Preface

Both the real-variable theory of function spaces and the boundedness of operators are always one of the core contents of harmonic analysis, while the Lebesgue spaces are the basic function spaces. However, due to the need for more inclusive classes of function spaces than the $L^p(\mathbb{R}^n)$ families from applications, Orlicz spaces were introduced by Birnbaum-Orlicz in [13] and Orlicz in [154], which is widely used in various branches of analysis. As the Orlicz spaces, Musielak-Orlicz spaces are also defined via the growth functions. Compared with the growth functions of Orlicz spaces, the growth functions of Musielak-Orlicz spaces may vary in both the spatial variable and the growth variable. Thus, by choosing special growth functions, Musielak-Orlicz spaces may have subtler and finer structures, which play a key role in solving the endpoint or the sharp problems of analysis.

The real-variable theory of Hardy spaces on the n-dimensional Euclidean space $\mathbb{R}^n$ was initiated by Stein and Weiss [178] and systematically developed by Fefferman and Stein in a seminal paper [58]. Since the Hardy space $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$ is, especially when studying the boundedness of operators, a suitable substitute of the Lebesgue space $L^p(\mathbb{R}^n)$, it plays an important role in various fields of analysis and partial differential equations.

Moreover, Musielak-Orlicz Hardy spaces are also suitable substitutes of Musielak-Orlicz spaces in dealing with many problems of analysis; see, for example, [106, 107, 199]. It is worth noticing that some special Musielak-Orlicz Hardy spaces appear naturally in the study of the products of functions in $\mathrm{BMO}(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ and the endpoint estimates for the div-curl lemma and the commutators of Calderón-Zygmund operators with $\mathrm{BMO}(\mathbb{R}^n)$ functions.

Recall that a famous result of Charles Fefferman and Elias M. Stein (see [58]) states that $\mathrm{BMO}(\mathbb{R}^n)$, the class of functions of bounded mean oscillation introduced by Fritz John and Louis Nirenberg in 1961 (see [109]), is indeed the dual of the real Hardy space $H^1(\mathbb{R}^n)$ studied by Elias M. Stein and Guido Weiss in 1960 (see [178]). However, this duality is not like that the dual space of $L^p(\mathbb{R}^n)$ with $p \in (1, \infty)$ is $L^q(\mathbb{R}^n)$ with $q \in (1, \infty)$ and $1/q + 1/p = 1$. More precisely, the pointwise product fg of a function $f \in \mathrm{BMO}(\mathbb{R}^n)$ and a function $g \in H^1(\mathbb{R}^n)$ is not locally integrable in general. So, a natural question is what we can say about the product

fg. This question has firstly been considered by Aline Bonami, Tadeusz Iwaniec, Peter Jones, and Michel Zinsmeister in 2007 (see [15]). Therein, they showed that, although the product fg is, in general, not in $L^1(\mathbb{R}^n)$, however, it can be viewed as a Schwartz distribution $f \times g$ and can be written as a sum of an integrable function and a Schwartz distribution in the weighted Orlicz-Hardy space $H^\phi_w(\mathbb{R}^n)$ associated with the Orlicz function

$$\phi(t) := \frac{t}{\log(e+t)}, \qquad \forall\, t \in (0, \infty),$$

and the Muckenhoupt weight

$$w(x) = \frac{1}{\log(e+|x|)}, \qquad \forall\, x \in \mathbb{R}^n;$$

see [15] for the details. Another motivation for investigating the distribution $f \times g$ comes from dealing with the following operator:

$$\mathfrak{L}(f) := f \log|f|, \quad f \in H^1(\mathbb{R}^n),$$

and a result of Elias M. Stein (see [175]) states that, if $f \in H^1(\mathbb{R}^n)$ and $f \geq 0$ in an open ball B, then $f \log f \in L^1_{\mathrm{loc}}(B)$. For every $f \in H^1(\mathbb{R}^n)$, Tadeusz Iwaniec and Anne Verde [101] showed that $f \log|f|$ is a Schwartz distribution.

Also, there are several natural reasons for investigating the distribution $f \times g$. *First*, in PDEs we find various nonlinear differential expressions identified by the theory of compensated compactness; see the seminal work of François Murat [147] and Luc Tartar [188] and the subsequent developments [53, 54, 89]. New and unexpected phenomena concerning higher integrability of the Jacobian determinants and other null Lagrangians have been discovered [71, 96, 97, 102, 144] and used in the geometrical function theory [8, 95, 103], calculus of variations [98, 182], and some areas of applied mathematics [143, 146, 231]. Recently a viable theory of the existence and the improved regularity for solutions of PDEs, where the uniform ellipticity is lost, has been built out of the distributional div-curl products and null Lagrangians [89, 99]. *Second*, these investigations bring us to new classes of functions, distributions, and measures [100], just to mention the grand L^p-spaces [79, 97, 170]. Subtler and clever ideas of the convergence in these spaces have been adopted from probability and measure theory, biting convergence for instance [11, 12, 22, 231]. Recent investigations of so-called very weak solutions of nonlinear PDEs [79, 98] rely on these new classes of functions. *Thirdly*, it seems likely that these methods will shed some new light on harmonic analysis with more practical applications.

Recently, Aline Bonami, Sandrine Grellier, and Luong Dang Ky [16] gave an answer for a question posted by Aline Bonami, Tadeusz Iwaniec, Peter Jones, and Michel Zinsmeister [15] by showing that there exist continuous bilinear operators that allow to split the product $f \times g$ of a function $f \in \mathrm{BMO}(\mathbb{R}^n)$ and a function

$g \in H^1(\mathbb{R}^n)$ into an $L^1(\mathbb{R}^n)$ part and a part in $H_w^\phi(\mathbb{R}^n)$. Therein, they also showed that $H_w^\phi(\mathbb{R}^n)$ can be replaced by a Hardy space of Musielak-Orlicz type $H^{\log}(\mathbb{R}^n)$ associated with the Musielak-Orlicz function

$$\varphi(x,t) = \frac{t}{\log(e+t) + \log(e+|x|)}, \quad \forall\, x \in \mathbb{R}^n,\ \forall\, t \in (0,\infty). \tag{$*$}$$

Moreover, in some sense, $H^{\log}(\mathbb{R}^n)$ is the smallest space and could not be replaced by a smaller space. Indeed, in the setting of holomorphic functions on the upper half-plane, it has been established very recently that the pointwise product fg of a holomorphic function $f \in \mathrm{BMO}A(\mathbb{C}_+)$ and a holomorphic function $g \in H_a^1(\mathbb{C}_+)$ is in the Musielak-Orlicz Hardy space $H_a^{\log}(\mathbb{C}_+)$ and, conversely, every holomorphic function in $H_a^{\log}(\mathbb{C}_+)$ can be written as such a product; see [14] for the details. Observe that the logarithmic terms of φ in $(*)$ make the corresponding Musielak-Orlicz Hardy-type space $H^{\log}(\mathbb{R}^n)$ have subtler and finer structure, compared with other function spaces (e.g., $H_w^\phi(\mathbb{R}^n)$), which are just the advantage of this space in solving the aforementioned product problems. Motivated by the study of the product of functions in $\mathrm{BMO}(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ in many contexts, the theory of Musielak-Orlicz Hardy spaces has been introduced, studied, and developed widely in recent years.

The main purpose of this book is to give a detailed and complete survey of the recent progress related to the real-variable theory and its applications of Musielak-Orlicz-type function spaces, which may lay the foundation for further applications of these function spaces.

To be precise, the whole book consists of eleven chapters. In Chap. 1, we recall the definition of the growth function and Musielak-Orlicz Hardy spaces $H^\varphi(\mathbb{R}^n)$, which generalize the Orlicz-Hardy spaces of Svante Janson [106] and the weighted Hardy spaces of Jose García-Cuerva [69] and Jan-Olov Strömberg and Alberto Torchinsky [181]. Here, $\varphi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is a function such that $\varphi(x,\cdot)$ is an Orlicz function and $\varphi(\cdot,t)$ is a Muckenhoupt A_∞ weight. A Schwartz distribution f belongs to $H^\varphi(\mathbb{R}^n)$ if and only if its non-tangential grand maximal function f^* is such that

$$x \mapsto \varphi(x, |f^*(x)|)$$

is integrable. We then establish their atomic decomposition. The class of pointwise multipliers for $\mathrm{BMO}(\mathbb{R}^n)$ characterized by Nakai and Yabuta can be seen as the dual of $L^1(\mathbb{R}^n) + H^{\log}(\mathbb{R}^n)$, where $H^{\log}(\mathbb{R}^n)$ denotes the Musielak-Orlicz Hardy space related to the Musielak-Orlicz function φ in $(*)$. Furthermore, under an additional assumption on φ, we prove that, if T is a sublinear operator and maps all atoms into uniformly bounded elements of a quasi-Banach space $\mathcal{B}$, then T can uniquely be extended to a bounded sublinear operator from $H^\varphi(\mathbb{R}^n)$ to $\mathcal{B}$.

Chapters 2 through 4 are devoted to establishing some new real-variable characterizations of $H^\varphi(\mathbb{R}^n)$ in terms of the vertical or the non-tangential maximal functions or the Littlewood-Paley functions or the molecular decomposition. We also characterize $H^{\varphi}(\mathbb{R}^n)$ via all the first-order Riesz transforms when $\frac{i(\varphi)}{q(\varphi)} > \frac{n-1}{n}$ and via all the Riesz transforms with the order not bigger than $m \in \mathbb{N}$ when $\frac{i(\varphi)}{q(\varphi)} > \frac{n-1}{n+m-1}$. Moreover, we also establish the Riesz transform characterizations of $H^{\varphi}(\mathbb{R}^n)$ by means of the higher-order Riesz transforms defined via the homogenous harmonic polynomials, respectively, via the odd order Riesz transforms.

In Chap. 5, we recall the Musielak-Orlicz Campanato space $\mathcal{L}_{\varphi,q,s}(\mathbb{R}^n)$, and, as an application, we prove that some of them is the dual space of the Musielak-Orlicz Hardy space $H^{\varphi}(\mathbb{R}^n)$. We also establish a John-Nirenberg inequality for functions in $\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$, and, as an application, we also obtain several equivalent characterizations of $\mathcal{L}_{\varphi,q,s}(\mathbb{R}^n)$, which, in return, further induce the φ-Carleson measure characterization of $\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$.

In Chap. 6, we establish the s-order intrinsic square function characterizations of $H^{\varphi}(\mathbb{R}^n)$ in terms of the intrinsic Lusin area function $S_{\alpha,s}$, the intrinsic g-function $g_{\alpha,s}$, and the intrinsic g_λ^*-function $g^*_{\lambda,\alpha,s}$, which are defined via $\mathrm{Lip}_{\alpha}(\mathbb{R}^n)$ functions supporting in the unit ball. A φ-Carleson measure characterization of the Musielak-Orlicz Campanato space $\mathcal{L}_{\varphi,1,s}(\mathbb{R}^n)$ is also established via the intrinsic function.

Chapter 7 is about the weak Musielak-Orlicz Hardy space $WH^{\varphi}(\mathbb{R}^n)$ which is defined via the grand maximal function. We then obtain its vertical or its non-tangential maximal function characterizations and other real-variable characterizations of $WH^{\varphi}(\mathbb{R}^n)$, respectively, in terms of the atom, the molecule, the Lusin area function, the Littlewood-Paley g-function, or g_λ^*-function.

In Chap. 8, we recall a local Musielak-Orlicz Hardy space $h^{\varphi}(\mathbb{R}^n)$ by the local grand maximal function and a local BMO-type space $\mathrm{bmo}^{\varphi}(\mathbb{R}^n)$ which is further proved to be the dual space of $h^{\varphi}(\mathbb{R}^n)$. As an application, we prove that the class of pointwise multipliers for the local BMO-type space $\mathrm{bmo}^{\phi}(\mathbb{R}^n)$, characterized by E. Nakai and K. Yabuta, is just the dual of

$$L^1(\mathbb{R}^n) + h^{\Phi_0}(\mathbb{R}^n),$$

where ϕ is an increasing function on $(0,\infty)$ satisfying some additional growth conditions and Φ_0 a Musielak-Orlicz function induced by ϕ. Characterizations of $h^{\varphi}(\mathbb{R}^n)$, including the atom, the local vertical, or the local non-tangential maximal functions, are presented. Using the atomic characterization, we prove the existence of finite atomic decompositions achieving the norm in some dense subspaces of $h^{\varphi}(\mathbb{R}^n)$, from which we further deduce some criterions for the boundedness on $h^{\varphi}(\mathbb{R}^n)$ of some sublinear operators. Finally, we show that the local Riesz transforms and some pseudo-differential operators are bounded on $h^{\varphi}(\mathbb{R}^n)$.

Let $s \in \mathbb{R}$, $q \in (0,\infty]$, $\varphi_1,\ \varphi_2 : \ \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ be two Musielak-Orlicz functions that, on the space variable, belong to the Muckenhoupt class $\mathbb{A}_\infty(\mathbb{R}^n)$ uniformly in the growth variable. In Chap. 9, we recall Musielak-Orlicz Besov-type spaces $\dot{B}^{s,\tau}_{\varphi_1,\varphi_2,q}(\mathbb{R}^n)$ and Musielak-Orlicz Triebel-Lizorkin-type spaces $\dot{F}^{s,\tau}_{\varphi_1,\varphi_2,q}(\mathbb{R}^n)$ and establish their φ-transform characterizations in the sense of Frazier and Jawerth. The embedding and lifting properties, characterizations via Peetre maximal functions, local means, Lusin area functions, and smooth atomic and molecular decompositions of these spaces are also presented. As applications, the boundedness on these spaces of Fourier multipliers with symbols satisfying some generalized Hörmander condition is obtained. These spaces have wide generality, which unify Musielak-Orlicz Hardy spaces, unweighted and weighted Besov(-type), and Triebel-Lizorkin(-type) spaces as special cases.

As an application of Musielak-Orlicz Hardy spaces, in Chap. 10, we prove that the product (in the distribution sense) of two functions, which are respectively from $\mathrm{BMO}(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$, may be written as a sum of two continuous bilinear operators, one from $H^1(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ and the other one from $H^1(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^n)$ into a special Musielak-Orlicz Hardy space $H^{\log}(\mathbb{R}^n)$. The two bilinear operators can be defined in terms of paraproducts. As a consequence, we find an endpoint estimate involving the space $H^{\log}(\mathbb{R}^n)$ for the div-curl lemma.

Let b be a BMO function. It is well known that the linear commutator $[b, T]$ of a Calderón-Zygmund operator T does not, in general, map continuously $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. However, Carlos Pérez showed that, if $H^1(\mathbb{R}^n)$ is replaced by a suitable atomic subspace $\mathcal{H}^1_b(\mathbb{R}^n)$, then the commutator is continuous from $\mathcal{H}^1_b(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. As another application of Musielak-Orlicz-type function spaces, in Chap. 11, we find the largest subspace $H^1_b(\mathbb{R}^n)$ such that all commutators of Calderón-Zygmund operators are continuous from $H^1_b(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. Some equivalent characterizations of $H^1_b(\mathbb{R}^n)$ are also given. We also study the commutators $[b, T]$ for T in a class $\mathcal{K}$ of sublinear operators containing almost all important operators in harmonic analysis. When T is linear, we prove that there exists a bilinear operator $\mathfrak{R} := \mathfrak{R}_T$ mapping continuously $H^1(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ such that, for all $(f, b) \in H^1(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^n)$, we have

$$[b, T](f) = \mathfrak{R}(f, b) + T(\mathfrak{S}(f, b)), \qquad (**)$$

where $\mathfrak{S}$ is a bounded bilinear operator from $H^1(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ which is independent of T. In the particular case when T is a Calderón-Zygmund operator satisfying $T1 = 0 = T^*1$ and $b \in \mathrm{BMO}^{\log}(\mathbb{R}^n)$, a special case of Musielak-Orlicz BMO spaces, we prove that the commutator $[b, T]$ maps continuously $H^1_b(\mathbb{R}^n)$ into $h^1(\mathbb{R}^n)$. Also, if b is in $\mathrm{BMO}(\mathbb{R}^n)$ and $T^*1 = T^*b = 0$, then the commutator $[b, T]$ maps continuously $H^1_b(\mathbb{R}^n)$ into $H^1(\mathbb{R}^n)$. When T is sublinear, we prove that there exists a bounded subbilinear operator $\mathfrak{R} := \mathfrak{R}_T : H^1(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ such that, for all $(f, b) \in H^1(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^n)$, we have

$$|T(\mathfrak{S}(f, b))| - \mathfrak{R}(f, b) \le |[b, T](f)| \le \mathfrak{R}(f, b) + |T(\mathfrak{S}(f, b))|. \qquad (***)$$

The bilinear decomposition $(**)$ and the subbilinear decomposition $(***)$ allow us to give a general overview of all known weak and strong L^1 estimates.

Throughout the book, we always let $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ and

$$\mathbb{R}^{n+1}_+ := \{(x, t) : x \in \mathbb{R}^n, t \in (0, \infty)\}.$$

We also use $\vec{0} := (\overbrace{0, \ldots, 0}^{n \text{ times}})$ denote the origin of $\mathbb{R}^n$. We use C to denote a *positive constant*, independent of the main parameters involved, but whose value may differ from line to line. *Constants with subscripts*, such as $C_{(8.3.1)}$, do not change in different occurrences, where the sub-index (8.3.1) indicates that $C_{(8.3.1)}$ is the first fixed positive constant in Sect. 8.3. We also use $C_{(\alpha, \beta, \ldots)}$ to denote a positive constant depending on the indicated parameters α, β, ... If $f \le Cg$, we write $f \lesssim g$ and, if $f \lesssim g \lesssim f$, we then write $f \sim g$. For any set $E \subset \mathbb{R}^n$, we use $E^{\complement}$ to denote the *set* $\mathbb{R}^n \setminus E$ and χ_E its *characteristic function*. For any index $q \in [1, \infty]$, we denote by q' its *conjugate index*, namely, $1/q + 1/q' = 1$. The symbol $\lfloor s \rfloor$ for any $s \in \mathbb{R}$ denotes the biggest integer not bigger than s.

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