Chapter 2
Hyperhamiltonian Dynamics

In this Chapter we introduce hyperhamiltonian dynamics and study some of the main features of hyperhamiltonian vector fields [60, 62–65, 120]. In the second part of the Chapter we will also discuss the notion of canonical transformations in hyperhamiltonian dynamics (we will see that the natural generalizations of the two equivalent ways of defining these in Hamiltonian dynamics are *not* equivalent in this context) and the relation between hyperhamiltonian vector fields and canonical transformations, generalizing a well known result in standard Hamiltonian dynamics.

2.1 The Hyperhamiltonian Evolution Equations

We start with the definition of the main character in our story, i.e. hyperhamiltonian dynamics.

Definition 2.1 Given a hyperkahler structure $\{\omega_1, \omega_2, \omega_3; g\}$ on M we define the hyperhamiltonian vector field associated to a triple of Hamiltonians $\{\mathcal{H}^1, \mathcal{H}^2, \mathcal{H}^3\}$ on M as the vector field

$$X = X_1 + X_2 + X_3 \tag{2.1}$$

where each of the X_α is the Hamiltonian vector field associated to $\mathcal{H}^\alpha$ via the symplectic form ω_α, see (1.2). That is (with no sum on α)

$$X_\alpha \lrcorner \omega_\alpha = \mathrm{d}\mathcal{H}^\alpha . \tag{2.2}$$

Remark 2.1 Note that as X is the sum of Liouville vector fields, it is also Liouville itself. ⊙

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Let us now consider the case where local coordinates are defined in the open set $U \subseteq M$; then the symplectic forms are represented by matrices K^α,

$$\omega_\alpha \ = \ \frac{1}{2} K^\alpha_{ij} \, \mathrm{d}x^i \wedge \mathrm{d}x^j \ , \tag{2.3}$$

and the components f^i of the vector field

$$X \ = \ f^i(x) \, \partial_i \tag{2.4}$$

are written as

$$f^i \ = \ \sum_{\alpha=1}^3 M^{ij}_\alpha \, (\partial_j \mathcal{H}^\alpha) \ , \tag{2.5}$$

where we have used the notation

$$M_\alpha \ := \ g^{-1} \, K_\alpha \, g^{-1} \ , \ \ \text{i.e.} \ \ M^{ij}_\alpha \ = \ g^{ip} \, K^\alpha_{pq} \, g^{qj} \ . \tag{2.6}$$

2.2 Hamiltonian versus Hyperhamiltonian Dynamics

A natural question immediately arises upon defining hyperhamiltonian dynamics: is this really more general than Hamiltonian one? The answer to this is positive, and can be obtained with little effort, as we show here.

First of all, we note that every Hamiltonian system is trivially hyperhamiltonian: in the hyperhamiltonian framework, it suffices to set two of the three Hamiltonian functions $\mathcal{H}^\alpha$ equal to zero to recover the standard Hamiltonian case.

On the other hand, let us check that there are systems which are hyperhamiltonian but cannot be written in Hamiltonian form with respect to *any* symplectic structure. In order to show this, we recall a result characterizing such vector fields [78].

Lemma 2.1 (Giordano-Marmo-Rubano) *Given a linear vector field*

$$X \ = \ A^i_j \, x^j \, \partial_i \ ,$$

if there is $k \in N$ such that

$$\mathrm{Tr}(A^{2k+1}) \ \neq \ 0 \ ,$$

then X is not Hamiltonian with respect to any symplectic structure.

The vanishing of $\mathrm{Tr}(A)$ *corresponds to the condition of zero divergence, which is also satisfied by hyperhamiltonian flows. Thus in the simplest case we are looking for cases where*

$$\mathcal{H}^\alpha \ = \ \frac{1}{2} \, (D_\alpha)_{ij} \, x^i \, x^j \tag{2.7}$$

(with D_α symmetric matrices; in the following we write all indices as lower ones to avoid confusion with powers) and

$$A \ := \ \sum_\alpha K_\alpha \, D_\alpha$$

satisfies $\mathrm{Tr}(A^3) = 0$.

This is obtained e.g. if [59]

$$\begin{aligned}
\mathcal{H}_1 &= \frac{1}{2} \, [x_1^2 - x_2^2 + x_3^2 - x_4^2 \ + \ 2(x_1 x_4 - x_2 x_3)] \ , \\
\mathcal{H}_2 &= \frac{1}{2} (x_1^2 + x_2^2 + x_3^2 + x_4^2) \ , \\
\mathcal{H}_3 &= 0 \ .
\end{aligned}$$

Thus we have shown (by explicit example) that:

Lemma 2.2 *There are hyperhamiltonian vector fields which are not Hamiltonian with respect to any symplectic structure.*

2.3 Alternative Approach to the Evolution Equations

Our definition of the hyperhamiltonian vector field was as the sum of three Hamiltonian vector fields. This is in a way not satisfactory, as such a sum does not appear to have any intrinsic meaning. It is thus appropriate to devote some page (this section) to an alternative formulation of the hyperhamiltonian evolution equations which is free from this drawback.[1]

To each symplectic form ω we associate a $(4n-2)$-form ζ via

$$\zeta \ := \ \omega \wedge ... \wedge \omega \quad (2n-1 \text{ factors}) \ . \tag{2.8}$$

In particular, to each of the three symplectic forms ω_α is thus associated a form ζ_α.

Moreover, we consider the volume form Ω in M; recall this can also be expressed, for each ω_α, as

$$\Omega \ = \ \frac{s}{(2n)!} \, \omega_\alpha \wedge ... \wedge \omega_\alpha \quad (s = \pm 1) \ . \tag{2.9}$$

[1] Actually, this was the *original* definition of hyperhamiltonian vector fields in [60]; our discussion in this section follows the one provided in that paper.

Then, given a hypersymplectic structure, i.e. an ordered triple of symplectic forms ω_α, to any triple of Hamiltonians $\mathcal{H}^\alpha$ is uniquely associated a vector field X defined by

$$X \lrcorner \Omega = \frac{1}{(2n-1)!} \sum_{\alpha=1}^{3} \mathrm{d}\mathcal{H}^\alpha \wedge \zeta_\alpha . \tag{2.10}$$

Using (2.9), this can also be rewritten as

$$X \lrcorner \sum_{\alpha=1}^{3} \omega_\alpha \wedge \zeta_\alpha = (6n\, s) \sum_{\alpha=1}^{3} \mathrm{d}\mathcal{H}^\alpha \wedge \zeta_\alpha . \tag{2.11}$$

Note that (2.10) defines X uniquely. On the other hand, it is immediately checked that the X defined by (2.1) and (2.2) satisfies (2.10). Thus the two definitions of X are equivalent.

Note also that (2.10) shows at once that X is Liouville, as already remarked right after Definition 2.1.

In the case where the symplectic forms admit a symplectic potential, we can consider still another form associated to any ω; recalling the definition of ζ considered above, and assuming σ is a symplectic potential for ω, we define the $(4n-1)$-form

$$\varphi := \sigma \wedge \zeta . \tag{2.12}$$

In particular, to each of the ω_α is thus associated a $(4n-1)$-form φ_α; we will then define new forms φ and ϑ, the latter also involving the time variable t:

$$\varphi = \sum_{\alpha=1}^{3} \sigma_\alpha \wedge \zeta_\alpha \ ; \quad \vartheta = \varphi + (6ns) \sum_{\alpha=1}^{3} \mathcal{H}^\alpha \, (\zeta_\alpha \wedge \mathrm{d}t) . \tag{2.13}$$

We stress that $\mathrm{d}\varphi$ is proportional to the volume form Ω, and that $\mathrm{d}\vartheta$ is non singular. We also stress that this construction is always possible locally; and globally if $H^2(M) = 0$. Even when this condition is not satisfied, it will be possible if the symplectic forms we are considering are exact.

The forms defined in (2.13) allow to provide yet another definition of the hyperhamiltonian vector field, now seen as (the spatial component of) a vector field Z in $\widetilde{M} = M \times \mathbf{R}$, where the $\mathbf{R}$ factor corresponds to the time variable. In fact, one uniquely identifies Z by

$$Z \lrcorner \mathrm{d}\vartheta = 0 \ , \quad Z \lrcorner \mathrm{d}t = 1 . \tag{2.14}$$

The second condition just means that we can write

$$Z = \partial_t + Y \ ,$$

where Y is a (possibly time-dependent) vector field on M. With this, the first equation in (2.14) decomposes (considering terms which contain or do not contain a $\mathrm{d}t$ factor) into two equations, i.e.

$$\begin{aligned} Y \lrcorner \sum_\alpha \omega_\alpha \wedge \zeta_\alpha \ &= \ (6ns) \ \sum_\alpha \mathrm{d}\mathcal{H}^\alpha \wedge \zeta_\alpha \ ; \\ Y \lrcorner \sum_\alpha \mathrm{d}\mathcal{H}^\alpha \wedge \zeta_\alpha \ &= \ 0 \, . \end{aligned} \tag{2.15}$$

The first of these corresponds to (2.11) and, in view of the uniqueness of X, shows that in fact $Y = X$. The second equation in (2.15) is just a consequence of the first one and thus carries no further information.

Remark 2.2 The hyperhamiltonian dynamics can also be characterized in terms of a variational principle [62–64]; this is briefly discussed in Appendix B. It turns out the hyperhamiltonian dynamics defined here is also natural from the point of view of generalizing (to the quaternionic setting) the description of Hamiltonian dynamics in terms of complex analysis; this is discussed in [120]. $\odot$

2.4 Hyperhamiltonian Flows and Dual Structures

In Sect. 1.5 we have introduced, for a given hyperkahler structure, the notion of dual hyperkahler structures. Here we consider the hyperhamiltonian dynamics associated to such dual structures, and (some of) its relations with the dynamics associated to the given hyperkahler structure.

2.4.1 Dual Hyperkahler Structures and Dual Hyperhamiltonian Dynamics

As discussed in Sect. 1.5, to any hyperkahler structure are associated one or more *dual* ones, with opposite orientation in M or at least in some of the minimal hyperkahler components $M_{(k)}$ of the ambient hyperkahler manifold M.

If the hyperkahler structure is characterized by the metric g and the symplectic forms $\{\omega_1, \omega_2, \omega_3\}$, any dual hyperkahler structure has the same metric and its symplectic forms will be denoted by $\{\widehat{\omega}_1, \widehat{\omega}_2, \widehat{\omega}_3\}$. The corresponding dual hyperkahler structure is then determined by g and the $\widehat{\omega}_\alpha$ through the Kahler relation.

If we have a given triple of symplectic forms $\{\omega_1, \omega_2, \omega_3\}$ and a triple of Hamiltonians $\{\mathcal{H}^1, \mathcal{H}^2, \mathcal{H}^3\}$, thus defining a hyperhamiltonian dynamical vector field

$$X \ = \ \sum X_a \ , \quad X_a \lrcorner \omega_a = \mathrm{d}\mathcal{H}^a , \tag{2.16}$$

we can consider a dual hyperhamiltonian vector field

$$\widehat{X} = \sum \widehat{X}_a , \quad \widehat{X}_a \lrcorner \widehat{\omega}_a = \mathrm{d}\mathcal{H}^a; \tag{2.17}$$

note this is characterized by the *same* Hamiltonians $\mathcal{H}^\alpha$ and by the dual symplectic forms $\widehat{\omega}_\alpha$.

We stress once again that there are many hyperkahler structures dual to any given one, as there are many duality maps ϱ. Thus the dual vector field to a given hyperhamiltonian vector field is surely *not* uniquely defined.

Lemma 2.3 *For X and $\widehat{X}$ as above, their commutator $Z = [X, \widehat{X}] = f^i \partial_i$ satisfies*

$$f = \left(\widehat{M}_\beta H^\beta M_\alpha - M_\beta H^\beta \widehat{M}_\alpha\right) \nabla\mathcal{H}^\alpha := A_\alpha \nabla\mathcal{H}^\alpha . \tag{2.18}$$

Proof This follows by direct computation. Writing $\partial_{ij}\mathcal{H}^\alpha = H^\alpha_{ij}$ for ease of notation, we have

$$\begin{aligned} f^i &= \left(M_\alpha^{jk} (\partial_k \mathcal{H}^\alpha)\right) \partial_j \left(\widehat{M}_\beta^{i\ell} (\partial_\ell \mathcal{H}^b)\right) - \left(\widehat{M}_\alpha^{jk} (\partial_k \mathcal{H}^\alpha)\right) \partial_j \left(M_\beta^{i\ell} (\partial_\ell \mathcal{H}^b)\right) \\ &= \left(\widehat{M}_\beta^{i\ell} H^\beta_{\ell j} M_\alpha^{jk} - M_\beta^{i\ell} H^\beta_{\ell j} \widehat{M}_\alpha^{jk}\right) \partial_k \mathcal{H}^\alpha ; \end{aligned}$$

this is just our statement. △

Remark 2.3 Note that A_α is in general not antisymmetric: in fact (using $M^T = -M$, $\widehat{M}^T = -\widehat{M}$ and $H^T = H$)

$$A_\alpha^T = -(\widehat{M}_\alpha H^\beta M_\beta - M_\alpha H^\beta \widehat{M}_\beta) ,$$

and in general $A^T \neq -A$. This implies, in particular, that in general A_α is *not* a combination of the M_α and/or the $\widehat{M}_\alpha$.

On the other hand, if H^α_{ij} is a multiple of the identity—as in particular in the case of *quaternionic oscillators*, to be considered in Chap. 5—then by $[M_\alpha, \widehat{M}_\beta] = 0$ we get $[X, \widehat{X}] = 0$. ⊙

2.4.2 Dirac Vector Fields

We have introduced the notion of dual hyperkahler structures, and correspondingly we have a notion of vector fields which are hyperhamiltonian with respect to dual hyperkahler structures.

As stressed above, dual structures are originated from maps preserving the volume form up to a sign, and reversing the orientation in any number of the minimal hyperkahler submanifolds $M_{(k)}$ in which the the hyperkahler manifold M under study can be decomposed.

Definition 2.2 Given a hyperkahler structure $\mathbf{J}$, we say that any vector field which is hyperhamiltonian w.r.t. either $\mathbf{J}$ or any of the dual hyperkahler structures is a *Dirac vector field* for $\mathbf{J}$.

In the same way as we usually say "hyperhamiltonian vector field" without specifying "w.r.t. the hyperkahler structure $\mathbf{J}$", we will just speak of "Dirac vector fields".

This notion will play a substantial role in our forthcoming discussion. In particular, while in the standard Hamiltonian case the vector fields which preserve the symplectic structure are all (and only) those which are Hamiltonian—with any Hamiltonian function—under the given symplectic structure, we anticipate (see Sect. 3.2.4 for details) that in the hyperhamiltonian case the vector fields which preserve the hyperkahler form are not only those which are hyperhamiltonian w.r.t. the given hyperkahler structure, but also those which are hyperhamiltonian w.r.t. dual hyperkahler structures; that is, Dirac vector fields.

Definition 2.3 Given a hyperkahler structure $\mathbf{J}$ in (M, g), let $\widehat{\mathbf{J}}$ be the dual hyperkahler structure corresponding to a duality map which reverses orientation in *each* of the irreducible hyperkahler components of $(M, g; \mathbf{J})$. We say that a vector field which is the sum of vector fields which are hyperhamiltonian w.r.t. the $\mathbf{J}$ and the $\widehat{\mathbf{J}}$ structures is a *strictly Dirac vector field*.

2.5 First Integrals, Conservation Laws, and Poisson-like Brackets

As always when dealing with a dynamics, we are specially interested in conservation laws and first integrals for the hyperhamiltonian vector field X.

2.5.1 First Integrals

Definition 2.4 A *first integral* for the vector field X is a smooth function $F : M \to \mathbf{R}$, such that

$$\mathcal{L}_X(F) \;=\; 0\,. \tag{2.19}$$

Lemma 2.4 *The smooth function $F : M \to \mathbf{R}$ is a first integral for the hyperhamiltonian vector field X if and only if the sum of its Poisson brackets with the three Hamiltonians (each w.r.t. the corresponding symplectic structure) vanishes.*

Proof The time evolution of a scalar smooth function $F : M \to \mathbf{R}$ under the hyperhamiltonian vector field X is, in view of (2.1), given by

$$\mathcal{L}_X(F) \;=\; X(F) \;=\; \sum_\alpha X_\alpha(F) \,; \tag{2.20}$$

recalling also (2.2) and working in local coordinates, we get

$$\begin{aligned} \mathcal{L}_X(F) &= \sum_\alpha M^{ij}_\alpha \, (\partial_j \mathcal{H}^\alpha) \, (\partial_i F) \\ &= \sum_\alpha [(\partial_i F) \, M^{ij}_\alpha \, (\partial_j \mathcal{H}^\alpha)] \;=\; \sum_\alpha \{F, \mathcal{H}^\alpha\}_\alpha \,, \end{aligned}$$

where we have denoted by $\{.,.\}_\alpha$ the Poisson bracket defined by the symplectic form ω_α. △

Remark 2.4 A special case is of course the one in which these three Poisson brackets are separately zero, but in general this is not required. ⊙

In Hamiltonian dynamics, given two first integrals F_1 and F_2, we obtain a new first integral (which might happen to be trivial or dependent on the first two) by applying the Poisson bracket, i.e. as

$$F_3 \;=\; \{F_1, F_2\} \,.$$

It is natural to wonder if there is a result of this kind also in the framework of hyperhamiltonian dynamics; it appears this is not the case, at least if we require to have the new first integral as a homogeneous bilinear function, independent of the chosen Hamiltonians, of the derivatives of the known ones.

In fact, any such function can be written as

$$F_3 \;=\; (\partial_i F_1) \; P^{ij} \; (\partial_j F_2) \,, \tag{2.21}$$

with P a matrix.

With integration by parts, writing $X = f^i \partial_i$, and using the assumption $X(F_1) = X(F_2) = 0$, we have

$$\begin{aligned} X(F_3) &= [X(\partial_i F_1)] \; P^{ij} \; (\partial_j F_2) \;+\; (\partial_i F_1) \; P^{ij} \; [X(\partial_j F_2)] \\ &= [\partial_i X(F_1)] \; P^{ij} \; (\partial_j F_2) \;+\; (\partial_i F_1) \; P^{ij} \; [\partial_j X(F_2)] \\ &\quad - \big[(\partial_i f^k)(\partial_k F_1) \; P^{ij} \; (\partial_j F_2) \;+\; (\partial_i F_1) \; P^{ij} \; (\partial_j f^k)(\partial_k F_2) \big] \\ &= - \big[(\partial_i f^k)(\partial_k F_1) \; P^{ij} \; (\partial_j F_2) \;+\; (\partial_i F_1) \; P^{ij} \; (\partial_j f^k)(\partial_k F_2) \big] \;; \end{aligned}$$

note we have already used the condition that F_1, F_2 are first integrals for X.

If now we write

$$\mathcal{F}^i_{\ k} \;=\; \partial_k f^i \,, \tag{2.22}$$

the above reads

$$\begin{aligned} X(F_3) &= -\left[(\partial_k F_1)\, \mathcal{F}^k_i\, P^{ij}\, (\partial_j F_2) \ + \ (\partial_i F_1)\, P^{ij}\, (\mathcal{F}^T)^k_j\, (\partial_k F_2) \right] \\ &= -\left[(\nabla F_1 \ \cdot\ \mathcal{F} P \nabla F_2) \ + \ (\nabla F_1 \ \cdot\ P \mathcal{F}^T \nabla F_2) \right] \\ &= - \ \left(\nabla F_1 \ \cdot\ (\mathcal{F} P \ + \ P \mathcal{F}^T)\, \nabla F_2 \right) \ . \end{aligned} \tag{2.23}$$

Remark 2.5 One could be tempted to require

$$\mathcal{F}\, P \ + \ P\, \mathcal{F}^T \ = \ 0\,, \tag{2.24}$$

but this would actually be too much, as discussed in a moment. Moreover, an explicit computation made with the standard hyperkahler structure in $\mathbf{R}^4$ shows that this condition is satisfied only for $P = 0$.

In fact, with the standard hyperkahler structure in $\mathbf{R}^4$ and P a skew-symmetric matrix, (2.24) is a homogeneous linear system with six unknowns (the independent components of P) and six equations (and $\mathcal{F}P + P\mathcal{F}^T$ will in general be nonzero); then the only solution is $P = 0$.

Actually, the coefficient matrix (which is a 6×6 matrix) has a peculiar structure, being the sum of a symmetric plus a skew-symmetric matrix, the two having non-null elements in different places. Due to this structure, its determinant is a perfect square, in fact the square of the sum of products of second derivatives of $\mathcal{H}_i$, each product having three factors; thus it is always non-zero. ⊙

When requiring

$$\left(\nabla F_1 \ \cdot\ (\mathcal{F} P \ + \ P \mathcal{F}^T)\, \nabla F_2 \right) \ = \ 0\,,$$

see (2.23) above, we should recall that F_1, F_2 are first integrals for X, i.e. their gradients are necessarily orthogonal to the stream lines of X; on the other hand, these are determined by the f^i and are hence embodied in the matrix $\mathcal{F}$.

Thus we should actually require that

$$\left(\xi \ \cdot\ (\mathcal{F} P \ + \ P \mathcal{F}^T)\, \eta \right) \ = \ 0 \tag{2.25}$$

for all vectors ξ, η orthogonal to the kernel of X. This requirement is weaker than (2.24).

2.5.2 Conservation Laws

By a *conservation law* we mean a "conserved form of submaximal degree" (this corresponds via Hodge duality to a conserved vector, similar to the Runge-Lenz vector for the Kepler problem), i.e. an object with $m = 4n$ components which is preserved under X. Rather than seeing this as a vector on M, it is more convenient to adopt the dual point of view and see it as a differential form. This in turn can be a one-form $\theta \in \Lambda^1(M)$ or, using Hodge duality [123, 124], a form of submaximal degree, $\Theta \in \Lambda^{(4n-1)}(M)$. It turns out that the latter point of view is often somehow more convenient.

Definition 2.5 A *conservation law* for the vector field X in the $4n$-dimensional manifold M is a form $\Theta \in \Lambda^{(4n-1)}(M)$ such that $\mathcal{L}_X(\Theta) = 0$.

In particular, given a triple of symplectic structures ω_α, to any triple $\mathcal{H}^\alpha$ of Hamiltonians (which uniquely define a hyperhamiltonian vector field X) is canonically associated a conserved form $\Theta \in \Lambda^{(4n-1)}(M)$; this is just

$$\Theta \ := \ \sum_\alpha \mathrm{d}\mathcal{H}^\alpha \wedge \zeta_\alpha \ . \tag{2.26}$$

Lemma 2.5 *The form Θ is preserved by the vector field X defined by the Hamiltonians $\mathcal{H}^\alpha$,*

$$\mathcal{L}_X(\Theta) \ = \ 0 \ . \tag{2.27}$$

Proof In fact, Θ is obviously closed; thus

$$\mathcal{L}_X(\Theta) \ = \ \mathrm{d}(X \lrcorner \Theta).$$

The form of Θ and the equations (2.1), (2.2) yield

$$(X \lrcorner \Theta) \ = \ (2n-1)! \, [X \lrcorner (X \lrcorner \Omega)] \ = \ 0 \, ;$$

thus *a fortiori* $\mathrm{d}(X \lrcorner \Theta) = 0$ and hence (2.27). △

Remark 2.6 Actually, the construction via Hodge duality canonically associates a $(4n-1)$-form χ to any vector field Y on M; this is written as

$$Y \lrcorner \Omega \ = \ \chi. \tag{2.28}$$

Conversely, this relation associates a vector field Y to any form $\chi \in \Lambda^{(4n-1)}(M)$; that is, there is a map $F : \Lambda^{(4n-1)}(M) \to \mathcal{X}(M)$.

Given two forms $\alpha, \beta \in \Lambda^{(4n-1)}(M)$, we have vector fields $Y_\alpha = F(\alpha)$ and $Y_\beta = F(\beta)$ on M. We can then take the commutator of these vector fields, and consider the form in $\Lambda^{(4n-1)}(M)$ associated to it, i.e. the form

$$\gamma \ = \ F^{-1} \left([F(\alpha), F(\beta)] \right) \ . \tag{2.29}$$

This construction defines a natural (antisymmetric) binary operation $\{.,.\}$ on $\Lambda^{(4n-1)}(M)$, so that the above can be rewritten as

$$\gamma \ = \ \{\alpha, \beta\} \, . \tag{2.30}$$

This is just the corresponding of the commutator when considering (both) the duality between vector fields and one forms, and the Hodge duality between $\Lambda^1(M)$ and $\Lambda^{(4n-1)}(M)$. ⊙

Remark 2.7 The construction of the above remark can be used to generate new conservation laws from known ones. That is, if α and β are conserved $(4n-1)$-forms, then $\gamma = \{\alpha, \beta\}$ is also conserved. In this respect, as mentioned in [60] (see Sect. 3 in there), $\{.,.\}$ is reminiscent of the Poisson bracket.

It should be stressed, however, that $\{.,.\}$ is just based on standard and Hodge duality; hence it only uses the metric structure in M and its volume form, and *not* the symplectic or hyperkahler structures.

In fact, a vector field Y_i is preserved under X if $\mathcal{L}_X(Y) = 0$; but

$$\mathcal{L}_X(Y) \ = \ [X, Y] \, ;$$

thus the conservation of γ just amounts to the fact that $[X, Y_\alpha] = 0$ and $[X, Y_\beta] = 0$ entail $[X, [Y_\alpha, Y_\beta]] = 0$. In other words the fact that, for γ as in (2.30), $\mathcal{L}_X(\alpha) = 0 = \mathcal{L}_X(\beta)$ entails $\mathcal{L}_X(\gamma) = 0$ is just a consequence of the Jacobi identity. ⊙

2.5.3 Combining First Integrals and Conservation Laws

If we have a first integral F and a conservation law $\Xi \in \Lambda^{(4n-1)}(M)$, we can readily produce a new scalar conserved quantity, i.e. a new first integral (as in the Hamiltonian case, this might be dependent on the quantities mentioned above, or even turn out to be trivial).

Lemma 2.6 *Let X be a hyperhamiltonian vector field in M, Ω the volume form in M, $\Xi \in \Lambda^{(4n-1)}(M)$ a conservation law for X, F a conserved quantity for X, and $\phi = \mathrm{d}F$. Then the scalar quantity σ defined by*

$$\Xi \wedge \phi \ = \ \sigma \, \Omega \tag{2.31}$$

is a first integral for X.

Proof In fact, $X(F) = 0$ can also be written as $X \lrcorner \mathrm{d}F = 0$; thus $\phi := \mathrm{d}F \in \Lambda^1(M)$ is a conserved one-form. As the form $\Xi \wedge \phi \in \Lambda^{(4n)}(M)$ is of maximal degree, this defines indeed a scalar function $\sigma : M \to \mathbf{R}$ through (2.31).

Taking into account that X is Liouville, we readily have

$$\mathcal{L}_X(\Xi \wedge \phi) \ = \ X(\sigma) \, \Omega \ + \ \sigma \, \mathcal{L}_X(\Omega) \ = \ X(\sigma) \, \Omega \ ; \tag{2.32}$$

thus the form $\Xi \wedge \phi$ is conserved if and only if $X(\sigma) = 0$, i.e. if σ is a first integral for X. On the other hand, by assumption Ξ and ϕ are both conserved forms, hence

$$\mathcal{L}_X(\Xi \wedge \phi) \ = \ [\mathcal{L}_X(\Xi)] \wedge \phi \ - \ \Xi \wedge [\mathcal{L}_X(\phi)] \ = \ 0 \, . \tag{2.33}$$

That is, σ defined by (2.31) is indeed a first integral for X. △

2.6 The Hyperkahler Form

2.6.1 General Setting

We have seen above that any hyperhamiltonian vector field is Liouville, i.e. it preserves the volume form Ω on M, see Remark 2.1.

On the other hand, each of the X_α, see (2.1) and (2.2), preserves the corresponding symplectic form[2] ω_α,

$$\mathcal{L}_{X_\alpha}(\omega_\alpha) \ = \ 0 \quad (\alpha = 1, 2, 3) \ , \tag{2.34}$$

but it will in general *not* preserve the other two symplectic forms, $\mathcal{L}_{X_\alpha}(\omega_\beta) \neq 0$ for $\alpha \neq \beta$. This also means that—except in very special cases, i.e. for very special choices of the Hamiltonians $\mathcal{H}^\alpha$—the hyperhamiltonian vector field X will not preserve the three symplectic forms: the hyperhamiltonian dynamics is in general *not* three-holomorphic.

One can and should also consider the *hyperkahler four-form*

$$\Psi \ = \ \frac{1}{2} \sum_{\alpha=1}^{3} \omega_\alpha \wedge \omega_\alpha \ . \tag{2.35}$$

Lemma 2.7 *Equivalent hyperkahler structures have the same hyperkahler four-form.*

Proof Let us consider a set of different forms $\widetilde{\omega}_\alpha$ obtained from the set ω_α by an SO(3) rotation:

$$\widetilde{\omega}_\alpha \ = \ \sum_{\beta=1}^{3} R_{\alpha\beta} \, \omega_\beta \ . \tag{2.36}$$

We can compute the hyperkahler four-form, which we will denote by $\widetilde{\Psi}$, based on these symplectic forms, and show it is just the same as the one based on the ω_α forms. In fact,

[2] And hence is canonical (in the sense of standard Hamiltonian dynamics) for the corresponding symplectic form.

$$
\begin{aligned}
\widetilde{\Psi} &= \frac{1}{2}\sum_{\alpha=1}^{3} \widetilde{\omega}_\alpha \wedge \widetilde{\omega}_\alpha \ = \ \frac{1}{2}\sum_{\alpha=1}^{3} \left(\sum_{\beta=1}^{3} R_{\alpha\beta}\omega_\beta \right) \wedge \left(\sum_{\gamma=1}^{3} R_{\alpha\gamma}\omega_\gamma \right) \\
&= \frac{1}{2} \sum_{\alpha,\beta,\gamma=1}^{3} R_{\alpha\beta}R_{\alpha\gamma}\omega_\beta \wedge \omega_\gamma \ = \ \frac{1}{2} \sum_{\beta,\gamma=1}^{3} \delta_{\beta\gamma}\omega_\beta \wedge \omega_\gamma \\
&= \frac{1}{2}\sum_{\beta=1}^{3} \omega_\beta \wedge \omega_\beta \ = \ \Psi
\end{aligned}
$$

This concludes the proof. △

Remark 2.8 We have thus shown that Ψ is actually not associated to the given triple of symplectic form but to the **Q**-structure these identify. ⊙

In the case of fully decomposable hyperkahler manifolds one can show that if two hyperkahler structures (induce the same decomposition of the manifold and) have the same hyperkahler form, then are equivalent. This will be done in the following subsection.

2.6.2 Hyperkahler Form for Low-Dimensional Standard Structures

It may be worth giving explicit expression for the standard hyperkahler structures in $\mathbf{R}^4$ and in $\mathbf{R}^8$, see Sect. 1.4.3 for the explicit expression of the ω_α and $\widehat{\omega}_\alpha$ in this case.

By a trivial computation, we obtain that the hyperkahler forms $\Psi^{(\pm)}$ for the positively and negatively oriented standard hyperkahler structure in $\mathbf{R}^4$ are

$$
\Psi^{(\pm)} \ = \ \pm 3 \ \left(\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 \right) \ . \tag{2.37}
$$

The situation in $\mathbf{R}^8$ is slightly more complex, but also more representative of the general situation in $\mathbf{R}^{4n}$ (this is why we report the explicit formulas below). We will denote the basis symplectic structures as $\omega_\alpha^{(\pm\pm)}$, where the upper indices refer to the orientation in the two basic four-dimensional blocks. Thus we will have in explicit terms

$$
\begin{aligned}
\omega_1^{(++)} &= \mathrm{d}x^1 \wedge \mathrm{d}x^2 \ + \ \mathrm{d}x^3 \wedge \mathrm{d}x^4 \ + \ \mathrm{d}x^5 \wedge \mathrm{d}x^6 \ + \ \mathrm{d}x^7 \wedge \mathrm{d}x^8 \ , \\
\omega_2^{(++)} &= \mathrm{d}x^1 \wedge \mathrm{d}x^4 \ + \ \mathrm{d}x^2 \wedge \mathrm{d}x^3 \ + \ \mathrm{d}x^5 \wedge \mathrm{d}x^8 \ + \ \mathrm{d}x^6 \wedge \mathrm{d}x^7 \ , \\
\omega_3^{(++)} &= \mathrm{d}x^1 \wedge \mathrm{d}x^3 \ + \ \mathrm{d}x^2 \wedge \mathrm{d}x^4 \ + \ \mathrm{d}x^5 \wedge \mathrm{d}x^7 \ - \ \mathrm{d}x^6 \wedge \mathrm{d}x^8 \ ; \\
\omega_1^{(+-)} &= \mathrm{d}x^1 \wedge \mathrm{d}x^2 \ + \ \mathrm{d}x^3 \wedge \mathrm{d}x^4 \ + \ \mathrm{d}x^5 \wedge \mathrm{d}x^6 \ - \ \mathrm{d}x^7 \wedge \mathrm{d}x^8 \ , \\
\omega_2^{(+-)} &= \mathrm{d}x^1 \wedge \mathrm{d}x^4 \ + \ \mathrm{d}x^2 \wedge \mathrm{d}x^3 \ + \ \mathrm{d}x^5 \wedge \mathrm{d}x^8 \ - \ \mathrm{d}x^6 \wedge \mathrm{d}x^7 \ , \\
\omega_3^{(+-)} &= \mathrm{d}x^1 \wedge \mathrm{d}x^3 \ + \ \mathrm{d}x^2 \wedge \mathrm{d}x^4 \ + \ \mathrm{d}x^5 \wedge \mathrm{d}x^7 \ + \ \mathrm{d}x^6 \wedge \mathrm{d}x^8 \ ;
\end{aligned}
$$

$$\begin{aligned}
\omega_1^{(-+)} &= dx^1 \wedge dx^2 \,-\, dx^3 \wedge dx^4 \,+\, dx^5 \wedge dx^6 \,+\, dx^7 \wedge dx^8 \,, \\
\omega_2^{(-+)} &= dx^1 \wedge dx^4 \,-\, dx^2 \wedge dx^3 \,+\, dx^5 \wedge dx^8 \,+\, dx^6 \wedge dx^7 \,, \\
\omega_3^{(-+)} &= dx^1 \wedge dx^3 \,-\, dx^2 \wedge dx^4 \,+\, dx^5 \wedge dx^7 \,-\, dx^6 \wedge dx^8 \,; \\
\omega_1^{(--)} &= dx^1 \wedge dx^2 \,-\, dx^3 \wedge dx^4 \,+\, dx^5 \wedge dx^6 \,-\, dx^7 \wedge dx^8 \,, \\
\omega_2^{(--)} &= dx^1 \wedge dx^4 \,-\, dx^2 \wedge dx^3 \,+\, dx^5 \wedge dx^8 \,-\, dx^6 \wedge dx^7 \,, \\
\omega_3^{(--)} &= dx^1 \wedge dx^3 \,-\, dx^2 \wedge dx^4 \,+\, dx^5 \wedge dx^7 \,+\, dx^6 \wedge dx^8 \,.
\end{aligned}$$

We will correspondingly write

$$\Psi^{(s_1 s_2)} \;=\; \frac{1}{2} \sum_{\alpha=1}^{3} \omega_\alpha^{(s_1 s_2)} \wedge \omega_\alpha^{(s_1 s_2)} \;.$$

By simple (and rather boring) computations we get the explicit expressions for the $\Psi^{(s_1 s_2)}$:

$$\begin{aligned}
\Psi^{(++)} = 3 \, \big(& dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \,+\, dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8\big) \\
& + dx^1 \wedge dx^2 \wedge dx^5 \wedge dx^6 \,+\, dx^1 \wedge dx^2 \wedge dx^7 \wedge dx^8 \\
& + dx^1 \wedge dx^3 \wedge dx^5 \wedge dx^7 \,-\, dx^1 \wedge dx^3 \wedge dx^6 \wedge dx^8 \\
& + dx^1 \wedge dx^4 \wedge dx^5 \wedge dx^8 \,+\, dx^1 \wedge dx^4 \wedge dx^6 \wedge dx^7 \\
& + dx^2 \wedge dx^3 \wedge dx^5 \wedge dx^8 \,+\, dx^2 \wedge dx^3 \wedge dx^6 \wedge dx^7 \\
& - dx^2 \wedge dx^4 \wedge dx^5 \wedge dx^7 \,+\, dx^2 \wedge dx^4 \wedge dx^6 \wedge dx^8 \\
& + dx^3 \wedge dx^4 \wedge dx^5 \wedge dx^6 \,+\, dx^3 \wedge dx^4 \wedge dx^7 \wedge dx^8 \,; \\
\Psi^{(+-)} = 3 \, \big(& dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \,-\, dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8\big) \\
& + dx^1 \wedge dx^2 \wedge dx^5 \wedge dx^6 \,-\, dx^1 \wedge dx^2 \wedge dx^7 \wedge dx^8 \\
& + dx^1 \wedge dx^3 \wedge dx^5 \wedge dx^7 \,+\, dx^1 \wedge dx^3 \wedge dx^6 \wedge dx^8 \\
& + dx^1 \wedge dx^4 \wedge dx^5 \wedge dx^8 \,-\, dx^1 \wedge dx^4 \wedge dx^6 \wedge dx^7 \\
& + dx^2 \wedge dx^3 \wedge dx^5 \wedge dx^8 \,-\, dx^2 \wedge dx^3 \wedge dx^6 \wedge dx^7 \\
& - dx^2 \wedge dx^4 \wedge dx^5 \wedge dx^7 \,-\, dx^2 \wedge dx^4 \wedge dx^6 \wedge dx^8 \\
& + dx^3 \wedge dx^4 \wedge dx^5 \wedge dx^6 \,-\, dx^3 \wedge dx^4 \wedge dx^7 \wedge dx^8 \,; \\
\Psi^{(-+)} = -3 \, \big(& dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \,-\, dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8\big) \\
& + dx^1 \wedge dx^2 \wedge dx^5 \wedge dx^6 \,+\, dx^1 \wedge dx^2 \wedge dx^7 \wedge dx^8 \\
& + dx^1 \wedge dx^3 \wedge dx^5 \wedge dx^7 \,-\, dx^1 \wedge dx^3 \wedge dx^6 \wedge dx^8 \\
& + dx^1 \wedge dx^4 \wedge dx^5 \wedge dx^8 \,+\, dx^1 \wedge dx^4 \wedge dx^6 \wedge dx^7 \\
& - dx^2 \wedge dx^3 \wedge dx^5 \wedge dx^8 \,-\, dx^2 \wedge dx^3 \wedge dx^6 \wedge dx^7 \\
& + dx^2 \wedge dx^4 \wedge dx^5 \wedge dx^7 \,-\, dx^2 \wedge dx^4 \wedge dx^6 \wedge dx^8 \\
& - dx^3 \wedge dx^4 \wedge dx^5 \wedge dx^6 \,-\, dx^3 \wedge dx^4 \wedge dx^7 \wedge dx^8 \,;
\end{aligned}$$

$$\begin{aligned}\Psi^{(--)} = & -3\left(dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 + dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8\right) \\ & + dx^1 \wedge dx^2 \wedge dx^5 \wedge dx^6 - dx^1 \wedge dx^2 \wedge dx^7 \wedge dx^8 \\ & + dx^1 \wedge dx^3 \wedge dx^5 \wedge dx^7 + dx^1 \wedge dx^3 \wedge dx^6 \wedge dx^8 \\ & + dx^1 \wedge dx^4 \wedge dx^5 \wedge dx^8 - dx^1 \wedge dx^4 \wedge dx^6 \wedge dx^7 \\ & - dx^2 \wedge dx^3 \wedge dx^5 \wedge dx^8 + dx^2 \wedge dx^3 \wedge dx^6 \wedge dx^7 \\ & + dx^2 \wedge dx^4 \wedge dx^5 \wedge dx^7 + dx^2 \wedge dx^4 \wedge dx^6 \wedge dx^8 \\ & - dx^3 \wedge dx^4 \wedge dx^5 \wedge dx^6 + dx^3 \wedge dx^4 \wedge dx^7 \wedge dx^8 .\end{aligned}$$

The net message to be extracted from these fully explicit formulas is that (at difference with possible, but over-optimistic, expectations) already for $n = 2$, and hence *a fortiori* for general $n > 1$, there is no simple relation—that is, no relation amounting just to an overall sign switch—between the hyperkahler forms of mutually dual hyperkahler structures.

This also provide evidence (albeit not a formal proof, see below for that) for the following statement:

Lemma 2.8 *Two hyperkahler structures in Euclidean* $\mathbf{R}^{4n}$*, generating the same splitting of* $\mathbf{R}^{4n} = \mathbf{R}^4 \oplus \ldots \oplus \mathbf{R}^4$ *into* $\mathbf{R}^4$ *invariant subspaces, have the same hyperkahler form if and only if they are equivalent.*

Proof As we have seen above, any hyperkahler structure in $(\mathbf{R}^{4n}, \delta)$ can be reduced to a standard one, i.e. to one which is the direct sum of positively or negatively oriented one in each of the $\mathbf{R}^4$ components.

The statement can be (and was, see above) explicitly checked for $n = 1$. In the case $n > 1$ the only possibility to have non-equivalent structures sharing the same hyperkahler form is through sign compensations, which was ruled out by our explicit computation for $n = 2$. In more formal terms if two hyperkahler structures $\mathbf{J}_1$ and $\mathbf{J}_2$ in $\mathbf{R}^{4n}$ (with $n \geq 2$) share the same hyperkahler form, i.e. (with obvious notation) $\Psi_1 = \Psi_2 = \Psi$, we can just consider the restriction of $\mathbf{J}_k$ and Ψ_k to a $\mathbf{R}^4$ subspace, invariant under $\mathbf{J}_k$. We have seen above that the restrictions of Ψ_k will coincide if and only if the restrictions of the $\mathbf{J}_k$ are equivalent. Thus if $\Psi_1 = \Psi_2$, and hence their restrictions also coincide for any choice of the $\mathbf{R}^4$ subspace, the $\mathbf{J}_1$ and $\mathbf{J}_2$ are equivalent in any $\mathbf{R}^4$ subspace, and hence for the full $\mathbf{R}^{4n}$ space. △

2.7 Hyperkahler and Canonical Maps

It is well known that in the standard Hamiltonian case, Hamiltonian vector fields generate a one-parameter (local) group of symplectic, i.e. canonical, transformations [10, 12, 109]; if the vector field is complete, we have a global group.

We will generalize this result (in this case, with substantial differences) to the hyperhamiltonian case.[3] It will actually turn out that there are *two* generalization of

[3]The discussion of this Section will follow our paper [68].

the notion of canonical transformations to the hyperhamiltonian framework; these will be called *canonical* and *hyperkahler* maps. In the next Chap. 3 we will discuss in detail canonical maps, while the study of hyperkahler ones is postponed to the subsequent Chap. 4.

We start by recalling the relevant notions and results in the standard Hamiltonian case.

2.7.1 Symplectic and Canonical Maps in Standard Hamiltonian Dynamics

Let (M, ω) be a symplectic manifold (of dimension $2n$); we say that a map $\phi : M \to M$ is *symplectic* if it preserves the symplectic form ω, i.e. if

$$\phi^*(\omega) \ = \ \omega \ . \tag{2.38}$$

An equivalent characterization is also quite common (we refer e.g. to Sect. 44 of [10] for detail). As well known, by Darboux theorem [10] one can introduce local coordinates (p_a, q^a) (for $a = 1, ..., n$) in a neighborhood $U \subset M$, such that $\omega = \mathrm{d}p_a \wedge \mathrm{d}q^a$. Then, one considers local manifolds of minimal dimension on which ω is non-degenerate; these are two-dimensional and are spanned by q^a and p_a (with same a). They are known as *Darboux submanifolds* and denoted as U_a; these also correspond to leaves of the Abelian distribution generated by the Hamiltonian vector fields associated with canonical coordinates.

Let us consider a given point in U and the manifolds U_a through this. Denote by ι_a the embedding $\iota_a : U_a \hookrightarrow U \subseteq M$; then the restriction $\iota_a^* \omega$ of the symplectic form to U_a provides a volume form $\Omega_a = \mathrm{d}p_a \wedge \mathrm{d}q^a$ (no sum on a) on U_a. Then, for any two-chain A in U and with $\pi_a A$ the projection of A to U_a,

$$\int_A \omega \ = \ \int_A \sum_{a=1}^n \mathrm{d}p_a \wedge \mathrm{d}q^a \ = \ \sum_{a=1}^n \int_A \Omega_a \ = \ \sum_{a=1}^n \mathrm{area}[\pi_a A] \ ;$$

thus preservation of ω is *equivalent* to preservation of the sum of oriented areas of projection of any A to Darboux submanifolds. That is, the map ϕ is canonical if and only if

$$\sum_{a=1}^n \mathrm{area}[\pi_a A] \ = \ \sum_{a=1}^n \mathrm{area}[\pi_a(\phi A)] \ .$$

It should be noted that if we start from a manifold equipped with a Riemannian metric, passing to Darboux coordinates will in general not preserve the representation of the metric tensor in coordinates, i.e. not preserve the (matrix g_{ij} representing the) metric. Thus this construction is general not viable if one requires preservation of the metric, as in the Kahler case.

In the case of a Kahler manifold, the symplectic form ω corresponds to a complex structure J through the Kahler relation (1.14). This satisfies $J^2 = -I$, and provides a splitting of $\mathrm{T}_0 M$ (at any point $m_0 \in M$) into two-dimensional invariant subspaces; the volume form Ω defined in M induces volume forms Ω_a in each of these, and $\omega = \sum \Omega_a$. Thus again canonical transformations can be characterized as those satisfying[4]

$$\sum_{a=1}^{n} \Omega_a = \sum_{a=1}^{n} \phi^*(\Omega_a) . \tag{2.39}$$

Remark 2.9 Note this construction does *not* make use of Darboux coordinates or submanifolds, but only of the splitting of $\mathrm{T}M$ induced by the action of the complex structure; moreover, we only consider volume forms. $\odot$

2.7.2 Hyperkahler Maps

Let us now pass to consider hyperhamiltonian dynamics. We will first focus our attention on maps which preserve the **Q**-structure; as we have seen just above this amounts to mapping a hyperkahler structure into an equivalent one. We will thus use the name *hyperkahler* for such maps (we will also use the name **Q**-*map*).

Definition 2.6 Let $(M, g; \mathbf{J})$ be a hyperkahler manifold. We say that the orthogonal map $\phi : M \to M$ is *hyperkahler*—or a **Q**-*map*—if it maps the hyperkahler structure into an equivalent one, i.e. if $\phi^* : \mathbf{S} \to \mathbf{S}$. It is *strongly hyperkahler* if it leaves the three complex structures J_α invariant, $\phi^*(J_\alpha) = J_\alpha$ for $\alpha = 1, 2, 3$.

These also have, of course, a symplectic counterpart:

Definition 2.6' Let $(M, g; \omega_1, \omega_2, \omega_3)$ be a hypersymplectic manifold. We say that the orthogonal map $\phi : M \to M$ is *hypersymplectic* if it maps the hypersymplectic structure into an equivalent one, i.e. if $\phi^* : \mathcal{S} \to \mathcal{S}$. It is *strongly hypersymplectic* if it leaves the hypersymplectic structures invariant, i.e. if $\phi^*(\omega_\alpha) = \omega_\alpha$ for $\alpha = 1, 2, 3$.

It is easily seen that a map is hyperkahler if and only if it is hypersymplectic (recall that such maps are required to be orthogonal, i.e. to preserve g).

The two definitions above should be seen as the generalization (from the point of view of complex and symplectic structures respectively) to the hyperkahler framework of the familiar condition of preservation of the symplectic form, relevant in Symplectic Geometry and Hamiltonian Mechanics [10, 12, 30, 79, 82, 109].

Remark 2.10 In the standard Hamiltonian dynamics framework, one also considers the condition to preserve the sum of the oriented areas of projections to the Darboux submanifolds. In the hyperkahler context, the generalization of this condition presents

---

[4]The reader is referred to Arnold [10] (see in particular Sect. 41.E there) for a discussion of the interrelations between orthogonal, symplectic and unitary transformations.

an obvious problem: that is, now there is no analogue of Darboux theorem, and hence no natural notion of Darboux manifolds. ⊙

Remark 2.11 Note that a Hamiltonian flow (i.e. a hyperhamiltonian flow in which only one of the three Hamiltonians is non-zero) will generate a one-parameter group of hyperkahler maps, but these are *not* strongly hyperkahler. To see this, it suffices to consider the case with $(M, g) = (\mathbf{R}^4, \delta)$ with standard hyperkahler structure and $\mathcal{H}_1 = |\beta|^2/2$, $\mathcal{H}_2 = \mathcal{H}_3 = 0$. Then ω_1 is preserved, while ω_2 and ω_3 change according to

$$\widetilde{\omega}_2 = \cos(\theta)\omega_2 - \sin(\theta)\omega_3, \ \widetilde{\omega}_3 = \sin(\theta)\omega_2 + \cos(\theta)\omega_3 \ ;$$

here θ is an angle, linearly depending on time. Thus the forms ω_2, ω_3 are rotated in the plane they span in $\mathcal{Q}$. In other words, the hypersymplectic structure (and hence the hyperkahler one) is in this case mapped into an equivalent—but different—one. This shows indeed we have hyperkahler, but not strongly hyperkahler, maps. ⊙

2.7.3 Canonical Maps

The problem mentioned in Remark 2.10 is not present for Euclidean hyperkahler manifolds. In fact, in this case the complex structures (which, as mentioned above, see Remark 1.6 in Chap. 1, can always be reduced at a single point—and globally in the Euclidean case—to the standard forms seen in Sect. 1.4.3) provide a natural splitting of $\mathrm{T}M = \mathbf{R}^{4n}$ into the sum of $\mathbf{R}^4$ subspaces,

$$\mathbf{R}^{4n} \ = \ \mathbf{R}^4_{(1)} \oplus ... \oplus \mathbf{R}^4_{(n)} \ .$$

We will denote by ι_k the embedding of $\mathbf{R}^4_{(k)}$ into $\mathbf{R}^{4n}$; thus $\iota_k^* : \Lambda(\mathbf{R}^{4n}) \to \Lambda(\mathbf{R}^4_{(k)})$ will represent the restriction of forms in $\Lambda(\mathbf{R}^{4n})$ to the $\mathbf{R}^4_{(k)}$ subspace.

Correspondingly, given a symplectic form ω we will write

$$\omega^{(k)} \ := \ \iota_k^*(\omega) \quad (k = 1, ..., n) \ . \tag{2.40}$$

Note that $\iota_k^*(\omega \wedge \omega) = \omega^{(k)} \wedge \omega^{(k)}$ (no sum on k); this is the volume form $\Omega^{(k)}$ in the $\mathbf{R}^4_{(k)}$ subspace.

Remark 2.12 As just recalled, our construction and in particular the splitting of $\mathrm{T}_{m_0}M$ can be done *at a single point* $m_0 \in M$ for any manifold, not just Euclidean ones. The obstacle to an extension of this construction to the general case is not only that the splitting is ill-defined in any neighborhood, no matter how small, of the reference point m_0: in fact, even at the reference point the splitting is in general not invariant under the holonomy group, and thus not intrinsic (see also the discussion in Sects. 2.7.3 and 3.2.3, as well as Appendix A). This problem is of course absent for the Euclidean case, where the holonomy group reduces to the identity.

Note also that in this respect any holonomy action in Sp(1) amounts to mapping the hyperkahler structure into an equivalent one; and any action in $\mathrm{Sp}(1) \times ... \times \mathrm{Sp}(1) \subset \mathrm{Sp}(n)$, where each of the Sp(1) factors is acting in one of the irreducible hyperkahler components of $\mathrm{T}_{m_0}M \simeq \mathbf{R}^{4n} = \mathbf{R}^4 \oplus ... \oplus \mathbf{R}^4$ (the splitting being of course that induced by the $\mathbf{J}$ structure, see above) does not alter the splitting of $\mathrm{T}M = \mathbf{R}^{4n}$ into the sum of $\mathbf{R}^4$ subspaces, hence the $\Omega^{(a)}$ four-forms.

In other words, our subsequent discussion will apply not only to the case where the holonomy group reduces to the identity (as for Euclidean spaces) but will also apply to the cases where the holonomy group lies in $\mathrm{Sp}(1) \times ... \times \mathrm{Sp}(1)$.

This is maybe an appropriate point to also recall that, by the Ambrose-Singer theorem [5, 106, 112, 123, 135] (see Appendix A for a statement), the holonomy group is generated by the curvature of the connection; thus if the Levi-Civita connection on (M, g) has a curvature form lying in $\mathrm{sp}(1) \times ... \times \mathrm{sp}(1)$, we are guaranteed to be in the case where our discussion applies. ⊙

We are now ready to give our definition of canonical maps for Euclidean hyperkahler manifolds, which can be given in two equivalent ways.

Definition 7a Let $(M = \mathbf{R}^{4n}, g = \delta; \mathbf{J})$ be an Euclidean hyperkahler manifold, with ι_k $(k = 1, ..., n)$ as above, and $\mathcal{S}$ the corresponding symplectic Kahler sphere. We say that the map $\phi : M \to M$ is *canonical* for the hyperkahler structure $(g = \delta; \mathbf{J})$ if, for any $\omega \in \mathcal{S}$ and any $k = 1, .., n$, it satisfies

$$\iota_k^*[\phi^*(\omega \wedge \omega)] \ = \ \iota_k^*(\omega \wedge \omega) \ \equiv \ \omega^{(k)} \wedge \omega^{(k)} \ . \tag{2.41}$$

Definition 7b Let $(M = \mathbf{R}^{4n}, g = \delta; \mathbf{J})$ be an Euclidean hyperkahler manifold, with ι_k $(k = 1, ..., n)$ as above, and let ω_α be the symplectic structures associated to the J_a. The map $\phi : M \to M$ is *canonical* for the hyperkahler structure $(g = \delta; \mathbf{J})$ if (with no sum on k), for any $\alpha = 1, 2, 3$ and $k = 1, ..., n$,

$$\iota_k^*[\phi^*(\omega_\alpha \wedge \omega_\alpha)] \ = \ \iota_k^*(\omega_\alpha \wedge \omega_\alpha) \ . \tag{2.42}$$

A stronger notion of canonicity, which implies the previous one, can also be defined (and was proposed in earlier works of ours [70]), but turns out to be too restrictive and hence of little use (see Appendix A in [71]); this is given below for the sake of completeness.

Definition 2.8 Let $(M, g; \mathbf{J})$ be a hyperkahler manifold, and $\mathcal{Q}$ the corresponding symplectic Kahler sphere. We say that the map $\phi : M \to M$ is *strongly canonical* if, for any $\omega \in \mathcal{S}$, it preserves the form $\omega \wedge \omega$, i.e. $\phi^*(\omega \wedge \omega) = \omega \wedge \omega$. Equivalently, if and only if (with no sum on a)

$$\phi^*(\omega_\alpha \wedge \omega_\alpha) \ = \ \omega_\alpha \wedge \omega_\alpha \quad \alpha = 1, 2, 3 \ .$$

As mentioned above, it is immediate to check that if a map is strongly canonical it is also canonical, while the converse is not true.

Remark 2.13 It is clear that the two notions of canonical and hyperkahler maps (or **Q**-maps) proposed here are *not* equivalent (at difference with the notion holding in the symplectic or Kahler case which they generalize). In a way, **Q**-maps preserve the **Q**-structure (that is, the hyperkahler form), while canonical ones only preserve the restriction of this to any irreducible hyperkahler component; moreover, note that we are *not* requiring canonical maps to be orthogonal.

Consider e.g. the standard ω_1 in $(\mathbf{R}^4, \delta)$ (see Sect. 1.4.3): under the map $x^1 \to \lambda x^1$, $x^2 \to \lambda x^2$, $x^3 \to \lambda^{-1} x^3$, $x^4 \to \lambda^{-1} x^4$, the form ω_1 is not preserved (note g is not preserved as well) nor mapped to a different form in $\mathcal{S}$, but $\omega_1 \wedge \omega_1$ is invariant (the forms ω_2 and ω_3 are instead invariant themselves, and *a fortiori* we get invariance of $\omega_2 \wedge \omega_2$ and of $\omega_3 \wedge \omega_3$). More generally, a canonical map could even mix the positively and negatively oriented structures. ⊙

Remark 2.14 Our Definition implies that canonical vector fields preserve the volume forms $\Omega^{(k)} = (1/2)(\omega^{(k)} \wedge \omega^{(k)})$ (no sum on k) in the four dimensional $\mathbf{R}^4_{(k)}$ subspaces.[5] ⊙

Remark 2.15 A hyperkahler map transform the triple of the ω_α into an equivalent one, and hence preserves the hyperkahler four-form $\Phi = (1/2)(\omega \wedge \omega)$. Obviously preservation of the form $\omega \wedge \omega$ implies preservation of its restriction to any subspace, and we immediately have that hyperkahler maps are also canonical. The converse is not true, as shown by the simple explicit example in Remark 2.13 above. Note also that the definition of canonical transformation does *not* require to preserve the metric g; canonical maps which are not orthogonal are definitely not **Q**-maps (the map considered in Remark 2.13 is indeed an example of this case). ⊙

The relation between hyperhamiltonian dynamics and hyperkahler or canonical transformations will be discussed in detail in the following two Chapters.

We anticipate that we have a *partial* generalization of the familiar results holding in the Hamiltonian case, where any Hamiltonian vector field generates a one-parameter (local) group of canonical transformations, and any such group admits a Hamiltonian vector field as generator.

In fact, in the hyperhamiltonian case we will obtain that any hyperhamiltonian vector field generates a one-parameter (local) group of canonical transformations, and that any such group admits a Dirac vector field (rather than a hyperhamiltonian one) as generator. Moreover—as already stressed—in the hyperkahler case hyperkahler and canonical transformations (in the sense of the definitions above) are *not* the same.

Remark 2.17 Finally, we note that $\Psi \wedge \ldots \wedge \Psi$ (with n factors) is proportional to the volume form Ω; thus preservation of Ψ implies, once again, preservation of Ω (the converse is in general not true).

We will also see that in Euclidean $\mathbf{R}^4$ with standard hyperkahler structure, any divergence-free vector field is a Dirac field. As in this case Ψ is just the volume form in $\mathbf{R}^4$, obviously Dirac fields preserve Ψ. ⊙

[5] And hence also the volume form $\Omega = \Omega^{(1)} \wedge \ldots \wedge \Omega^{(n)}$ in $\mathbf{R}^{4n}$.

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