Chapter 2
Maximal Function Characterizations of Musielak-Orlicz Hardy Spaces

In this chapter, we establish some real-variable characterizations of $H^{\varphi}(\mathbb{R}^n)$ in terms of the vertical or the non-tangential maximal functions, via first establishing a Musielak-Orlicz Fefferman-Stein vector-valued inequality.

2.1 Musielak-Orlicz Fefferman-Stein Vector-Valued Inequality

This section is devoted to establishing an interpolation theorem of operators, in the spirit of the Marcinkiewicz interpolation theorem, associated with a growth function. In what follows, for any non-negative locally integrable function w on $\mathbb{R}^n$ and $p \in (0, \infty)$, the *weighted Lebesgue space* $L^p_w(\mathbb{R}^n)$ is defined to be the space of all measurable functions f such that

$$\|f\|_{L^p_w(\mathbb{R}^n)} := \left\{\int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx\right\}^{1/p} < \infty.$$

Theorem 2.1.1 *Let $p_1, p_2 \in (0, \infty)$, $p_1 < p_2$ and φ be a Musielak-Orlicz function with uniformly lower type p_φ^- and uniformly upper type p_φ^+. If $0 < p_1 < p_\varphi^- \le p_\varphi^+ < p_2 < \infty$ and T is a sublinear operator defined on $L^{p_1}_{\varphi(\cdot,1)}(\mathbb{R}^n) + L^{p_2}_{\varphi(\cdot,1)}(\mathbb{R}^n)$ satisfying that, for $i \in \{1, 2\}$, all $\alpha \in (0, \infty)$ and $t \in (0, \infty)$,*

$$\varphi(\{x \in \mathbb{R}^n : \ |T(f)(x)| > \alpha\}, t) \le C_{(2.1.i)}\, \alpha^{-p_i} \int_{\mathbb{R}^n} |f(x)|^{p_i} \varphi(x, t)\, dx, \tag{2.1}$$

where $C_{(2.1.i)}$ is a positive constant independent of f, t and α. Then T is bounded on $L^{\varphi}(\mathbb{R}^n)$ and, moreover, there exists a positive constant C such that, for all

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$f \in L^{\varphi}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \varphi(x, |T(f)(x)|)\, dx \le C \int_{\mathbb{R}^n} \varphi(x, |f(x)|)\, dx.$$

Proof First observe that, for all $t \in (0, \infty)$,

$$\int_{\mathbb{R}^n} |f(x)|^p \varphi(x, t)\, dx < \infty \quad \text{if and only if} \quad \int_{\mathbb{R}^n} |f(x)|^p \varphi(x, 1)\, dx < \infty.$$

Thus, the spaces $L^p_{\varphi(\cdot,t)}(\mathbb{R}^n)$ and $L^p_{\varphi(\cdot,1)}(\mathbb{R}^n)$ coincide as sets. Now we show that

$$L^{\varphi}(\mathbb{R}^n) \subset \left[L^{p_1}_{\varphi(\cdot,1)}(\mathbb{R}^n) + L^{p_2}_{\varphi(\cdot,1)}(\mathbb{R}^n)\right].$$

For any given $t \in (0, \infty)$, we decompose $f \in L^{\varphi}(\mathbb{R}^n)$ as

$$f = f\chi_{\{x\in\mathbb{R}^n:\ |f(x)|>t\}} + f\chi_{\{x\in\mathbb{R}^n:\ |f(x)|\le t\}} =: f^{(t)} + f_{(t)}.$$

Then, by the fact that φ is of uniformly lower type p_{φ}^- and $p_1 < p_{\varphi}^-$, we conclude that

$$\begin{aligned}\int_{\mathbb{R}^n} |f^{(t)}(x)|^{p_1} \varphi(x, 1)\, dx &\lesssim \int_{\{x\in\mathbb{R}^n:\ |f(x)|>t\}} |f(x)|^{p_1} \left[\frac{t}{|f(x)|}\right]^{p_{\varphi}^-} \varphi\left(x, \frac{|f(x)|}{t}\right) dx \\ &\lesssim t^{p_1} \int_{\mathbb{R}^n} \varphi\left(x, \frac{|f(x)|}{t}\right) dx < \infty,\end{aligned}$$

namely, $f^{(t)} \in L^{p_1}_{\varphi(\cdot,1)}(\mathbb{R}^n)$. Similarly, we have $f_{(t)} \in L^{p_2}_{\varphi(\cdot,1)}(\mathbb{R}^n)$ and hence $T(f)$ is well defined.

By the fact that T is sublinear and Lemma 1.1.6(ii), we further know that

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi(x, |T(f)(x)|)\, dx &\sim \int_0^{\infty} \frac{1}{t} \int_{\{x\in\mathbb{R}^n:\ |T(f)(x)|>t\}} \varphi(x, t)\, dx\, dt \\ &\lesssim \int_0^{\infty} \frac{1}{t} \int_{\{x\in\mathbb{R}^n:\ |T(f)^{(t)}(x)|>t/2\}} \varphi(x, t)\, dx\, dt \\ &\quad + \int_0^{\infty} \frac{1}{t} \int_{\{x\in\mathbb{R}^n:\ |T(f)_{(t)}(x)|>t/2\}} \cdots \\ &=: \mathrm{I}_1 + \mathrm{I}_2.\end{aligned}$$

On I_1, since T is of weak type (p_1, p_1) (namely, (2.1) with $i = 1$), φ is of uniformly lower type p_φ^- and $p_1 < p_\varphi^-$, it follows that

$$\begin{aligned}
\mathrm{I}_1 &\lesssim \int_0^\infty \frac{1}{t}\left(\frac{t}{2}\right)^{-p_1} \int_{\mathbb{R}^n} |f^{(t)}(x)|^{p_1} \varphi(x,t)\,dx\,dt \\
&\sim \int_0^\infty \frac{1}{t^{1+p_1}} \int_{\{x\in\mathbb{R}^n:\ |f(x)|>t\}} |f(x)|^{p_1} \varphi(x,t)\,dx\,dt \\
&\sim \int_0^\infty \frac{1}{t^{1+p_1}} \int_{\{x\in\mathbb{R}^n:\ |f(x)|>t\}} \varphi(x,t)\left[\int_t^{|f(x)|} p_1 s^{p_1-1}\,ds + t^{p_1}\right] dx\,dt \\
&\sim \int_0^\infty s^{p_1-1} \int_{\{x\in\mathbb{R}^n:\ |f(x)|>s\}} \int_0^s \frac{\varphi(x,t)}{t^{1+p_1}}\,dt\,dx\,ds \\
&\quad + \int_0^\infty \frac{1}{t} \int_{\{x\in\mathbb{R}^n:\ |f(x)|>t\}} \varphi(x,t)\,dx\,dt \\
&\lesssim \int_0^\infty s^{p_1-1} \int_{\{x\in\mathbb{R}^n:\ |f(x)|>s\}} \varphi(x,s) s^{-p_\varphi^-} \int_0^s \frac{1}{t^{1+p_1-p_\varphi^-}}\,dt\,dx\,ds \\
&\quad + \int_{\mathbb{R}^n} \varphi(x,|f(x)|)\,dx \\
&\sim \int_0^\infty \frac{1}{s} \int_{\{x\in\mathbb{R}^n:\ |f(x)|>s\}} \varphi(x,s)\,dx\,ds + \int_{\mathbb{R}^n} \varphi(x,|f(x)|)\,dx \\
&\sim \int_{\mathbb{R}^n} \varphi(x,|f(x)|)\,dx.
\end{aligned}$$

Also, from the weak type (p_2, p_2) of T (namely, (2.1) with $i = 2$), the uniformly upper type p_φ^+ property of φ and $p_\varphi^+ < p_2$, we deduce that

$$\begin{aligned}
\mathrm{I}_2 &\lesssim \int_0^\infty \frac{1}{t}\left(\frac{t}{2}\right)^{-p_2} \int_{\mathbb{R}^n} |f_{(t)}(x)|^{p_2} \varphi(x,t)\,dx\,dt \\
&\sim \int_0^\infty \frac{1}{t^{1+p_2}} \int_{\{x\in\mathbb{R}^n:\ |f(x)|\le t\}} |f(x)|^{p_2} \varphi(x,t)\,dx\,dt \\
&\sim \int_0^\infty \frac{1}{t^{1+p_2}} \int_{\{x\in\mathbb{R}^n:\ |f(x)|\le t\}} \varphi(x,t) \int_0^{|f(x)|} p_2 s^{p_2-1}\,ds\,dx\,dt \\
&\sim \int_0^\infty s^{p_2-1} \int_{\{x\in\mathbb{R}^n:\ |f(x)|>s\}} \int_s^\infty \frac{\varphi(x,t)}{t^{1+p_2}}\,dt\,dx\,ds \\
&\lesssim \int_0^\infty s^{p_2-1} \int_{\{x\in\mathbb{R}^n:\ |f(x)|>s\}} \varphi(x,s) s^{-p_\varphi^+} \int_s^\infty \frac{1}{t^{1+p_2-p_\varphi^+}}\,dt\,dx\,ds
\end{aligned}$$

$$\sim \int_0^\infty \frac{1}{s} \int_{\{x\in\mathbb{R}^n:\ |f(x)|>s\}} \varphi(x,s)\,dx\,ds$$
$$\sim \int_{\mathbb{R}^n} \varphi(x,|f(x)|)\,dx.$$

Thus, T is bounded on $L^\varphi(\mathbb{R}^n)$, which completes the proof of Theorem 2.1.1. $\square$

Let $q(\varphi)$ be as in (1.13). As a simple corollary of Theorem 2.1.1, together with the fact that, for any $p \in (q(\varphi),\infty)$ when $q(\varphi) \in (1,\infty)$ or when $q(\varphi) = 1$ and $\varphi \notin \mathbb{A}_1(\mathbb{R}^n)$, or for any $p \in [1,\infty)$ when $q(\varphi) = 1$ and $\varphi \in \mathbb{A}_1(\mathbb{R}^n)$, there exists a positive constant $C_{(p,\varphi)}$ such that, for all $f \in L^p_{\varphi(\cdot,t)}(\mathbb{R}^n)$ and $t \in (0,\infty)$,

$$\varphi(\{x \in \mathbb{R}^n :\ |\mathcal{M}f(x)| > \alpha\}, t) \le C_{(p,\varphi)}\alpha^{-p} \int_{\mathbb{R}^n} |f(x)|^p \varphi(x,t)\,dx,$$

we immediately obtain the following boundedness of $\mathcal{M}$ on $L^\varphi(\mathbb{R}^n)$, the details being omitted.

Corollary 2.1.2 *Let φ be a Musielak-Orlicz function with uniformly lower type p_φ^- and uniformly upper type p_φ^+ satisfying $q(\varphi) < p_\varphi^- \le p_\varphi^+ < \infty$, where $q(\varphi)$ is as in* (1.13)*. Then the Hardy-Littlewood maximal function $\mathcal{M}$ is bounded on $L^\varphi(\mathbb{R}^n)$ and, moreover, there exists a positive constant C such that, for all $f \in L^\varphi(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} \varphi(x,\mathcal{M}f(x))\,dx \le C \int_{\mathbb{R}^n} \varphi(x,|f(x)|)\,dx.$$

The *space* $L^\varphi(\ell^r,\mathbb{R}^n)$ is defined to be the set of all $\{f_j\}_{j\in\mathbb{Z}}$ satisfying

$$\left[\sum_j |f_j|^r\right]^{1/r} \in L^\varphi(\mathbb{R}^n),$$

equipped with the (quasi-)norm

$$\|\{f_j\}_j\|_{L^\varphi(\ell^r,\mathbb{R}^n)} := \left\|\left[\sum_j |f_j|^r\right]^{1/r}\right\|_{L^\varphi(\mathbb{R}^n)}.$$

We have the following vector-valued interpolation theorem of Musielak-Orlicz type.

Theorem 2.1.3 *Let p_1, p_2 and φ be as in Theorem 2.1.1 and $r \in [1,\infty]$. Assume that T is a sublinear operator defined on $L^{p_1}_{\varphi(\cdot,1)}(\mathbb{R}^n) + L^{p_2}_{\varphi(\cdot,1)}(\mathbb{R}^n)$ satisfying that, for*

$i \in \{1,2\}$ and all $\{f_j\}_j \in L^{p_i}_{\varphi(\cdot,1)}(\ell^r, \mathbb{R}^n)$, $\alpha \in (0,\infty)$ and $t \in (0,\infty)$,

$$\varphi\left(\left\{x \in \mathbb{R}^n : \left[\sum_j |T(f_j)(x)|^r\right]^{\frac{1}{r}} > \alpha\right\}, t\right)$$
$$\leq C_i \alpha^{-p_i} \int_{\mathbb{R}^n} \left[\sum_j |f_j(x)|^r\right]^{\frac{p_i}{r}} \varphi(x,t)\,dx, \tag{2.2}$$

where C_i is a positive constant independent of $\{f_j\}_j$, t and α. Then there exists a positive constant C such that, for all $\{f_j\}_j \in L^\varphi(\ell^r, \mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \varphi\left(x, \left[\sum_j |T(f_j)(x)|^r\right]^{1/r}\right) dx \leq C \int_{\mathbb{R}^n} \varphi\left(x, \left[\sum_j |f_j(x)|^r\right]^{1/r}\right) dx.$$

Proof For all $\{f_j\}_j \in L^\varphi(\ell^r, \mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let

$$n_j(x) := \frac{f_j(x)}{[\sum_j |f_j(x)|^r]^{1/r}} \quad \text{when} \quad \left[\sum_j |f_j(x)|^r\right]^{1/r} \neq 0,$$

and $n_j(x) := 0$ otherwise. Then $[\sum_j |n_j(x)|^r]^{1/r} = 1$ for all $x \in \mathbb{R}^n$. Consider the operator

$$A(g) := \left[\sum_j |T(gn_j)|^r\right]^{1/r},$$

where $g \in L^{p_1}_{\varphi(\cdot,1)}(\mathbb{R}^n) + L^{p_2}_{\varphi(\cdot,1)}(\mathbb{R}^n)$. Then, for all $g_1,\ g_2 \in L^{p_1}_{\varphi(\cdot,1)}(\mathbb{R}^n) + L^{p_2}_{\varphi(\cdot,1)}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, by the sublinear property of T and the Minkowski inequality, we know that

$$A(g_1+g_2)(x) = \left[\sum_j |T((g_1+g_2)n_j)(x)|^r\right]^{1/r}$$
$$\leq \left\{\sum_j \left[|T(g_1n_j)(x)| + |T(g_2n_j)(x)|\right]^r\right\}^{1/r}$$

$$\leq \left[\sum_j |T(g_1 n_j)(x)|^r\right]^{1/r} + \left[\sum_j |T(g_2 n_j)(x)|^r\right]^{1/r}$$
$$= A(g_1)(x) + A(g_2)(x).$$

Thus, A is sublinear. Moreover, by (2.2), we further conclude that, for all $i \in \{1, 2\}$, $\alpha \in (0, \infty)$, $t \in (0, \infty)$ and $g \in L^{p_1}_{\varphi(\cdot,1)}(\mathbb{R}^n) + L^{p_2}_{\varphi(\cdot,1)}(\mathbb{R}^n)$,

$$\varphi(\{x \in \mathbb{R}^n : |A(g)(x)| > \alpha\}, t) = \varphi\left(\left\{x \in \mathbb{R}^n : \left[\sum_j |T(gn_j)(x)|^r\right]^{1/r} > \alpha\right\}, t\right)$$
$$\lesssim \alpha^{-p_i} \int_{\mathbb{R}^n} \left[\sum_j |gn_j(x)|^r\right]^{p_i/r} \varphi(x, t)\, dx$$
$$\lesssim \alpha^{-p_i} \int_{\mathbb{R}^n} |g(x)|^{p_i} \varphi(x, t)\, dx,$$

which implies that A satisfies (2.1). Thus, if let $g := [\sum_j |f_j|^r]^{1/r}$, from Theorem 2.1.1, we deduce that

$$\int_{\mathbb{R}^n} \varphi\left(x, \left[\sum_j |T(f_j)(x)|^r\right]^{1/r}\right) dx = \int_{\mathbb{R}^n} \varphi(x, |A(g)(x)|)\, dx$$
$$\lesssim \int_{\mathbb{R}^n} \varphi(x, |g(x)|)\, dx$$
$$\lesssim \int_{\mathbb{R}^n} \varphi\left(x, \left[\sum_j |f_j(x)|^r\right]^{1/r}\right) dx,$$

which completes the proof of Theorem 2.1.3. □

By using Theorem 2.1.3 and [7, Theorem 3.1(a)], we immediately obtain the following Musielak-Orlicz Fefferman-Stein vector-valued inequality. We point out that, to apply Theorem 2.1.3, we need $r \in (1, \infty]$, the details being omitted.

Theorem 2.1.4 *Let* $r \in (1, \infty]$, φ *be a Musielak-Orlicz function with uniformly lower type* p_φ^- *and upper type* p_φ^+, $q \in (1, \infty)$ *and* $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$. *If* $q(\varphi) < p_\varphi^- \leq p_\varphi^+ < \infty$, *then there exists a positive constant C such that, for all* $\{f_j\}_{j\in\mathbb{Z}} \in$

$L^{\varphi}(\ell^r, \mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \varphi\left(x, \left\{\sum_{j\in\mathbb{Z}} \left[\mathcal{M}(f_j)(x)\right]^r\right\}^{1/r}\right) dx \le C \int_{\mathbb{R}^n} \varphi\left(x, \left[\sum_{j\in\mathbb{Z}} |f_j(x)|^r\right]^{1/r}\right) dx.$$

2.2 Maximal Function Characterizations of $H^{\varphi}(\mathbb{R}^n)$

In this section, we establish some maximal function characterizations of $H^{\varphi}(\mathbb{R}^n)$. First, we recall the notions of the vertical and the non-tangential maximal functions.

Definition 2.2.1 Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} \psi(x)\, dx = 1. \tag{2.3}$$

Let $f \in \mathcal{S}'(\mathbb{R}^n)$. The *vertical maximal function* $\psi_+^*(f)$ of f associated to ψ is defined by setting, for all $x \in \mathbb{R}^n$,

$$\psi_+^*(f)(x) := \sup_{t\in(0,\infty)} |\psi_t * f(x)| \tag{2.4}$$

and the *non-tangential maximal function* $\psi_\nabla^*(f)$ of f associated to ψ is defined by setting, for all $x \in \mathbb{R}^n$,

$$\psi_\nabla^*(f)(x) := \sup_{|x-y|<t} |\psi_t * f(y)|. \tag{2.5}$$

Obviously, for all $x \in \mathbb{R}^n$, we have

$$\psi_+^*(f)(x) \le \psi_\nabla^*(f)(x) \lesssim f^*(x), \tag{2.6}$$

where the implicit equivalent positive constants are independent of f and x.

In order to establish the vertical or the non-tangential maximal function characterizations of $H^{\varphi}(\mathbb{R}^n)$, we first establish some inequalities in the norm of $L^{\varphi}(\mathbb{R}^n)$ involving the maximal functions $\psi_\nabla^*(f)$, $\psi_+^*(f)$ and f^*.

Theorem 2.2.2 *Let φ be a growth function as in Definition 1.1.4 and ψ as in Definition 2.2.1. Then there exists a positive constant C, depending only on ψ, φ and n, such that, for all $f \in \mathcal{S}'(\mathbb{R}^n)$,*

$$\left\|\psi_\nabla^*(f)\right\|_{L^{\varphi}(\mathbb{R}^n)} \le C \left\|\psi_+^*(f)\right\|_{L^{\varphi}(\mathbb{R}^n)} \tag{2.7}$$

and

$$\|f^*\|_{L^\varphi(\mathbb{R}^n)} \le C \left\|\psi_+^*(f)\right\|_{L^\varphi(\mathbb{R}^n)}. \tag{2.8}$$

Proof Let $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfy $\psi_+^*(f) \in L^\varphi(\mathbb{R}^n)$. We first show (2.7). Indeed, for any $\epsilon \in (0, 1)$, $N \in \mathbb{N}$ sufficiently large and $x \in \mathbb{R}^n$, let

$$\mathcal{M}^*_{\psi,\epsilon,N}(f)(x) := \sup_{|x-y|<t<\frac{1}{\epsilon}} |(f * \psi_t)(y)| \left(\frac{t}{t+\epsilon}\right)^N (1+\epsilon|y|)^{-N}.$$

Obviously, for all $x \in \mathbb{R}^n$,

$$\lim_{\epsilon\to 0^+,\, N\to\infty} \mathcal{M}^*_{\psi,\epsilon,N}(f)(x) = \psi^*_\nabla(f)(x).$$

We first claim that, for all $\lambda \in (0, \infty)$, there exists a positive constant $C_{(N,n,\varphi,\psi)}$, depending only on N, n, φ and ψ, such that

$$\int_{\mathbb{R}^n} \varphi\left(x, \frac{\mathcal{M}^*_{\psi,\epsilon,N}(f)(x)}{\lambda}\right) dx \le C_{(N,n,\varphi,\psi)} \int_{\mathbb{R}^n} \varphi\left(x, \frac{\psi_+^*(f)(x)}{\lambda}\right) dx. \tag{2.9}$$

To prove this claim, for all $x \in \mathbb{R}^n$, let

$$\tilde{\mathcal{M}}^*_{\psi,\epsilon,N}(f)(x) := \sup_{|x-y|<t<\frac{1}{\epsilon}} t\left|\nabla_y (f * \psi_t)(y)\right| \left(\frac{t}{t+\epsilon}\right)^N (1+\epsilon|y|)^{-N}.$$

From the proof of [74, (6.4.22)], we deduce that, for any $p \in (0, \infty)$, $\epsilon \in (0, 1)$ and $N \in \mathbb{N}$, there exists a positive constant $C_{(N,n,\varphi,\psi)}$ such that, for all $x \in \mathbb{R}^n$,

$$\tilde{\mathcal{M}}^*_{\psi,\epsilon,N}(f)(x) \le C_{(N,n,\varphi,\psi)} \left\{\mathcal{M}\left(\left[\mathcal{M}^*_{\psi,\epsilon,N}(f)\right]^p\right)(x)\right\}^{1/p}, \tag{2.10}$$

where $\mathcal{M}$ denotes the Hardy-Littlewood maximal function as in (1.7).

Now, let

$$E_{\epsilon,N} := \left\{x \in \mathbb{R}^n : \tilde{\mathcal{M}}^*_{\psi,\epsilon,N}(f)(x) \le C_0 \mathcal{M}^*_{\psi,\epsilon,N}(f)(x)\right\},$$

where C_0 is a sufficiently large constant whose size is determined later. For all $(x, t) \in \mathbb{R}^{n+1}_+$, let

$$\varphi_p(x, t) := \varphi(x, t^{1/p}).$$

By the definition of $i(\varphi)$, we know that there exists $p_0 \in (0, i(\varphi))$ such that, for any $x \in \mathbb{R}^n$, $\varphi(x, \cdot)$ is of lower type p_0. It is easy to see that $i(\varphi_p) = \frac{i(\varphi)}{p}$ and, for any $x \in \mathbb{R}^n$, $\varphi_p(x, \cdot)$ is of lower type $\frac{p_0}{p}$. Thus, by taking p sufficiently small, we obtain $q(\varphi_p) < i(\varphi_p)$, which, together with (2.10), Corollary 2.1.2 and the lower type p_0 property of $\varphi(x, \cdot)$, implies that there exists a positive constant $C_{(\varphi)}$ satisfying that, for any $\lambda \in (0, \infty)$,

$$\begin{aligned}
&\int_{(E_{\epsilon,N})^{\complement}} \varphi\left(x, \frac{\mathcal{M}^*_{\psi,\epsilon,N}(f)(x)}{\lambda}\right) dx \\
&\quad\le C_{(\varphi)} \left(\frac{1}{C_0}\right)^{p_0} \int_{(E_{\epsilon,N})^{\complement}} \varphi\left(x, \frac{\tilde{\mathcal{M}}^*_{\psi,\epsilon,N}(f)(x)}{\lambda}\right) dx \\
&\quad\le C_{(N,n,\varphi,\psi)} \left(\frac{1}{C_0}\right)^{p_0} \int_{(E_{\epsilon,N})^{\complement}} \varphi_p\left(x, \frac{\mathcal{M}([\mathcal{M}^*_{\psi,\epsilon,N}(f)]^p)(x)}{\lambda^p}\right) dx \\
&\quad\le C_{(N,n,\varphi,\psi)} \left(\frac{1}{C_0}\right)^{p_0} \int_{\mathbb{R}^n} \varphi\left(x, \frac{\mathcal{M}^*_{\psi,\epsilon,N}(f)(x)}{\lambda}\right) dx. \qquad (2.11)
\end{aligned}$$

By taking C_0 in (2.11) sufficiently large so that $C_{(N,n,\varphi,\psi)}(\frac{1}{C_0})^{p_0} < \frac{1}{2}$, we know that

$$\int_{\mathbb{R}^n} \varphi\left(x, \frac{\mathcal{M}^*_{\psi,\epsilon,N}(f)(x)}{\lambda}\right) dx \le 2\int_{E_{\epsilon,N}} \varphi\left(x, \frac{\mathcal{M}^*_{\psi,\epsilon,N}(f)(x)}{\lambda}\right) dx. \qquad (2.12)$$

Moreover, from [74, (6.4.27)], it follows that, for all $r < i(\varphi)$ and $x \in E_{\epsilon,N}$,

$$\mathcal{M}^*_{\psi,\epsilon,N}(f)(x) \le C_{(N,n,\varphi,\psi)} \left\{\mathcal{M}\left(\left[\psi^*_+(f)\right]^r\right)(x)\right\}^{1/r},$$

which, together with (2.12) and an argument similar to that used in the estimate (2.11), implies that (2.9) holds true.

Now, we finish the proof of Theorem 2.2.2 by using the above claim. Observe that, for $x \in \mathbb{R}^n$,

$$\mathcal{M}^*_{\psi,\epsilon,N}(f)(x) \ge \frac{2^{-N}}{(1+\epsilon|x|)^N} \sup_{|x-y|<t<\frac{1}{\epsilon}} |(f * \psi_t)(y)| \left(\frac{t}{t+\epsilon}\right)^N =: F_{\epsilon,N}(x).$$

It is easy to see, for each N and x, $F_{\epsilon,N}(x)$ is increasing to $2^{-N}\psi^*_{\triangledown}(f)(x)$ as $\epsilon \to 0^+$, which, combined with (2.9) and the Lebesgue monotone convergence theorem, implies that

$$\int_{\mathbb{R}^n} \varphi\left(x, \frac{\psi_\nabla^*(f)(x)}{\lambda}\right) dx \le C_{(N,\,n,\,\varphi,\,\psi)} \int_{\mathbb{R}^n} \varphi\left(x, \frac{\psi_+^*(f)(x)}{\lambda}\right) dx.$$

Here and hereafter, $\epsilon \to 0^+$ means $\epsilon > 0$ and $\epsilon \to 0$.

In particular, $\psi_+^*(f) \in L^\varphi(\mathbb{R}^n)$ implies that $\psi_\nabla^*(f) \in L^\varphi(\mathbb{R}^n)$. This, together with a repetition of the above argument used in the proof of the estimate (2.9) with $\epsilon := 0$ and $N := \infty$ in $\mathcal{M}_{\psi,\,\epsilon,\,N}^*(f)$ and $\tilde{\mathcal{M}}_{\psi,\,\epsilon,\,N}^*(f)$, implies that

$$\int_{\mathbb{R}^n} \varphi\left(x, \frac{\psi_\nabla^*(f)(x)}{\lambda}\right) dx \le C_{(n,\,\varphi,\,\psi)} \int_{\mathbb{R}^n} \varphi\left(x, \frac{\psi_+^*(f)(x)}{\lambda}\right) dx.$$

This finishes the proof of (2.7).

Now we show (2.8).

For $\lambda \in (0,\infty)$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let

$$\psi_T^\lambda(f)(x) := \sup_{y\in\mathbb{R}^n,\, t\in(0,\infty)} |f * \psi_t(y)| \left(\frac{t}{|x-y|+t}\right)^\lambda.$$

Then, from the estimate in [74, p. 51], it follows that, for all $\lambda \in (0,\infty)$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$f^*(x) \lesssim \psi_T^\lambda(f)(x). \tag{2.13}$$

On the other hand, choose $\lambda \in (n/p,\infty)$ and let $r := n/\lambda$. It follows, from the definition of $\psi_\nabla^*(f)$, that, if $z \in B(y,t)$, then $|f * \psi_t(y)| \le \psi_\nabla^*(f)(z)$. Since $B(y,t) \subset B(x, |x-y|+t)$, it follows that

$$|f * \psi_t(y)|^r \le \frac{1}{|B(y,t)|} \int_{B(y,t)} [\psi_\nabla^*(f)(z)]^r\, dz \lesssim \left(\frac{|x-y|+t}{t}\right)^n \mathcal{M}([\psi_\nabla^*(f)]^r)(x).$$

By this, we conclude that, for all $\lambda \in (n/p,\infty)$, $r = n/\lambda$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$[\psi_T^\lambda(f)(x)]^r \lesssim \mathcal{M}([\psi_\nabla^*(f)]^r)(x),$$

which, together with the same argument as that used in (2.11), further implies that

$$\|\psi_T^\lambda(f)\|_{L^\varphi(\mathbb{R}^n)} \lesssim \|\psi_\nabla^*(f)\|_{L^\varphi(\mathbb{R}^n)}.$$

Thus, by this and (2.13), we have $\|f^*\|_{L^\varphi(\mathbb{R}^n)} \lesssim \|\psi_\nabla^*(f)\|_{L^\varphi(\mathbb{R}^n)}$, which completes the proof of (2.8) and hence Theorem 2.2.2. $\square$

From Theorem 2.2.2, we immediately deduce the following vertical and the non-tangential maximal function characterizations of $H^{\varphi}(\mathbb{R}^n)$, the details being omitted.

Theorem 2.2.3 *Let φ be a growth function as in Definition 1.1.4, and ψ_+^* and ψ_∇^* as in Definition 2.2.1. Then the followings are mutually equivalent:*

(i) $f \in H^{\varphi}(\mathbb{R}^n)$;
(ii) $f \in \mathcal{S}'(\mathbb{R}^n)$ *and* $\psi_+^*(f) \in L^{\varphi}(\mathbb{R}^n)$;
(iii) $f \in \mathcal{S}'(\mathbb{R}^n)$ *and* $\psi_\nabla^*(f) \in L^{\varphi}(\mathbb{R}^n)$.

Moreover, for all $f \in H^{\varphi}(\mathbb{R}^n)$,

$$\|f\|_{H^{\varphi}(\mathbb{R}^n)} \sim \|\psi_+^*(f)\|_{L^{\varphi}(\mathbb{R}^n)} \sim \left\|\psi_\nabla^*(f)\right\|_{L^{\varphi}(\mathbb{R}^n)},$$

where the implicit equivalent positive constants are independent of f.

2.3 Notes and Further Results

2.3.1 The main results of this chapter are from [126]. It worth to point out that there is a gap in the proof of the maximal function characterizations of $H^{\varphi}(\mathbb{R}^n)$ in [126, Theorem 3.6], and we now fix it in Theorem 2.2.2.

2.3.2 Let A be an expansive dilation. Li et al. [122] introduced the anisotropic Hardy space of Musielak-Orlicz type, $H_A^{\varphi}(\mathbb{R}^n)$, via the grand maximal function. They then obtained some real-variable characterizations of $H_A^{\varphi}(\mathbb{R}^n)$ by means of the radial, the non-tangential, or the tangential maximal functions. Finally, they characterized these spaces by anisotropic atomic decompositions. They also obtained the finite atomic decomposition characterization of $H_A^{\varphi}(\mathbb{R}^n)$ and, as an application, they proved that, for a given admissible triplet (φ, q, s), if T is a sublinear operator and maps all (φ, q, s)-atoms with $q < \infty$ (or all continuous (φ, q, s)-atoms with $q = \infty$) into uniformly bounded elements of some quasi-Banach space B, then T can uniquely be extended to a bounded sublinear operator from $H_A^{\varphi}(\mathbb{R}^n)$ to B.

2.3.3 Let $A := -(\nabla - ia) \cdot (\nabla - ia) + V$ be a magnetic Schrödinger operator on $L^2(\mathbb{R}^n)$, $n \geq 2$, where $a := (a_1, a_2, \ldots, a_n) \in L_{\mathrm{loc}}^2(\mathbb{R}^n, \mathbb{R}^n)$ and $0 \leq V \in L_{\mathrm{loc}}^1(\mathbb{R}^n)$. Da. Yang and Do. Yang [216] established the equivalent characterizations of the Musielak-Orlicz-Hardy space $H_A^{\varphi}(\mathbb{R}^n)$, defined by the Lusin area function associated with $\{e^{-t^2 A}\}_{t\in(0,\infty)}$, by means of the Lusin area function associated with $\{e^{-t\sqrt{A}}\}_{t\in(0,\infty)}$, the radial maximal functions or the non-tangential maximal functions associated with $\{e^{-t^2 A}\}_{t\in(0,\infty)}$ and $\{e^{-t\sqrt{A}}\}_{t\in(0,\infty)}$, respectively. The boundedness of the Riesz transforms $L_k A^{-1/2}$, $k \in \{1, 2, \ldots, n\}$, from $H_A^{\varphi}(\mathbb{R}^n)$ to $L^{\varphi}(\mathbb{R}^n)$ was also presented, where L_k is the closure of $\frac{\partial}{\partial x_k} - ia_l$ in $L^2(\mathbb{R}^n)$.

2.3.4 Let $n \geq 3$, Ω be a strongly Lipschitz domain of $\mathbb{R}^n$ and $L_\Omega := -\Delta + V$ a Schrödinger operator on $L^2(\Omega)$ with the Dirichlet boundary condition, where Δ is the Laplace operator and the non-negative potential V belongs to the reverse Hölder class $\mathrm{RH}_{q_0}(\mathbb{R}^n)$ for some $q_0 > n/2$. Assume the uniformly critical lower type index $i(\varphi)$ of the growth function satisfies $i(\varphi) \in (\frac{n}{n+\delta}, 1]$, where $\delta := \min\{\mu_0, 2 - \frac{n}{q_0}\}$ and $\mu_0 \in (0, 1]$ denotes the critical regularity index of the heat kernels of the Laplace operator Δ on Ω. Chang et al. [36] showed that the heat kernels of L satisfy the Gaussian upper bound estimates and the Hölder continuity. They then introduced the geometrical Musielak-Orlicz-Hardy space $H_{\varphi,L_{\mathbb{R}^n},r}(\Omega)$ via $H_{\varphi,L_{\mathbb{R}^n},r}(\mathbb{R}^n)$, the Hardy space associated with $L_{\mathbb{R}^n} := -\Delta + V$ on $\mathbb{R}^n$, and established its several equivalent characterizations, respectively, by means of the non-tangential or the vertical maximal functions or the Lusin area functions associated with L.

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