant c depending only on n, λ and Λ.

Proof For every $x \in F$, since F is open, there exists some maximal section which is contained in F and contains x. We choose one of those sections for each $x \in F$ and call it $S_u(x, \bar{h}(x))$.

If $S_u(x, \bar{h}(x)) = S_u(0, h)$ for any $x \in F$, then the result of the lemma follows immediately since $|E| \leq (1 - \delta)|S_u(0, h)|$, so let us assume that it is not the case.

We claim that $|S_u(x, \bar{h}(x)) \cap E| \leq (1 - \delta)|S_u(x, \bar{h}(x))|$. Otherwise, we could find a slightly larger section $\tilde{S}$ containing $S_u(x, \bar{h}(x))$ such that $|\tilde{S} \cap E| > (1 - \delta)|\tilde{S}|$ and $\tilde{S} \not\subset F$, contradicting the first hypothesis.

The family of sections $S_u(x, \bar{h}(x))$ covers the set F. By the Vitali covering Lemma 2.20, we can select a subcollection of non overlapping sections $S_j := S_u(x_j, \bar{h}(x_j))$ such that $F \subset \bigcup_{j=1}^{\infty} S_u(x_j, K\bar{h}(x_j))$ for some universal constant K depending only on n, λ and Λ. The volume estimates in Lemma 3.42 then imply that

$$|S_u(x_j, K\bar{h}(x_j))| \leq C(n, \lambda, \Lambda)|S_u(x_j, \bar{h}(x_j))|$$

for each j.

By construction, $S_j \subset F$ and $|S_j \cap E| \leq (1-\delta)|S_j|$. Thus, we have that $|S_j \cap (F \setminus E)| \geq \delta |S_j|$. Therefore

$$|F \setminus E| \geq \sum_{j=1}^{\infty} |S_j \cap (F \setminus E)| \geq \sum_{j=1}^{\infty} \delta |S_j|$$

$$\geq \frac{\delta}{C(n,\lambda,\Lambda)} \sum_{j=1}^{\infty} |S_u(x_j, K\bar{h}(x_j))| \geq \frac{\delta}{C(n,\lambda,\Lambda)} |F|.$$

Hence $|E| \leq (1-c\delta)|F|$ where $c = C(n,\lambda,\Lambda)^{-1}$. □

Lemma 2.20 (Vitali Covering) *Suppose that $\lambda \leq \det D^2 u \leq \Lambda$ in a bounded on convex set $\Omega \subset \mathbb{R}^n$. Then there exists a universal constant $K > 1$ depending only on n, λ and Λ with the following properties.*

(i) Let $\mathcal{S}$ be a collection of sections $S^x = S_u(x, h(x)) \subset\subset \Omega$. Then there exists a countable subcollection of disjoint sections $\bigcup_{i=1}^{\infty} S_u(x_i, h(x_i))$ such that

$$\bigcup_{S^x \in \mathcal{S}} S^x \subset \bigcup_{i=1}^{\infty} S_u(x_i, Kh(x_i)).$$

(ii) Let D be a compact set in Ω and assume that to each $x \in D$ we associate a corresponding section $S_u(x, h(x)) \subset\subset \Omega$. Then we can find a finite number of these sections $S_u(x_i, h(x_i)), i = 1, \cdots, m$, such that

$$D \subset \bigcup_{i=1}^{m} S_u(x_i, h(x_i)), \text{ with } S_u(x_i, K^{-1}h(x_i)) \text{ disjoint.}$$

Proof of Lemma 2.20 We use the following fact for sections compactly included in Ω: There exists a universal constant $K > 1$ such that if $S_u(x_1, h_1) \cap S_u(x_2, h_2) \neq \emptyset$ and $2h_1 \geq h_2$ then $S_u(x_2, h_2) \subset S_u(x_1, Kh_1)$. The proof of this fact is based on the engulfing property of sections in Theorem 3.54. Suppose that $x \in S_u(x_1, h_1) \cap S_u(x_2, h_2)$ and $2h_1 \geq h_2$. Then we have $S_u(x_2, h_2) \subset S_u(x, \theta_0 h_2) \subset S_u(x, 2\theta_0 h_1)$ and $x_1 \in S_u(x_1, h_1) \subset S_u(x, 2\theta_0 h_1)$. Again, by the engulfing property, we have $S_u(x, 2\theta_0 h_1) \subset S_u(x_1, 2\theta_0^2 h_1)$. It follows that $S_u(x_2, h_2) \subset S_u(x_1, 2\theta_0^2 h_1)$. The result follows by choosing $K = 2\theta_0^2$.

(i) From the volume estimate for sections in Lemma 3.42 and $S_u(x, h(x)) \subset\subset \Omega$, we find that

$$H \equiv \sup\{h(x) | S^x \in \mathcal{S}\} \leq C(n,\lambda,\Lambda,\Omega) < \infty.$$

Define

$$\mathcal{S}_i \equiv \{S^x \in \mathcal{S} | \frac{H}{2^i} < h(x) \le \frac{H}{2^{i-1}}\} \ (i = 1, 2, \cdots).$$

We define $\mathcal{F}_i \subset \mathcal{S}_i$ as follows. Let $\mathcal{F}_1$ be any maximal disjoint collection of sections in $\mathcal{S}_1$. By the volume estimate in Lemma 3.42, $\mathcal{F}_1$ is finite. Assuming $\mathcal{F}_1, \cdots, \mathcal{F}_{k-1}$ have been selected, we choose $\mathcal{F}_k$ to be any maximal disjoint subcollection of

$$\left\{ S \in \mathcal{S}_k | S \cap S^x = \emptyset \text{ for all } S^x \in \bigcup_{j=1}^{k-1} \mathcal{F}_j \right\}.$$

Each $\mathcal{F}_k$ is again a finite set.

We claim that the countable subcollection of disjoint sections $S_u(x_i, h(x_i))$ where $S^{x_i} \in \mathcal{F} := \bigcup_{k=1}^{\infty} \mathcal{F}_k$ satisfies the conclusion of the lemma. To see this, it suffices to show that for any section $S^x \in \mathcal{S}$, there exists a section $S^y \in \mathcal{F}$ such that $S^x \cap S^y \ne \emptyset$ and $S^x \subset S_u(y, Kh(y))$. The proof of this fact is simple. There is an index j such that $S^x \subset \mathcal{S}_j$. By the maximality of $\mathcal{F}_j$, there is a section $S^y \in \bigcup_{k=1}^{j} \mathcal{F}_k$ with $S^x \cap S^y \ne \emptyset$. Because $h(y) > \frac{H}{2^j}$ and $h(x) \le \frac{H}{2^{j-1}}$, we have $h(x) \le 2h(y)$. By the fact established above, we have $S^x \subset S_u(y, Kh(y))$.

(ii) We apply (i) to the collection of sections $S_u(x, K^{-1}h(x))$ where $x \in D$. Then there exists a countable subcollection of disjoint sections $\{S_u(x_i, K^{-1}h(x_i))\}_{i=1}^{\infty}$ such that

$$D \subset \bigcup_{x \in D} S_u(x, K^{-1}h(x)) \subset \bigcup_{i=1}^{\infty} S_u(x_i, h(x_i)).$$

By the compactness of D, we can choose a finite number of sections $S_u(x_i, h(x_i))$ $(i = 1, \cdots, m)$ which cover D. □

Proof of Lemma 2.13 Suppose $v(x_0) \le 1$ at $x_0 \in S_\alpha$ where $\alpha \in (0, 1/2)$. Consider the set of vertices $V = S_\alpha$. We claim there is a large constant a (called the opening) such that, for each $y \in V$, there is a constant c_y such that the generalized paraboloid $-a[u(x) - Du(y) \cdot (x - y) - u(y)] + c_y$ touches the graph of v from below at some point x (called the contact point) in S_1. Indeed, for each $y \in V$, we consider the function

$$P(x) = v(x) + a[u(x) - Du(y) \cdot (x - y) - u(y)]$$

and look for its minimum points on $\overline{S_1}$. On the boundary ∂S_1, we have

$$P \ge a[u(x) - Du(y) \cdot (x - y) - u(y)] \ge aC_1(n, \lambda, \Lambda)$$

by the Aleksandrov maximum principle. At x_0, we have

$$P(x_0) \le 1 + a[u(x_0) - Du(y) \cdot (x_0 - y) - u(y)] \le 1 + a\alpha\theta_0.$$

The last inequality follows from the engulfing property. Indeed, we have $x_0, y \in S_\alpha$ and hence by the engulfing property in Theorem 3.54, $x_0, y \in S_u(0, \alpha) \subset S_u(y, \theta_0\alpha)$. Consequently,

$$u(x_0) - Du(y) \cdot (x_0 - y) - u(y) \le \theta_0\alpha.$$

Thus, we can fix $\alpha > 0$ small, universal and a, M_1 large such that

$$M_1 = 2 + a\alpha\theta_0 < aC_1.$$

Therefore, P attains its minimum at a point $x \in S_1$. Furthermore

$$v(x) \le P(x_0) < M_1.$$

At the contact point $x \in S_1$, we have

$$Dv(x) = a(Du(y) - Du(x))$$

which gives

$$Du(y) = Du(x) + \frac{1}{a}Dv(x).$$

We also have

$$D^2v(x) \ge -aD^2u(x). \tag{2.12}$$

Hence

$$D^2u(y)D_xy = D^2u(x) + \frac{1}{a}D^2v(x) \ge 0. \tag{2.13}$$

Now using the equation at only x, we find that

$$\mathrm{trace}((D^2u)^{-1}D^2v(x)) = 0.$$

This together with (2.12) gives

$$C(a, n)D^2u(x) \ge D^2v(x) \ge -aD^2u(x). \tag{2.14}$$

Here we use the following basic estimates. If $A \ge -B$ and $\mathrm{trace}(B^{-1}A) = 0$ then

$$CB \ge A \ge -B.$$

Indeed, we can rewrite

$$B^{-1/2}AB^{-1/2} \geq -I_n, \ \text{trace}(B^{-1/2}AB^{-1/2}) = 0.$$

Hence

$$B^{-1/2}AB^{-1/2} \leq C(n)I_n.$$

Now, taking the determinant in (2.13) and invoking (2.14), we obtain

$$\det D^2u(y)\,|\det D_x y| = \det(D^2u(x) + \frac{1}{a}D^2v(x)) \leq C(a,n)\det D^2u(x).$$

This implies the bound

$$|\det D_x y| \leq C(a,n,\Lambda,\lambda).$$

Then, by the area formula, the set E of contact points x satisfies

$$|S_\alpha| = |V| = \int_E |\det D_x y| \leq C(a,n,\Lambda,\lambda)\,|E| \leq C|\{v < M_1\} \cap S_1|.$$

Using the volume estimate of sections in Lemma 3.42, we find that $|S_1| \leq C^*|\{v < M_1\} \cap S_1|$ for some $C^* > 1$ universal. The conclusion of the Lemma holds with $\delta = 1/C^*$. □

Proof of Lemma 2.14 Recall that $u(0) = 0, Du(0) = 0$ and $B_1(0) \subset S_u(0,4) \subset B_n(0)$. To prove the lemma, it suffices to construct a subsolution $w : S_2 \backslash S_\alpha \longrightarrow \mathbb{R}$, i.e., $U^{ij}w_{ij} \geq 0$, with the following properties

(i) $w \leq 0$ on ∂S_2
(ii) $w \leq 1$ on ∂S_α
(iii) $w \geq c(n,\Lambda,\lambda)$ in $S_1 \backslash S_\alpha$.

Our first guess is

$$w = C(\alpha,m)(u^{-m} - 2^{-m})$$

where m is large.

Let $(u^{ij})_{1\leq i,j\leq n}$ be the inverse matrix $(D^2u)^{-1}$ of the Hessian matrix D^2u. We can compute for $W = u^{-m} - 2^{-m}$

$$u^{ij}W_{ij} = mu^{-m-2}[(m+1)u^{ij}u_iu_j - uu^{ij}u_{ij}] = mu^{-m-2}[(m+1)u^{ij}u_iu_j - nu]. \quad (2.15)$$

By Lemma 3.64

$$u^{ij}u_iu_j \geq \frac{|Du|^2}{\text{trace}(D^2u)}.$$

If $x \in S_2 \setminus S_\alpha$ and $y = 0$ then from the convexity of u, we have $0 = u(y) \geq u(x) + Du(x) \cdot (0 - x)$ and therefore,

$$|Du(x)| \geq \frac{u(x)}{|x|} \geq \frac{\alpha}{n} \equiv 2c_n$$

for some constant c_n depending only on n, λ and Λ.

In order to obtain $u^{ij}W_{ij} \geq 0$ using (2.15), we only have trouble when $\|D^2u\|$ is unbounded. But the set of bad points, i.e., where $\|D^2u\|$ is large, is small. Here is how we see this. Because $S_u(0, 4)$ is normalized, we can deduce from the Aleksandrov maximum principle, Theorem 3.20 applied to $u - 4$, that

$$\text{dist}(S_u(0, 3), \partial S_u(0, 4)) \geq c(n, \lambda, \Lambda)$$

for some universal $c(n, \lambda, \Lambda) > 0$. By Lemma 3.11, Du is bounded on S_3. Now let ν denote the outernormal unit vector field on ∂S_3. Then, using the convexity of u, we have $\|D^2u\| \leq \Delta u$ and thus, by the divergence theorem,

$$\int_{S_3} \|D^2u\| \leq \int_{S_3} \Delta u = \int_{\partial S_3} \frac{\partial u}{\partial \nu} \leq C(n, \lambda, \Lambda).$$

Therefore, given $\varepsilon > 0$ small, the set

$$H_\varepsilon = \{x \in S_3 \mid \|D^2u\| \geq \frac{1}{\varepsilon}\}$$

has measure bounded from above by

$$|H_\varepsilon| \leq C\varepsilon.$$

To construct a proper subsolution bypassing the bad points in H_ε, we only need to modify w at bad points. Roughly speaking, the modification involves the solution to

$$\det D^2u_\varepsilon = \Lambda\chi_{H_\varepsilon}, \; u_\varepsilon = 0 \text{ on } \partial S_4.$$

Here we use χ_E to denote the characteristic function of the set $E : \chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if otherwise. The problem with this equation is that the solution is not in general smooth while we need two derivatives to construct the subsolution. But this problem can be fixed, using approximation, as follows.

We approximate H_ε by an open set $\tilde{H}_\varepsilon$ where $H_\varepsilon \subset \tilde{H}_\varepsilon \subset S_4$ and the measure of their difference is small, that is

$$|\tilde{H}_\varepsilon \setminus H_\varepsilon| \leq \varepsilon.$$

We introduce a smooth function φ with the following properties:

$$\varphi = 1 \text{ in } H_\varepsilon,\ \varphi = \varepsilon \text{ in } S_4 \setminus \tilde{H}_\varepsilon,\ \varepsilon \leq \varphi \leq 1 \text{ in } S_4.$$

Let h_ε be the solution to

$$\det D^2 h_\varepsilon = \Lambda\varphi,\ h_\varepsilon = 0 \text{ on } \partial S_4;$$

see Theorem 3.27. By Caffarelli's $C^{2,\alpha}$ estimates [6], $h_\varepsilon \in C^{2,\alpha}(S_4)$ for all $\alpha \in (0,1)$. From the Aleksandrov maximum principle, Theorem 3.20, we have on S_4

$$|h_\varepsilon| \leq C_n \mathrm{diam}(S_4) \left(\int_{S_4} \Lambda\varphi\right)^{1/n}.$$

We need to estimate the above right hand side. From the definitions of $\tilde{H}_\varepsilon$ and φ, we can estimate

$$\int_{S_4} \Lambda\varphi = \int_{H_\varepsilon} \Lambda + \int_{\tilde{H}_\varepsilon\setminus H_\varepsilon} \Lambda\varphi + \int_{S_4\setminus\tilde{H}_\varepsilon} \varepsilon \leq \Lambda|H_\varepsilon| + \Lambda|\tilde{H}_\varepsilon\setminus H_\varepsilon| + \varepsilon C(n,\lambda,\Lambda) \leq C(n,\lambda,\Lambda)\varepsilon.$$

It follows that for some universal constant $C_1(n,\lambda,\Lambda)$,

$$|h_\varepsilon| \leq C_1(n,\lambda,\Lambda)\varepsilon^{1/n}.$$

By the gradient estimate in Lemma 3.11, we have on S_2

$$|Dh_\varepsilon(x)| \leq \frac{-h_\varepsilon(x)}{\mathrm{dist}(S_3,\partial S_4)} \leq C_2(n,\lambda,\Lambda)\varepsilon^{1/n}.$$

We choose ε small so that

$$C_1(n,\lambda,\Lambda)\varepsilon^{1/n} \leq 1/4,\ C_2(n,\lambda,\Lambda)\varepsilon^{1/n} \leq c_n. \tag{2.16}$$

Let

$$\tilde{V} = (u - h_\varepsilon) \text{ and } \tilde{W} = \tilde{V}^{-m} - 2^{-m}.$$

Then

$$|\tilde{V}| \leq 3 \text{ and } |D\tilde{V}| \geq c_n \text{ on } S_2 \setminus S_\alpha;\ \alpha \leq \tilde{V} \leq 1 + 1/4 = 5/4 \text{ on } S_1\setminus S_\alpha. \tag{2.17}$$

Now, compute as before

$$u^{ij}\tilde{W}_{ij} = m\tilde{V}^{-m-2}[(m+1)u^{ij}\tilde{V}_i\tilde{V}_j - \tilde{V}u^{ij}\tilde{V}_{ij}] = m\tilde{V}^{-m-2}[(m+1)u^{ij}\tilde{V}_i\tilde{V}_j + \tilde{V}(u^{ij}(h_\varepsilon)_{ij} - n)].$$

We note that, by Lemma 3.63,

$$u^{ij}(h_\varepsilon)_{ij} = \operatorname{trace}((D^2u)^{-1}D^2h_\varepsilon) \geq n(\det(D^2u)^{-1}\det D^2h_\varepsilon)^{1/n} \geq n \text{ on } H_\varepsilon.$$

It follows that

$$u^{ij}\tilde{W}_{ij} \geq 0 \text{ on } H_\varepsilon.$$

On $(S_2 \setminus S_\alpha)\backslash H_\varepsilon$, we have $\operatorname{trace}(D^2u) \leq n\varepsilon^{-1}$ and from (2.17)

$$\begin{aligned} u^{ij}\tilde{W}_{ij} &\geq m\tilde{V}^{-m-2}[(m+1)u^{ij}\tilde{V}_i\tilde{V}_j - n\tilde{V}] \\ &\geq m\tilde{V}^{-m-2}[(m+1)\frac{|D\tilde{V}|^2}{\operatorname{trace}(D^2u)} - n\tilde{V}] \\ &\geq m\tilde{V}^{-m-2}[(m+1)n^{-1}\varepsilon c_n - n\tilde{V}] \geq 0 \end{aligned}$$

if we choose m large, universal. Therefore,

$$u^{ij}\tilde{W}_{ij} \geq 0 \text{ on } S_2 \setminus S_\alpha$$

and hence $\tilde{W} = \tilde{V}^{-m} - 2^{-m}$ is a subsolution to $u^{ij}v_{ij} \geq 0$ on $S_2 \setminus S_\alpha$.

Finally, by (2.17) and $\tilde{W} \leq 0$ on ∂S_2, we choose a suitable $C(\alpha, n, \lambda, \Lambda)$ so that the subsolution of the form

$$\tilde{w} = C(\alpha, n, \lambda, \lambda)(\tilde{V}^{-m} - 2^{-m})$$

satisfies $\tilde{w} \leq 1$ on ∂S_α. Now, we obtain the desired universal lower bound for v in S_1 from $v \geq \tilde{w}$ on $S_1 \backslash S_\alpha$ and $v \geq 1$ on S_α. □

2.2.2 Proof of the Interior Hölder Estimates for the Inhomogeneous Linearized Monge-Ampère Equation

In this section, we prove Theorem 2.9, following an argument of Trudinger and Wang [40].

The following lemma is a refined version of the Aleksandrov-Bakelman-Pucci (ABP) maximum principle for convex domains.

Lemma 2.21 *Assume that Ω is a bounded, convex domain in $\mathbb{R}^n$. Let*

$$Lu(x) = \operatorname{trace}(A(x)D^2u(x))$$

where A is an $n \times n$ symmetric and positive definite matrix in $\overline{\Omega}$. Then, for all $u \in C^2(\Omega) \cap C(\overline{\Omega})$,

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u + C(n)|\Omega|^{1/n} \left\| \frac{Lu}{(\det A)^{1/n}} \right\|_{L^n(\Omega)}.$$

Proof We use the ABP estimate, Theorem 2.3, and John's lemma, Lemma 3.23. According to this lemma, there is an affine transformation $T(x) = Mx + b$ where M is an $n \times n$ invertible matrix and $b \in \mathbb{R}^n$ such that

$$B_1(0) \subset T(\Omega) \subset B_n(0). \tag{2.18}$$

For $x \in T(\Omega)$, we define

$$v(x) = u(T^{-1}x) \text{ and } \tilde{L}v = \operatorname{trace}(\tilde{A}(x)D^2v(x))$$

where $\tilde{A}(x) = MA(T^{-1}x)M^t$. We then compute $D^2v(x) = (M^{-1})^t D^2u(T^{-1}x)M^{-1}$ and hence

$$\tilde{L}v(x) = Lu(T^{-1}(x)).$$

Applying the ABP to v and $\tilde{L}v(x)$ on $T(\Omega)$, we find

$$\max_{\overline{T(\Omega)}} v \leq \max_{\partial T(\Omega)} v + C_1(n)\operatorname{diam}(T(\Omega)) \left\| \frac{\tilde{L}v}{(\det \tilde{A})^{1/n}} \right\|_{L^n(T(\Omega))}. \tag{2.19}$$

By changing variables $x = T(y)$ for $x \in T(\Omega)$, we find from $\det \tilde{A} = (\det M)^2 \det A$ that

$$\left\| \frac{\tilde{L}v}{(\det \tilde{A})^{1/n}} \right\|_{L^n(T(\Omega))} = \frac{1}{(\det M)^{1/n}} \left\| \frac{Lu}{(\det A)^{1/n}} \right\|_{L^n(\Omega)} \tag{2.20}$$

From (2.18), we have $\det M \geq c(n)|\Omega|^{-1}$ and $\operatorname{diam}(T(\Omega)) \leq 2n$. Using these estimates in (2.19) and (2.20), we obtain the conclusion of the lemma. □

By employing Lemma 2.21 and the interior Harnack inequality in Theorem 2.7 for nonnegative solutions to the homogeneous linearized Monge-Ampère equations, we get:

Lemma 2.22 (Harnack Inequality for Inhomogeneous Linearized Monge-Ampère) *Assume that* $\lambda \le \det D^2u \le \Lambda$ *in a convex domain* $\Omega \subset \mathbb{R}^n$. *Let* $f \in L^n(\Omega)$ *and* $v \in W^{2,n}_{loc}(\Omega)$ *satisfy* $U^{ij}v_{ij} = f$ *almost everywhere in* Ω. *Then if* $S_u(x,t) \subset\subset \Omega$ *and* $v \ge 0$ *in* $S_u(x,t)$, *we have*

$$\sup_{S_u(x,\frac{t}{2})} v \le C(n,\lambda,\Lambda)\Big(\inf_{S_u(x,\frac{t}{2})} v + t^{\frac{1}{2}}\,\|f\|_{L^n(S_u(x,t))}\Big). \tag{2.21}$$

Proof Let w be the solution of

$$U^{ij}w_{ij} = f \ \text{ in } S_u(x,t), \ \text{ and } w = 0 \ \text{ on } \partial S_u(x,t).$$

Then, by Lemma 2.21 and the volume bound on sections in Theorem 3.42, we get

$$\sup_{S_u(x,t)} |w| \le C(n,\lambda)|S_u(x,t)|^{\frac{1}{n}}\|f\|_{L^n(S_u(x,t))} \le Ct^{1/2}\|f\|_{L^n(S_u(x,t))}. \tag{2.22}$$

Furthermore, we have $U^{ij}(v-w)_{ij} = 0$ in $S_u(x,t)$ and $v - w \ge 0$ on $\partial S_u(x,t)$. Thus we conclude from the ABP maximum principle that $v - w \ge 0$ in $S_u(x,t)$. Hence, we can apply the interior Harnack inequality, Theorem 2.7, to obtain

$$\sup_{S_u(x,\frac{t}{2})} (v-w) \le C \inf_{S_u(x,\frac{t}{2})} (v-w),$$

for some constant C depending only on n, λ, and Λ, which then implies

$$\sup_{S_u(x,\frac{t}{2})} v \le C'\Big(\inf_{S_u(x,\frac{t}{2})} v + \sup_{S_u(x,\frac{t}{2})} |w|\Big) \le C\Big(\inf_{S_u(x,\frac{t}{2})} v + t^{\frac{1}{2}}\,\|f\|_{L^n(S_u(x,t))}\Big).$$

□

As a consequence of Lemma 2.22, we obtain the following oscillation estimate:

Corollary 2.23 *Assume that* $\lambda \le \det D^2u \le \Lambda$ *in a convex domain* $\Omega \subset \mathbb{R}^n$. *Let* $f \in L^n(\Omega)$ *and* $v \in W^{2,n}_{loc}(\Omega)$ *satisfy* $U^{ij}v_{ij} = f$ *almost everywhere in* Ω. *Then if* $S_u(x,h) \subset\subset \Omega$, *we have*

$$osc_{S_u(x,\rho)} v \le C\Big(\frac{\rho}{h}\Big)^{\alpha}\Big[osc_{S_u(x,h)} v + h^{\frac{1}{2}}\,\|f\|_{L^n(S_u(x,h))}\Big] \quad \textit{for all} \quad \rho \le h,$$

where C, $\alpha > 0$ *depend only on* n, λ, *and* Λ, *and* $osc_E v := \sup_E v - \inf_E v$.

Proof Let us write S_t for the section $S_u(x,t)$. Set

$$m(t) := \inf_{S_t} v, \quad M(t) := \sup_{S_t} v, \quad \text{and} \quad \omega(t) := M(t) - m(t).$$

Let $\rho \in (0, h]$ be arbitrary. Then since $\tilde{v} := v - m(\rho)$ is a nonnegative solution of $U^{ij}\tilde{v}_{ij} = f$ in S_ρ, we can apply Lemma 2.22 to $\tilde{v}$ to obtain

$$\frac{1}{C}\sup_{S_{\frac{\rho}{2}}}\tilde{v} \leq \inf_{S_{\frac{\rho}{2}}}\tilde{v} + \rho^{\frac{1}{2}}\,\|f\|_{L^n(S_\rho)}.$$

It follows that for all $\rho \in (0, h]$, we have

$$\omega(\frac{\rho}{2}) = \sup_{S_{\frac{\rho}{2}}}\tilde{v} - \inf_{S_{\frac{\rho}{2}}}\tilde{v} \leq \big(1 - \frac{1}{C}\big)\sup_{S_{\frac{\rho}{2}}}\tilde{v} + \rho^{\frac{1}{2}}\,\|f\|_{L^n(S_\rho)} \leq \big(1 - \frac{1}{C}\big)\omega(\rho) + \rho^{\frac{1}{2}}\,\|f\|_{L^n(S_h)}.$$

Thus, by the standard iteration we deduce that

$$\omega(\rho) \leq C'\big(\frac{\rho}{h}\big)^{\alpha}\Big[\omega(h) + h^{\frac{1}{2}}\,\|f\|_{L^n(S_h)}\Big],$$

giving the conclusion of the corollary. □

Proof of Theorem 2.9 By Lemma 3.11, there is a constant $M > 1$ depending only on n, λ and Λ such that $|Du(z)| \leq M$ for all $z \in B_{3/4}(0)$. By Theorem 3.50, there exists a constant $r_0 > 0$ depending only on n, λ and Λ such that $S_u(z, r_0) \subset B_{3/4}(0)$ for all $z \in B_{1/2}(0)$. The gradient bound implies that $B(z, \frac{r}{2M}) \subset S_u(z, r)$ for all $z \in B_{1/2}(0)$ and $r \leq r_0$. Fix $x \in B_{1/2}(0)$. It suffices to prove the lemma for $y \in S_u(x, r_0/4)$. Let $r \in (0, r_0/2)$ be such that $y \in S_u(x, r)\backslash S_u(x, r/2)$. Then $|y - x| \geq \frac{r}{4M}$. The above corollary gives

$$\begin{aligned}|v(y) - v(x)| \leq \operatorname{osc}_{S_u(x,r)} v &\leq C(\frac{r}{r_0})^{\alpha}\left[\|v\|_{L^\infty(S_\phi(x,r_0))} + r_0^{\frac{1}{2}}\,\|f\|_{L^n(S_u(x,r_0))}\right]\\ &\leq C|x - y|^{\alpha}\left[\|v\|_{L^\infty(B_1(0))} + \|f\|_{L^n(B_1(0))}\right].\end{aligned}$$

□

Remark 2.24 The proof of Theorem 2.7 follows the presentation in [25] where the case of lower order terms was treated. For related results, see also [28].

2.3 Global Hölder Estimates for the Linearized Monge-Ampère Equations

In this section, we prove Proposition 1.14 and Theorem 1.13.

2.3.1 Boundary Hölder Continuity for Solutions of Non-uniformly Elliptic Equations

Proof of Proposition 1.14 By considering the equation satisfied by $\frac{v}{\|\varphi\|_{C^\alpha(\partial\Omega)}+\|g\|_{L^n(\Omega)}}$, we can assume that

$$\|\varphi\|_{C^\alpha(\partial\Omega)} + \|g\|_{L^n(\Omega)} = 1$$

and we need to prove that

$$|v(x) - v(x_0)| \leq C|x - x_0|^{\frac{\alpha}{\alpha+2}} \text{ for all } x \in \Omega \cap B_\delta(x_0).$$

Moreover, without loss of generality, we assume that $\lambda = 1$ and

$$\Omega \subset \mathbb{R}^n \cap \{x_n > 0\},\ 0 \in \partial\Omega.$$

Take $x_0 = 0$. By the ABP estimate in Theorem 2.3 and the assumption $\det(a^{ij}) \geq 1$, we have

$$|v(x)| \leq \|\varphi\|_{L^\infty(\partial\Omega)} + C_n \text{diam}(\Omega)\|g\|_{L^n(\Omega)} \leq C_0 \ \forall\, x \in \Omega$$

for a constant $C_0 > 1$ depending only on n and $\text{diam}(\Omega)$, and hence, for any $\varepsilon \in (0, 1)$

$$|v(x) - v(0) \pm \varepsilon| \leq 3C_0 := C_1. \tag{2.23}$$

Consider now the functions

$$h_\pm(x) := v(x) - v(0) \pm \varepsilon \pm C_1(\inf\{y_n : y \in \overline{\Omega} \cap \partial B_{\delta_2}(0)\})^{-1} x_n$$

in the region $A := \Omega \cap B_{\delta_2}(0)$ where δ_2 is small to be chosen later.

Note that, if $x \in \partial\Omega$ with

$$|x| \leq \delta_1(\varepsilon) := \varepsilon^{1/\alpha}$$

then, we have from $\|\varphi\|_{C^\alpha(\partial\Omega)} \leq 1$ that

$$|v(x) - v(0)| = |\varphi(x) - \varphi(0)| \leq |x|^\alpha \leq \varepsilon. \tag{2.24}$$

It follows that, if we choose $\delta_2 \leq \delta_1$ then from (2.23) and (2.24), we have

$$h_- \leq 0, h_+ \geq 0 \text{ on } \partial A.$$

On the other hand,

$$a^{ij}(h_\pm)_{ij} = g \text{ in } A.$$

The ABP estimate in Theorem 2.3 applied in A gives

$$h_- \leq C_n \mathrm{diam}(A)\|g\|_{L^n(A)} \leq C_n\delta_2 \text{ in } A$$

and

$$h_+ \geq -C_n \mathrm{diam}(A)\|g\|_{L^n(A)} \geq -C_n\delta_2 \text{ in } A.$$

By restricting $\varepsilon \leq C_n^{\frac{-\alpha}{1-\alpha}}$, we can assume that

$$\delta_1 = \varepsilon^{1/\alpha} \leq \frac{\varepsilon}{C_n}.$$

Then, for $\delta_2 \leq \delta_1$, we have $C_n\delta_2 \leq \varepsilon$ and thus, for all $x \in A$, we have

$$|v(x) - v(0)| \leq 2\varepsilon + C_1(\inf\{y_n : y \in \overline{\Omega} \cap \partial B_{\delta_2}(0)\})^{-1}x_n.$$

The uniform convexity of Ω gives

$$\inf\{y_n : y \in \overline{\Omega} \cap \partial B_{\delta_2}(0)\} \geq C_2^{-1}\delta_2^2. \tag{2.25}$$

Therefore, choosing $\delta_2 = \delta_1$, we obtain

$$|v(x) - v(0)| \leq 2\varepsilon + C_1(\inf\{y_n : y \in \overline{\Omega} \cap \partial B_{\delta_2}(0)\})^{-1}x_n = 2\varepsilon + \frac{2C_1C_2}{\delta_2^2}x_n \text{ in } A.$$

As a consequence, we have just obtained the following inequality

$$|v(x) - v(0)| \leq 2\varepsilon + \frac{2C_1C_2}{\delta_2^2}|x| = 2\varepsilon + 2C_1C_2\varepsilon^{-2/\alpha}|x| \tag{2.26}$$

for all x, ε satisfying the following conditions

$$|x| \leq \delta_1(\varepsilon) := \varepsilon^{1/\alpha}, \varepsilon \leq C_n^{\frac{-\alpha}{1-\alpha}} := c_1(\alpha, L, K, n). \tag{2.27}$$

Finally, let us choose $\varepsilon = |x|^{\frac{\alpha}{\alpha+2}}$. It satisfies the conditions in (2.27) if

$$|x| \leq \min\{c_1^{\frac{\alpha+2}{\alpha}}, 1\} := \delta.$$

Then, by (2.26), we have for all $x \in \Omega \cap B_\delta(0)$

$$|v(x) - v(0)| \leq C|x|^{\frac{\alpha}{\alpha+2}}, \; C = 2 + 2C_1C_2.$$

□

Proposition 1.14 gives the boundary Hölder continuity for solutions to the linearized Monge-Ampère equation

$$U^{ij}v_{ij} = g$$

where (U^{ij}) is the cofactor matrix of the Hessian matrix D^2u of the convex function u satisfying

$$\lambda \leq \det D^2 u \leq \Lambda.$$

This combined with the interior Hölder continuity estimates of Caffarelli-Gutiérrez in Theorem 2.9 gives the global Hölder estimates for solutions to the linearized Monge-Ampère equations on uniformly convex domains as stated in Theorem 1.13. The rest of this section will be devoted to the proof of these global Hölder estimates.

The main tool to connect the interior and boundary Hölder continuity for solutions to the linearized Monge-Ampère equation is Savin's Localization Theorem at the boundary for the Monge-Ampère equation.

2.3.2 Savin's Localization Theorem

We now state the main tool used in the proof of Theorem 1.13, the localization theorem.

Let $\Omega \subset \mathbb{R}^n$ be a bounded convex set with

$$B_\rho(\rho e_n) \subset \Omega \subset \{x_n \geq 0\} \cap B_{\frac{1}{\rho}}(0), \tag{2.28}$$

for some small $\rho > 0$. Here $e_n = (0, \cdots, 0, 1) \in \mathbb{R}^n$. Assume that

$$\text{for each } y \in \partial\Omega \cap B_\rho(0) \text{ there is a ball } B_\rho(z) \subset \Omega \text{ that is tangent to } \partial\Omega \text{ at y.} \tag{2.29}$$

Let $u : \overline{\Omega} \to \mathbb{R}$, $u \in C^{0,1}(\overline{\Omega}) \cap C^2(\Omega)$ be a convex function satisfying

$$\det D^2 u = f, \qquad 0 < \lambda \leq f \leq \Lambda \quad \text{in } \Omega, \tag{2.30}$$

and assume that

$$u(0) = 0, \quad \nabla u(0) = 0. \tag{2.31}$$

If the boundary data has quadratic growth near $\{x_n = 0\}$ then, as $h \to 0$, the section $S_u(0,h)$ of u at 0 with level h is equivalent to a half-ellipsoid centered at 0; here we recall that

$$S_u(x,h) := \{y \in \overline{\Omega} : u(y) < u(x) + \nabla u(x) \cdot (y - x) + h\}.$$

This is the content of Savin's Localization Theorem proved in [34, 35]. Precisely, this theorem reads as follows.

Theorem 2.25 (Localization Theorem [34, 35]) *Assume that Ω satisfies (2.28)–(2.29) and u satisfies (2.30), (2.31) above and,*

$$\rho|x|^2 \le u(x) \le \rho^{-1}|x|^2 \quad on\ \partial\Omega \cap \{x_n \le \rho\}. \tag{2.32}$$

Then, for each $h < k$ there exists an ellipsoid E_h of volume $\omega_n h^{n/2}$ such that

$$kE_h \cap \overline{\Omega} \subset S_u(0,h) \subset k^{-1}E_h \cap \overline{\Omega}.$$

Moreover, the ellipsoid E_h is obtained from the ball of radius $h^{1/2}$ by a linear transformation A_h^{-1} (sliding along the $x_n = 0$ plane)

$$A_h E_h = h^{1/2} B_1, \quad \det A_h = 1,$$

$$A_h(x) = x - \tau_h x_n, \quad \tau_h = (\tau_1, \tau_2, \ldots, \tau_{n-1}, 0),$$

with

$$|\tau_h| \le k^{-1}|\log h|.$$

The constant k above depends only on $\rho, \lambda, \Lambda, n$.

The ellipsoid E_h, or equivalently the linear map A_h, provides useful information about the behavior of u near the origin. From Theorem 2.25 we also control the shape of sections that are tangent to $\partial\Omega$ at the origin.

Proposition 2.26 *Let u and Ω satisfy the hypotheses of the Localization Theorem 2.25 at the origin. Assume that for some $y \in \Omega$ the section $S_u(y,h) \subset \Omega$ is tangent to $\partial\Omega$ at 0 for some $h \le c$ with c universal. Then there exists a small constant $k_0 > 0$ depending on λ, Λ, ρ and n such that*

$$Du(y) = ae_n \quad for\ some \quad a \in [k_0 h^{1/2}, k_0^{-1} h^{1/2}],$$

$$k_0 E_h \subset S_u(y,h) - y \subset k_0^{-1} E_h, \qquad k_0 h^{1/2} \le dist(y, \partial\Omega) \le k_0^{-1} h^{1/2},$$

with E_h the ellipsoid defined in the Localization Theorem 2.25.

Proposition 2.26, proved in [36], is a consequence of Theorem 2.25. We sketch its proof here.

Proof of Proposition 2.26 Assume that the hypotheses of the Localization Theorem 2.25 hold at the origin. For $a \geq 0$ we denote

$$S'_a := \{x \in \overline{\Omega} | \quad u(x) < ax_n\},$$

and clearly $S'_{a_1} \subset S'_{a_2}$ if $a_1 \leq a_2$. The proposition easily follows once we show that $S'_{ch^{1/2}}$ has the shape of the ellipsoid E_h for all small h.

From Theorem 2.25 we know

$$S_u(0,h) := \{u < h\} \subset k^{-1}E_h \subset \{x_n \leq k^{-1}h^{1/2}\}$$

and since $u(0) = 0$ we use the convexity of u and obtain

$$S'_{kh^{1/2}} \subset S_u(0,h) \cap \Omega. \tag{2.33}$$

This inclusion shows that in order to prove that $S'_{kh^{1/2}}$ is equivalent to E_h it suffices to bound its volume by below

$$|S'_{kh^{1/2}}| \geq c|E_h|.$$

From Theorem 2.25, there exists $y \in \partial S_{\theta h}$ such that $y_n \geq k(\theta h)^{1/2}$. We evaluate $\tilde{u} := u - kh^{1/2}x_n$, at y and find

$$\tilde{u}(y) \leq \theta h - kh^{1/2}k(\theta h)^{1/2} \leq -\delta h,$$

for some $\delta > 0$ provided that we choose θ small depending on k. Since $\tilde{u} = 0$ on $\partial S'_{kh^{1/2}}$ and $\det D^2\tilde{u} \leq \Lambda$, we apply Lemma 2.21 to $-\tilde{u}$ which solves $U^{ij}(-\tilde{u})_{ij} = -n \det D^2 u$. We have

$$\delta h \leq \max_{S'_{kh^{1/2}}} -\tilde{u} \leq C(\Lambda, n)|S'_{kh^{1/2}}|^{2/n},$$

hence

$$ch^{n/2} \leq |S'_{kh^{1/2}}|.$$

□

The quadratic separation from tangent planes on the boundary for solutions to the Monge-Ampère equation is a crucial assumption in the Localization Theorem 2.25. This is the case for u in Theorem 1.13 as proved in [35, Proposition 3.2].

Proposition 2.27 *Let u be as in Theorem 1.13. Then, on $\partial\Omega$, u separates quadratically from its tangent planes on $\partial\Omega$. This means that if $x_0 \in \partial\Omega$ then*

$$\rho\,|x-x_0|^2 \le u(x) - u(x_0) - \nabla u(x_0)\cdot(x-x_0) \le \rho^{-1}\,|x-x_0|^2\,, \tag{2.34}$$

for all $x \in \partial\Omega$, for some small constant ρ universal.

Proof We prove the Proposition for the case $x_0 \in \partial\Omega$. By rotation of coordinates, we can assume that $x_0 = 0$ and

$$\Omega \subset \{x \in \mathbb{R}^n : x_n > 0\}.$$

We denote a point $x = (x_1, \cdots, x_{n-1}, x_n) \in \mathbb{R}^n$ by $x = (x', x_n)$ where $x' = (x_1, \cdots, x_{n-1})$. By the Aleksandrov maximum principle, we have that u is universally bounded. Since Ω is uniformly convex at the origin and $\det D^2u$ is bounded from above, we can use barriers and obtain that l_0, the tangent plane at the origin, has bounded slope. The proof of this fact is quite similar to that of Lemma 1.19. After subtracting this linear function from u and $\phi = u|_{\partial\Omega}$, we may assume $l_0 = 0$. Thus, $u \ge 0$ and it suffices to show that

$$\rho\,|x-x_0|^2 \le u(x) \le \rho^{-1}\,|x-x_0|^2\,, \tag{2.35}$$

for all $x \in \partial\Omega$. Since u is universally bounded, we only need to prove (2.35) for $|x|$ universally small.

Since $\phi = u|_{\partial\Omega}$, $\partial\Omega$ are C^3 at the origin, we find that

$$\phi(x) = Q_0(x') + o(|x'|^3) \text{ for } x = (x', x_n) \in \partial\Omega\,, \tag{2.36}$$

with Q_0 a cubic polynomial. Indeed, locally around 0, $\partial\Omega$ is given by the graph of a C^3 function ψ: for some c small,

$$\partial\Omega \cap B_c(0) = \{(x', x_n) : x_n = \psi(x')\}.$$

Thus, we can write for $(x', x_n) \in \partial\Omega \cap B_c(0)$:

$$x_n = Q_1(x') + o(|x'|^3) \tag{2.37}$$

with Q_1 a cubic polynomial. Since $\phi \in C^3(\overline{\Omega})$, we can again write around 0:

$$\phi(x) = Q_2(x) + o(|x|^3) \text{ for } x = (x', x_n) \in \overline{\Omega}$$

with Q_2 a cubic polynomial. Substituting (2.37) into this equation, we obtain (2.36) as claimed.

Now we use (2.36). Because $u = \phi \geq 0$ on $\partial\Omega$, Q_0 has no linear part and its quadratic part is given by, say

$$\sum_{i<n} \frac{\mu_i}{2} x_i^2, \quad \text{with} \quad \mu_i \geq 0.$$

We need to show that $\mu_i > 0$.

If $\mu_1 = 0$, then the coefficient of x_1^3 is 0 in Q_0. Thus, if we restrict to $\partial\Omega$ in a small neighborhood near the origin, then for all small h the set $\{\phi < h\}$ contains

$$\{|x_1| \leq r(h)h^{1/3}\} \cap \{|x'| \leq ch^{1/2}\}$$

for some $c > 0$ and with

$$r(h) \to \infty \quad \text{as } h \to 0.$$

Now $S_u(0, h)$ contains the convex set generated by $\{\phi < h\}$ thus, since Ω is uniformly convex,

$$|S_u(0, h)| \geq c'(r(h)h^{1/3})^3 h^{(n-2)/2} \geq c' r(h)^3 h^{n/2}.$$

On the other hand, since $\det D^2u \geq \lambda$ and

$$0 \leq u \leq h \quad \text{in } S_u(0, h)$$

we obtain from Lemma 3.44 that

$$|S_u(0, h)| \leq C(\lambda, n)h^{n/2},$$

and we contradict the inequality above as $h \to 0$. □

2.3.3 Proof of Global Hölder Estimates for the Linearized Monge-Ampère Equation

Proof of Theorem 1.13 We recall from Proposition 2.27 that u separates quadratically from its tangent planes on $\partial\Omega$. Therefore, Proposition 2.26 applies. Let $y \in \Omega$ with $r := \operatorname{dist}(y, \partial\Omega) \leq c$, for c universal, and consider the maximal section $S_u(y, \bar{h}(y))$ centered at y, i.e.,

$$\bar{h}(y) = \max\{h \mid \quad S_u(y, h) \subset \Omega\}.$$

When it is clear from the context, we write $\bar{h}$ for $\bar{h}(y)$. By Proposition 2.26 applied at the point $x_0 \in \partial S_u(y, \bar{h}) \cap \partial\Omega$, we have

$$\bar{h}^{1/2} \approx r, \tag{2.38}$$

and $S_u(y, \bar{h})$ is equivalent to an ellipsoid E i.e

$$cE \subset S_u(y, \bar{h}) - y \subset CE,$$

where

$$E := \bar{h}^{1/2} A_{\bar{h}}^{-1} B_1(0), \quad \text{with} \quad \|A_{\bar{h}}\|, \|A_{\bar{h}}^{-1}\| \leq C |\log \bar{h}|; \det A_{\bar{h}} = 1. \tag{2.39}$$

We denote

$$u_y := u - u(y) - Du(y) \cdot (x - y).$$

The rescaling $\tilde{u} : \tilde{S}_1 \to \mathbb{R}$ of u

$$\tilde{u}(\tilde{x}) := \frac{1}{\bar{h}} u_y(T\tilde{x}) \qquad x = T\tilde{x} := y + \bar{h}^{1/2} A_{\bar{h}}^{-1} \tilde{x},$$

satisfies

$$\det D^2 \tilde{u}(\tilde{x}) = \tilde{f}(\tilde{x}) := f(T\tilde{x}),$$

and

$$B_c(0) \subset \tilde{S}_1 \subset B_C(0), \qquad \tilde{S}_1 = \bar{h}^{-1/2} A_{\bar{h}} (S_u(y, \bar{h}) - y), \tag{2.40}$$

where $\tilde{S}_1$ represents the section of $\tilde{u}$ at the origin at height 1.

We define also the rescaling $\tilde{v}$ for v

$$\tilde{v}(\tilde{x}) := v(T\tilde{x}) - v(x_0), \quad \tilde{x} \in \tilde{S}_1.$$

Then $\tilde{v}$ solves

$$\tilde{U}^{ij} \tilde{v}_{ij} = \tilde{g}(\tilde{x}) := \bar{h} g(T\tilde{x}).$$

Now, we apply Caffarelli-Gutiérrez's interior Hölder estimates in Theorem 2.9 to $\tilde{v}$ to obtain

$$|\tilde{v}(\tilde{z}_1) - \tilde{v}(\tilde{z}_2)| \leq C\, |\tilde{z}_1 - \tilde{z}_2|^{\beta} \{\|\tilde{v}\|_{L^\infty(\tilde{S}_1)} + \|\tilde{g}\|_{L^n(\tilde{S}_1)}\}, \quad \forall \tilde{z}_1, \tilde{z}_2 \in \tilde{S}_{1/2},$$

for some small constant $\beta \in (0, 1)$ depending only on n, λ, Λ.

By (2.40), we can decrease β if necessary and thus we can assume that

$$2\beta \leq \frac{\alpha}{\alpha+2} := 2\gamma.$$

Note that, by (2.39)

$$\|\tilde{g}\|_{L^n(\tilde{S}_1)} = \bar{h}^{1/2} \|g\|_{L^n(S_u(y,\bar{h}))}.$$

We observe that (2.38) and (2.39) give

$$B_{Cr|\log r|}(y) \supset S_u(y,\bar{h}) \supset S_u(y,\bar{h}/2) \supset B_{c\frac{r}{|\log r|}}(y)$$

and

$$\operatorname{diam}(S_u(y,\bar{h})) \leq Cr\,|\log r|\,.$$

By Proposition 1.14, we have

$$\|\tilde{v}\|_{L^\infty(\tilde{S}_1)} \leq C\operatorname{diam}(S_u(y,\bar{h}))^{2\gamma} \leq C(r\,|\log r|)^{2\gamma}.$$

Hence

$$|\tilde{v}(\tilde{z}_1) - \tilde{v}(\tilde{z}_2)| \leq C\,|\tilde{z}_1 - \tilde{z}_2|^\beta \left\{(r\,|\log r|)^{2\gamma} + \bar{h}^{1/2}\,\|g\|_{L^n(S_u(y,\bar{h}))}\right\} \forall \tilde{z}_1, \tilde{z}_2 \in \tilde{S}_{1/2}.$$

Rescaling back and using

$$\tilde{z}_1 - \tilde{z}_2 = \bar{h}^{-1/2} A_{\bar{h}}(z_1 - z_2),$$

and the fact that

$$|\tilde{z}_1 - \tilde{z}_2| \leq \left\|\bar{h}^{-1/2} A_{\bar{h}}\right\| |z_1 - z_2| \leq C\bar{h}^{-1/2} \left|\log \bar{h}\right| |z_1 - z_2| \leq Cr^{-1}\,|\log r|\,|z_1 - z_2|\,,$$

we find

$$|v(z_1) - v(z_2)| \leq |z_1 - z_2|^\beta \quad \forall z_1, z_2 \in S_u(y, \bar{h}/2). \tag{2.41}$$

Notice that this inequality holds also in the Euclidean ball $B_{c\frac{r}{|\log r|}}(y) \subset S_u(y,\bar{h}/2)$. Combining this with Proposition 1.14, we easily obtain that

$$\|v\|_{C^\beta(\bar{\Omega})} \leq C,$$

for some $\beta \in (0,1)$, C universal.

For completeness, we include the details. By rescaling the domain, we can assume that $\Omega \subset B_{1/100}(0)$. We estimate $\frac{|v(x)-v(y)|}{|x-y|^\beta}$ for x and y in Ω. Let $r_x = \operatorname{dist}(x, \partial\Omega)$ and $r_y = \operatorname{dist}(y, \partial\Omega)$. Suppose that $r_y \le r_x$, say. Take $x_0 \in \partial\Omega$ and $y_0 \in \partial\Omega$ such that $r_x = |x - x_0|$ and $r_y = |y - y_0|$. From the interior Hölder estimates of Caffarelli-Gutiérrez, we only need to consider the case $r_y \le r_x \le c$.

Assume first that $|x-y| \le c\frac{r_x}{|\log r_x|}$. Then $y \in B_{c\frac{r_x}{|\log r_x|}}(x) \subset S_u(x, \bar{h}(x)/2)$. By (2.41), we have

$$\frac{|v(x)-v(y)|}{|x-y|^\beta} \le 1.$$

Assume finally that $|x-y| \ge c\frac{r_x}{|\log r_x|}$. We claim that $r_x \le C\,|x-y|\,|\log|x-y||$. Indeed, if

$$1 > r_x \ge |x-y|\,|\log|x-y|| \ge |x-y|$$

then

$$r_x \le \frac{1}{c}\,|x-y|\,|\log r_x| \le \frac{1}{c}\,|x-y|\,|\log|x-y||\,.$$

Now, we have

$$|x_0 - y_0| \le r_x + |x-y| + r_y \le C\,|x-y|\,|\log|x-y||\,.$$

Hence, by Proposition 1.14 and recalling $2\gamma = \frac{\alpha}{\alpha+2}$,

$$\begin{aligned} |v(x)-v(y)| &\le |v(x)-v(x_0)| + |v(x_0)-v(y_0)| + |v(y_0)-v(y)| \\ &\le C\left(r_x^{2\gamma} + |x_0-y_0|^\alpha + r_y^{2\gamma}\right) \\ &\le C\,(|x-y|\,|\log|x-y||)^{2\gamma} \le C\,|x-y|^\beta\,. \end{aligned}$$

□

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